# SUPPORT THEOREMS FOR CERTAIN RAY TRANSFORMS

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# Abstract

The main object of study in this thesis are integral transforms over simple, real analytic, Riemannian manifolds.

Let $(M,g)$ be a simple, real analytic, Riemannian manifold with boundary and of dimension $n \geq 3$. The first result that we present here establishes a support theorem for the transverse ray transform of tensor fields of rank 2 defined over such manifolds. More specifically, given a symmetric tensor field $f$ of rank 2, we show that if the transverse ray transform of $f$ vanishes over an appropriate open set of maximal geodesics of $M$, then the support of $f$ vanishes on the points of $M$ that lie on the union of the aforementioned open set of geodesics.

The second problem concerns integral moments transform of a symmetric $m$-tensor field on a simple, real analytic, Riemannian manifold as above. Integral moments of m-tensor field were first introduced by Sharafutdinov. At first we generalize a Helgason type support theorem proven by Krishnan and Stefanov in "A support theorem for the geodesic ray transform of symmetric tensor fields", Inverse Problems and Imaging, 3(3):453-464,2009. We use this extended result along with the first $m+1$-integral moments of the $m$-tensor field to prove an injectivity result and support theorem for such transforms.

Dedication

To my parents.

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# Contents

SUPPORT THEOREMS FOR CERTAIN RAY TRANSFORMS

# Chapter 1

# Introduction

The objects of study in this dissertation are the *transverse ray transform* of a symmetric tensor field of rank 2 and the *integral moments transform* of a symmetric tensor field of arbitrary rank. Let $f$ be a compactly supported symmetric tensor field of rank 2, then the transverse ray transform of such fields in $\mathbb{R}^n$ is given by:

$$Jf(x,\xi,\eta) = \int_{\mathbb{R}} f_{ij}(x+t\xi)\eta^i\eta^j\,dt. \tag{1.1}$$

Here, $x+t\xi$ is the equation of a line passing through point $x$ and in the direction $\xi$ and $\eta$ is a vector field which is perpendicular to the line for all $t$. More generally, we can study such transforms in the setting of Riemannian manifolds where lines will be replaced by geodesics $\gamma(t)$ and the vector field $\eta(t)$ will be such that it is a vector field whose covariant derivative along the geodesic vanishes and $\langle \eta(t), \dot{\gamma}(t)\rangle = 0$ for all $t$. Such transforms arise quite naturally in the study of polarization tomography, see [30, Chapter 5].

On the other hand, the *integral moments transform* of a compactly supported symmetric tensor field $f$ of rank $m$ over a set of lines in $\mathbb{R}^n$ is given by, see [29]:

$$If(x,\xi) = \int_{\mathbb{R}} t^q \langle f(x+t\xi), \xi^m(t)\rangle dt = \int_{\mathbb{R}} t^q f_{i_1\dots i_m}(x+t\xi)\xi^{i_1}\cdots\xi^{i_m}(t)\,dt. \tag{1.2}$$

When $q = 0$, the integral moments transform reduce to the well known *X- ray transform*. In this thesis, we will study the integral moments transform in the more general setting of a Riemannian manifold, with lines replaced by geodesics.

An object of ubiquitous interest to mathematicians and scientists alike is what is referred to as "Support theorem for ray transforms". Loosely the support theorems for the ray transforms have the following basic structure: *If we know that a ray transform (e.g transverse ray transform) of a symmetric tensor field vanishes over* *an appropriate open set of geodesics, does the tensor field itself vanish over the set of points that lie on the union of geodesics that belong to this open set of geodesics?* Having a support theorem for an integral transform of a function or a tensor field tells us that we can reconstruct the desired function or the tensor field in the exterior of a given region solely by tomographic measurements in the exterior of the given region. We also note that injectivity results for such transforms follows as a result of the more general support theorems.

In this thesis, we consider the following problems related to these transforms:

- *Support theorem for transverse ray transform of rank 2 tensor fields*: In this work, we prove a support theorem for transverse ray transform of symmetric 2-tensor fields defined on a compact, simple, real analytic Riemannian manifold.
- *Support theorems and an injectivity result for integral moments of symmetric $m$-tensor field*: In this joint work with Dr. Rohit Kumar Mishra, a postdoctoral researcher at University of California, Santa Cruz, we prove an injectivity result and support theorems for the integral moments tranform defined on a compact, real analytic Riemannian manifold. This work also forms a part of Dr. Mishra's dissertation.

The organization of this thesis is as follows: We state definitions and basic concepts that we need for this thesis in Chapter 2. In Chapter 3, we will provide a brief overview about integral transforms of the kind considered in this work, in Chapter 4 we state and prove a support theorem for the transverse ray transform and in Chapter 5 we state and prove an injectivity result and support theorems for the integral moments transform.

# Chapter 2

# Mathematical preliminaries

In this chapter we develop concepts and notations that will be needed to prove our results.

## 2.1 Tensor fields on a manifold

For the presentation in this subsection, we follow the classic texts of Boothby [7], Lee [21] and also the book by Sharfutdinov on "Integral geometry of tensor fields" [30].

Let $M$ be a $C^\infty$ manifold of dimension $n$, $p \in M$ be any arbitrary point on the manifold and $U \subset M$ be an open subset. $C^\infty(p)$ are the germs of $C^\infty$ functions which are defined on some neighbourhood of $p$ such that functions are identical on their common domain of definition. The algebra of smooth functions on $U$ will be denoted by $C^\infty(U)$. We can give an algebraic definition of the tangent space and tangent vectors as follows:

**Definition 2.1.1** *[7, Chapter IV, Definition 1.1] The tangent space $T_p(M)$ to $M$ at $p$ is the set of all mappings $v_p : C^\infty(p) \to \mathbb{R}$ satisfying for all $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $f \in C^\infty(p)$, $g \in C^\infty(p)$, the following two conditions:*

1. $v_p(\alpha f + \beta g) = \alpha(v_p f) + \beta(v_p g)$ *(Linearity)*
2. $v_p(fg) = (v_p f)g + f(v_p g)$ *(Leibniz Rule)*

*with the vector space operations in $T_p(M)$ given by:*

1. $(v_p + w_p)f = v_p f + w_p f$
2. $(\alpha v_p)f = \alpha(v_p f)$

*The mapping $v_p \in T_p(M)$ is called a tangent vector.*

For every local coordinate system near $p \in M$ given by $(U; x^1, \ldots, x^n)$, there exists a natural basis for the $T_p(M)$, $\partial_i|_p = \frac{\partial}{\partial x^i}|_p$ for $1 \le i \le n$. This is defined by the equality $\partial_i|_p f = \frac{\partial f}{\partial x^i}(p)$ for any smooth function $f$. Therefore, every tangent vector $v_p = (v^1, \ldots, v^n) \in T_p M$ can be represented uniquely as:

$$v_p = v^i \frac{\partial}{\partial x^i}|_p.$$

**Definition 2.1.2** *[7, Chapter 4, Definition 2.1] A vector field $v$ on $M$ is a function assigning to each point $p \in M$, a vector $v_p \in T_p(M)$. In the domain $U$ of a local coordinate system, for any $p \in U$ we have, $v(p) = v^i(p)\frac{\partial}{\partial x^i}|_p$. A vector field $v$ is said to be smooth if the component functions $v^i \in C^\infty(U)$.*

Under a change of basis, vector fields transform in a special way. This transformation law can be used to give an equivalent definition for the vector fields. Let $v = \tilde{v}^i \frac{\partial}{\partial \tilde{x}^i}$ in a different local coordinate system $(\tilde{U}; \tilde{x}^1, \ldots, \tilde{x}^n)$ near $p$ on $M$. Then in $U \cap \tilde{U}$, the components of $v$, which can be found by a change of variables, are given by the following equality:

$$\tilde{v}^i = \frac{\partial \tilde{x}^i}{\partial x^j} v^j. \tag{2.1}$$

Now we define a vector field $v$ on a manifold $M$ to be an assignment which assigns $n$ functions $v^i \in C^\infty(U)(1 \le i \le n)$ to every local coordinate system $(U; x^1, \ldots, x^n)$ such that under a change of coordinates they are transformed by the above formula (2.1).

It is possible to define the vector fields in yet another manner. Let $\tau_M = \bigcup_{p \in M} T_p M$ be the tangent bundle of $M$. Let the canonical projection map from the tangent bundle to the underlying manifold $M$ be given by: $p : \tau_M \to M$. A smooth section of $\tau_M$ is a $C^\infty$ mapping $F : M \to \tau_M$ such that $p \circ F = id|_M$. Then the $C^\infty$ vector fields can also be thought of as smooth sections of $\tau_M$. By $C^\infty(\tau_M)$, we represent is the space of smooth sections of the tangent bundle, i.e. the space of smooth vector fields on $M$. And $C^\infty(\tau_M;U)$ denotes the set of all vector fields on an open subset $U\subset M$. It can be easily shown that for a local coordinate system $(U;(x^1,\dots,x^n))$, the coordinate vector fields $\partial_i=\frac{\partial}{\partial x^i}(1\le i\le n)$ forms a basis of $C^\infty(\tau_M;U)$.

A *covector field* $v$ on a manifold $M$ is an assignment which assigns $n$ functions $v_i\in C^\infty(U)(1\le i\le n)$ to every local coordinate system $(U;x^1,\dots,x^n)$ such that under a change of coordinates they are transformed by the formula:

$$\tilde{v}_i=\frac{\partial x^j}{\partial\tilde{x}^i}v_j. \tag{2.2}$$

By $\tau'_M$, we denote the cotangent bundle of $M$ and $C^\infty(\tau'_M)$ denotes the set of all covector fields. Like we did for vector fields, one can understand covector fields as smooth sections of the cotangent bundle. Let $(U;x^1,\dots,x^n)$ be a local coordinate system, then it can be easily checked, that for a given function $f\in C^\infty(M)$, the differentials $(df)_i=\partial_i f$ transform according to the transformation law 2.2 and not 2.1. This shows that differentials are covector fields rather than vector fields. In particular, the differentials $dx^i\in C^\infty(\tau'_M;U)$ of the coordinate functions $x^i$ can be shown to form a basis of $C^\infty(\tau'_M;U)$. These covector fields are called *coordinate covector fields.*

One can generalize the notions discussed above to define what are called mixed tensor fields of arbitrary order. Among the above definitions for vector fields, the one definition which we would especially like to generalize with an eye to the computations done in this thesis is the transformation law. The notation and descriptions in this section follow closely the book by Sharafutdinov, [30]. Hence, a *mixed tensor field* of order $(r,s)$ on $M$, for any two nonnegative integers $r$ and $s$, is defined as a rule associating functions $v^{i_1\dots i_r}_{j_1\dots j_s}\in C^\infty(U)$(all indices vary from 1 to $n$) to every local coordinate system $(U;x^1,\dots,x^n)$ which satisfy the following transformation rule under a change of coordinates:

$$\tilde{v}^{i_1\dots i_r}_{j_1\dots j_s}=\frac{\partial\tilde{x}^{i_1}}{\partial x^{k_1}}\cdots\frac{\partial\tilde{x}^{i_r}}{\partial x^{k_r}}\frac{\partial x^{l_1}}{\partial\tilde{x}^{j_1}}\cdots\frac{\partial x^{l_s}}{\partial\tilde{x}^{j_s}}v^{k_1\dots k_r}_{l_1\dots l_s}. \tag{2.3}$$

The set of all tensor fields of degree $(r,s)$ on an open subset $U$ of $M$ is denoted by $C^\infty(\tau^r_s M;U)$ and $C^\infty(\tau^r_s M)$ will denote the set of tensor fields on $M$. A tensor field $v \in C^\infty(\tau^r_s M)$ is said to be $r$ times contravariant and $s$ times covariant. Tensor fields of degrees $(0,0)$, $(1,0)$ and $(0,1)$ are smooth functions, vector fields and covector fields respectively.

The *tensor product* $v \otimes w \in C^\infty(\tau^{r+r'}_{s+s'} M)$ of $v \in C^\infty(\tau^r_s M)$ and $w \in C^\infty(\tau^{r'}_{s'} M)$ is defined as

$$(v \otimes w)^{i_1 \dots i_{r+r'}}_{j_1 \dots j_{s+s'}} = v^{i_1 \dots i_r}_{j_1 \dots j_s} w^{i_{r+1} \dots i_{r+r'}}_{j_{s+1} \dots j_{s+s'}}.$$

Let $(U; x^1, \dots, x^n)$ be a local coordinate system on $M$, then a basis for $C^\infty(\tau^r_s M)$ is given by: $\{\frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}\}$ where each index ranges over $\{1, \dots, n\}$. Hence any $v \in C^\infty(\tau^r_s M;U)$ can be represented as

$$v = v^{i_1 \dots i_r}_{j_1 \dots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}$$

where the component functions $v^{i_1 \dots i_r}_{j_1 \dots j_s} \in C^\infty(U)$ and *Einstein summation convention* of adding over repeated indices is assumed, e.g for $f \in \tau^1_1 M$, $f = \sum_i \sum_j f_{ij} \frac{\partial}{\partial x^i} \otimes dx^j$ will be briefly written as $f = f_{ij} \frac{\partial}{\partial x^i} \otimes dx^j$ and summing over indices $i$ and $j$ will be implicitly assumed. If there is no confusion about the coordinates, we will simply write:

$$v = (v^{i_1 \dots i_r}_{j_1 \dots j_s}).$$

The spaces $C^\infty(\tau^r_s M)$ and $C^\infty(\tau^s_r M)$ can be thought of as mutual dual to each other with respect to the pairing $\langle v, w \rangle = v^{i_1 \dots i_r}_{j_1 \dots j_s} w^{j_1 \dots j_s}_{i_1 \dots i_r}$. In particular this implies that a covariant tensor field $v \in C^\infty(\tau^0_s M)$ can be also thought as a $C^\infty(M)$-multilinear mapping over $s$ copies of $C^\infty(\tau_M)$, i.e. $v : C^\infty(\tau_M) \times \dots \times C^\infty(\tau_M) \to C^\infty(M)$. Finally, the *trace* or the *contraction* over indices $k, l$ denoted as

$$C^k_l : C^\infty(\tau^r_s M) \to C^\infty(\tau^{r-1}_{s-1} M)$$

for $1 \le k \le r$ and $1 \le l \le s$ is defined by the equality

$$(C_l^k v)^{i_1 \dots i_{r-1}}_{j_1 \dots j_{s-1}} = \sum_{p=1}^{n} v^{i_1 \dots i_{k-1} p i_k \dots i_{r-1}}_{j_1 \dots j_{l-1} p j_l \dots j_{s-1}}.$$

## 2.2 Riemannian Manifolds

**Definition 2.2.1** *[21, Chapter 3] A Riemannian metric on a smooth manifold $M$ is a tensor field $g \in C^\infty(\tau_2 M)$ that has the following properties:*

1. *As a bilinear map, $g : C^\infty(\tau_M) \times C^\infty(\tau_M) \to C^\infty(M)$, $g$ is symmetric i.e., $g(\xi, \eta) = g(\eta, \xi)$,*

2. *$g$ is positive definite i.e., $g(\xi, \xi) > 0$ when $\xi \neq 0$.*

*A Riemanniamn metric determines an inner product on $T_pM$, $\langle \xi, \eta \rangle = g(\xi, \eta) = g_{ij}\xi^i \eta^j$. A smooth manifold with a Riemannian metric will be called a Riemannian manifold.*

Using the inner product structure introduced on the manifold by the Riemannian metric, one can define canonical isomorphisms: $C^\infty(\tau_s^r M) \simeq C^\infty(\tau_0^{r+s} M) \simeq C^\infty(\tau_{r+s}^0 M)$. As a result, we will not distinguish between covariant and contravariant tensor fields on the Riemannian manifold and will think of these as different representations of the same tensor field. In particular, for a tensor field $f$ of rank $m$, $f^{i_1 \dots i_m}$ and $f_{i_1 \dots i_m}$ will be thought of as equivalent representations of the same tensor field which are related by the following equation: $f_{i_1 \dots i_m} = g_{i_1 j_1} \dots g_{i_m j_m} f^{j_1 \dots j_m}$. In fact, the inner product can be extended as a bilinear mapping:

$$g : C^\infty(\tau_m^0 M) \times C^\infty(\tau_m^0 M) \to C^\infty(M), \quad g(u, v) = \langle u, v \rangle = u_{i_1 \dots i_m} v^{i_1 \dots i_m}.$$

This enables us to define an inner product on the space of compactly supported tensor fields, $C_c^\infty(\tau_m^0 M)$ in the following manner:

$$(f, g)_{L_2(\tau_m^0(M))} = \int_M \langle f, g \rangle dV^n(x) \tag{2.4}$$

where, $dV^n(x)$ is the volume form on the manifold.

## 2.3 Covariant Differentiation

*Connections* on $M$ provide a way to compute covariant derivatives.

**Definition 2.3.1** *[21, Chapter 4] A linear connection on $M$ is a mapping: $\nabla : C^\infty(\tau_M) \times C^\infty(\tau_M) \to C^\infty(\tau_M)$, sending a pair of vector fields $u$ an $v$ to a third field $\nabla_u v$ such that it satisfies the following properties:*

1. *$\nabla_u v$ is linear over $C^\infty(M)$ in $u$: $\nabla_{fu_1+gu_2} v = f\nabla_{u_1} v + g\nabla_{u_2} v$.*

2. *$\nabla_u v$ is $\mathbb{R}-$ linear in $v$: $\nabla_u(av_1 + bv_2) = a\nabla_u v_1 + b\nabla_u v_2$.*

3. *(Product Rule): $\nabla_u(fv) = f\nabla_u v + (uf)v$ for any $f \in C^\infty(M)$.*

*$\nabla_u v$ is called the **covariant derivative** of $v$ in the direction of $u$.*

The concept of connections and covariant derivatives having been defined for vector fields can be extended to tensor fields of arbitrary order.

**Theorem 2.3.2** *[21, Lemma 4.6] Given a connection $\nabla$, there exist uniquely determined $\mathbb{R}$-linear mappings*

$$\nabla : C^\infty(\tau^r_s M) \to C^\infty(\tau^r_{s+1} M) \tag{2.5}$$

*for all integers $r$ and $s$, such that*

*(a) $\nabla_u f = uf$, for $f \in C^\infty(M) = C^\infty(\tau^0_0 M)$.*

*(b) On $\tau_M$, $\nabla$ agrees with the given connection.*

*(c) The operator $\nabla$ commutes with the operation of taking the "trace" over any pair of indices, i.e. $\nabla_u(\operatorname{tr} v) = \operatorname{tr}(\nabla_u v)$.*

*(d) $\nabla$ obeys the product rule with respect to tensor products:*

$$\nabla_u(v \otimes w) = (\nabla_u v) \otimes w + v \otimes \nabla_u w.$$

If $(U; x^1, \dots, x^n)$ is a local coordinate system defined on $M$, then we define the Christoffel symbols of the connection $\nabla$ in the following way:

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k. \tag{2.6}$$

Let $\nabla$ be any linear connection. In local coordinates, the components of the total covariant derivative of a $(r,s)$- tensor field $F$ is given by:

$$\nabla_m F^{j_1\dots,j_r}_{i_1\dots i_s} = \partial_m F^{j_1\dots,j_r}_{i_1\dots i_s} + \sum_{k=1}^{r} F^{j_1\dots p\dots j_r}_{i_1\dots i_s}\Gamma^{j_k}_{mp} - \sum_{k=1}^{s} F^{j_1\dots j_r}_{i_1\dots p\dots i_s}\Gamma^{p}_{mi_k} \tag{2.7}$$

$\nabla_m F^{j_1\dots,j_r}_{i_1\dots i_s}$ will alternatively be denoted by $F^{j_1\dots,j_r}_{i_1\dots i_s;m}$

On Riemannian manifolds there is a unique connection called the *Levi-Civita connection* that is compatible with the metric, i.e. $u\langle\xi,\eta\rangle = \langle\nabla_u\xi,\eta\rangle + \langle\xi,\nabla_u\eta\rangle$. The Christoeffel symbols of this connection are given by:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kp}\big(\partial_i g_{jp} + \partial_j g_{ip} - \partial_p g_{ij}\big). \tag{2.8}$$

For the rest of this thesis, whenever we talk of connections and covariant differentiation in the context of Riemannian manifolds, we will assume this metric compatible connection.

## 2.4 Geodesics

A smooth, parametrized curve on a manifold is a smooth map $\gamma : I \to M$ where $I \subset \mathbb{R}$ is an interval. In local coordinates, the curve is given by $x^i = \gamma^i(t)$, for $i = \{1,\dots,n\}$. A smooth vector field along a curve $\gamma : I \to M$ is a smooth map $u : I \to \tau_M$ such that $u(t) \in T_{\gamma(t)}M$. In particular, the velocity vector field $\dot\gamma(t) = \dot\gamma^i(t)\partial_i$ of a curve is a vector field along it as $\dot\gamma(t) \in T_{\gamma(t)}M$. We will denote by $\tau_\gamma$, the space of vector fields along the curve $\gamma$. We can get a large number of vector fields along any curve by the following construction: Let $\gamma : I \to M$ be any curve and consider any vector field $\tilde v \in \tau_M$. For each $t \in I$, let $v(t) = \tilde v_{\gamma_t}$. One can easily check that $v$ is smooth.

A vector field $v$ along $\gamma$ is said to be *extendible* if there exists a vector field $\tilde{v}$ on $M$ related to $v$ in the above manner. The *covariant differentiation along a curve* is given by the following lemma:

**Lemma 2.4.1** *[21, Lemma 4.9] Let $\nabla$ be a linear connection on $M$. For each curve $\gamma : I \to M$, $\nabla$ determines a unique operator*

$$D_t : \tau_\gamma \to \tau_\gamma$$

*satisfying the following properties:*

1. *Linearity over $\mathbb{R}$:*

$$D_t(aV + bW) = aD_tV + bD_tW \qquad a, b \in \mathbb{R}$$

2. *Product Rule:*

$$D_t(fV) = \dot{f}V + fD_tV \quad \text{for } f \in C^\infty(I)$$

3. *If $V$ is extendible, then for any extension $\tilde{V}$ of $V$,*

$$D_tV(t) = \nabla_{\dot{\gamma}(t)}\tilde{V}$$

*For any $V \in \tau_\gamma$, $D_tV$ is called the covariant derivative of $V$ along $\gamma$.*

In particular, one can show that the following relation holds in any choice of a local coordinate system [21, equation 4.10]:

$$D_tV(t_0) = \big(\dot{V}^k(t_0) + V^j(t_0)\dot{\gamma}^i(t_0)\Gamma^k_{ij}(\gamma(t_0))\big)\partial_k \tag{2.9}$$

Now we are ready to define *geodesics*. We will call a curve a geodesic of the manifold, if its *acceleration* given by $D_t\dot{\gamma} \equiv 0$

**Theorem 2.4.2 (Existence and Uniqueness of Geodesics)** *[21, Theorem 4.10] Let $M$ be a manifold with a connection $\nabla$. For any point $p \in M$, any $V \in T_pM$ and* *for any* $t_0 \in \mathbb{R}$*, there exists an open interval* $I \subset \mathbb{R}$ *containing* $t_0$ *and a geodesic* $\gamma : I \to M$ *satisfying* $\gamma(t_0) = p, \dot{\gamma}(t_0) = V$*. Any two such geodesics agree on their common domain.*

*Proof:* Let us choose a local coordinate system $(U; x^1, \dots, x^n)$ near any point $p \in M$. A curve $\gamma : I \to M$ is a geodesic iff it satisfies $D_t \dot{\gamma} \equiv 0$. By equation (2.9), this means that the component functions $\gamma(t) = (x^1(t), \dots, x^n(t))$ satisfy:

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}(x(t)) = 0. \tag{2.10}$$

The above equation is often referred to as the ***geodesic equation***. Note that the geodesic equation is a second order system of ODEs. We can recast it as a first order system by introducing auxilliary variables $v^i = \dot{x}^i$ and rewriting the above as:

$$\begin{aligned} \dot{x}^k(t) &= v^k(t), \\ \dot{v}^k(t) &= -v^i(t)v^j(t)\Gamma^k_{ij}(x(t)) \end{aligned}$$

By the theorem of existence and uniqueness of solution for first order ODEs, for any given $(p, V) \in U \times \mathbb{R}^n$, there exists an $\epsilon > 0$ and a unique solution $\eta : (t_0 - \epsilon, t_0 + \epsilon) \to U \times \mathbb{R}^n$ to the first order ODE system with initial value $\eta(t_0) = (p, V)$. If we write $\eta(t) = (x(t), v(t)$, then the curve $\gamma(t) = (x^1(t), \dots, x^n(t)$ is easily seen to satisfy the existence claim in the theorem.

Uniqueness can be proved by the method of contradiction. Suppose there are two geodesics $\gamma$ and $\sigma$ defined on the open interval $I$ such $\gamma(t_0) = \sigma(t_0)$ and $\dot{\gamma}(t_0) = \dot{\sigma}(t_0)$. Due to the uniqueness of the solution for first order ODEs, they must agree on some open interval containing the point $t_0$. Let $\beta$ be the supremum of numbers $b$ such that they agree on $[t_0, b]$. If $\beta \in I$, then by continuity $\sigma(\beta) = \gamma(\beta)$ and $\dot{\gamma}(\beta) = \dot{\sigma}(\beta)$. Now, by existence and uniqueness of solution of the first order system in a neighbourhood of $\beta$, this means that $\gamma$ and $\sigma$ agree on a slightly larger neighbourhood thus contradicting our assumption. □

**Example 2.4.3 (Geodesics in Euclidean space)** In the Euclidean space of dimension $n$, Christoeffel symbols vanish by equation 2.8 because the Euclidean metric is a constant everywhere. If $\gamma(t) = (x^1(t), \dots, x^n(t))$, then the geodesic equation in this case is written as

$$\ddot{x}^k(t) = 0.$$

With the initial conditions: $\gamma(0) = p$ and $\dot{\gamma}(0) = v$, we get the solution to the geodesic equation as $x^k(t) = p^k + tv^k$. This shows that geodesics in Euclidean space are just straight lines. ◊

The above theorem also helps us in establishing that for any $p \in M$ and $V \in T_pM$, there exists a unique geodesic $\gamma : I \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = V$ which is maximal in the sense that it can not be extended to a larger interval. To get the maximal interval, we just take the union of all such intervals on which the geodesic with the given initial conditions is defined.

## 2.5 Parallel Translation

Let $M$ be a manifold with a given linear connection $\nabla$. A vector field $v$ along any curve $\gamma$ is said to be parallel along the curve iff $D_t v = 0$. In local coordinates, if $v$ is parallel along a curve $\gamma$, then it satisfies the following linear ODE system:

$$\dot{v}^k(t) = -v^j(t)\dot{\gamma}^i(t)\Gamma^k_{ij}(\gamma(t)) \quad \text{for } k = \{1, \dots, n\}. \tag{2.11}$$

One can easily check that the velocity vector field of a geodesic is translated in a parallel manner along it.

We have the following existence and uniqueness theorem about parallel translations.

**Theorem 2.5.1** *Given a curve $\gamma : I \to M$, $t_0 \in I$ and any vector $V_0 \in T_{\gamma(t_0)}M$, there exists a unique parallel vector field $V$ along $\gamma$ such that $V(t_0) = V$.*

For the proof of the above theorem, we refer to the reader [21, Theorem 4.11].

## 2.6 Exponential Map

By theorem 2.4.2, we already know that given a point $p \in M$ and a vector $v \in T_pM$, there exists a unique maximal geodesic $\gamma_{(p,v)}$ that satisfies the initial conditions $\gamma_{(p,v)}(0) = p$ and $\dot{\gamma}_{(p,v)}(0) = v$. This introduces a mapping between the tangent bundle $\tau_M$ and the set of geodesics. In fact, it allows us to define a map from a subset of $\tau_M$ to $M$ such that $(p,v) \mapsto \gamma_{(p,v)}(1)$. To make these notions more precise, let us introduce the following subset

$$\mathcal{G} := \{(p,v) \in \tau_M : \gamma_{(p,v)} \text{ is defined on } [0,1]\}$$

Now we can define the *exponential map* $\exp : \mathcal{G} \to M$ by

$$\exp(p,v) = \gamma_{(p,v)}(1)$$

**Lemma 2.6.1** *[21, Lemma 5.8] For any $(p,v) \in \tau_M$ and $c,t \in \mathbb{R}$, $\gamma_{(p,cv)}(t) = \gamma_{(p,v)}(ct)$ whenever both sides are defined.*

*Proof:* We will first show that $\gamma_{(p,cv)}(t)$ exists and the equality stated in the lemma holds if the right hand side is defined. To show the converse, we can then replace $(p,v)$ by $(p,cv)$, $t$ by $ct$ and $c$ by $\frac{1}{c}$. To prove the forward direction we will invoke the theorem of existence and uniqueness of geodesics. We will first define a curve $\tilde{\gamma}$ such that $\tilde{\gamma}(t) = \gamma_{(p,v)}(ct)$ and defined on the interval $c^{-1}I : \{t : ct \in I\}$. We will show that $\tilde{\gamma}$ satisfies the geodesic equation and the initial conditions and then by uniqueness we will get $\tilde{\gamma} \equiv \gamma_{c(p,v)}$. Note that $\tilde{\gamma}(0) = \gamma_{(p,v)}(0) = p$. Also $\dot{\tilde{\gamma}}^i(t) = \frac{d}{dt}\gamma^i(ct) = c\dot{\gamma}^i(ct)$. Hence, $\dot{\tilde{\gamma}}(0) = cv$.

Let $D_t$ and $\tilde{D}_t$ represent the covariant differentiation along $\gamma$ and $\tilde{\gamma}$ respectively.

$$\begin{aligned}
\tilde{D}_t\dot{\tilde{\gamma}}(t) &= \Big(\frac{d}{dt}\tilde{\gamma}^k(t) + \Gamma^k_{ij}(\tilde{\gamma}(t))\tilde{\gamma}^i(t)\tilde{\gamma}^j(t)\Big)\partial_k \\
&= \Big(c^2\ddot{\gamma}^k(ct) + \Gamma^k_{ij}(\gamma(ct))\dot{\gamma}^i(ct)\dot{\gamma}^j(ct)\Big)\partial_k
\end{aligned}$$

$$= c^2 D_t \dot{\gamma}(ct)$$

$$= 0$$

Hence $\tilde{\gamma}$ is a geodesic and must be equal to $\gamma_{c(p,v)}$ by uniqueness. □

Now we will state without proof, the properties of the exponential map:

**Theorem 2.6.2** *[21, Proposition 5.7]*

1. *$\mathcal{G}$ is an open subset of $\tau_M$ containing the zero section and for each point $p \in M$, the restriction $\mathcal{G}_p = \mathcal{G} \cap T_pM$ is star shaped with respect to 0.*

2. *For any $(p,v) \in \tau_M$, $\gamma_{(p,v)}(t) = \exp(p, tv)$ for all $t$ such that both sides are defined.*

3. *The exponential map is smooth.*

## 2.7 Normal Neighbourhood

A subset $\mathcal{S}$ of a vector space is called star shaped about the origin, if $v \in \mathcal{S} \implies tv \in \mathcal{S}, \quad \forall t \in [0,1]$, i.e. $\mathcal{S}$ is the union of radial line segments. Any open neighbourhood $\mathcal{U}$ of $p \in M$ that is diffeomorphic to a star shaped neighbourhood $\mathcal{V}$ of origin in $T_pM$ is called a *normal neighbourhood*. Let $\epsilon > 0$ be such that the map $\exp_p$ is a diffeomorphism on the ball $B_\epsilon(0) \subset T_pM$, then the image set $\exp_p(B_\epsilon(0))$ is called a *geodesic ball* in $M$. Furthermore, if the closed ball $\bar{B}_\epsilon(0)$ is contained in the set $\mathcal{V}$ for which $\exp_p$ is a diffeomorphism, then we call the image set $\exp_p(\bar{B}_\epsilon(0))$ as *closed geodesic ball* and the image set $\exp_p(\partial\bar{B}_\epsilon(0))$ as the *geodesic sphere*.

Consider the restricted exponential map $\exp_p : \mathcal{G}_p \to M$. We have the following lemma for *normal neighbourhoods*.

**Lemma 2.7.1 (Normal Neighbourhood Lemma)** *[21, Lemma 5.10] For any $p \in M$, there is a neighbourhood $\mathcal{V}$ of the origin in $T_pM$ and a neighbourhood $\mathcal{U}$ of $p \in M$ such that the exponential map $\exp_p : \mathcal{V} \to \mathcal{U}$ is a diffeomorphism.*

*Proof:* We will show that the push forward $(\exp_p)_*$ is invertible at 0 and then by an application of inverse function theorem we conclude the proof of the lemma. First of all, note that there is a natural identification between the vector spaces $T_pM \sim T_0(T_pM)$. Under this identification, it is easy to show that $(\exp_p)_* : T_0(T_pM) = T_pM \to T_pM$ is identity. To that end, consider a curve $c \subset T_pM$ starting at 0 in the direction $v$ given by $c(t) = tv$. Then

$$\begin{aligned}(\exp_p)_* v &= \frac{d}{dt}|_{t=0} \exp_p \circ c(t) \\ &= \frac{d}{dt}|_{t=0} \exp_p(tv) \\ &= \frac{d}{dt}|_{t=0} \gamma_{(p,v)}(t) \\ &= v\end{aligned}$$

Hence $(\exp_p)_*$ is identity and in particular invertible at 0. □

## 2.8 Normal Coordinates

$T_pM$ is a vector space of dimension $n$. Let us choose a basis of $T_pM$ as $\{E_i\}_{i=1}^n$. Then we have the follwoing isomorphism between $E : \mathbb{R}^n \to T_pM$ given by: $E(x) = x^i E_i$. This isomorphism can be used to define a coordinate chart for the normal neighbourhoods in the following way. Let $\mathcal{U}$ be a norpmal neighbourhood around $p \in M$. We define a local coordinate system on $\mathcal{U}$ by $\phi := E^{-1} \circ \exp_p^{-1} : \mathcal{U} \to \mathbb{R}^n$. In any normal neighbourhood centered at $p \in M$, we will call $r(x) = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ as the radial distance function and $\frac{\partial}{\partial_r} = \frac{x^i}{r}\frac{\partial}{\partial_{x_i}}$ as the radial vector field. It is clear that for Euclidean space, $r(x)$ is simply the distance of the point to the origin and $\frac{\partial}{\partial_r}$ is the radial vector field tangent along the radial straight lines. One can also show that [21, Proposition 5.11], if $(\mathcal{U}; y^i)$ is a normal coordinate chart centered at $p \in M$ then:

1. In this chart, $p = (0, \dots, 0)$.

2. Any geodesic $\gamma_{(p,v)}$ is given in these coordinates by: $\gamma_{p,v}(t) = (tv^1, \dots, tv^n)$ as

long as the geodesic stays within $\mathcal{U}$.

## 2.9 Simple Riemannian Manifolds

**Definition 2.9.1 (Simple Manifold)** *A compact Riemannian manifold (M,g) with smooth boundary is said to be simple if:*

*(i) The boundary of the manifold $\partial M$ is strictly convex: $\langle \nabla_\xi \nu(p), \xi \rangle > 0$ for all $\xi \in T_p(\partial M)$ and where $\nu(p)$ is the unit outward pointing normal at $p \in U \cap \partial M$ such that $(U, x^1, \ldots x^n)$ is a boundary chart containing the point $p$.*

*(ii) The map $\exp_p : \exp_p^{-1}(M) \to M$ is a diffeomorphism for all $p \in M$.*

Here the second condition implies that any two points $x$ and $y$ in the manifold $M$ are connected by a unique geodesic which depends smoothly on the points $x$ and $y$.

**Theorem 2.9.2** *[30] Every simple manifold is diffeomorphic to the closed unit ball in $\mathbb{R}^n$*

*Proof:* [15] Let $p \in \text{int}(M)$ and $x \in \partial M$. Note that $M$ is diffeomorphic to to $\exp^{-1}(M) \subset T_pM$. Let $v \in \exp_p^{-1} M$ be the unique vector such that $\exp_p(v) = x$. By the rescaling lemma 2.6.1, $\forall t \in [0,1]$ we have that $tv \in \exp_p^{-1} M$. Now consider any vector $u$ in the sphere bundle at $p$. Since $p \in \text{int}(M)$, so there exists a scalar multiple of every such vector in $\exp_p^{-1} M$, i.e. for any $u$ we let $r > 0$ such that $\exp_p(ru) \in \partial M$. This shows that $\exp_p^{-1} M$ is in fact a star shaped neigbourhood of $T_pM \sim \mathbb{R}^n$ with smooth boundary. This establishes that $M$ is diffeomorphic to a star shaped neighbourhood of $\mathbb{R}^n$ (with smooth boundary) which in turn is diffeomorphic to a ball in $\mathbb{R}^n$. $\square$

Due to the above theorem, henceforth we can assume that the manifold is some domain in $\mathbb{R}^n$.

**Definition 2.9.3** *A compact Riemannian manifold $(M, g)$ is said to be non-trapping if its boundary is strictly convex and all maximal geodesics are of finite length.*

The end-points of the maximal geodesic lie on the boundary of any such manifold in the following sense: If the maximal geodesic is defined on the interval $\gamma : [l^-, l^+] \to M$, then $\gamma(l^-), \gamma(l^+) \in \partial M$. This allows us to parametrize the maximal geodesics in $M$ by points on the following set:

$$\Gamma_- := \left\{(x, \xi) \in \tau_M \text{ such that } x \in \partial M, |\xi| = 1, \langle \xi, \nu(x) \rangle < 0\right\},$$

where $\nu(x)$ is the outer unit normal to $\partial M$ at $x$.

**Proposition 2.9.4** *A simple manifold $M$ is non-trapping.*

*Proof:* We will first show that all geodesics of $M$ are of finite length. For any $p \in M($ or $\partial M)$, consider a point $x \in \partial M$. Since $\exp_p : \exp_p^{-1} M \to M$ is a diffeomorphism, hence there exists a vector $v \in T_p M($ or $T_p \partial M)$ such that $\exp_p(v) = x$. The geodesic connecting $p$ and $x$ is then given by $\gamma_{(p,v)}(t) = \exp_p(tv)$ for $t \in [0, 1]$. Hence, the geodesics are of finite length. Note that a simple manifold is strictly convex by assumption. This completes our proof.

## 2.10 Boundary Normal Coordinates

Consider a boundary chart, $(U; x^i)$ containing $p_0 \in \partial M$ such that $U \cap \partial M = \{x_n = 0\}$. For any point $p \in \partial M$, let $\nu(p)$ represent the inner normal to $M$. Consider the map $E : U \to M$ by $E(p, t) = exp_p(t\nu(p))$. $E_*(p, 0) = \frac{d}{dt}|_0 exp_p(t\nu(p)) = Id$. Hence for $U$ small, $E$ is a diffeomorphism. So there exists a coordinate system $(U; y^i)$ such that $(y^1, \ldots, y^n) = (x^1, \ldots, x^{n-1}, r)$ where $r$ is the Riemannian distance of $(x^1, \ldots, x^{n-1}, r)$ to $\partial M$.

## 2.11 Microlocal Analysis in the Smooth Category

Let us first introduce some standard notation that will be useful for the rest of this work.

We will denote *multi-indices* by greek letters. Hence if $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, then $\alpha_j$ is a non-negative integer for all $j$. For any vector $x \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. If we denote the partial derivative $\frac{\partial}{\partial x_j} = \partial_j$ and $\partial = (\partial_1, \dots, \partial_n)$, then $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$. The length of $\alpha$ will be denoted by $|\alpha|$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$. Additionally, for any vector $x \in \mathbb{R}^n$, $|x|^2 = x_1^2 + \dots + x_n^2$. We will also write $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. Similarly for $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$, we represent $(1 + D_1^2 + \dots + D_n^2)^{\frac{1}{2}}$ by $\langle D \rangle$. Now we give a brief account of the theory of microlocal analysis in smooth category. The presentation here follows [32, Chapter 1]

### 2.11.1 Oscillatory integrals

Let $\mathcal{S}(\mathbb{R}^n)$ represent the space of Schwartz functions. Recall that

$$\mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |x^\alpha (\partial^\beta u)(x)| < \infty \text{ for all multi-indices } \alpha, \beta \text{ as above } \}.$$

With the semi-norms in $\mathcal{S}((R)^n)$ given by $q_N(u) = \sup_{x \in \mathbb{R}^n, |\alpha| \le N} |x^\alpha (\partial^\beta u)(x)|$, one can turn $\mathcal{S}(\mathbb{R}^n)$ in to a Frechet space by usual methods. The *Fourier transform* of any function $u(x) \in \mathcal{S}(\mathbb{R}^n)$ is given by:

$$(\mathcal{F}u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix.\xi} u(x) dx.$$

Here $dx = dx_1 \dots dx_n$ is the Lebesue measure on $\mathbb{R}^n$. We also know that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear isomorphism. Hence, the inverse Fourier transform of such functions exist and is given by:

$$(\mathcal{F}^{-1}\hat{u})(x) = u(x) = \int_{\mathbb{R}^n} e^{ix.\xi} \hat{u}(\xi) d^\dagger \xi.$$

where $d^\dagger \xi = (2\pi)^{-n} d\xi_1 \dots d\xi_n$.

Next, we show how to extend the definition of Fourier transform to continuous functions $u(x)$ satisfying the condition: $\exists$ constants $C > 0, N > 0$ such that,

$$|u(x)| \le C \langle x \rangle^N. \tag{2.12}$$

We will do so using the standard procedure of defining the Fourier transform of such functions by duality. Hence, if $\mathcal{S}'(\mathbb{R}^n)$ represents the dual space of tempered distributions then, for the $u(x)$ with aforementioned growth estimate 2.12, we will show that $\hat{u}(\xi) \in \mathcal{S}'(\mathbb{R}^n)$. Let us formally write,

$$(\hat{u}, \Psi) = \int \int e^{-ix.\xi} u(x) \Psi(\xi) dx d\xi. \tag{2.13}$$

where $\Psi(\xi) \in \mathcal{S}(\mathbb{R}^n)$. We take note that the LHS of 2.13 makes sense when $u$ is a tempered distribution. In fact, we will justify that the RHS of 2.13 converges as well for $u$ and $\Psi$ as above. Let $D_j = \frac{1}{i}\partial_j$ and $D = (D_1, \ldots, D_n)$. Now it is easy to check that for $k$ even,

$$e^{ix.\xi} = \langle x \rangle^{-k} \langle D_\xi \rangle^k e^{-ix.\xi}. \tag{2.14}$$

First of all note that when $u(x)$ is in Schwartz class, then RHS of 2.13 is simply $\int \hat{u}(\xi)\Psi(\xi)d\xi$ which converges absolutely. In fact, for $u$ in Schwartz class, by using (2.14), we can write:

$$\begin{aligned}(\hat{u}, \Psi) &= \int \int \langle x \rangle^{-k} \langle D_\xi \rangle^k e^{-ix.\xi} u(x) \Psi(\xi) dx d\xi \\ &= \int \int e^{-ix.\xi} u(x) \langle x \rangle^{-k} \langle D_\xi \rangle^k \Psi(\xi) dx d\xi \quad \text{(by integration by parts)}\end{aligned}$$

Now the RHS in the equation just above makes sense for $u(x)$ that satisfies the estimate (2.12). To see this we just take $k > N + n$. The integral in (2.13) is an example of what we will *oscillatory integral*, which we will define in the subsequent paragraphs.

**Definition 2.11.1** *Let $m, \rho, \delta$ be real numbers; $0 \leq \rho, \delta \leq 1$. The* ***symbol class*** *$S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ consists of functions $a(x, \theta) \in C^\infty(X \times \mathbb{R}^n)$ such that for any multi-indices $\alpha, \beta$ and any compact set $K \in X$, there exists a constant $C_{\alpha,\beta,K}\langle \theta \rangle^{m-\rho|\alpha|+\delta|\beta|}$ where $x \in K$ and $\theta \in \mathbb{R}^n$. a is also refereed to as an* ***amplitude*** *function.*

Symbols class $S^m_{1,0}$ will simply be written as $S^m$. We also note that $S^{-\infty} = \cap_m S^m$.

**Definition 2.11.2** *A function $\Phi(x,\theta) \in C^\infty(X \times (\mathbb{R}^n \setminus 0))$ is called a* ***phase function*** *if :*

1. *$\Phi(x,\theta)$ is real valued an positively homogeneous of degree 1 in $\theta$ i.e. $\Phi(x,t\theta) = t\Phi(x,\theta)$ for every $x \in X, \theta \in \mathbb{R}^n$ and $t > 0$.*

2. *It does not have any critical points for $\theta \neq 0$, i.e. $\Phi'_{x,\theta} \neq 0$ for all $x \in X, \theta \in \mathbb{R}^n \setminus 0$.*

Now we are ready to define *oscillatory integrals.*

**Definition 2.11.3 (Oscillatory integrals)** *Let $a(x,\theta)$ be an amplitude function and $\Phi(x,\theta)$ be a phase function. An integral of the form:*

$$I_\Phi(au) = \iint e^{i\Phi(x,\theta)} a(x,\theta) u(x) dx d\theta \tag{2.15}$$

*where $u(x) \in C_c^\infty(X)$, is called an oscillatory integral.*

To show that the integral on the RHS of (2.15) converges we state the following lemma without proof:

**Lemma 2.11.4** *[32, Lemma 1.1] For every phase function $\Phi(x,\theta)$, there exists on $X \times \mathbb{R}^n$, an operator*

$$L = \sum_{j=1}^{N} a_j(x,\theta)\partial_{\theta_j} + \sum_{k=1}^{n} b_k(x,\theta)\partial_{x_k} + c(x,\theta) \tag{2.16}$$

*such that $a_j(x,\theta) \in S^0 X \times \mathbb{R}^n, b_k(x,\theta) \in S^{-1}(X \times \mathbb{R}^n), c \in S^{-1}(X \times \mathbb{R}^n)$ for which the formal adjoint is defined by the formula:*

$$L^t u(x,\theta) = -\sum_{j=1}^{N} \partial_{\theta_j}(a_j u) - \sum_{k=1}^{n} \partial_{x_k}(b_k u) + cu \tag{2.17}$$

*and it satisfies*

$$L^t e^{i\Phi} = e^{i\Phi}. \tag{2.18}$$

Now the regularization of the oscillatory integrals of the kind (2.15) can be carried out in the following way:

$$I_\Phi(au) = \iint (L^t)^k e^{i\Phi(x,\theta)} a(x,\theta)u(x)dxd\theta \tag{2.19}$$

$$= \iint e^{i\Phi(x,\theta)} L^k(a(x,\theta)u(x))dxd\theta \quad \text{by integration by parts} \tag{2.20}$$

which converges absolutely for $k > m + N$ and for $a(x,\theta), u(x)$ and $\Phi(x,\theta)$ as above.

**Example 2.11.5** Consider the RHS of equation (2.13), $I_\Phi(au) = \iint e^{-ix.\xi}u(x)\Psi(\xi)dxd\xi$ is an oscillatory integral with $\Phi(x,\xi) = x.\xi$ and $a(x,\xi) = \Psi(\xi) \in \mathcal{S}(\mathbb{R}^n)$. ◊

In fact, for fixed $a$ and $\Phi$, $I_\Phi(au)$ considered as a functional of $u \in C_c^\infty(X)$, defines a distribution on $X$ i.e. an element of $\mathcal{D}'(X)$ and we can define:

$$(A, u) := I_\phi(au), \; A \in \mathcal{D}'(X).$$

### 2.11.2 Fourier integral operators

**Definition 2.11.6** *[32, Definition 2.1, Chapter 1] Let $X \subset \mathbb{R}^{n_X}, Y \subset \mathbb{R}^{n_Y}$. Consider the expression:*

$$Au(x) = \int e^{i\Phi(x,y,\theta)} a(x,y,\theta)u(y)dyd\theta \tag{2.21}$$

*where $u(y) \in C_c^\infty$, $x \in X$, $\Phi(x,y,\theta)$ is a phase function $X \times Y \times \mathbb{R}^n$ and $a(x,y,\theta) \in S^m(X \times Y \times \mathbb{R}^n)$. Under these conditions, the integral*

$$\langle Au, v\rangle = \iiint e^{i\phi(x,y,\theta)} a(x,y,\theta)u(y)v(x)dxdyd\theta, \quad v \in C_c^\infty(X) \tag{2.22}$$

*is defined and is an ordinary **oscillatory integral**. Viewed as a functional of $v$, (2.22) defines a distribution $Au \in \mathcal{D}'(X)$. Therefore a linear operator*

$$A : C_c^\infty(Y) \to \mathcal{D}'(X) \tag{2.23}$$

*is defined. We call such an operator a **Fourier Integral Operator** or **FIO** in short.*

Furthermore, if the phase function $\Phi(x, y, \theta)$ satisfies,

1. $\Phi'_{y,\theta}(x, y, \theta) \neq 0$ for $\theta \neq 0$, $x \in X$ and $y \in Y$.

2. $\Phi'_{x,\theta}(x, y, \theta) \neq 0$ for $\theta \neq 0$, $x \in X$ and $y \in Y$.

then we can extend the FIO as a continuous map, [32, Proposition 2.3]:

$$A : \mathcal{E}'(Y) \to \mathcal{D}'(X) \tag{2.24}$$

**Remark 2.11.7** In the above, continuity is understood in the sense of weak topologies on $\mathcal{E}'(Y)$ and $\mathcal{D}'(X)$. ◊

**Example 2.11.8 (Linear differential operators)** Let $A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ where $a_\alpha(x) \in C_c^\infty(X)$, $X \subset \mathbb{R}^n$ is open and $D = \frac{1}{i}\partial_x$. Using the Fourier transformation properties we can write

$$D^\alpha u(x) = \iint \xi^\alpha e^{i(x-y).\xi} u(y) dy d^\dagger \xi$$

. Hence we can write the above as a FIO

$$Au(x) = \iint e^{i(x-y).\xi} \sigma_A(x, \xi) u(y) dy d^\dagger \xi$$

where $\sigma_A(x, \xi) = \sum_{|\alpha \leq m|} a_\alpha(x) \xi^\alpha$ is the symbol of the operator and $\sigma_A(x, \xi) \in S^m(X \times \mathbb{R}^n)$ and phase function $\Phi(x, y, \xi) = (x - y).\xi$. ◊

**Example 2.11.9** Let $n_x = n_y = N = n$ and $X = Y$. Then an FIO with a phase function $\Phi(x, y, \xi) = (x - y).\xi$ is a pseudodifferential operator ($\Psi$DO) in short. ◊

## 2.12 Analytic Microlocal Analysis

For proving the support theorems in this thesis we will make use tools and techniques from *analytical microlocalm analysis*. We describe briefly the concepts that will be pertinent to this thesis.

### 2.12.1 Analytic functions

**Theorem 2.12.1** *A $C^\infty$ (smooth) function $f$ defined on $\Omega$ is real-analytic (or, simply analytic in what follows) if any of the following equivalent conditions are satisfied:*

1. *$\forall x \in \Omega$ there exists a neighbourhood of $x$, $U(x)$ such that the Taylor series of $f$ at $x$ converges uniformly to $f$ in $U(x)$.*

2. *For any compact set $\mathcal{K} \subset \Omega$ there exist constants $C_0, C_1$ such that for all multi-indices $\alpha \in \mathbb{Z}_+^n$,*
$$|\partial^\alpha f(x)| \leq C_0 C_1^{|\alpha|} \alpha!$$

3. *There is a complex neighbourhood $\Omega^{\mathbb{C}}$ of $\Omega$ in $\mathbb{C}^n$ such that $f$ can be extended as a holomorphic function there.*

Before we define the analytic wavefront sets we will recall the definition of smooth $C^\infty$ wavefront sets:

**Definition 2.12.2 (Smooth wavefront sets)** *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ then the wavefront set, $WF(f)$, of $f$ is the complement of the set of $(x_0, \xi_0) \in T^*(\mathbb{R}^n) \setminus \{0\}$ with the following property: there exists a function $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi(x_0) \neq 0$, a conic neighborhood $\Gamma$ of $\xi_0$, and constants $C_m$ such that*

$$|\widehat{\psi f}(\xi)| \leq C_m (1 + |\xi|)^{-m}, \qquad m = 1, 2, \ldots \text{ and } \xi \in \Gamma.$$

In the above definition $\Psi$ is a smooth cutoff function that enables us to localize in the base space. We would like to have a similar definition for analytic wavefront sets, too. However, there can be no “analytic cutoff functions". This is because one can not have analytic functions with nonempty compact support. To get around this problem one uses a sequence of cutoff functions with specific growth bounds on the derivatives.

**Lemma 2.12.3** *[38, Chapter 5, Lemma 1.1] There exists a constant $C_n > 0$ such that for any open set $U \subset \mathbb{R}^n$, $d \in \mathbb{R}^+$, $N \in \mathbb{Z}^+$, there is a $C^\infty$ function $g_N$ with the following properties:*

*1.* $0 \leq g_N \leq 1$, $g_N = 1$ *in* $U$ *and* $g_N(x) = 0$ *if* $\text{dist}(x, U) > d$*;*

*2.* $|D^\alpha g_N| \leq (C_n N/d)^{|\alpha|}$ *for all multi-index* $|\alpha| \leq N$*.*

Let us contrast property 2. for cut-off functions $g_N$ as in lemma 2.12.3 above with the growth estimate for analytic functions as mentioned in theorem 2.12.1. The cut offs satisfy estimates of the kind that analytic functions satisfy not for all multi-index but only for multi-index such that $|\alpha| < N$.

Now we are ready to define the analytic wavefront sets in consonance with the definition for smooth wavefront sets.

**Definition 2.12.4 (Analytic wavefront sets)** *Let* $(x_0, \xi_0) \in T^*\mathbb{R}^n \smallsetminus \{0\}$*. We say that a distribution* $f \in \mathcal{D}'(\mathbb{R}^n)$ *is analytic near* $(x_0, \xi_0)$ *if there is an open neighbourhood* $U \ni x_0$ *and an open cone* $\Gamma \in \mathbb{R}^n$ *containing* $\xi_0$ *such that for some constant* $C > 0$ *and for each integer* $N = \{0, 1, \dots\}$ *on can find functions* $\phi_N$ *as in lemma 2.12.3 for which the following estimate holds:*

$$|\widehat{\phi_N f}(\xi)| \leq C^{N+1} N! (1 + |\xi|)^{-N}, \qquad \forall \xi \in \Gamma. \tag{2.25}$$

*The complement of the set of points in* $T^*\mathbb{R}^n \smallsetminus \{0\}$ *near which* $f$ *is analytic is called the analytic wavefront set of* $f$ *and denoted as* $WF_a(f)$*.*

Analytic Wavefront set of a distribution is in general larger than the smooth $C^\infty$ wavefront set. The following example illustrates this point.

**Example 2.12.5** Let

$$f(x) = e^{-\frac{1}{x^2}} \qquad x > 0$$
$$= 0 \qquad x \leq 0$$

It is clear that the $C^\infty$ wavefront set of the function is empty because the function is smooth. However, $(0, \pm 1) \in WF_a(f)$. To see this, let us assume to the contrary that $(0, \pm 1) \notin WF_a(f)$. Then there exists a sequence of cut-off functions $\phi_N$ near $x_0$

as in the definition 2.12.4 such that we have the following:

$$\phi_N(x)e^{-\frac{1}{x^2}} = \frac{1}{2\pi}\int e^{-ix.\xi}\widehat{\phi_n e^{-1/x^2}}d\xi \qquad \text{(By inverse fourier transform)}$$

Now differentiating the LHS $k = N - 2$ times and using the estimate (2.25), we get:

$$D^k(\phi_N e^{-1/x^2}) \leq C^{N+1}N!\int \frac{1}{\langle\xi\rangle^2} \leq C_0C_1^k k!$$

for some constants $C_0, C_1$. By statement 2 in characterization of analytic functions, see theorem 2.12.1, it shows that $e^{-\frac{1}{x^2}}$ has a Taylor series expansion in some neighbourhood of $x = 0$ which is false. Hence $(0, \pm 1) \in WF_a(f)$. ◊

In the following paragraphs we will make another definition of analytic wavefront sets due to Bros-Iagolnitzer and Hörmander. This definition has been shown to be equivalent to the one above by Bony. The presentation follows the one in [33]. For further reading, we point the reader to [33], [6], [14] [8].

### 2.12.2 Alternative approach to analytic wavefront sets

Let $\Omega \subset \mathbb{C}^n$ be open and $\phi : \Omega \to \mathbb{R}$ be a continuous function.

A function $u(z;\lambda)$ on $\Omega \times \mathbb{R}_+$ is said to be in the space $H_\phi^{loc}(\Omega)$ if for each $\lambda > 0$, $u$ is holomorphic in $z$ and for each compact subset $K \subset \Omega$ and every $\epsilon > 0$ there is a constant $C_{K,\epsilon} > 0$ such that

$$|u(z;\lambda)| \leq C_{K,\epsilon}e^{\lambda(\phi(z)+\epsilon)}, \text{ for } z \in K, \quad 1 \leq \lambda < \infty. \tag{2.26}$$

Any $u \in H_0^{loc}(\Omega)$ will be called an analytic symbol. We will say that the analytic symbol has a finite order $m \in \mathbb{R}$ if for each compact $K \subset \Omega$ there is a constant $C_K > 0$ such that

$$|u(z;\lambda)| \leq C_K\lambda^m, \quad z \in K, \quad 1 \leq \lambda < \infty. \tag{2.27}$$

We will define an equivalence relation on the space $H_\phi^{loc}(\Omega)$ in the following manner. $u, v \in H_\phi^{loc}(\Omega)$ are said to be equivalent (written as $u \sim v$), if there exists a continuous function $\phi_1 < \phi$ on $\Omega$ such that

$$u - v \in H^{loc}_{\phi_1}(\Omega). \tag{2.28}$$

A formal element of $H^{loc}_{\phi}(\Omega)$ is given by:

- A covering $\Omega = \bigcup_{\alpha \in A\, \alpha}$ where the $\Omega_\alpha \subset \Omega$ are open.
- For each $\Omega_\alpha$ an element $u_\alpha \in H^{loc}_{\phi}(\Omega_\alpha)$ (a local representative) such that $u_\alpha \sim u_\beta$ in $H^{loc}_{\phi}(\Omega_\alpha \cap \Omega_\beta)$ for each $\alpha, \beta \in A$.

Elements of $H_0$ are called *formal analytic symbol.*

**Example 2.12.6** [33, Example 1.1 ] Consider a sequence of holomorphic functions, $\{a_k(z)\}$, $k = 0, 1, 2, \ldots$ on $\Omega$ such that for every $\tilde{\Omega} \Subset \Omega$, we have

$$|a_k(z)| \leq C^{k+1} k^k, \qquad k = 0, 1, 2, \ldots \quad z \in \tilde{\Omega},$$

where $C = C_{\tilde{\Omega}} > 0$.

Then up to an equivalence class we can define a formal analytic symbol on $\Omega$ as:

$$a_{\tilde{\Omega}}(z; \lambda) = \sum_{0 \leq k \leq \lambda (e C_{\tilde{\Omega}})^{-1}} \lambda^{-k} a_k(z), \qquad z \in \tilde{\Omega}.$$

In fact, $a_{\tilde{\Omega}}$ is an analytic symbol of order 0 on $\tilde{\Omega}$ because for $z \in \tilde{\Omega}$, $0 \leq k \leq \lambda (eC)^{-1}$:

$$|\lambda^{-k} a_k(z)| \leq C (Ck)^k \lambda^{-k} \leq C e^{-k}, \quad C = C_{\tilde{\Omega}}.$$

If $C_1 > C$ and $\lambda (eC_1)^{-1} < k \leq \lambda (eC)^{-1}$, $z \in \tilde{\Omega}$, then

$$|\lambda^{-k} a_k(z)| \leq C e^{-\lambda (eC_1)^{-1}},$$

which shows the equivalence of the local representatives in $\tilde{\Omega}_\alpha \cup \tilde{\Omega}_\beta$. We write the formal analytic symbol as:

$$a(z;\lambda) = \sum_{0}^{\infty} \lambda^{-k} a_k(z), \tag{2.29}$$

◇

Let $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$ and let $\phi(x, y, \alpha)$ is an analytic function defined in a neighborhood of $(x_0, x_0, \xi_0)$ such that

$$\phi = 0, \text{ and} \phi'_x = \alpha_\xi \qquad \text{for } x = \alpha_x$$

$$\text{Im } \phi \geq C(|x - \alpha_x|^2) \qquad \text{for real } x, \alpha.$$

**Definition 2.12.7** *[33, Definition 6.1] Let $X \subset \mathbb{R}^n$ be an open set containing $x_0$ and $a(x, \alpha; \lambda)$ be an elliptic analytic symbol defined near $(x_0, x_0, \xi_0)$ (ellipticity means $a_0$ is non-zero everywhere). Let $\chi \in C_0^\infty(X)$ is equal to 1 near $x_0$. Then, we say a distribution $u$ on $X$ is microlocally 0 at $(x_0, \xi_0)$ if*

$$\int e^{i\lambda\phi(x,\alpha)} a(x, \alpha; \lambda) \chi(x) \overline{u(x)} dx$$

*is exponentially decreasing when $\lambda \to \infty$, uniformly for $\alpha$ in a real neighborhood of $(x_0, \xi_0)$.*

We recall that this definition does not depend on a choice of $\chi$, see [33, Proposition 6.2].

**Definition 2.12.8 (Analytic wavefront set)** *The analytic wavefront set of $u \in \mathcal{D}'(X)$ is the complement of the conic set of points $(x, \xi) \in T^*\mathbb{R}^n \setminus 0$ where $u$ is 0 microlocally in sense of above definition. The analytic wavefront set of $u$ will be denoted by $WF_A(u)$.*

**Remark 2.12.9** When $f$ is a $m$- tensor field, we say that $(x_0, \xi_0) \notin WF_A(f)$ if and only if $(x_0, \xi_0) \notin WF_A(f_{i_1 \dots i_m})$ where $i_1, i_2, \dots, i_m \in \{1, \dots, n\}$. ◇

Now we state a theorem due to Sato, Kawai and Kashiwara that will be most useful in proving the support theorems in this work.

**Theorem 2.12.10 (Sato-Kawai-Kashiwara theorem)** *Let $u \in \mathcal{D}'(X)$. Let $x_0 \in X$ and let $U$ be a neighborhood of $x_0$. Assume that $S$ is a $C^2$ hypersurface of $X$ and $x_0 \in supp(f) \cap S$. Furthermore, let $S$ divide $U$ into two open connected sets and assume that $f = 0$ on one of these open sets. Let $\xi \in N^*_{x_0}(S) \setminus 0$, then $(x_0, \xi) \in WF_A(u)$.*

Finally we recall the following two theorems that will be crucial in proving some estimates that are needed in this thesis.

**Theorem 2.12.11** *[33, Lemma 2.7] Let $\phi(z)$ be a holomorphic function defined near $z_0$ such that $z_0$ is a non-degenarate critical point, i.e. $\phi'(z_0) = 0$ and $det(\phi''(z_0)) \neq 0$. Then we can choose holomorphic local coordinates $\tilde{z}$ near $z_0$ such that :*

$$\phi(z) = \phi(z_0) + \frac{1}{2}(\tilde{z_1}^2 + \cdots + \tilde{z_n}^2)$$

**Theorem 2.12.12 (Complex stationary phase theorem)** *[33, Theorem 2.8] Let $U \subset \mathbb{C}^n$ be an open neighborhood of $0$, and let $\phi$ be a function holomorphic on $U$ with $z = 0$ as its only non-degenerate critical point. Let $V \subset\subset U$ be an open neighborhood of $0$, and suppose that $Re\phi(x) \geq 0$ for each $x \cup V_{\mathbb{R}} = V \cup \mathbb{R}^n$ and that $Re\phi(x) > 0$ on $V_{\mathbb{R}}$. Then there exist constants $C > 0$ and $\epsilon > 0$ such that for each bounded holomorphic function $u$ on $U$ we have:*

$$I(h) = \int_{V_{\mathbb{R}}} e^{-\phi(x)/h} u(x)dx \tag{2.30}$$

$$= \sum_{0 \leq k \leq \frac{1}{ch}} (2\pi)^{n/2} \frac{1}{k!} \left(\frac{\tilde{\Delta}}{2}\right)^k \left(\frac{u}{\mathcal{T}}\right)(0) + R(h) \tag{2.31}$$

*where,*

$$|R(h)| \leq \frac{1}{\epsilon} e^{\frac{-\epsilon}{h}} \sup_U |u(z)|, \quad 0 \leq h \leq 1$$

*and $\tilde{\Delta} = \sum \frac{\partial^2}{\partial_{z_j^2}}$, $\mathcal{T} = det(\frac{\tilde{z}}{z})$. Note that $\tilde{z}$ are the same coordinates as in lemma 2.12.11 and $\mathcal{T}(0) = \phi''(0)$.*

# Chapter 3

# A brief overview of ray transforms

Tomography refers to non-destructive imaging modalities that attempt to reconstruct some information about the interior of an object by measuring how the object interacts with signals (e.g. electromagnetic waves) passing through it. The mathematical foundations of tomography can be traced back to the seminal article [26] of Radon published in 1917 in which he considered the problem of finding a function defined in $\mathbb{R}^2$ from its integral over the set of lines in a plane. In doing so he defined a transform $Rf$, which takes a function $f$ defined in the plane to another function $Rf$ that is defined on a set of lines and such that the value of $Rf$ on each line is equal to the line integral of the function over that particular line. The transform $Rf$ is now commonly referred to as the Radon Transform of the function $f$. In $n$ dimensions, one can study a $d$- plane transform where integrals are taken over $d$-dimensional planes and $d < n$. When $d = n - 1$, such hyperplane transforms are still called Radon Transforms, however if $d = 1$, it is customary in the literature to call such line-integral transforms as X-ray Transforms as they appear most commonly in X-ray computerized tomography applications. Much more generally, the area of mathematics in which we study transforms that take a tensor field defined on some manifold (e.g. $\mathbb{R}^n$) to its integrals over some collection of submanifolds (e.g. hypersurfaces) is referred to as Integral Geometry. Over the course of the past hundred years, transforms as described above have been studied not just as important theoretical constructs in the field of integral geometry but have also found a wide variety of useful applications especially in the field of medical imaging [19].

In this thesis, we study two different kinds of ray transforms on simple, real-analytic, Riemannian manifolds: (1) longitudinal or geodesic ray transform of a tensor field of arbitrary order and (2) the transverse ray transform of tensor fields of rank 2. We will now present a brief account of the known results for such transforms.

Let $(M, g)$ be an $n$-dimensional smooth, compact, Riemannian manifold with boundary given by $\partial M$. Let $\gamma$ be a geodesic of this manifold such that its end points lie on the boundary and $l(\gamma)$ denotes the length of this geodesic. We make these concepts precise in section 2.4. For a symmetric tensor field of order 2 the geodesic ray transform is defined as:

$$If(\gamma) := \int_0^{l(\gamma)} \langle f(\gamma(t)), \dot{\gamma}^2(t)\rangle dt = \int_0^{l(\gamma)} f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)dt \tag{3.1}$$

where *Einstein summation convention* of summing over repeated indices is assumed. The geodesic ray transform of a symmetric tensor field of order 2 arises naturally in the context of lens and boundary rigidity problems and has been studied in e.g. [30], [28], [34], [35]. In the subsequent paragraphs, we show how the above transform (3.1) arises as a linearization of the boundary rigidity problem. The presentation here follows [30, Chapter 1].

Suppose that $(M, g)$ is a compact, simple, Riemannian manifold. By simplicity, we mean that the boundary $\partial M$ is strictly convex and any two points $p, q \in M$ are joined by a unique geodesic $\gamma_{pq}$ which can be extended maximally in the sense that their end points lie on the boundary. A more precise definition of a simple manifold is given in this thesis in Section 2.9. If $p, q \in \partial M$ then we denote the length of geodesic $\gamma_{pq}$ by $\Gamma_{pq}$. Now consider the following real valued function called *hodograph*:

$$\Gamma_g : \partial M \times \partial M \to \mathbb{R}.$$

This function measures the geodesic distance between any two boundary points on the manifold. We will now concern ourselves with the following question: *To what extent can the metric $g$ be recovered from the hodograph?* One source of non-uniqueness of the solution is the following: Let $\phi$ be a diffeomorphism of $M$ in to itself such that it preserves the boundary i.e. $\phi|_{\partial M} = \mathbb{I}$. The map $\phi$ induces a mapping between the tensor bundles of order 2 such that we get a new simple metric $g_1$ which is a pull back of the metric $g$, i.e. $g_1 = \phi^* g$. Here $\phi^*$ represents a pull back map. This means that $\forall x \in M$, and for every pair of vectors $\xi, \eta \in T_xM$ we have the following equality:

$$\langle \xi, \eta \rangle_g = \langle \phi_* \xi, \phi_* \eta \rangle_{g_1}$$

where $\phi_* : T_xM \to T_{\phi(x)}M$ is the *pushforward* map. This equality shows that the two different metrics have different families of geodesics and the same hodograph. Hence one is led to the following refined problem, called the *boundary rigidity problem*:

**Determining a metric from its *hodograph***: *Let $g$ and $g_1$ are two different metrics on a compact, simple, Riemannian manifold with smooth boundary. If $\Gamma_g = \Gamma_{g_1}$, then does there exist a diffeomorphism $\phi : M \to M$ such that $\phi|_{\partial M} = \mathbb{I}$ and $g_1 = \phi^* g$?*

Let us linearize the above problem. Introduce a family of simple Riemannian metrics, $g_\tau$ smoothly depending on $\tau \in (-\epsilon, \epsilon)$. For fixed $p, q \in \partial M$, let $l = \Gamma_{g_0}(p,q)$. Let $\gamma_\tau : [0, l] \to M$ be a geodesic of the metric $g_\tau$ such that $\gamma_\tau(0) = p, \gamma_\tau(l) = q$. In some local coordinates, we write $\gamma_\tau(t) = (\gamma^1(t;\tau)), \ldots, \gamma^n(t,\tau)$ and $g_\tau = (g_\tau)_{ij}$. Then

$$\frac{1}{l}\big(\Gamma_{g_\tau}(p,q)\big)^2 = \int_0^l (g_\tau)_{ij}(\gamma_\tau(t))\dot{\gamma}^i(t;\tau)\dot{\gamma}^j(t;\tau)dt. \tag{3.2}$$

Differentiating LHS of equation 3.2 with respect to $\tau$ and evaluating at $\tau = 0$ we get,

$$\begin{aligned}\frac{1}{l}\partial_\tau|_{\tau=0}\big(\Gamma_{g_\tau}(p,q)\big)^2 = &\int_0^l f_{ij}(\gamma_0(t))\dot{\gamma}^i(t;0)\dot{\gamma}^j(t;0)dt \\ &+\bigg(\int_0^l \big[\partial_{x_k}(g_0)_{ij}(\gamma_0(t))\dot{\gamma}^i(t;0)\dot{\gamma}^j(t;0)\partial_\tau\gamma^k(t,0) \\ &\quad + 2(g_0)_{ij}(\gamma_0(t))\dot{\gamma}^i(t;0)\partial_\tau\dot{\gamma}^j(t;0)\big]dt\bigg)\end{aligned} \tag{3.3}$$

where $f_{ij} = \partial_\tau|_{\tau=0}(g_\tau)_{ij}$. Now the second integral in equation (3.3) is zero as it the extremal of the function:

$$E_0(\gamma) = \int_0^l (g_\tau)_{ij}\dot{\gamma}^i(t)\dot{\gamma}^j(t)$$

and $\partial_\tau \gamma^i(0,0) = \partial_\tau \gamma^i(0,l) = 0$. Hence we have the following equality

$$\frac{1}{l}\partial_\tau|_{\tau=0}\big(\Gamma_{g_\tau}(p,q)\big)^2 = \int_0^l f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)dt. \tag{3.4}$$

The RHS of equation (3.4) is easily recognized as the aforementioned geodesic ray transform of 2−tensor fields. Hence the linearized version of the hodograph problem will be to determine a tensor $f$ from its geodesic ray transform.

Recall that the problem of determining the metric from its hodograph has a non-uniqueness built in to it, i.e. if $g$ is a solution to the hodograph problem then so is a metric $g_1 = \phi^* g$ where $\phi : M \to M$ was a diffeomorphism fixing the boundary $\partial M$. This non-uniqueness appears in the linearized version in the form of existence of a non-trivial kernel of the geodesic ray transform. In fact, for any tensor field $v$ of rank 1 such that $v|_{\partial M} = 0$, the *symmetrized covariant derivative* written in local coordinates as $(dv)_{ij} = \frac{\nabla_i v_j + \nabla_j v_i}{2}$ is in the kernel of the transform.

Let us now define the geodesic ray transform for symmetric tensor fields of arbitrary order. The geodesic ray transform of a compactly supported symmetric tensor field $f$ of order $m$ is a function $If(\gamma)$ given by (e.g. see [30]),

$$If(\gamma) = \int_0^{l(\gamma)} \langle f(\gamma(t)), \dot{\gamma}^m(t)\rangle dt = \int_0^{l(\gamma)} f_{i_1\ldots i_m}(\gamma(t))\dot{\gamma}^{i_1}(t)\cdots\dot{\gamma}^{i_m}(t)dt \tag{3.5}$$

where, $\gamma : [0, l(\gamma)] \to M$ is a geodesic of $M$ parametrized by its arc length and with end points on the boundary $\partial M$. Note that the above definition of the geodesic ray transform is invariantly defined. As for order 2, it is easy to see that such transforms are not injective. Indeed, for any symmetric tensor field $f$ of order $m$, if $f = dv$ where $v$ is another symmetric $m-1$ tensor field and $v|_{\partial M} = 0$, then such a tensor field $f$ is in the kernel of the geodesic ray transform.

Let us now introduce some notation that will be used very often in this thesis. We will represent by $\tau_M$, the tangent bundle of the manifold $M$ and by $\tau'_M$, its cotangent bundle. Note that their fibers $T_xM$ and $T'_xM$ respectively are the spaces tangent and cotangent to $M$ at the point $x$. Their sections are called real vector (or, contravariant) and covector (or, covariant) fields on M. $\tau_s^r M$ will be used to represent the bundle of tensors of degree $(r,s)$ on M, and its sections will be the tensor fields of degree $(r,s)$. Recall that a tensor field is said to be of degree $(r,s)$ if it is $r$ times contravariant and $s$ times covariant. By $S^m\tau'_M$ we denote the subbundle of $\tau_m^0$ which consists of tensors that are invariant with respect to all transpositions of the indices. The elements of $S^m\tau'_M$ will be briefly referred to as symmetric tensor fields of order $m$.

**Definition 3.0.1** *Consider a compact Riemannian manifold $M$ with metric $g$ and let $k > 0$ be an integer. Then the real Hilbert space $H^k(S^m\tau'_M)$ is defined as a completion of $C^\infty(S^m\tau'_M)$, (i.e. space of smooth symmetric tensor fields of order $m$) with respect to the Sobolev norm $\|.\|_k$.*

Now we recall the following decomposition for the symmetric $m$- tensor fields.

**Theorem 3.0.2** *[30, Theorem 3.3.2] Let $M$ be a compact Riemannian manifold with boundary; let $k \geq 1$ and $m \geq 0$ be integers. For every symmetric field $f$ of order $m$ such that $f \in H^k(S^m\tau'_M)$, there exists a uniquely determined symmetric tensor field $f^s \in H^k(S^m\tau'_M)$ and $v \in H^{k+1}(S^{m-1}\tau'_M)$such that:*

$$f = f^s + \mathrm{d}\, v, \quad v|_{\partial M=0}, \quad \delta(f^s) = 0$$

*where $dv$ is the symmetrized covariant derivative of $v$ and $\delta(f^s)$ is the divergence of $f^s$.*

The above theorem can be seen as an extension of the well known fact about decomposition of vector fields in to potential and solenoidal parts. As such the tensor field $f^s$ is referred to as the solenoidal part of the tensor field and $\mathrm{d}\, v$ is called the potential part of the tensor field. As stated earlier, the potential part of symmetric tensor fields is easily seen to be in the kernel of the geodesic ray transform. One can then hope to recover only the solenoidal part of such symmetric tensor fields. In fact, in $\mathbb{R}^n$ Sharafutdinov gives an explicit reconstruction of the solenoidal part of the field from its X- ray transform in [30].

This leads us to ask the following question: What data are needed to get an injectivity result for the tensor field $f$ itself? Sharafutdinov considers this question in [29] wherein he defines the "integral moments of symmetric $m-$ tensor field" in the Euclidean setting. The $q^{th}$ **integral moment** of a compactly supported symmetric $m$-tensor field $f$ is a function $I^q f(x,\xi)$ defined on the space of lines (more generally on the space of geodesics) and is given by (for e.g. see [30]),

$$I^q f(x,\xi) = \int_{\mathbb{R}} t^q \langle f(x+t\xi), \xi^m \rangle dt = \int_{\mathbb{R}} f_{i_1 \dots i_m}(x+t\xi)\xi^{i_1} \cdots \xi^{i_m} dt.$$

He shows that if the first $m+1$ integral moments of the tensor field $f$ vanishes over the set of all lines in $\mathbb{R}^n$, then the tensor field $f$ itself vanishes in $\mathbb{R}^n$. Hence full recovery of the symmetric $m-$ tensor field is possible from its $m+1$ integral moments. We show in this thesis that such a result holds in greater generality, too. More specifically, we show that for the appropriately defined "integral moments of symmetric $m$- tensor fields" in a compact, simple, real analytic Riemannian manifold $M$, if the first $m+1$ integral moments of such a field vanish over the set of geodesics of the manifold $M$, then the field $f$ also vanishes over the manifold. Indeed, we establish a stronger result for such integral moments in form of a support theorem. These results are stated and proved in Chapter 4 of this thesis. In proving these results , we rely heavily on the tools and techniques of microlocal analyis. After the pioneering work of Guillemin and Sternberg, introducing the microlocal approach for Radon transforms in [12], Boman and Quinto first used analytic microlocal analysis to prove a support theorem for Radon transforms with positive real analytic weights, see [4]. Stefanov and Uhlmann and Stefanov and Krishnan have used analytic microlocal analyis to prove injectivity results and support theorems for geodesic ray transforms of tensor fields defined on Riemannian manifolds, see [35], [36], [34], [18], [16].

We also take up the study of transverse ray transform of symmetric $2-$ tensor fields in this thesis. This transform comes up in the study of a wide variety of vector-tomography problems like diffraction-strain tomography, polarization tomography, etc. We will now briefly describe the physical model of polarization tomography and the associated linear problem that gives rise to transforms of this kind.

Anisotropy (directional irregularities) in the medium characteristics like dielectric permeability results in polarization of the electromagnetic wave passing through the system. We would like to know *whether it's possible to detect and measure the anisotropy of the medium characteristic by measuring the polarization of each electromagnetic ray that passes through the medium?* In the following paragraphs we will see how to state this problem in a mathematically precise fashion.

Let $D \subset \mathbb{R}^3$ be a domain with Riemannian metric given by $g_{ij}(x)$. Let us represent the dielectric permeability in $D$ by $\varepsilon_{ij}(x)$. We assume further that the dielectric permeability tensor is actually *quasi-isotropic* i.e. we can write $\varepsilon_{ij}(x) = \varepsilon(x)\delta_{ij} + \frac{1}{k}\chi_{ij}(x)$ where $\varepsilon(x)$ is a positive function, $k$ is the wave number corresponding to the electromagnetic wave passing through the medium and $\chi_{ij}(x)$ is some other tensor field that signifies anisotropicity of the medium. For such cases, the *refraction coefficient* of the medium is given by $n = \sqrt{\varepsilon(x)}$. In the realm of geometrical optics, we can write the following series expansion for the electric field $\mathbf{E}(x)$:

$$\mathbf{E}(x) = e^{ik\phi(x)} \sum_{m=0}^{\infty} \frac{\mathbf{E}_m(x)}{(ik)^m}$$

which is assumed to be valid in the topology of compactly supported functions, $\mathcal{E}(\mathbb{R}^3)$. Here $\phi(x)$ is the phase and $k$ is the wave number as per usual convention. By using Maxwell equations, one can show that for the field $\mathbf{E}(x)$ defined as above, the phase function $\phi(x)$ satisfies the *eikonal equation*, see [30, Equation 5.1.16]:

$$|\nabla\phi|^2 = n^2 = \varepsilon(x).$$

The characteristics of the *eikonal equation* are the directions along which the electromagnetic waves propagate. They are in fact the geodesics of the following Riemannian metric, see [30, Equation 5.1.17]:

$$h_{jk}(x) = n^2(x) g_{jk}(x).$$

Let $\gamma : [0, l(\gamma)] \to M$ be a geodesic in $M$ parametrized by $t$ corresponding to the known metric $h$ along which an electromagnetic wave is traveling with end points of the geodesic on the boundary $\partial M$ as before. Hence, $l(\gamma)$ is the length of geodesic $\gamma$ with end points on the boundary of $M$. Due to the transverse nature of such electromagnetic waves, the electric field (and the magnetic field) oscillate in a direction transverse (perpendicular) to the direction of propagation. $\dot{\gamma}(t)$ denotes the velocity of the geodesic and $P_{\dot{\gamma}}$ is the orthogonal projection on to a plane perpendicular to $\dot{\gamma}$. The covariant derivative operator along $\gamma$ [21, Chapter 4] will be written as $\frac{D}{dt}$. Then the polarization vector field $\eta$ along $\gamma$ is given by *Rytov's Law* [30, Equation 5.1.56]:

$$\frac{D\eta}{dt} = P_{\dot{\gamma}} \left( \frac{i\chi}{(2\varepsilon(x))} \right) \eta. \tag{3.6}$$

If we write $\frac{i\chi}{(2\varepsilon(x))} = f$, then the equation (3.6) can be written simply as:

$$\frac{D\eta}{dt} = P_{\dot{\gamma}} f \eta. \tag{3.7}$$

Now we are ready to state the inverse problem of polarization tomography. Suppose that for every geodesic (electromagnetic ray), $\gamma : [0, l(\gamma)] \to M$, with end points on the boundary of $D$, we know the value of $\eta(l(\gamma))$ of the solution to the system (3.7) as a function of the initial value $\eta(0)$ and the geodesic $\gamma$:

$$\eta(l(\gamma)) = U(\gamma)\eta(0). \tag{3.8}$$

In other words, the fundamental matrix $U(\gamma)$ of the system (3.7) is known.

**Inverse problem of polarization tomography**: *Determine $f$ (or equivalently the anisotropic part of the electrical permeability tensor, $\chi$) from $U(\gamma)$.* This problem is nonlinear, so let's linearize it. In fact we will generalize the problem to $n$-dimensions now. Let $\gamma : [0, 1] \to D \subset \mathbb{R}^n$ be a fixed geodesic with end points on the boundary of $D$. Let $e_1(t), \ldots, e_{n-1}(t), e_n(t) = \dot{\gamma}(t)$ be an orthonormal basis along $\gamma$. In this basis, equation (3.7) is written as:

$$\dot{\eta}^i = \sum_{j=1}^{n-1} f_{ij}\eta^j.$$

The solution to the Cauchy problem of the system is given by the Neumann series:

$$\eta(1) = \left[ I + \int_0^{l(\gamma)} F(t)dt + \text{other nonlinear terms} \right]\eta(0)$$

where $F = (f_{ij})_{i,j=1}^{n-1}$. Neglecting nonlinearities,

$$\eta^i(l(\gamma)) - \eta^i(0) = \int_0^{l\gamma} \sum_{i,j=1}^{n-1} f_{ij}(t)\eta^j(0)dt. \tag{3.9}$$

Multiplying each of the equalities in (3.9) by $\xi^i$ and summing them up, we get

$$Jf(\gamma;\xi,\eta) = \int_0^{l(\gamma)} \sum_{i,j=1}^{n-1} f_{ij}(t)\xi^i\eta^j(0)dt. \tag{3.10}$$

Here $Jf(\gamma;\xi,\eta) = \sum_{i,j=1}^{n} \xi^i(\eta^i(l(\gamma)) - \eta^i(0))$ is a known function. We can write (3.10) invariantly by introducing $\xi^n = \eta^n = 0$ and increase the upper summation limit in (3.10) to $n$. Now we also note that the fields $\xi(t) = \xi^i e_i(t)$ and $\eta(t) = \eta^i e_i(t)$ are parallel along $\gamma$ and perpendicular to $\dot{\gamma}$. So (3.10) takes the form:

$$Jf(\gamma;\xi,\eta) = \int_0^{l(\gamma)} f_{ij}(\gamma(t))\xi^i(t)\eta^j(t)dt. \tag{3.11}$$

If we restrict ourselves to symmetric tensor fields $f$, then that leads us to consider the quadratic form:

$$Jf(\gamma,\eta) = \int_0^{l(\gamma)} f_{ij}(\gamma(t))\eta^i(t)\eta^j(t)dt. \tag{3.12}$$

The above transform $Jf(\gamma,\eta)$ is called the *transverse ray transform* of symmetric $2-$ tensor field $f$.

In this work, we will consider the transverse ray transform of symmetric $2-$ tensor fields in a more general setting. Let $(M,g)$ represent a compact, simple, real analytic

Riemannian manifold of dimension $\geq 3$ and $f$ be a symmetric tensor field of order 2. Let $[0, l(\gamma)] \ni t \mapsto \gamma(t)$ be a geodesic of the manifold $M$ with end points on the boundary and $\eta(t)$ be a vector field parallel along $\gamma(t)$ and orthogonal to $\dot{\gamma}(t)$ for every $t$. Sharafutdinov defines the transverse ray transform of a symmetric tensor field $f$ of rank 2 as [30, Chapter 5]:

$$Jf(\gamma, \eta) = \int_0^{l(\gamma)} f_{ij}(\gamma(t))\eta^i(t)\eta^j(t)dt \qquad (\eta(t) \in \gamma^{\perp}(t)).$$

Such transforms have been studied by several authors, see e.g. [30], [24], [9], [22], [13], [31], in the context of a multitude of physical problems like polarization tomography, diffraction strain tomography and other such imaging modalities. In [30], Sharafutdinov proves an injectivity result for the Transverse Ray Transform on a Compact Dissipative Riemannian manifold (CDRM). Recently, Desai and Lionheart have given an improved reconstruction algorithm for Transverse Ray Transform for symmetric 2- tensor fields in the Euclidean setting, see [9]. In this thesis, we prove a support theorem for transverse ray tranforms of symmetric 2- tensor field in the aforementioned setting.

In the next two chapters we provide a detailed account of our work.

# Chapter 4

# Support theorem for transverse ray transform of rank 2 tensor fields

## 4.1 Introduction

In this chapter, our main goal is to prove a support theorem for the transverse ray transform (TRT) of a symmetric tensor field of rank 2. Transverse ray transforms of such tensor fields appear quite naturally in the study of polarization tomography. The general physical principle behind polarization tomography is as follows. The anisotropy in the medium characteristics, such as magnetic permeability tensor and dielectric permeability tensor, polarizes the electromagnetic waves passing through it. By measuring the polarization of a large number of rays passing through the medium, one is then able to detect and measure the anisotropy in the medium characteristics. Due to the transverse nature of electromagnetic rays, the polarization measurements along a ray which is obtained in the form of an integral along that ray depends only on the component of the desired medium characteristic that is "transverse" to the ray direction. Hence the central problem of polarization tomography is to reconstruct a medium characteristic from the data which is in the form of the transverse ray transform of the medium characteristic. For a more detailed discussion, see [30, Chapter 5].

Consider a simple, real analytic, Riemannian manifold $M$ of dimension $n \geq 3$ with an analytic metric $g$. Let $[0, l(\gamma)] \ni t \mapsto \gamma(t)$ be a geodesic of the manifold $M$ with end points on the boundary and $\eta(t)$ be a vector field parallel along $\gamma(t)$ (recall from section 2.5 that this implies that the covariant derivative of $\eta(t)$ vanishes along $\gamma(t)$) and orthogonal to $\dot{\gamma}(t)$ for every $t$. We will denote this by writing $\eta(t) \in \gamma(t)^{\perp}$ in the subsequent paragraphs. Sharafutdinov defines the transverse ray transform of a symmetric tensor field $f$ of rank 2 as [30, Chapter 5]:

$$Jf(\gamma,\eta) = \int_0^{l(\gamma)} f_{ij}(\gamma(t))\eta^i(t)\eta^j(t)dt \qquad (\eta(t) \in \gamma^{\perp}(t))$$

Such transforms have been studied by several authors, see e.g. [30], [24], [9], [22], [13], [31], in the context of a multitude of physical problems like polarization tomography, diffraction strain tomography and other such imaging modalities. In [30], Sharafutdinov proves an injectivity result for the transverse ray transform on a Compact Dissipative Riemannian manifold (CDRM). After the pioneering work of Sharafutdinov [30] and Lionheart and Withers [22] who provided reconstruction methods for transverse ray transforms, recently, Desai and Lionheart have given an improved reconstruction algorithm for transverse ray transform for symmetric 2- tensor fields in the Euclidean setting, see [9].

In this work, we prove a support theorem for transverse ray transform of symmetric 2- tensor fields defined on a compact, simple, real analytic Riemannian manifold. Apart from their theoretical significance, support theorems are useful for practical reasons in various tomography problems. Having a support theorem for an integral transform of a function or a tensor field tells us that we can reconstruct the desired function or the tensor field in the exterior of a given region solely by tomographic measurements in the exterior of the given region. The injectivity result for such transforms follows as a corollary of our more general result. We use the tools of analytic microlocal analysis to prove our results. Such techniques have been extensively used to prove injectivity results and support theorems for very general Radon Transforms and X-ray transforms by several authors, among which we give below a partial list. Analytic microlocal techniques were first used by Boman and Quinto to prove a support theorem for Radon Transforms with real analytic weights in [4]. Stefanov and Uhlmann have used analytic microlocal analysis to prove an s-injectivity result for geodesic ray transform of symmetric 2-tensor fields in [35]. Krishnan in [16] and Krishnan and Stefanov in [18] prove support theorems for geodesic ray transform of functions and symmetric 2- tensor fields respectively. Expanding on these works, the authors in [1] proved a support theorem for integral moments of symmetric $m$-tensor fields.

The organization of the current chapter is as follows: In section 3.2, we give some definitions and our main theorem in this work. In section 3.3, we prove a preliminary result that shows the analytic dependence of a normal vector field on the geodesic along which it is translated in a parallel manner. In section 3.4, we prove a microlocal proposition which is an analogue of [36, Proposition 2]. The proof of our main theorem is given in section 3.5 along the lines of the proof given by Krishnan in [16, section 3].

## 4.2 Definitions and the Main Theorem

**Definition 4.2.1 (Simple Manifold)** *A compact Riemannian manifold (M,g) with smooth boundary is said to be simple if:*

*(i) The boundary of the manifold $\partial M$ is strictly convex: $\langle \nabla_\xi \nu(x), \xi \rangle > 0$ for all $\xi \in T_x(\partial M)$ and where $\nu(x)$ is the unit outward pointing normal to the boundary.*

*(ii) The map $\exp_x : \exp_x^{-1}(M) \mapsto M$ is a diffeomorphism for all $x \in M$.*

Here the second condition implies that any two points $x$ and $y$ in the manifold $M$ are connected by a unique geodesic which depends smoothly on the points $x$ and $y$. It is well-known that any such simple manifold is necessarily diffeomorphic to a ball in $\mathbb{R}^n$, see [30]. Hence in the following analysis, we can assume that the manifold is some domain in $\mathbb{R}^n$. In this article, we work with a fixed simple Riemannian manifold $M$ with a fixed real analytic atlas and a given metric $g$ which is also assumed to be real analytic. Note that a tensor field is said to be real analytic on a set $U$ if it is real analytic in a neighbourhood of the set $U$. Furthermore we will work with symmetric tensor fields $f$ of order 2 on $M$ i.e. $f \in S^2(M)$. In coordinate representation, the tensor field $f = f_{ij}$. We will also assume Einstein convention of summing over any repeated indices and raise or lower index on a tensor field via the metric $g$. As such,$f_{ij}$ and $f^{ij} = f_{kl} g^{ik} g^{jl}$ will be thought of as equivalent representations of the same tensor field.

Let $\widetilde{M}$ be a real analytic extension of $M$ such that the simple metric $g$ also extends analytically to $\widetilde{M}$, see [35, section 1]. We extend the tensor fields $f \in S^2(M)$ by 0 in $\widetilde{M} \setminus M$. We will parametrize the maximal geodesics in $\widetilde{M}$ with endpoints on $\partial\widetilde{M}$ by their starting points and directions.

Set

$$\Gamma_- := \left\{(x,\xi) \in T\widetilde{M} |\ x \in \partial\widetilde{M}, |\xi| = 1, \langle \xi, \nu(x)\rangle < 0\right\},$$

where $\nu(x)$ is the outer unit normal to $\partial\widetilde{M}$ at $x$. We will think of maximal geodesics in $M$ as restriction of maximal geodesics of $\widetilde{M}$. A geodesic will be denoted by $\gamma$. Let $\mathcal{A}$ be an open set of such geodesics in $\widetilde{M}$. We assume that any geodesic in this set $\mathcal{A}$ is homotopically deformable within the set $\mathcal{A}$ to a geodesic outside $M$. By this we mean that there exists a continuous map which takes a geodesic of $\widetilde{M}$, say $\gamma_0 \in \mathcal{A}$ to some geodesic $\gamma_1 \in \mathcal{A}$ which lies completely outside $M$. We will denote by $M_{\mathcal{A}}$ the set of points of $M$ that belong to $\mathcal{A}$, i.e. $M_{\mathcal{A}} = \underset{\gamma\in\mathcal{A}}{\cup} \gamma$ and similarly $\partial_{\mathcal{A}} M = M_{\mathcal{A}} \cap \partial M$. Finally, with a slight abuse of notation we will denote by $\mathcal{E}'(\widetilde{M})$, the set of tensor fields whose components are compactly supported distributions in the $int(\widetilde{M})$. Now we are ready to state our main theorem in this work:

**Theorem 4.2.2** *Let $(M,g)$ be a simple real analytic Riemannian manifold of dimension $n \geq 3$ and $\widetilde{M}$ be a real analytic extension of $M$. Let $\mathcal{A}$ be any open set of geodesics of $\widetilde{M}$ such that each geodesic $\gamma \in \mathcal{A}$ is continuously deformable within the set $\mathcal{A}$ to some geodesic outside $M$. Let $f \in \mathcal{E}'(\widetilde{M})$ be a symmetric tensor field of order 2 supported in $M$. If $Jf(\gamma,\eta) = 0$ for every $\gamma \in \mathcal{A}$ and for every $\eta \in \gamma^\perp$, then $f = 0$ on $M_{\mathcal{A}}$.*

**Remark:** For the case $n = 2$, observe that the data available by taking the transverse ray transform is the same as the data from the geodesic ray transform up to a diffeomorphism. We also recall that the geodesic ray transform has a non-trivial kernel, see e.g. [30].

## 4.3 Preliminary Results

Let $f$ be a symmetric tensor field defined on $M$, i.e. $f_{ij} = f_{ji}$. Then for each geodesic $\gamma$ with end points on the boundary $\partial M$, and for each $\eta \in \gamma^{\perp}$ where $\gamma^{\perp}$ is the space of vector fields formed by parallel translation along $\gamma$ and orthogonal to $\dot{\gamma}$, the transverse ray transform is given by the bilinear form:

$$Jf(\gamma,\eta) = \int_0^{l(\gamma)} f_{ij}(\gamma(t))\eta^i(t)\eta^j(t)dt \qquad (\eta(t) \in \gamma^{\perp}(t))$$

**Lemma 4.3.1** *Let $\gamma(t)$ be a maximal geodesic of the Riemannian manifold $M$ determined by its starting point $\gamma(0)$ on the boundary and initial direction $\dot{\gamma}(0)$. Furthermore, let $\eta_0$ be a vector orthogonal to $\dot{\gamma}(0)$ at $\gamma(0)$. Any field $\eta(t)$ that is got by the parallel translation of $\eta_0$ along the geodesic $\gamma$ and is orthogonal to it depends analytically on $\gamma(0), \dot{\gamma}(0)$ and $\eta_0$.*

**Remark 4.3.2** From theorem 2.5.1, $\eta(t)$ is the *unique* vector field along $\gamma$ that is formed by the parllel translation of the initial vector $\eta(0) = \eta_0$ along $\gamma$. Also, for $\eta(t)$ as in this lemma, we will say that $\eta(t)$ depends analytically on $\gamma(t)$ and $\eta_0$. ◇

*Proof:* Note that $\eta(t)$ is a parallel translate along $\gamma$ [30, Page 151] and satisfies [21, Theorem 4.13]:

$$\dot{\eta}^k(t) = -\eta^j(t)\dot{\gamma}(t)\Gamma^k_{ij}(\gamma(t)); \qquad \eta(0) = \eta_0$$

where $\Gamma^k_{ij}$ represents Christoffel symbols. Let us rewrite this equation as:

$$\dot{\eta}^k(t) = F(\eta,\gamma,\dot{\gamma}); \qquad \eta(0) = \eta_0 \tag{4.1}$$

$F(\eta,\dot{\gamma},\gamma)$ is analytic in its arguments because we assume that metric $g$ is analytic and hence Christoffel symbols $\Gamma^k_{ij}$ are analytic as well. Let us rename $\dot{\gamma}(t)$ as $\zeta(t)$ and $\gamma(t)$ as $z(t)$. Then we recast equation 4.1 in to the following system:

$$\dot{\eta}^k(t) = F(\eta,\zeta,z); \qquad \eta(0) = \eta_0$$
$$\dot{\zeta}^k(t) = \Gamma^k_{ij}z^i(t)z^j(t); \qquad \zeta(0) = \zeta_0 (= \dot{\gamma}(0))$$

$$\dot{z}^k(t) = \zeta^k(t); \qquad z(0) = z_0(= \gamma(0))$$

(In writing $\dot{\zeta}^k(t) = \Gamma^k_{ij} z^i(t) z^j(t)$, we have made use of the geodesic equation, since $z(t) = \gamma(t)$ and $\zeta(t) = \dot{\gamma}(t)$.) Together the three equations can be rewritten as a new system:

$$\dot{\tilde{\eta}}(t) = \tilde{F}(\tilde{\eta}); \qquad \tilde{\eta}(0) = \tilde{\eta}_0 \tag{4.2}$$

where $\tilde{\eta} = (\eta, \zeta, z)$ and $\tilde{F}$ is the RHS of the system written above. Clearly $\tilde{F}$ is also analytic. Hence by [37, Proposition 6.2], $\tilde{\eta}(t)$ depends analytically on initial conditions $\tilde{\eta}_0$. In particular, $\eta(t)$ depends analytically on $\gamma(0)$ and $\dot{\gamma}(0)$. But $\gamma(0)$ and $\dot{\gamma}(0)$ are the parameters (starting point and initial direction respectively) which uniquely determine a geodesic on the manifold. This shows that $\eta(t)$ depends on geodesics $\gamma$ analytically. □

Before we move on further, we would like to show how to extend the definition of the transverse ray transform to distribution valued tensor fields.

### Extension of the definition of the transverse ray transform to distribution valued tensor fields

Let $\gamma = \gamma_{x,\theta} : [\tau_-(x,\theta), 0] \to M$ be a maximal geodesic of $M$ with initial conditions $\gamma(0) = x$ and $\dot{\gamma}(0) = \theta$. Recall that these maximal geodesics can be thought of as restriction of geodesics of $\widetilde{M}$ with end points in $\widetilde{M} \setminus M$. Let us denote by $I_\gamma^{t,0}$, the operator of parallel translation along the geodesic $\gamma$ i.e. $I_\gamma^{t,0} : T_{\gamma(t)}M \to T_{\gamma(0)}M$. We also recall that this operator is a linear isomorphism between the respective tangent spaces. Since $\eta(t)$ is a parallel translate along $\gamma(t)$, hence there exists a unique $\eta_0 \in T_{\gamma(0)}M$ such that $\eta(t) = I_\gamma^{0,t}(\eta_0)$. Let

$$\Gamma_- := \left\{(x,\xi) \in TM \mid x \in \partial M, |\xi| = 1, \langle \xi, \nu(x)\rangle < 0\right\}$$

where $\nu(x)$ is the unit outer normal to $\partial M$ at $x$. Next, consider the space of symmetric tensor fields of rank 2 on $T^*M$ which will be denoted by: $S^2(T^*M)$.

We know that there is a canonical embedding that identifies tensor fields on $M$ with tensor fields on $T^*M$ which are independent of the second argument, see [30, 3.4.7]. Under this identification, the field $f \in S^2(M)$ will be identified with the corresponding field in $S^2(T^*M)$ which we will again refer to as $f$. Further, if we take the restriction of the projection operator for the tangent bundle, $p : \Gamma_- \to M$, then this induces a smooth map between the space of tensor fields, $p^* : S^2M \to S^2(\Gamma_-)$ [20, Proposition 11.9]. Now using the above mentioned identification of tensor fields in $S^2(M)$ with tensor fields in $S^2(T^*M)$ which are independent of the second argument, let us consider the space of pullback of such tensor fields $p^*(S^2(T^*M))$ and denote it by $S^2(\Pi_M)$. Following Sharafutdinov, we define the operator:

$$\widetilde{J} : C^\infty(S^2(T^*M)) \mapsto C^\infty(S^2\Pi_M)$$

where $\widetilde{J}$ is given by the relation

$$\widetilde{J}f(x,\theta) = \int_{\tau_-(x,\theta)}^{0} I_\gamma^{t,0}(P_{\dot\gamma(t)}f(\gamma(t))dt \qquad (x,\theta) \in \Gamma_-$$

Here, $(P_{\dot\gamma(t)}f)_{ij} = \big(\delta_i^k - \frac{1}{|\dot\gamma(t)|^2}\dot\gamma(t)_i\dot\gamma(t)^k\big)\big(\delta_j^l - \frac{1}{|\dot\gamma(t)|^2}\dot\gamma(t)_j\dot\gamma(t)^l\big)f_{kl} = \big((Id - \frac{\dot\gamma\dot\gamma^t}{|\dot\gamma|^2})f(Id - \frac{\dot\gamma\dot\gamma^t}{|\dot\gamma|^2})\big)_{ij}$. From [30, equation 5.2.5], we have the following:

$$\langle \widetilde{J}f(x,\theta), \eta_0 \otimes \eta_0 \rangle = Jf(\gamma,\eta) \tag{4.3}$$

Thus in order to make sense of the transverse ray transform for distribution valued tensor fields, all we need to do is to interpret $\widetilde{J}f(x,\theta)$ by duality for compactly supported distribution valued tensor fields $f$. For this we will need an expression for the adjoint $(\widetilde{J})^*$. First let $f \in L^2(M)$ and take any $\phi(x,\xi) \in C_c^\infty(S^2\Pi_M)$. We will consider the following inner product:

$$(\widetilde{J}f,\phi)_{\Gamma_-} = \int_{\Gamma_-} \bar\phi(x,\theta) \int_{\tau_-(x,\theta)}^{0} I_\gamma^{t,0}(P_{\dot\gamma(t)}f(\gamma(t)))dtd\mu(x,\theta)$$

Let us now define a function $\phi^\sharp(\gamma(t),\dot\gamma(t))$ such that it is constant along the geodesic and is equal to $\bar\phi(x,\theta)$ on $\Gamma_-$, i.e.

$$\nabla_{\dot\gamma}\phi^\sharp(\gamma(t),\dot\gamma(t)) = 0, \qquad \phi^\sharp(\gamma(0),\dot\gamma(0)) = \bar\phi(x,\theta).$$

This means that $\phi^\sharp(\gamma(t),\dot\gamma(t))$ is formed by parallel translation of $\bar\phi(x,\theta)$ along the geodesic $\gamma(t)$. Then the above can be rewritten as:

$$\begin{aligned}
(\widetilde{J}f,\phi)_{\Gamma_-} &= \int_{\Gamma_-}\int_{\tau_-(x,\theta)}^{0} I_\gamma^{t,0}(P_{\dot\gamma(t)}f(\gamma(t)))\bar\phi(x,\theta)\,dt\,d\mu(x,\theta)\\
&= \int_{\Gamma_-}\int_{\tau_-(x,\theta)}^{0} P_{\dot\gamma(t)}f(\gamma(t))I_\gamma^{0,t}\bar\phi(x,\theta)\,dt\,d\mu(x,\theta)\\
&= \int_{\Gamma_-}\int_{\tau_-(x,\theta)}^{0} P_{\dot\gamma(t)}\big(f(\gamma(t))\big)\phi^\sharp(\gamma(t),\dot\gamma(t))\,dt\,d\mu(x,\theta)\\
&= \int_{\Gamma_-}\int_{\tau_-(x,\theta)}^{0} f(\gamma(t))P_{\dot\gamma(t)}\big(\phi^\sharp(\gamma(t),\dot\gamma(t))\big)\,dt\,d\mu(x,\theta)
\end{aligned}$$

Now we apply Santalo's formula [28, Lemma 3.3.2] to the above and get:

$$\begin{aligned}
(\widetilde{J}f,\phi)_{\Gamma_-} &= \int_{SM} f(x)P_\xi\big(\phi^\sharp\big)\,d\sigma \qquad \text{where } d\sigma \text{ is a measure on } SM\\
&= \int_M f(x)\int_{S_xM} P_\xi\big(\phi^\sharp\big)\,d\sigma_x(\xi)\,d\,\mathrm{Vol}(x)\\
&= (f,(\widetilde{J})^*\phi)_{L^2(M)}
\end{aligned}$$

where we have the adjoint of $\widetilde{J}$ given by

$$\begin{aligned}
(\widetilde{J})^*\phi &= \int_{S_xM} P_\xi\big(\phi^\sharp\big)\,d\sigma_x(\xi)\\
&= \int_{S_xM}\Big(Id-\frac{\xi\xi^t}{|\xi|^2}\Big)\phi^\sharp\Big(Id-\frac{\xi\xi^t}{|\xi|^2}\Big)\,d\sigma_x(\xi)
\end{aligned}$$

Now for a compactly supported distribution valued tensor field $f$ and for $\phi\in C_c^\infty(S^2\Pi_M)$ we define,

$$\langle\widetilde{J}f,\phi\rangle := \langle f,(\widetilde{J})^*\phi\rangle.$$

**Remark:** To understand $Jf(\gamma,\eta) = \langle \widetilde{J}f(x,\theta), \eta_0 \otimes \eta_0 \rangle$ when $f$ is a compactly supported distribution, we multiply $\eta_0 \otimes \eta_0$ by a compactly supported function $\Psi(x,\theta)$ which is identically 1 on the support of $f$ such that $\Psi \times (\eta_0 \otimes \eta_0)$ is in $C_c^\infty(S^2\Pi_M)$. Then we follow the procedure described above to interpret the tranverse ray transform for such fields.

## 4.4 A microlocal proposition

The following proposition is an analogue of [36, Proposition 2].

**Proposition 4.4.1** *Let $M$ be as above. Let $(x_0,\xi_0) \in T^*M \smallsetminus 0$ and let $\gamma_0$ be a fixed geodesic through $x_0$ normal to $\xi_0$. Let $Jf(\gamma,\eta) = 0$ for some symmetric 2-tensor $f \in \mathcal{E}'(\widetilde{M})$ supported in $M$ and for all $\gamma \in nbd.(\gamma_0)$ and for all $\eta \in \gamma^\perp$. Then*

$$(x_0,\xi_0) \notin WF_A(f).$$

*Proof:* The following argument is adapted from the proof of [36, Proposition 2]. For the compact, simple, real analytic Riemannian manifold $M$, we can construct analytical semi geodesic coordinates $(x',x^n)$ in some tubular neighbourhood $U$ of $\gamma_0$ such that on the boundary $x_n = 0$ and the lines corresponding to $x' =$ constant correspond to the geodesics of $M$. This construction has been worked out and used by Stefanov and Uhlmann in many articles, see e.g. [36, section 2]. Furthermore, we assume $x_0 = 0$ and $x' = 0$ on $\gamma_0$. We can represent $U$ in these coordinates by: $U = \{x : |x'| \le \epsilon\}$ for some $\epsilon << 1$. We can choose $\epsilon$ to be so that the $\{|x'| \le \epsilon; x^n = l^\pm\}$ is outside $M$. Then $\xi_0 = ((\xi_0)',0)$ with $(\xi_0)_n = 0$. Our goal is to show:

$$(0,\xi_0) \notin WF_A(f).$$

Consider $Z = \{|x| < \frac{7\epsilon}{8} : |x_n| = 0\}$ and let $x'$ variable be denoted on $Z$ by $z'$. Then $(z',\theta')$ are local coordinates in nbd.$(\gamma_0)$ given by $(z',\theta') \to \gamma_{(z',0),(\theta',1)}$. Here, $|\theta'| << 1$ (where, the geodesic is in the direction $(\theta',1)$).

Let $\{\chi_N(z')\}$ be a sequence of cutoff functions satisfying the estimates:

$$\partial_\alpha(\chi_N) \le (CN)^{|\alpha|}; \qquad \text{for some } C \text{ and for } |\alpha| < N.$$

Let $\theta = (\theta', 1)$. We multiply

$$Jf(\gamma_{(z',0;\theta)}, \eta_{(z',0;\theta)}) = 0$$

by $\chi_N(z')e^{i\lambda z'.\xi'}$, where $\lambda > 0$, $\xi'$ is in a complex neighbourhood of $(\xi_0)'$ and integrate it with respect to $z'$ to get:

$$\int e^{i\lambda z'(x,\theta').\xi'} \chi_N(z') f_{ij}(\gamma_{(z',0;\theta)}(t)) {\eta^i}_{(z',0;\theta)}(t) {\eta^j}_{(z',0;\theta)}(t) dt dz' = 0$$

By following the arguments in [36, Proposition 2] (which involves a change of variables), we get the following equation :

$$\int e^{i\lambda z'(x,\theta').\xi'} a_N(x,\theta') f_{ij}(x) b^i(x,\theta') b^j(x,\theta') dx = 0 \tag{4.4}$$

Here, $(x,\theta') \to a_N$ (which is obtained from $\chi_N$ by a change of variables) is analytic and satisfies the estimate:

$$|\partial^\alpha a_N| \le (CN)^{|\alpha|}, \quad \alpha \le N, \tag{4.5}$$

Also, note that $b(x,\theta)$ is similarly got from $\eta_{(z',0;\theta)}$ and satisfies $b(0,\theta') = \eta_{(0,0;\theta)} \in \theta^\perp$ and $a_N(0,\theta') = 1$.

To fix ideas let $\xi_0 = e_{n-1}$ and the $\theta_0$ corresponding to the geodesic $\gamma_0$ be $e_n$ as in the proof of [36, Proposition 2]. Further, let us choose $\theta(\xi)$ to be a vector depending analytically on $\xi$ in a neighbourhood of the geodesic $\gamma_0$ in the following way:

$$\theta(\xi) = \left(\xi_1, \dots, \xi_{n-2}, -\frac{\xi_1^2 + \dots + \xi_{n-2}^2 + \xi_n}{\xi_{n-1}}, 1\right).$$

Since, $\theta$ can be made to analytically depend on $\xi$, we can also make the hyperplane

$\theta^{\perp}$ depend analytically on $\xi$. This is to say that every $\eta$ which is orthogonal to $\gamma$ and is parallel along it also depends analytically on $\xi$. Indeed, this is to be expected as we showed in the proof of Lemma 4.3.1, that $\eta(t)$ depends analytically on the initial conditions, in particular on $\theta$ and since $\theta$ depends on $\xi$ analytically, so will $\eta$ depend on $\xi$ analytically. In fact, this can be achieved by carrying out a Gram Schmidt orthogonalization procedure. Consider the vectors: $v_1, \dots, v_{n-1}, \theta(\xi)$ such that these form a basis at $T_{\gamma(0)}M$. Next, let $\eta_1(0;\xi) = v_1 - \mathrm{proj}_{\theta(\xi)} v_1$. This depends analytically on $\xi$ because $\theta(\xi)$ does and because the projection map is also analytic. Now $\eta_2(0;\xi) = v_2 - \mathrm{proj}_{\eta_1(0;\xi)} v_2 - \mathrm{proj}_{\theta(\xi)} v_2$. We can construct $\eta_3(0;\xi), \dots, \eta_{n-1}(0;\xi)$ in a similar fashion, and all $\eta_i(0;\xi)$ will depend on $\xi$ analytically. Now the parallel translate of $\eta$ along $\gamma$ can also be shown to depend on $\xi$ analytically through its analytical dependence on initial conditions. It is also clear that if $\eta_i(t)$ and $\eta_j(t)$ depend analytically on $\xi$, then any linear combination also depends analytically on $\xi$.

Now with the choice of $\theta(\xi)$ made as above near $\xi = \xi_0$, it is easy to check that the following conditions are satisfied :

$$\theta(\xi)\cdot\xi = 0, \quad \theta^n(\xi) = 1 \qquad \text{and}$$
$$\theta(\xi_0) = (0, \dots, 1) = e_n$$

We will rewrite (4.4) using the above mapping in the following form:

$$\int e^{i\lambda\phi(x,\xi)} \tilde{a_N}(x,\xi) f_{ij}(x) \tilde{b}^i(x,\xi) \tilde{b}^j(x,\xi)dx = 0. \tag{4.6}$$

Here $\phi(x,\xi) = z'\cdot\xi'$ is the phase function, $\tilde{b}(x,\xi) = \eta_{(z',0;\theta(\xi))}(t)$, $t = t(x,\theta(\xi))$ and $z' = z'(x,\theta(\xi))$, see [36]. This phase function has been shown in [36] to be non-degenerate in a neighborhood of the geodesic $\gamma_0$.

Now we quote the following lemma from [36]:

**Lemma 4.4.2** *[36, Lemma 5] Let, $\theta(\xi)$ and $\phi(x,\xi)$ be as above. Then, $\exists\ \delta > 0$*

*such that if*

$$\phi_\xi(x,\xi) = \phi_\xi(y,\xi)$$

*for some* $x \in U$, $|y| < \delta$, $|\xi - \xi_0| < \delta$ *where* $\xi$ *is complex, then* $y = x$.

We will study the analytic wavefront set of $f$ using Sjöstrand's complex stationary phase method. For this assume $x$, $y$ as in Lemma 4.4.2 and $|\xi_0 - \upsilon| < \frac{\delta}{\tilde{C}}$ with $\tilde{C} >> 2$ and $\delta << 1$. Multiply (4.6) by

$$\tilde{\chi}(\xi - \upsilon)e^{i\lambda\left(i\frac{(\xi-\upsilon)^2}{2} - \phi(y,\xi)\right)}$$

where $\tilde{\chi}$ is the characteristic function of the ball $B(0,\delta) \subset \mathbb{C}^n$ and then integrate w.r.t. $\xi$ to get :

$$\iint e^{i\lambda\Phi(y,x,\xi,\upsilon)}\tilde{\tilde{a_N}}(x,\xi)f_{ij}(x)\tilde{b}^i(x,\xi)\tilde{b}^j(x,\xi)dxd\xi = 0. \tag{4.7}$$

In the above equation, $\tilde{\tilde{a_N}} = \tilde{\chi}(\xi - \upsilon)\tilde{a}_N$ is another analytic and elliptic amplitude for $x$ close to zero and $|\xi - \upsilon| < \frac{\delta}{\tilde{C}}$ and

$$\Phi = -\phi(y,\xi) + \phi(x,\xi) + \frac{i}{2}(\xi - \upsilon)^2.$$

Furthermore,

$$\Phi_\xi = \phi_\xi(x,\xi) - \phi_\xi(y,\xi) + i(\xi - \upsilon).$$

To apply the stationary phase method we need to know the critical points of $\xi \mapsto \Phi$. Using lemma 5.4.3 above we have :

1. If $y = x$, $\exists$ a unique real critical point $\xi_c = \upsilon$
2. If $y \neq x$, there are no real critical points
3. Also by Lemma 5.4.3, if $y \neq x$, there is a unique complex critical point if $|x - y| < \delta/C_1$ and no critical points for $|x - y| > \delta/C_0$ for some constants $C_0$ and $C_1$ with $C_1 > C_0$.

Define, $\psi(x,y,\upsilon) := \Phi(\xi_c)$. Then at $x = y$

(i) $\psi_y(x,x,\upsilon) = -\phi_x(x,\upsilon)$ (ii) $\psi_x(x,x,\upsilon) = \phi_x(x,\upsilon)$ (iii) $\psi(x,x,\upsilon) = 0$.

Now, we split the $x$ integral in (5.6) in to two parts : we integrate over $\{x : |x-y| > \delta/C_0\}$ for some $C_0 > 1$ and its complement. Since, $|\Phi_\xi|$ has a positive lower bound for $\{x : |x-y| > \delta/C_0\}$ and there are no critical points of $\xi \to \Phi$ in this set, we can estimate that integral in the following manner: First note that, $e^{i\lambda\Phi(x,\xi)} = \frac{\Phi_\xi \partial_\xi}{i\lambda|\Phi_\xi|^2} e^{i\lambda\Phi(x,\xi)}$. Using, (5.7) and integrating by parts $N$ times with respect to $\xi$ and the fact that on the boundary $|\xi - \upsilon| = \delta$, we get

$$\left| \iint_{|x-y|>\delta/C_0} e^{i\lambda\Phi(y,x,\xi,\upsilon)} \tilde{\tilde{a_N}}(x,\xi) f_{ij}(x) \tilde{b}^i(x,\xi)\tilde{b}^j(x,\xi)\,dx\,d\xi \right| \le C\left(\frac{CN}{\lambda}\right)^N + CNe^{-\frac{\lambda}{C}} \tag{4.8}$$

We choose $N \le \lambda/Ce \le N+1$ to get an exponential error on the right. Now in estimating the integral

$$\left| \int_{|x-y|\le\delta/C_0} e^{i\lambda\Phi(y,x,\xi,\upsilon)} \tilde{\tilde{a_N}}(x,\xi) f_{ij}(x) \tilde{b}^i(x,\xi)\tilde{b}^j(x,\xi)\,dx\,d\xi \right| \tag{4.9}$$

we use [33, Theorem 2.8] and the remark following that to conclude:

$$\int_{|x-y|\le\delta/C_0} e^{i\lambda\psi(x,\alpha)} f_{ij}(x) B^{ij}(x,\alpha;\lambda)\,dx = \mathcal{O}(e^{-\lambda/C}) \tag{4.10}$$

where $\alpha = (y,\upsilon)$ and $B$ is a classical analytical symbol with principal part $\tilde{b}\otimes\tilde{b}$. See appendix of [1] for a proof of estimates in (5.8) and (4.10).

Let, $\beta = (y,\mu)$ where, $\mu = \phi_y(y,\upsilon) = \upsilon + \mathcal{O}(\delta)$. At $y = 0$, we have $\mu = \upsilon$. Also $\alpha \to \beta$ is a diffeomorphism following similar analysis as in [36, Section 4]. If we write $\alpha = \alpha(\beta)$, then the above equation becomes:

$$\int_{|x-y|\le\delta/C_0} e^{i\lambda\psi(x,\beta)} f_{ij}(x) B^{ij}(x,\beta;\lambda)\,dx = \mathcal{O}(e^{-\lambda/C}) \tag{4.11}$$

where $\psi$ satisfies (i), (ii) and (iii), and $B$ is a classical analytical symbol as before and :

$$\psi_y(x,x,\upsilon) = -\mu, \quad \psi_x(x,x,\upsilon) = \mu \quad \text{ and } \quad \psi_y(x,x,\upsilon) = 0$$

The symbols in (4.11) satisfy :

$$\sigma_P(B)(0,0,\mu) = \eta(\mu) \otimes \eta(\mu) = \eta^{\otimes 2}(\mu)$$

where $\eta$ is a vector perpendicular to $\theta$ as before.

We will now show that there exists $N = \frac{n(n+1)}{2}$ unit vectors at $x_0$, say, $\eta_1, \eta_2, \ldots, \eta_N$ and such that for each $\eta_p$ there exists some vector perpendicular to $\eta_p$ in a small neighbourhood of $\theta_0 = e_n$ such that any constant symmetric 2- tensor $f_{ij}$ is uniquely determined by the numbers $f_{ij}(\eta_p)_i(\eta_p)_j$.

**Remark 4.4.3** This will show that the symbol of the tensor valued operator, given by $\eta_p \otimes \eta_p$, $p = \{1, 2, \ldots, N\}$ is elliptic at $(0, 0, \xi_0)$. Note also that the number $N = \frac{n(n+1)}{2}$ corresponds to the number of independent components of any symmetric 2- tensor. $\diamond$

First of all, let

$$S = \{\eta : \exists\theta \text{ in nbd.}(\theta_0) \text{ such that } \theta \in \xi_0^\perp, \langle\theta, \eta\rangle = 0\}$$

We will first show that for $n \geq 3$, $\text{Span}(S) = \mathbb{R}^n$. Let $\theta_0 = e_n$ and $\xi_0 = e_{n-1}$ as has been the case in the proof of this proposition. Then it is immediately obvious that $e_i = \eta_i \in S$ for $i = 1, \ldots, n-1$. Now consider a vector $v = \epsilon e_1 + e_n \in \xi_0^\perp$ such that $v \in nbd.(\theta_0)$. Then $\eta_n = e_1 - \epsilon e_n$ also belongs to $S$. Clearly now $\text{Span}(S) = \mathbb{R}^n$. Hence, $\exists \eta_1, \ldots, \eta_n$ in the set $S$ which are linearly independent. Now we make the following claim:

**Lemma 4.4.4** *There exists a linear combination $a_p\eta_p + a_q\eta_q$, $p, q = 1, \ldots, n$ and $p < q$ for each $p$ and $q$ such that $a_p\eta_p + a_q\eta_q \in S$. Hence there exists $\binom{n}{2}$ such linear combinations for different pairs of $p$ and $q$ which can be listed as $\eta_l$, $l = (n+1), \ldots, \frac{n(n+1)}{2}$.*

*Proof:* Our goal is to show that for each pair $\eta_p, \eta_q$ as stated in the claim, $\exists\theta$ in nbd.$(\theta_0)$ such that $\theta \in \xi_0^\perp, \langle\theta, (a_p\eta_p + a_q\eta_q)\rangle = 0$. Since, $\eta_p$,$\eta_q$ are in $S$, hence there must exist $\theta_p$ and $\theta_q$ in $nbd.(\theta_0) \cap \xi_0^\perp$such that $\langle\theta_p, \eta_p\rangle = 0$ and $\langle\theta_q, \eta_q\rangle = 0$. Now we have 3 cases to consider:

**Case 1**: $\langle\theta_q, \eta_p\rangle = 0$ and $\langle\theta_p, \eta_q\rangle = 0$.

In this case, it is clear that any linear combination $\eta_p$ and $\eta_q$ is in $S$. This is because $\langle\theta_p, (\eta_p + \eta_q)\rangle = \langle\theta_q(\eta_p + \eta_q)\rangle = 0$.

**Case 2**: Only one of $\langle\theta_q, \eta_p\rangle$ or $\langle\theta_p, \eta_q\rangle$ is zero.

Without the loss of generality, let $\langle\theta_q, \eta_p\rangle = 0$. In this case also, any linear combination $a_p\eta_p + a_q\eta_q \in S$. This is because, $\langle\theta_q, (a_p\eta_p + a_q\eta_q)\rangle = 0$.

**Case 3**: $\langle\theta_q, \eta_p\rangle \neq 0$ and $\langle\theta_p, \eta_q\rangle \neq 0$

Consider the vector $\theta_p + \epsilon\theta_q$ where $\epsilon \neq 0$ is chosen such that $\theta_p + \epsilon\theta_q \in nbd.(\theta_0) \cap \xi_0^\perp$. Now let $a_p\eta_p + a_q\eta_q$ be such that $a_p = 1$ and $a_q = -\frac{\langle\theta_p,\eta_q\rangle}{\epsilon(\langle\theta_q,\eta_p\rangle)}$. Then it can be readily seen that $\langle(\theta_p + \epsilon\theta_q), (a_p\eta_p + a_q\eta_q) = 0$ for the above mentioned choice for $a_p$ and $a_q$. Hence this $a_p\eta_p + a_q\eta_q \in S$.

This proves our claim. □

So consider the collection of $N = \frac{n(n+1)}{2}$ vectors $\eta_k$ as listed above for $k = 1, \dots, N$ (where we represent $\frac{\eta_k}{|\eta_k|}$ as $\eta_k$ by a slight abuse of notation). The set $\{\eta_k^{\otimes 2}\}_{k=1}^{k=N}$ is linearly independent and determines $f_{ij}$ from the numbers: $f_{ij}\eta_k^i\eta_k^j$ for $k = 1, \dots, N$. Coming back to the proof of Proposition 4.4.1, we can get $N$ equations of the kind 5.10, with symbols $B_k$ such that $\sigma_p(B_k) = (\eta_k)^{\otimes 2}$ for $k = 1, \dots, N$. Together these $N$ equations can be thought of as a tensor valued operator applied to tensor $f$, and then by similar analysis as in the proof of [36, Proposition 2], we conclude the proof of Proposition 4.4.1. □

## 4.5 Proof of the support theorem

*Proof (Proof of Theorem 4.2.2):* We will use similar ideas as in [16] for the proof. We extend $f$ outside $M$ by zero, i.e. $f = 0$ in $\widetilde{M} \setminus M$. Let $\gamma_0$ be a fixed geodesic in $\mathcal{A}$ which is continuously deformable within the set $\mathcal{A}$ to some geodesic $\gamma_1$ in $\mathcal{A}$

that lies in $\widetilde{M} \setminus M$. Let $\gamma_t$ be an intermediate geodesic in the deformation. As $f$ is compactly supported within $M$ and $\mathcal{A}$ is open, hence such a $\gamma_1$ can always be found. Furthermore, we parametrize these geodesics by their starting points $x \in \partial\widetilde{M}$ and their initial directions $\theta \in S^{n-1}$. Accordingly, let $(x_t, \theta_t)$ denote the starting point and the initial direction for the geodesic $\gamma_t$. Consider a "cone of geodesics" formed by geodesics around $\gamma_t$ having the same starting point as $\gamma_t$ i.e. $x_t$ and with initial directions that are sufficiently close to $\theta_t$, say within $\epsilon > 0$, such that this cone is still in $\mathcal{A}$. Clearly, by a compactness argument, there is one such cone that does not intersect $\mathrm{supp}(f)$. Now carry out the construction of such cones for each $\gamma_t$ in the aforementioned deformation, i.e. for all values of $t \in [0,1]$. We will call these cones $C_t$. Let

$$t_1 = \inf\{t \in [0,1] : C_{t_2} \cap \mathrm{supp}(f) = \varnothing, \quad \forall t_2 > t\}.$$

Suppose that $t_1 > 0$. Then the cone $C_{t_1}$ intersects $\mathrm{supp}(f)$ at some point, say $p_0$ that lies on the boundary of the cone. By the Sato- Kawai- Kashiwara Theorem, $(p_0, \zeta_0) \in WF_A(f)$ where $\zeta_0$ is normal to the cone $C_{t_1}$. But this counters Proposition 4.4.1. This shows that $f$ is zero in a neighborhood of $p_0$. By a compactness argument this shows that $f = 0$ for a positive distance away from the cone $C_{t_1}$, see [16]. But this would contradict the claim that $t_1 > 0$ is the infimum. Hence $t_1 = 0$ which completes the proof of our main theorem. $\square$

# Chapter 5

# Support theorems for integral moments of a symmetric $m$-tensor field

This chapter presents support theorems for integral moments of a symmetric $m$-tensor field. These results were obtained in collaboration with Dr. Rohit Kumar Mishra who is currently a post-doctoral researcher in the Department of Mathematics at University of California (Santa Cruz). It has also appeared previously as part of his doctoral thesis at Tata Institute of Fundamental Research- Centre for Applicable Mathematics (TIFR-CAM), Bangalore. In this work, the first two sections are of an introductory nature. Section 5.3, Subsection 5.4.2, Section 5.5, Subsection 5.6.1 have been worked out by Dr. Mishra and the rest is my contribution. We also note that many of the definitions that appear in this chapter have appeared elsewhere in this thesis. The problem set-up also bears some similarities to the problem set-up in Chapter 3. However, we recall them again in this chapter for the sake of completeness.

## 5.1 Introduction

Let $(\Omega, g)$ be a compact, simple, real-analytic Riemannian manifold of dimension $n$ with smooth boundary. We will parametrize the maximal geodesics in $\Omega$ with endpoints on $\partial\Omega$ by their starting points and directions.

Set

$$\Gamma_- := \left\{(x,\xi) \in T\Omega |\ x \in \partial\Omega, |\xi| = 1, \langle \xi, \nu(x)\rangle < 0\right\},$$

where $\nu(x)$ is the outer unit normal to $\partial\Omega$ at $x$. Then we will define the $q$-th integral moment of a symmetric $m$-tensor field $f$, $I^q f$ as a function on $\Gamma_-$ by

$$I^q f(x,\xi) = \int_0^{l(\gamma_{x,\xi})} t^q \langle f(\gamma_{x,\xi}(t)), \dot{\gamma}^m_{x,\xi}(t)\rangle dt = \int_0^{l(\gamma_{x,\xi})} t^q f_{i_1\dots i_m}(\gamma_{x,\xi}(t))\dot{\gamma}^{i_1}_{x,\xi}(t)\cdots\dot{\gamma}^{i_m}_{x,\xi}(t) dt.$$

where $\gamma_{x,\xi}(t)$ is the geodesic starting from $x$ in the direction $\xi$ and $l(\gamma_{x,\xi})$ is the value of the parameter $t$ at which this geodesic intersects the boundary again. The above definition of integral moments for a symmetric $m$-tensor fields was first introduced by Sharafutdinov in the context of $\mathbb{R}^n$, see [29]. In the same paper he proved that if the first $(m+1)$ integral moments $I^q f$ for $q = 0, 1, \ldots, m$ of a compactly supported symmetric $m$-tensor field $f$ are known along all straight lines, then $f$ can be uniquely recovered.

Zeroth integral moment coincides with the usual geodesic ray transform of a symmetric $m$-tensor field. In this work, we are interested in injectivity results and support theorem for integral moments defined above. Microlocal techniques play a very crucial role in proving such results. Guillemin first introduced the microlocal approach in the Radon transform setting, see [12]. Analytic microlocal techniques were used by Boman and Quinto in [4] to prove support theorems for Radon transforms with positive real-analytic weights. For more literature on such support theorems, we refer to the reader [2, 3, 5, 10, 11, 25, 27, 40] and references therein. For the analytic microlocal techniques used in this paper, we will mainly refer to [18, 33–36].

The geodesic ray transform of any symmetric tensor field of order 2, which in our notation will be denoted by $I^0(f)$ arises naturally in the context of lens and boundary rigidity problems and has been studied in e.g. [30], [28], [34], [35]. Support theorems for such transforms have been of independent interest among mathematicians. In [35], the authors prove a s-injectivity result for symmetric 2 tensors fields. The same proof works for a symmetric tensor field of any order. That is if $I^0(f) = 0$ for all the geodesics of $\Omega$ then its solenoidal part vanishes. A question arises as to what data is sufficient for us to conclude such an injectivity result for the tensor field $f$ itself. Using the result stated above, we show that if $I^q f = 0$ for $q = 0, 1, \ldots, m$ for all the geodesics of $\Omega$, then $f = 0$. Injectivity result for the local geodesic ray transform of a function has been proved in [39] using new techniques. We also treat the case in which the integral moments are known for the open set of geodesics that do not intersect a given geodesically convex set. We do so using the techniques laid out in [18], where the authors prove a Helgason type support theorem for symmetric tensor fields of order 2 over simple, real analytic Riemannian manifolds. We first extend the result in [18] for symmetric $m$ tensor fields. Using this new result, we prove a stronger version of such support type theorems, i.e. if we know $I^q f = 0$ for $q = 0, 1, \ldots, m$ over the open set of geodesics not intersecting a convex set, then it implies that the support of $f$ lies in the convex set. We would also like to mention that Krishnan already proved such a support theorem for the case of functions in [16]. The chapter is organized as follows. In Section 4.2 we give the definitions and our main theorems. Section 4.3 has some preliminary propositions and lemmas that are needed for the proof of the main theorems. In Section 4.4 we will prove a Helgason type support theorem which we state in Section 4.2 and prove the support theorem. In Section 4.5 we prove the s-injectivity result mentioned above and use it to prove the injectivity of integral moments. We will provide proof of some lemmas and inequalities in the Appendix.

## 5.2 Definitions and statements of the theorems

**Definition 5.2.1 (Simple Manifold)** *A compact Riemannian manifold* $(\Omega, g)$ *with boundary is said to be simple if*

*(i) The boundary* $\partial\Omega$ *is strictly convex:* $\langle \nabla_\xi \nu(x), \xi \rangle > 0$ *for each* $\xi \in T_x(\partial\Omega)$ *where* $\nu(x)$ *is the unit outward normal to the boundary.*

*(ii) The map* $\exp_x : \exp_x^{-1}(\Omega) \to \Omega$ *is a diffeomorphism for each* $x \in \Omega$*.*

The second condition ensures that any two points $x, y$ in $\Omega$ are connected by a unique geodesic in $\Omega$ that depends smoothly on $x, y$. Any simple manifold $M$ is necessarily diffeomorphic to a ball in $\mathbb{R}^n$, see [30]. Therefore, in the analysis of simple manifolds, we can assume that $\Omega$ is a domain $\Omega \subset \mathbb{R}^n$. We are going to work on a fixed simple Riemannian manifold $(\Omega, g)$ with a fixed real analytic atlas. A tensor field is said to be analytic on a set $U$ if it is real analytic in some neighborhood of $U$. Let $S^m(\Omega)$ be the collection of symmetric $m$-tensor fields defied on $\Omega$. We will work with symmetric $m$-tensor field $f = \{f_{i_1 \ldots i_m}\}$. We will assume the Einstein summation convention and raise and lower indexes using the metric tensor. The tensors $f$ and $f^{i_1\dots i_m} = f_{j_1\dots j_m} g^{i_1 j_1}\cdots g^{i_m j_m}$ will be thought of as same tensors with different representations.

It is well known from [30] that any symmetric $m$-tensor field can be decomposed uniquely in the following way:

**Theorem 5.2.2** *[30, Theorem 3.3.2] Let $\Omega$ be a compact Riemannian manifold with boundary; let $k \geq 1$ and $m \geq 0$ be integers. For every field $f \in H^k(S^m(\Omega))$, there exist uniquely determined $f^s \in H^k(S^m(\Omega))$ and $v \in H^{k+1}(S^{m-1}(\Omega))$ such that*

$$f = f^s + dv, \qquad \delta f^s = 0, \qquad v|_{\partial\Omega} = 0.$$

*We call the fields $f^s$ and $dv$ the solenoidal and potential parts respectively of the field $f$.*

Let $\widetilde{\Omega}$ be an open, real analytic extension of $\Omega$ such that $g$ can also be extended to a real analytic metric in $\widetilde{\Omega}$. We will also extend all symmetric tensor fields $f$ defined on $\Omega$ by 0 in $\widetilde{\Omega} \setminus \Omega$. We will think of each maximal geodesic in $\Omega$ as a restriction of a geodesic with distinct endpoints in $\widetilde{\Omega} \setminus \Omega$ to $\Omega$. Let $\gamma_{[x,y]}$ be the geodesic connecting $x$ and $y$.

Let $\mathcal{A}$ be an open set of geodesics with endpoints in $\widetilde{\Omega} \setminus \Omega$ such that any geodesic in $\mathcal{A}$ is homotopic, within the set $\mathcal{A}$, to a geodesic lying outside $\Omega$. Set of points lying on the geodesics in $\mathcal{A}$ is denoted by $\Omega_{\mathcal{A}}$ i.e. $\Omega_{\mathcal{A}} = \bigcup_{\gamma \in \mathcal{A}} \gamma$ and $\partial_{\mathcal{A}}\Omega = \Omega_{\mathcal{A}} \cap \partial\Omega$. Now we will define what we mean by a geodesically convex subset.

**Definition 5.2.3** *A subset $K$ of the Riemannian manifold $(\Omega, g)$ is said to be geodesically convex if for any two points $x \in K$ and $y \in K$, the geodesic connecting them lies entirely in the set $K$.*

Finally, let $\mathcal{E}'(\widetilde{\Omega})$ be the space of compactly supported tensor fields. We can then extend the definition of $I$ by duality on tensor fields which are distributions in $\widetilde{\Omega}$ supported in $\Omega$, see [18]. Now we are ready to state the main theorems that we will prove in this article.

**Theorem 5.2.4** *Let $f$ be a symmetric $m$-tensor field on a simple real analytic manifold $(\Omega, g)$ with components in $\mathcal{E}'(\widetilde{\Omega})$ and supported in $\Omega$ and $K$ be a closed geodesically convex subset of $\Omega$. If for each geodesic $\gamma$ not intersecting $K$, we have that $I^0 f(\gamma) = 0$ then we can find a $(m-1)$-tensor field $v$ with components in $\mathcal{D}'(int(\widetilde{\Omega}) \smallsetminus K)$ such that $f = dv$ in $int(\widetilde{\Omega}) \smallsetminus K$ and $v = 0$ in $int(\widetilde{\Omega}) \smallsetminus \Omega$.*

Here we would like to mention that this theorem has been shown to be true for the case $m = 2$ in [18].

**Theorem 5.2.5** *Let $f$ be a symmetric $m$-tensor field on a simple real analytic manifold $(\Omega, g)$ with components in $\mathcal{E}'(\widetilde{\Omega})$ and supported in $\Omega$ and $K$ be a closed geodesically convex subset of $\Omega$. If for each geodesic $\gamma$ not intersecting $K$, we have that $I^q f(\gamma) = 0$ for $q = 0, 1, \ldots, m$ then $supp(f) \subset K$.*

**Theorem 5.2.6** *Let $(\Omega, g)$ be a simple real analytic manifold and $g$ is real analytic in a neighborhood of $\mathrm{cl}(\Omega)$. If for a symmetric $m$-tensor field $f$ with components in $L^2(\Omega)$, we have that $I^q f = 0$ for $q = 0, 1, \ldots, m$. Then $f = 0$.*

Here we would like to comment that the Theorem 5.2.6 also follows as a corollary of Theorem 5.2.5 when $f$ is supported in $\Omega$, however as we show in this paper that it can also be proved independently using $s$-injectivity of ray transform where we say $I^0 = I$ is $s$-injective if $If = 0$ implies $f^s = 0$. In the next section we will prove a proposition and some lemmas that will be needed for the proofs of our main theorems.

## 5.3 Preliminaries

We will now prove some results which are analogues of some results already proved for the case of symmetric 2-tensor fields in [18]. These will be needed later in the proof of our main theorems. First of all, we will describe the construction of semigeodesic coordinates in a neighborhood of any given maximal geodesic. This construction has been worked out and used previously by Stefanov and Uhlmann in several articles, e.g. [36], [34].

Fix a maximal geodesic $\gamma_0$ connecting points $x_0 \neq y_0$ on $\partial\widetilde{\Omega}$. The map $\exp_{x_0}$ :

$\exp_{x_0}^{-1}(\widetilde{\Omega}) \to \widetilde{\Omega}$ is a diffeomorphism for $x_0 \in \widetilde{\Omega}$ due to the manifold being simple. Consider polar coordinates $(r,\theta), r > 0, |\theta| = 1$ on $\exp_{x_0}^{-1}(\widetilde{\Omega})$ given by $\xi = r\theta$. Now consider the Cartesian system on $\exp_{x_0}^{-1}(\widetilde{\Omega})$ such that $\{\xi_n = 0\}$ defines the plane tangent to $\exp_{x_0}^{-1}(\widetilde{\Omega})$ at the origin, $\xi = 0$. By the assumption of simplicity (in particular, that the boundary is strictly convex), $\theta_n > 0$ for the maximal geodesic $\gamma_0$ starting at $x_0$ in the direction $\theta_0$ which connects $x_0$ to $y_0 \neq x_0$. Consider $w = (w', w_n)$ such that $w' = \theta'/\theta_n$ and $w_n = r$. One can show that the map $\xi \mapsto (w', w_n)$ is a diffeomorphism on $\exp_{x_0}^{-1}(\widetilde{\Omega})$, see also [34, section 9]. Hence, we get a new coordinate system on $\exp_{x_0}^{-1}(\widetilde{\Omega})$. This can be now used to get semigeodesic coordinates $x = (x', x^n)$ near $x_0$ in $\widetilde{\Omega}$ so that $x^n$ is the distance to $x_0$, and $\frac{\partial}{\partial x^n}$ is normal to $\frac{\partial}{\partial x^\alpha}$, $\alpha < n$, see [34, Section 2]. In these coordinates, the metric $g$ satisfies $g_{ni} = \delta_{ni}$, for all $i$, and the Christoffel symbols satisfy $\Gamma^i_{nn} = \Gamma^n_{in} = 0$. Under these coordinates lines of the type $x' =$ constant are now geodesics with $x^n$ as arc length parameter.

We use boundary normal semi-geodesic coordinates near points $p \in \Omega$ close to the boundary $\partial\Omega$ just as mentioned in [36, paragraph 3, section 2.1]. If $x' \in \mathbb{R}^{n-1}$ are local coordinates near the boundary $\partial\Omega$ and $\nu(x')$ is the outer unit normal, then $p = \exp_{(x',0)}(-x_n\nu)$.

Let $U$ be a tubular neighborhood of $\gamma_0$ in $\Omega$, $U = \{(x', x^n) : |x'| < \epsilon,\ a(x') \le x^n \le b(x')\}$, where $\partial\Omega$ is locally given by $x^n = a(x')$ and $x^n = b(x')$. In the next proposition, we prove that for a symmetric $m$- tensor field $f$, one can always construct an $(m-1)$-tensor field $v$ in $U$ such that for

$$h := f - dv$$

one has

$$h_{i_1 \dots i_{m-1} n} = 0, \qquad \text{for all possible values of } i_j \text{ and } v(x', a(x')) = 0.$$

$\widetilde{U}$ denotes the tubular neighborhood of $\gamma_0$ of the same type but in $\widetilde{\Omega}$.

**Remark 5.3.1** Numbers of $n$ in the suffix of the tensor $v_{n \dots n i_1 \dots i_k}$ will be clear from the order of the tensor $v$. For example, if $v$ is a $m$- tensor then

$$v_{n\dots ni_1\dots i_k} = v_{\underbrace{n\dots n}_{m-k-times}\, i_1\dots i_k}.$$

**Proposition 5.3.2** *Let $f$ be a symmetric $m$-tensor field then there exists a unique $(m-1)$-tensor field $v$ such that $v(x', a(x')) = 0$ and for $h = f - dv$, we have*

$$h_{i_1\dots i_{m-1}n} = 0, \qquad \textit{for all possible values of } i_j.$$

To prove this proposition, we need the following lemma for which we provide a proof in the Appendix section:

**Lemma 5.3.3** *Let $v$ be a symmetric $(m-1)$-tensor field. Then for any $0 \le k \le m$, we have*

$$\begin{aligned}(dv)_{n\dots ni_k\dots i_1} = \frac{(m-k)}{m}\frac{\partial v_{n\dots ni_k\dots i_1}}{\partial x^n} - \frac{2(m-k)}{m}\sum_{l=1}^{k}\Gamma^p_{i_l n} v_{n\dots ni_k\dots\hat{i_l}\dots i_1 p} \\ + \frac{1}{m}\sum_{l=1}^{k}\frac{\partial v_{n\dots ni_k\dots\hat{i_l}\dots i_1}}{\partial x^{i_l}} - \frac{2}{m}\sum_{l,q=1,l\neq q}^{k}\Gamma^p_{i_l i_q} v_{n\dots ni_k\dots\hat{i_l}\dots\hat{i_q}\dots i_1 p}.\end{aligned}$$

Now, let us come back to the proof of Proposition 5.3.2.

*Proof (Proof of Proposition 5.3.2):* Let us first recall the following definition:

$$(dv)_{i_1\dots i_m} = \sigma(i_1,\dots,i_m)\left(\frac{\partial v_{i_1\dots i_{m-1}}}{\partial x^{i_m}} - \sum_{l=1}^{m-1}\Gamma^p_{i_m i_l} v_{i_1\dots i_{l-1} p i_{l+1}\dots i_{m-1}}\right)$$

where $\sigma$ is a the symmetrization operator.

Proving

$$h_{i_1\dots i_{m-1}n} = 0$$

is equivalent to proving the existence of a $(m-1)$–tensor field $v$ such that

$$(dv)_{i_1\dots i_{m-1}n} = f_{i_1\dots i_{m-1}n}.$$

First we consider

$$\frac{\partial v_{n\ldots n}}{\partial x^n} = f_{n\ldots n}.$$

We will solve this equation together with the initial condition $v_{n\ldots n}(x', a(x')) = 0$ to get $v_{n\ldots n}$. After solving for $v_{n\ldots n}$ we will consider

$$(dv)_{n\ldots ni} = f_{n\ldots ni}$$

$$\Rightarrow \qquad \frac{\partial v_{n\ldots ni}}{\partial x^n}(x) - 2\Gamma^p_{in} v_{n\ldots np}(x) = \frac{m}{m-1} f_{n\ldots ni}(x) - \frac{1}{m-1}\frac{\partial v_{n\ldots n}}{\partial x^i}(x).$$

Now we will solve this system of equations together with the initial conditions $v_{n\ldots ni}(x', a(x')) = 0$ to get $v_{n\ldots ni}$.

Proceeding in a similar manner let us assume that for a given $k$ such that $0 \le (k-1) \le (m-1)$, we have already found $v_{n\ldots n i_{k-1}\ldots i_1}$ for which $h_{n\ldots n i_{k-1}\ldots i_1} = f_{n\ldots n i_{k-1}\ldots i_1} - (dv)_{n\ldots n i_{k-1}\ldots i_1} = 0$. If $(k-1) = (m-1)$ then we are done and if not then we can find $v_{n\ldots n i_k\ldots i_1}$ in the following manner. Using Lemma 5.3.3, we can construct the following system of equations for $h_{n\ldots n i_k \ldots i_1} = 0$.

$$\frac{\partial v_{n\ldots n i_k\ldots i_1}}{\partial x^n}(x) - 2\sum_{l=1}^{k}\Gamma^p_{i_l n} v_{n\ldots n i_k\ldots \hat{i_l}\ldots i_1 p}(x) = \frac{1}{(m-k)}\Bigg\{ m f_{n\ldots n i_k\ldots i_1}(x) - \sum_{l=1}^{k}\frac{\partial v_{n\ldots n i_k\ldots \hat{i_l}\ldots i_1}}{\partial x^{i_l}}(x) + 2\sum_{l,q=1, l\neq q}^{k}\Gamma^p_{i_l i_q} v_{n\ldots n i_k\ldots \hat{i_l}\ldots \hat{i_q}\ldots i_1 p}(x)\Bigg\}.$$

Finally, we will solve the above system of equations with the initial conditions $v_{n\ldots n i_k\ldots i_1}(x', a(x')) = 0$ to get $v_{n\ldots n i_k\ldots i_1}$ uniquely. We repeat the same process till $k = (m-1)$ to prove the proposition. □

**Lemma 5.3.4** *Let $f$ be supported in $\Omega$, and $I^0 f(\gamma) = 0$ for all maximal geodesics in $\widetilde{U}$ belonging to some neighborhood of the geodesics $x' = const$. Then $v = 0$ in $int(\widetilde{U}) \setminus \Omega$.*

*Proof:* First let $f \in C^\infty(\Omega)$. We will give another invariant definition of $v$ and use it to conclude our lemma. For any $x \in \widetilde{U}$ and any $\xi \in T_x\widetilde{U} \setminus \{0\}$ so that $\gamma_{x,\xi}$ stays in

$\widetilde{U}$, we set

$$u(x,\xi) = \int_0^{l(x,\xi)} f_{i_1\ldots i_m}(\gamma_{x,\xi}(t))\dot{\gamma}^{i_1}_{x,\xi}(t)\cdots\dot{\gamma}^{i_m}_{x,\xi}(t)dt. \tag{5.1}$$

Extend the definition of $\gamma_{x,\xi}$ for $\xi \neq 0$ as a solution of the geodesic equation. Then $u(x,\xi)$ is homogeneous of order $(m-1)$ in $\xi$. Consider

$$u(x,\lambda\xi) = \lambda^{m-1}u(x,\xi)$$

$$\Rightarrow \qquad \xi^{j_1}\cdots\xi^{j_{m-1}}\frac{\partial^{m-1}}{\partial\xi^{j_1}\cdots\partial\xi^{j_{m-1}}}u(x,\lambda\xi) = (m-1)!\ u(x,\xi), \quad \text{diff. } (m-1) \text{ times w.r.t } \lambda$$

$$\Rightarrow \qquad \xi^{j_1}\cdots\xi^{j_{m-1}}\frac{\partial^{m-1}}{\partial\xi^{j_1}\cdots\partial\xi^{j_{m-1}}}u(x,\xi) = (m-1)!\ u(x,\xi), \text{ for } \lambda = 1.$$

Now, we shall define a symmetric $(m-1)$- tensor field $v$ as following:

$$v_{i_1\ldots i_{m-1}}(x) = \frac{1}{(m-1)!}\ \frac{\partial^{m-1}}{\partial\xi^{i_1}\cdots\partial\xi^{i_{m-1}}}u(x,\xi)\bigg|_{\xi=e_n}. \tag{5.2}$$

Consider for any $0 \leq l \leq (m-1)$

$$\begin{aligned} v_{i_1\ldots i_{m-1-l}n\ldots n}(x) &= \frac{1}{(m-1)!}\ \frac{\partial^{m-1}}{\partial\xi^{i_1}\cdots\partial\xi^{i_{m-1-l}}\partial\xi^{n}\cdots\partial\xi^{n}}u(x,\xi)\bigg|_{\xi=e_n} \\ &= \frac{1}{(m-1)!}\ \xi^{j_1}\cdots\xi^{j_l}\frac{\partial^{m-1}}{\partial\xi^{i_1}\cdots\partial\xi^{i_{m-1-l}}\partial\xi^{j_1}\cdots\partial\xi^{j_l}}u(x,\xi)\bigg|_{\xi=e_n} \\ &= \frac{l!}{(m-1)!}\ \frac{\partial^{m-1}}{\partial\xi^{i_1}\cdots\partial\xi^{i_{m-1-l}}}u(x,\xi)\bigg|_{\xi=e_n} \qquad (\text{ using homogenity of } u). \end{aligned}$$

Then, we have

$$v_{n\ldots n}(x) = u(x,e_n).$$

We will now show that with this definition of $v$, for $h = f - dv$, one has

$$h_{i_1\ldots i_{m-1}n} = 0, \qquad \text{for all possible values of } i_j.$$

Define

$$w(x,\xi) = \int_0^{l(x,\xi)} h_{i_1\dots i_m}(\gamma_{x,\xi}(t))\dot{\gamma}^{i_1}_{x,\xi}(t)\cdots\dot{\gamma}^{i_m}_{x,\xi}(t)dt. \tag{5.3}$$

**Lemma 5.3.5** *Let $0 \le l \le (m-1)$ and $w(x,\xi)$ is defined as above then*

$$\left.\frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} w(x,\xi)\right|_{\xi=e_n} = 0. \tag{5.4}$$

*Proof:* Consider for any $0 \le l \le (m-1)$,

$$\begin{aligned}
&\frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} w(x,\xi)\\
&= \frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} u(x,\xi) - \frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} \int_0^{l(x,\xi)} (dv)_{i_1\dots i_m}(\gamma_{x,\xi}(t))\dot{\gamma}^{i_1}_{x,\xi}(t)\cdots\dot{\gamma}^{i_m}_{x,\xi}(t)dt\\
&= \frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} u(x,\xi) - \frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} \int_0^{l(x,\xi)} \frac{d}{dt}\left(v_{i_1\dots i_{m-1}}(\gamma_{x,\xi}(t))\dot{\gamma}^{i_1}_{x,\xi}(t)\cdots\dot{\gamma}^{i_{m-1}}_{x,\xi}(t)\right)dt\\
&= \frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} u(x,\xi) - \frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} \left(v_{i_1\dots i_{m-1}}(x)\xi^{i_1}\cdots\xi^{i_{m-1}}\right)\\
&= \frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} u(x,\xi) - \frac{(m-1)!}{l!}\, v_{j_1\dots j_l n\dots n}(x)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\qquad \left.\frac{\partial^{m-1}}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} w(x,\xi)\right|_{\xi=e_n} &= \left.\frac{\partial^l}{\partial\xi^{j_1}\cdots\partial\xi^{j_l}} u(x,\xi)\right|_{\xi=e_n} - \frac{(m-1)!}{l!}\, v_{j_1\dots j_l n\dots n}(x)\\
&= \frac{(m-1)!}{l!}\, v_{j_1\dots j_l n\dots n}(x) - \frac{(m-1)!}{l!}\, v_{j_1\dots j_l n\dots n}(x)\\
&= 0. \qquad\square
\end{aligned}$$

Now let us recall the following relation [30, Section 1.2]

$$Gw(x,\xi) = h_{i_1\dots i_m}(x)\xi^{i_1}\cdots\xi^{i_m} \tag{5.5}$$

where $G = \xi^i\partial_{x^i} - \Gamma^k_{ij}\xi^i\xi^j\partial_{\xi^k}$ is the generator of the geodesic flow. After differentiating (5.5) $(m-1)$ times w.r.t. $\xi$, we get

$$\frac{\partial^{m-1}}{\partial\xi^{j_1}\cdots\partial\xi^{j_{m-1}}} Gw(x,\xi) = m!\; h_{j_1\dots j_{m-1}i}(x)\xi^i$$

$$\Rightarrow \qquad \left.\frac{\partial^{m-1}}{\partial\xi^{j_1}\ldots\partial\xi^{j_{m-1}}}Gw(x,\xi)\right|_{\xi=e_n} = m!\ h_{j_1\ldots j_{m-1}n}(x).$$

We will prove L.H.S. of the above equation is 0. This will prove our lemma. Consider

$$\begin{aligned}
\frac{\partial Gw(x,\xi)}{\partial\xi^{j_1}} &= \frac{\partial}{\partial\xi^{j_1}}\left(\xi^i\frac{\partial}{\partial x^i}w(x,\xi)\right) - \Gamma^k_{ij}\frac{\partial}{\partial\xi^{j_1}}\left(\xi^i\xi^j\frac{\partial}{\partial\xi^k}w(x,\xi)\right)\\
&= \frac{\partial w(x,\xi)}{\partial x^{j_1}} + \xi^i\frac{\partial^2 w(x,\xi)}{\partial\xi^{j_1}\partial x^i} - \Gamma^k_{ij}\frac{\partial}{\partial\xi^{j_1}}(\xi^i\xi^j)\frac{\partial w(x,\xi)}{\partial\xi^k} - \Gamma^k_{ij}\xi^i\xi^j\frac{\partial^2 w(x,\xi)}{\partial\xi^{j_1}\partial\xi^k}\\
\Rightarrow \frac{\partial^2 Gw(x,\xi)}{\partial\xi^{j_1}\partial\xi^{j_2}} &= \frac{\partial^2 w(x,\xi)}{\partial x^{j_1}\partial\xi^{j_2}} + \frac{\partial^2 w(x,\xi)}{\partial\xi^{j_1}\partial x^{j_2}} + \xi^i\frac{\partial^3 w(x,\xi)}{\partial\xi^{j_1}\partial\xi^{j_2}\partial x^i} - \Gamma^k_{ij}\frac{\partial^2}{\partial\xi^{j_1}\partial\xi^{j_2}}(\xi^i\xi^j)\frac{\partial w(x,\xi)}{\partial\xi^k}\\
&\quad - \Gamma^k_{ij}\frac{\partial}{\partial\xi^{j_1}}(\xi^i\xi^j)\frac{\partial^2 w(x,\xi)}{\partial\xi^{j_2}\partial\xi^k} - \Gamma^k_{ij}\frac{\partial}{\partial\xi^{j_2}}(\xi^i\xi^j)\frac{\partial^2 w(x,\xi)}{\partial\xi^{j_1}\partial\xi^k} - \Gamma^k_{ij}\xi^i\xi^j\frac{\partial^3 w(x,\xi)}{\partial\xi^{j_1}\partial\xi^{j_2}\partial\xi^k}\\
&= \frac{\partial^2 w(x,\xi)}{\partial x^{j_1}\partial\xi^{j_2}} + \frac{\partial^2 w(x,\xi)}{\partial\xi^{j_1}\partial x^{j_2}} + \xi^i\frac{\partial^3 w(x,\xi)}{\partial\xi^{j_1}\partial\xi^{j_2}\partial x^i} - 2\Gamma^k_{j_1j_2}\frac{\partial w(x,\xi)}{\partial\xi^k}\\
&\quad - 2\Gamma^k_{ij_1}\xi^i\frac{\partial^2 w(x,\xi)}{\partial\xi^{j_2}\partial\xi^k} - 2\Gamma^k_{ij_2}\xi^i\frac{\partial^2 w(x,\xi)}{\partial\xi^{j_1}\partial\xi^k} - \Gamma^k_{ij}\xi^i\xi^j\frac{\partial^3 w(x,\xi)}{\partial\xi^{j_1}\partial\xi^{j_2}\partial\xi^k}.
\end{aligned}$$

Using similar calculations, we get

$$\begin{aligned}
\frac{\partial^{m-1}Gw(x,\xi)}{\partial\xi^{j_1}\ldots\partial\xi^{j_{m-1}}} &= \xi^i\frac{\partial^m w(x,\xi)}{\partial\xi^{j_1}\ldots\partial\xi^{j_{m-1}}\partial x^i} - \sum_{l,k=1,l\neq k}^{m-1} 2\Gamma^k_{j_lj_p}\frac{\partial^{m-2}w(x,\xi)}{\partial\xi^k\partial\xi^{j_1}\ldots\partial\hat{\xi^{j_l}}\ldots\partial\hat{\xi^{j_p}}\ldots\partial\xi^{j_{m-1}}}\\
&\quad + \sum_{l=1}^{m-1}\frac{\partial^{m-1}w(x,\xi)}{\partial x^{j_l}\partial\xi^{j_1}\ldots\partial\hat{\xi^{j_l}}\ldots\partial\xi^{j_{m-1}}} - \sum_{l=1}^{m-1}2\Gamma^k_{ij_l}\xi^i\frac{\partial^{m-1}w(x,\xi)}{\partial\xi^k\partial\xi^{j_1}\ldots\partial\hat{\xi^{j_l}}\ldots\partial\xi^{j_{m-1}}}\\
&\quad - \Gamma^k_{ij}\xi^i\xi^j\frac{\partial^m w(x,\xi)}{\partial\xi^k\partial\xi^{j_1}\ldots\partial\xi^{j_{m-1}}}.
\end{aligned}$$

Which implies

$$\left.\frac{\partial^{m-1}Gw(x,\xi)}{\partial\xi^{j_1}\ldots\partial\xi^{j_{m-1}}}\right|_{\xi=e_n} = 0, \qquad \text{(Using Claim 5.3.5 and } \Gamma^k_{nn} = 0\text{)}.$$

Now that we have proved the proposition for the case when $f$ is smooth, it can be extended to the case when $f$ is a distribution by exactly the same reasoning as in [18, Lemma 3.1]. □

## 5.4 Proofs of Theorem 5.2.4 and Theorem 5.2.5

We will start with proving some lemmas and propositions required to prove our main theorems.

**Lemma 5.4.1** *Let $f$ be a symmetric $m$-tensor field as above. Let $\gamma_0$ be a geodesic of $\widetilde{\Omega}$ and $U$ be a neighborhood of $\gamma_0$ in $\widetilde{\Omega}$. Assume that $WF_A(f)\bigcap\pi^{-1}(U)$ does not contain covectors of the type $(\xi',0)$, then $h = f - dv$ also does not contain such covectors.*

*Proof:* Since $v$ and $dv$ have the same analytic wavefront set, so we will prove the lemma for $v$. We will prove this by induction by proving it for $v_{n\ldots ni_k\ldots i_1}$ for every $k \le (m-1)$. Let us first do the analysis for $v_{n\ldots n}$. Note that $v_{n\ldots n}$ can be rewritten as a convolution with the Heaviside function in the following manner

$$\begin{aligned} v_{n\ldots n}(x) &= \int_{-\infty}^{x^n} f_{n\ldots n}(x^{'},y^n)dy^n \\ &= \int_{-\infty}^{\infty} f_{n\ldots n}(x^{'},y^n)H(x^n-y^n)dy^n. \end{aligned}$$

The wavefront set of the convolution can be found by applying [14, 8.2.16]. Since we have assumed that $\mathrm{WF}_A(f)\bigcap\pi^{-1}(\mathrm{U})$ does not contain covectors of the type $(\xi',0)$, hence it will be true for $v_{n\ldots n}(x)$ as well. Now let us assume that the lemma holds for any $0 \le k-1 < (m-1)$ i.e. $v_{n\ldots ni_{k-1}\ldots i_1}$ satisfies the same wavefront conditions. We will show that this implies that the Lemma 5.4.1 is true for $k$. For this consider the system of ODEs from Lemma 5.3.3,

$$\begin{aligned} \frac{\partial v_{n\ldots ni_k\ldots i_1}}{\partial x^n}(x) - 2\sum_{l=1}^{k}\Gamma^p_{i_l n} v_{n\ldots ni_k\ldots\hat{i_l}\ldots i_1 p}(x) &= \frac{1}{(m-k)}\Bigg\{m f_{n\ldots ni_k\ldots i_1}(x) - \sum_{l=1}^{k}\frac{\partial v_{n\ldots ni_k\ldots\hat{i_l}\ldots i_1}}{\partial x^{i_l}}(x) \\ &\quad + 2\sum_{l,q=1,l\neq q}^{k}\Gamma^p_{i_l i_q} v_{n\ldots ni_k\ldots\hat{i_l}\ldots\hat{i_q}\ldots i_1 p}(x)\Bigg\}, \\ v_{n\ldots ni_k\ldots i_1}(x',a(x')) &= 0. \end{aligned}$$

This can be rewritten as :

$$\partial_n(\tilde{v}) - A(x', x^n)\tilde{v} = w,$$
$$\tilde{v}|_{x^n << 0} = 0$$

where $A$ is an analytic matrix, $\tilde{v} = v_{n\ldots ni_k\ldots i_1}$ and $\mathrm{WF}_A(w) \bigcap \pi^{-1}(\mathrm{U})$ does not have covectors of the type $(\xi', 0)$. By Duhamel's principle the solution to the above is given by:

$$\tilde{v}(x', x^n) = \int_{-\infty}^{x^n} \Phi(x', x^n, y^n) w(x', y^n) dy^n$$

where $\Phi$ is analytic. The expression given above for $\tilde{v}(x', x^n)$ can be rewritten as:

$$\tilde{v}(x', x^n) = \int_{R^n} \Phi(x', x^n, y^n) H(x^n - y^n)\delta(x' - y')w(y', y^n) dy' dy^n.$$

The kernel of the integral operator is given by : $\Phi(x', x^n, y^n)H(x^n - y^n)\delta(x' - y')$. Note that the frequency set of the analytic wavefront set of the Heaviside and delta distributions here are perpendicular to each other and hence satisfy Hörmander's non cancellation condition [14, 8.5.3]. The lemma then follows from the argument in [18]. □

### 5.4.1 Analyticity along Conormal Directions

Before moving further, we will need the following proposition which is an analogue of Proposition 2 from [36] and generalizes that proposition for the case when $f$ is a symmetric $m$-tensor. We will mimic the proof for the case when $m = 2$ as given in that paper and adapt the arguments wherever necessary to make it work for a symmetric tensor field of any order.

**Proposition 5.4.2** *Let $\Omega$ and $f$ be as above. Let $\gamma_0$ be a fixed geodesic through $x_0$ normal to $\xi_0$ where $(x_0, \xi_0) \in T^*\Omega \backslash 0$. Assume $(I^0 f)(\gamma) = 0$ for all $\gamma$ in a neighborhood of $\gamma_0$ and $g$ is analytic in this neighborhood. Let $\delta f = 0$ near $x_0$. Then*

$$(x_0, \xi_0) \notin WF_A(f).$$

*Proof:* For the given geodesic $\gamma_0$ that passes through $x_0$ and is normal to $\xi_0$, let us consider a tubular neighborhood $U$ of $\gamma_0$ endowed with analytic semi-geodesic coordinates $x = (x', x^n)$ on it. Without loss of generality, assume that $x_0 = 0$. Furthermore, $\forall x \in \gamma_0$, $x' = 0$. Note that $U = \{(x', x^n) : |x'| < \epsilon \text{ and } l^- < x_n < l^+; 0 < \epsilon << 1\}$ in this co-ordinate system. Choose $\epsilon$ such that $\{x : x_n = l^-, l^+ \text{ and } |x'| < \epsilon\}$ lies outside $\Omega$. Clearly $\xi_0 = (\xi_0', 0)$. Hence our goal is now to show:

$$(0, \xi_0) \notin \mathrm{WF}_A(f).$$

As stated earlier, we will reproduce the arguments from [36] here for the sake of completeness. Consider $Z = \{|x| < \frac{7\epsilon}{8} : x_n = 0\}$ and let $x'$ variable be denoted on $Z$ by $z'$. Then $(z', \theta')$ are local co-ordinates in $\mathrm{nbd}(\gamma_0)$ (in the set of geodesics) given by $(z', \theta') \to \gamma_{(z',0),(\theta',1)}$. Here, $|\theta'| << 1$ (where, the geodesic is in the direction $(\theta', 1)$). By following their arguments verbatim, we get the following equation:

$$\int e^{i\lambda z'(x,\theta').\xi'} a_N(x,\theta') f_{i_1\dots i_m}(x) b^{i_1}(x,\theta') \cdots b^{i_m}(x,\theta') dx = 0. \tag{5.6}$$

Here, $(x, \theta') \to a_N$ is compactly supported function with support in some compact subset of $U$, analytic in some neighborhood of $\gamma_0$ and satisfies

$$|\partial^\alpha a_N| \le (CN)^{|\alpha|}, \quad \alpha \le N, \tag{5.7}$$

see [36, Equation(38) and discussion below it]. Also, note that $b(0, \theta') = \theta$ and $a_N(0, \theta') = 1$.

Further, let us choose $\theta(\xi)$ to be a vector depending analytically on $\xi$ near $\xi = \xi_0$ and satisfying the following conditions:

$$\theta(\xi) \cdot \xi = 0, \quad \theta^n(\xi) = 1 \qquad \text{and}$$
$$\theta(\xi_0) = (0, \cdots, 1) = e_n$$

Now, we will rewrite (5.6) using the above mapping in the following form:

$$\int e^{i\lambda\phi(x,\xi)}\tilde{a_N}(x,\xi)f_{i_1\dots i_m}(x)\tilde{b}^{i_1}(x,\xi)\cdots\tilde{b}^{i_m}(x,\xi)dx = 0. \tag{5.8}$$

Here $\phi(x,\xi) = z'\cdot\xi'$. For $|\theta'| << 1$, the points of the manifold in a neighborhood of the geodesic $\gamma_0$ are given by $x = (z' + t\theta', t) + \mathcal{O}(|\theta'|)$, see [36, section 4]. Now recall that the $x'$ variables on $Z$ were denoted by $z'$. Hence $\frac{\partial z'}{\partial x'}(0,\theta(\xi)) = \mathbb{I}$. Furthermore, using the linearized expression for $x$ as before in this paragraph, $\frac{\partial z'}{\partial x_n}(0,\theta(\xi)) = -\theta'(\xi)$. So, $\phi_x(0,\xi) = (\xi', -\theta'(\xi).\xi)$. This in turn shows $\phi_{x\xi}(0,\xi) = \mathbb{I}$. This also implies that $x \to \phi_\xi(x,\xi)$ is a diffeomorphism in a neighborhood of $(0,\xi_0)$.

To establish the above condition in a neighborhood of the geodesic $\gamma_0$, one chooses the co-normal vector

$$\xi_0 = e_{n-1}, \quad \text{i.e. the covector } (0,0,\cdots,0,1,0) \tag{5.9}$$

and defines

$$\theta(\xi) = (\xi_1,\cdots,\xi_{n-2}, -\frac{\xi_1^2 + \cdots + \xi_{n-2}^2 + \xi_n}{\xi_{n-1}}, 1).$$

This definition of $\theta$ is consistent with the requirement put on $\theta(\xi)$ as above. One can then show that the differential of the map $\xi \to \theta(\xi)$ where $\xi \in S^{n-1}$ is invertible at $\xi_0 = e_{n-1}$, see [36, Equation (44)].

**Lemma 5.4.3** *[36, Lemma 5] Let, $\theta(\xi)$ and $\phi(x,\xi)$ be as above. Then, $\exists\ \delta > 0$ such that if*

$$\phi_\xi(x,\xi) = \phi_\xi(y,\xi)$$

*for some $x \in U$, $|y| < \delta$, $|\xi - \xi_0| < \delta$ where $\xi$ is complex, then $y = x$.*

We will study the analytic wavefront set of $f$ using Sjöstrand's complex stationary phase method. For this assume $x$, $y$ as in Lemma 5.4.3 and $|\xi_0 - \eta| < \frac{\delta}{\tilde{C}}$ with $\tilde{C} >> 2$ and $\delta << 1$. Multiply (5.8) by

$$\tilde{\chi}(\xi-\eta)e^{i\lambda\left(i\frac{(\xi-\eta)^2}{2} - \phi(y,\xi)\right)}$$

where $\tilde{\chi}$ is the characteristic function of the ball $B(0,\delta) \subset \mathbb{C}^n$ and then integrate w.r.t. $\xi$ to get:

$$\iint e^{i\lambda\Phi(y,x,\xi,\eta)}\tilde{\tilde{a_N}}(x,\xi)f_{i_1\dots i_m}(z)\tilde{b}^{i_1}(x,\xi)\cdots\tilde{b}^{i_m}(x,\xi)dxd\xi = 0. \tag{5.10}$$

In the above equation, $\tilde{\tilde{a_N}} = \tilde{\chi}(\xi-\eta)\tilde{a}_N$ is another analytic and elliptic amplitude for $x$ close to zero and $|\xi-\eta| < \frac{\delta}{C}$. As a result $\tilde{\tilde{a_N}}$ also satisfies an estimate of the same kind as (5.7). Furthermore,

$$\Phi = -\phi(y,\xi) + \phi(x,\xi) + \frac{i}{2}(\xi-\eta)^2.$$

and

$$\Phi_\xi = \phi_\xi(x,\xi) - \phi_\xi(y,\xi) + i(\xi-\eta).$$

To apply the stationary phase method we need to know the critical points of $\xi \mapsto \Phi$. Using the Lemma 5.4.3 above we have:

(i) If $y = x$, $\exists$ a unique real critical point $\xi_c = \eta$

(ii) If $y \neq x$, there are no real critical points

(iii) Also by Lemma 5.4.3, if $y \neq x$, there is a unique complex critical point if $|x-y| < \delta/C_1$ and no critical points for $|x-y| > \delta/C_0$ for some constants $C_0$ and $C_1$ with $C_1 > C_0$.

Define, $\psi(x,y,\eta) := \Phi(\xi_c)$. Then at $x = y$

(i) $\psi_y(x,x,\eta) = -\phi_x(x,\eta)$ (ii) $\psi_x(x,x,\eta) = \phi_x(x,\eta)$ (iii) $\psi(x,x,\eta) = 0$.

Now, we split the $x$ integral in (5.10) in to two parts : we integrate over $\{x : |x-y| > \delta/C_0\}$ for some $C_0 > 1$ and its complement. Since, $|\Phi_\xi|$ has a positive lower bound for $\{x : |x-y| > \delta/C_0\}$ and there are no critical points of $\xi \to \Phi$ in this set, we can estimate that integral in the following manner: First note that, $e^{i\lambda\Phi(x,\xi)} = \frac{\Phi_\xi \partial_\xi}{i\lambda|\Phi_\xi|^2}e^{i\lambda\Phi(x,\xi)}$. Using, (5.7) and integrating by parts $N$ times with respect to $\xi$ and the fact that on the boundary $|\xi - \eta| = \delta$, we get

$$\left| \iint_{|x-y|>\delta/C_0} e^{i\lambda\Phi(y,x,\xi,\eta)} \tilde{\tilde{a_N}}(x,\xi) f_{i_1\dots i_m}(x) \tilde{b}^{i_1}(x,\xi) \cdots \tilde{b}^{i_m}(x,\xi) dx d\xi \right| \le C\left(\frac{CN}{\lambda}\right)^N + CNe^{-\frac{\lambda}{C}}. \tag{5.11}$$

We choose $N \le \lambda/Ce \le N+1$ to get an exponential error on the right. Now in estimating the integral

$$\left| \iint_{|x-y|\le\delta/C_0} e^{i\lambda\Phi(y,x,\xi,\eta)} \tilde{\tilde{a_N}}(x,\xi) f_{i_1\dots i_m}(x) \tilde{b}^{i_1}(x,\xi) \cdots \tilde{b}^{i_m}(x,\xi) dx d\xi \right|, \tag{5.12}$$

we use [33, Theorem 2.8] and the [33, Remark 2.10] following that to conclude:

$$\int_{|x-y|\le\delta/C_0} e^{i\lambda\psi(x,\alpha)} f_{i_1\dots i_m}(x) B^{i_1\dots i_m}(x,\alpha;\lambda) dx = \mathcal{O}(e^{-\lambda/C}) \tag{5.13}$$

where $\alpha = (y,\eta)$ and $B$ is a classical analytical symbol with principal part $\tilde{b} \otimes \cdots \otimes \tilde{b}$. See appendix below for a proof of estimates in (5.11) and (5.13).

Let, $\beta = (y,\mu)$ where, $\mu = \phi_y(y,\eta) = \eta + \mathcal{O}(\delta)$. At $y = 0$, we have $\mu = \eta$. Also $\alpha \to \beta$ is a diffeomorphism following similar analysis as in [36, Section 4]. If we write $\alpha = \alpha(\beta)$, then the above equation becomes:

$$\int_{|x-y|\le\delta/C_0} e^{i\lambda\psi(x,\beta)} f_{i_1\dots i_m}(x) B^{i_1\dots i_m}(x,\beta;\lambda) dx = \mathcal{O}(e^{-\lambda/C}) \tag{5.14}$$

where $\psi$ satisfies (i), (ii) and (iii), and $B$ is a classical analytical symbol as before and

$$\psi_y(x,x,\eta) = -\mu, \quad \psi_x(x,x,\eta) = \mu \quad \text{and} \quad \psi_y(x,x,\eta) = 0.$$

The symbols in (5.14) satisfy :

$$\sigma_P(B)(0,0,\mu) = \theta(\mu) \otimes \cdots \otimes \theta(\mu) = \theta^{\otimes m}(\mu)$$

and in particular,

$$\sigma_P(B)(0,0,\xi_0) = e_n \otimes \cdots \otimes e_n.$$

Let, $\theta_1 = e_n$, $\theta_2, \ldots, \theta_N$ be $N = \binom{n+m-2}{m}$ unit vectors at $x_0 = 0$ lie in the hyperplane perpendicular to $\xi_0$. We will also assume that $\{\theta_i^{\odot m}\}_{i=1}^N$ are independent, where $\odot$ is a symmetrized product of vectors. Existence of such vectors in any open set in $\xi_0^\perp$ can be shown. We can therefore assume that $\theta_p$ belongs in a small neighborhood around $\theta_1 = e^n$. Then we can rotate the axes a little such that $\xi_0 = e^{n-1}$ and $\theta_p = e_n$ and do the same construction as above. This gives us $N = \binom{n+m-2}{m}$ phase functions $\psi_{(p)}$, and as many number of analytic symbols for which (5.14) is true i.e.

$$\int_{|x-y|\le\delta/C_0} e^{i\lambda\psi_{(p)}(x,\beta)} f_{i_1\ldots i_m}(x) B_{(p)}^{i_1\ldots i_m}(x,\beta;\lambda)dx = \mathcal{O}(e^{-\lambda/C}) \tag{5.15}$$

where

$$\sigma_P(B_p)(0,0,\mu) = \theta_p(\mu) \otimes \cdots \otimes \theta_p(\mu), \quad p = 1, \ldots, N \quad \text{up to elliptic factors.}$$

Now we use the fact that $\delta f = 0$ near $x_0$. So integrating

$$\frac{1}{\lambda}\exp(i\lambda\psi_{(1)}(x,\beta)\chi_0\delta f = 0$$

w.r.t. $x$ and after an integration by parts, we get

$$\int_{|x-y|\le\delta/C_0} e^{i\lambda\psi_{(1)}(x,\beta)} f_{i_1\ldots i_m}(x) C^{i_m}(x,\beta;\lambda)dx = \mathcal{O}(e^{-\lambda/C}) \tag{5.16}$$

where, $i_j \in \{1, \ldots, n\}$, $j = \{1, \ldots, (m-1)\}$ and for $\beta_x = y$ small enough, where $\sigma_P(C^{i_m})(0,0,\xi_0) = (\xi_0)^{i_m}$. This gives us additional $\tilde{N} = \binom{n+m-2}{m-1}$ equations such that the system of $N + \tilde{N} = \binom{n+m-1}{m}$ equations (5.15), (5.16) can be viewed as a tensor valued operator on $f$. We claim that the symbol for this operator is elliptic at $(0,0,\xi_0)$. Indeed, to show that the symbol is elliptic at $(0,0,\xi_0)$ amounts to showing that the only solution to following system of equations is $f = 0$:

$$\theta_p^{i_1} \dots \theta_p^{i_m} f_{i_1 \dots i_m} = 0, \quad \text{for all } p = \{1, \dots, N\} \tag{5.17}$$

$$\xi_0^{i_m} f_{i_1 \dots i_m} = 0, \quad \text{for } 1 \le i_1 \le \dots \le i_{m-1} \le n. \tag{5.18}$$

Using conditions on $\theta_p$ and $\xi_0$, it is proved in [17] that above system of equations will imply $f = 0$. □

For the more general case, when $\delta f$ is microlocally analytic at $(x_0, \xi_0)$, we use the same arguments as above, except that we multiply (5.14) by an appropriate cut-off near $(x_0, x_0, \xi_0)$ and use integration by parts as explained in [18, Section 4] to conclude the following proposition:

**Proposition 5.4.4** *Let $\widetilde{\Omega}$, $f$ and $\gamma_0$ be as in the statement of Proposition 5.4.2. If $(x_0, \xi_0) \notin WF_A(\delta f)$ (where $\xi_0$ is normal to the geodesic $\gamma_0$ at $x_0$), and $I^0 f(\gamma) = 0$ for all $\gamma$ in a nbd. of $\gamma_0$, then $(x_0, \xi_0) \notin WF_A(f)$.*

The rest of the argument from [18] applies as it is and thereby we prove Theorem 5.2.4. We will briefly outline the ideas here for the sake of completeness: We will first need to show that the following analogue of [18, Theorem 2.2(a)] holds for the case of symmetric $m$ tensor fields as well:

**Theorem 5.4.5** *Let $f$ be as above. Then $I^0 f(\gamma) = 0$ for each geodesic $\gamma$ in $\mathcal{A}$, if and only if for each geodesic $\gamma_0 \in \mathcal{A}$ there exists a a neighborhood $\mathcal{U}$ of $\gamma_0$ and a $(m-1)$-tensor field $v \in \mathcal{D}'(\widetilde{\Omega_{\mathcal{U}}})$ such that $f = dv$ in $\widetilde{\Omega_{\mathcal{U}}}$, and $v = 0$ outside $\Omega$.*

The “if" part follows from the Fundamental Theorem of Calculus. To prove the “only if" part of the theorem assume that $\gamma_0$ is a geodesic in the set $\mathcal{A}$, where $\mathcal{A}$ is defined in Section 5.3. This means that it can be continuously deformed within the set to a point. Hence by extending all geodesics in $\Omega$ to maximal geodesics in $\tilde{\Omega}$, we know that there must exist two continuous curves $a(t)$, $b(t)$, $t \in [0,1]$ such that $\gamma_{(a(0),b(0))}$ is tangent to $\partial\Omega$, $\gamma_{(a(t),b(t))} \in \mathcal{A}$ and $\gamma_{(a(1),b(1))}$ is $\gamma_0$. Using [23, Theorem A], one can show that the Theorem 5.4.5 is at least true in a small neighborhood of $\partial\Omega$ i.e.

in some neighborhood of the geodesics $\gamma_{(a(t),b(t))}$ for $0 \leq t \leq 2t_0$ for some $t_0 << 1$. More precisely,

**Lemma 5.4.6** *[18, Lemma 5.1] There exists an $\epsilon_0 > 0$ such that in a neighborhood $V$ of $\partial\Omega$ given by $V = \{x : \text{dist}(x, \partial\Omega) < \epsilon_0\}$, there is a unique $v_0$ which satisfies $f = dv_0$ in $V$, $v_0 = 0$ on $\partial\Omega$ and $v_0$ is analytic in $V$, up to the boundary $\partial\Omega$.*

Note that the above implies that in $V$, the tensor $h = f - dv$ as constructed in Proposition 5.3.2 is zero. We will now construct a sequence of neighborhoods beginning with a neighborhood of $\gamma_{(a(0),b(0))}$ and up to a neighborhood of $\gamma_{(a(1),b(1))}$ for which the locally defined tensor field $h = f - dv$ is zero. However to implement this program we will need the following theorem due to Sato-Kawai-Kashiwara, see e.g. [27] or [40]:

**Lemma 5.4.7** *[40, Lemma 3.1] Let $f \in \mathcal{D}'(\Omega)$. Let $x_0 \in \Omega$ and let $U$ be a neighborhood of $x_0$. Assume that $S$ is a $C^2$ submanifold of $\Omega$ and $x_0 \in supp(f) \cap S$. Furthermore, let $S$ divide $U$ into two open connected sets and assume that $f = 0$ on one of these open sets. Let $\xi \in N^*_{x_0}(S) \setminus 0$, then $(x_0, \xi) \in WF_A(f)$.*

Consider the cone of all vectors in $T_{a(t)}\widetilde{\Omega}$ at an angle less than $\epsilon$ with $\dot{\gamma}_{[a(t),b(t)]}$ for some small properly chosen $\epsilon$. The cone $C_\epsilon(t)$ with its vertex at $a(t) \in \partial\widetilde{\Omega}$ is then the image of the above cone of vectors under the exponential map. We will choose $\epsilon > 0$ such that

1. $C_{2\epsilon}(t) \subset \widetilde{\Omega}_{\mathcal{A}}$, $\forall t \in [0,1]$.

2. $C_\epsilon(t) \subset \tilde{V}$ for $0 \leq t \leq t_0$ where $\tilde{V} := V \cup (\widetilde{\Omega}/\Omega)$.

3. No geodesic inside the cone $cl(C_{2\epsilon}(t))$, $t_0 < t < 1$, with vertex at $a(t)$ is tangent to $\partial\Omega$.

For any $t$, let us construct a tensor field $h_t$ in $C_\epsilon(2t)$ just as in Proposition 5.3.2. Recall that the support of $h_t$ lies in $\Omega$. Since $C_\epsilon(t) \subset \tilde{V}$ for $0 \leq t \leq t_0$ then by Lemma 5.4.6 we have $h_t = 0$ in $C_\epsilon(t) \subset \tilde{V}$. Hence the set $\{t \in [0,1] : h_t = 0 \text{ in } C_\epsilon(t)\}$ is non empty. Let $t^* = \sup\{t \in [0,1] : h_t = 0 \text{ in } C_\epsilon(t)\}$. We will show: $t^* = 1$. This will imply that there exists a neighborhood $\mathcal{U}$ of $\gamma_0$ and a $(m-1)$ tensor field $v \in \mathcal{D}'(\widetilde{\Omega_{\mathcal{U}}})$ such that $h = f - dv = 0$ there.

Assume $t^* < 1$. Then $h_{t^*} = 0$ in $C_\epsilon(t^*)$ because $h_{t^*} = 0$ outside $\Omega$. Next we will show that $h_{t^*} = 0$ in $C_{2\epsilon}(t^*)$. This gives us a contradiction, because on increasing $t^*$ slightly to $t$, we can get $C_\epsilon(t) \cap \Omega \subset C_{2\epsilon}(t^*) \cap \Omega$ such that $h_t$ is zero in this $C_\epsilon(t)$. Here we would like to mention that as $h_t$ is found by solving an initial value problem for a system of ODEs, hence they are locally unique. In particular, if $h_{t^*} = 0$ in $C_{2\epsilon}(t^*)$ and $C_\epsilon(t) \cap \Omega \subset C_{2\epsilon}(t^*) \cap \Omega$, then $h_t = 0$ in $C_\epsilon(t)$ which contradicts the choice for $t^*$. To show this, consider $h_{t^*}$ in $C_{2\epsilon}(t^*)$. As stated earlier, $h_{t^*} = 0$ in $C_\epsilon(t^*)$. Let $\epsilon \le \epsilon_0 \le 2\epsilon$ be such that $C_{\epsilon_0}(t^*)$ is the first cone whose boundary intersects $\text{supp}(h_{t^*})$. If no such $\epsilon_0$ can be found then we are done. Let $q \in \text{supp}(h_{t^*}) \cap \partial C_{\epsilon_0}(t)$. Clearly $q \notin \partial\widetilde{\Omega}$, because $h_{t^*} = 0$ outside $\Omega$. So $q$ is an interior point of $\widetilde{\Omega}$. In $\widetilde{\Omega}$, $(\delta f)_{i_1 \dots i_{m-1}} = (\delta(f\chi))_{i_1 \dots i_{m-1}}$ where $\chi$ is the characteristic function of $\Omega$. But one can first prove the theorem for $f$ such that $f = f^s$ in $\Omega$ and then make the argument for any general $f$. Working first with such tensor fields for which $f = f^s$, one knows that such a tensor field is analytic in $\partial\Omega$ up to $\partial\Omega$, see [18, Section 5].

Now,

$$
\begin{aligned}
(\delta(f^s\chi))_{i_1 \dots i_{m-1}} &= \left(\nabla_k (f^s_{i_1 \dots i_{m-1} j}\chi)\right) g^{jk} \\
&= \left(\chi \nabla_k f^s_{i_1 \dots i_{m-1} j}\right) g^{jk} + f^s_{i_1 \dots i_{m-1} j} g^{jk} \nabla_k \chi \\
&= f^s_{i_1 \dots i_{m-1} j} \nabla^j \chi \\
&= -f^s_{i_1 \dots i_{m-1} j} \nu^j \delta_{\partial\Omega}
\end{aligned}
$$

where $\delta_{\partial\Omega}$ represents dirac delta concentrated at $\partial\Omega$. This shows that the analytic wavefront set of $\delta f$ is in $N^*(\partial\Omega)$. Let $\tilde{\gamma}$ be the geodesic in $\Omega$ on the surface of $\partial C_{\epsilon_0}(t^*)$ that contains $q$. Because $N^*\tilde{\gamma}$ does not intersect $N^*\partial\Omega$, by Proposition 5.4.4 and by Lemma 5.4.1, $h$ has no analytic singularities in $N^*\tilde{\gamma}$. Consider a small open set $W$ containing $q$ which is divided by the surface of $\partial C_{\epsilon_0}(t^*)$ into two open connected sets as in the statement of Lemma 5.4.7 and $h_{t^*} = 0$ in one of these open sets. Since the co-normals to $C_{\epsilon_0}(t^*)$ at $q$ are not in $WF_A(h_{t^*})$, this implies

$q \notin supp(h_{t^*})$ by the Sato-Kawai-Kashiwara theorem mentioned above. This shows that $h_{t^*} = 0$ in $C_{2\epsilon}(t^*)$ which in turn implies $t^* = 1$. This proves Lemma 5.4.5.

Using the condition that any closed path with a base point on $\partial\Omega$ is homotopic to a point lying on $\partial\Omega$ and using the geometric arguments in Section 6 of [18] along with Lemma 5.4.5, we conclude the proof of Theorem 5.2.4.

**Remark:** The symmetric $m-1$ tensor field $v$ also has components in $\mathcal{E}'(\widetilde{\Omega})$ and is supported in $\Omega$ just like the $m$-tensor field $f$.

### 5.4.2 Proof of Theorem 5.2.5

*Proof:* We will first prove the following lemma:

**Lemma 5.4.8** *For any $1 \le k \le m$, if $f = dv$ with $v|_{\partial\Omega} = 0$. Then $I^k f = -kI^{k-1}v$.*

*Proof:* Consider

$$\begin{aligned}
I^k f(\gamma) &= I^k(dv)(\gamma) \\
&= \int_0^{l(\gamma)} t^k (dv)_{i_1\ldots i_m}(\gamma(t))\dot{\gamma}^{i_1}(t)\ldots\dot{\gamma}^{i_m}(t)dt \\
&= \int_0^{l(\gamma)} t^k \frac{d}{dt}\{v_{i_1\ldots i_{m-1}}(\gamma(t))\dot{\gamma}^{i_1}(t)\ldots\dot{\gamma}^{i_{m-1}}(t)\}dt \\
&= \{t^k v_{i_1\ldots i_{m-1}}(\gamma(t))\dot{\gamma}^{i_1}(t)\ldots\dot{\gamma}^{i_{m-1}}(t)\}_0^{l(\gamma)} \\
&\qquad - k\int_0^{l(\gamma)} t^{k-1} v_{i_1\ldots i_{m-1}}(\gamma(t))\dot{\gamma}^{i_1}(t)\ldots\dot{\gamma}^{i_{m-1}}(t)dt \\
&= -kI^{k-1}v(\gamma),
\end{aligned}$$

where first term in the second last equality is 0 because of our assumption $v|_{\partial\Omega} = 0$. Thus, we have our lemma. □

Let us come back to the proof of Theorem 5.2.5. As we know from Theorem 5.2.4 that if $I^0 f(\gamma) = If(\gamma) = 0$ for each geodesic $\gamma$ not intersecting $K$ then there exist $(m-1)$-tensor field $v_1$ which is 0 on the boundary $\partial\Omega$ such that $f = dv_1$ on $\Omega \setminus K$. And from the Lemma 5.4.8, we know

$$I^1 f(\gamma) = I^1(dv_1)(\gamma) = -I^0 v_1(\gamma).$$

Again using Theorem 5.2.4 we conclude that there exist $(m-2)$–tensor field $v_2$ such that $v_1 = dv_2$ and $v_2|_{\partial\Omega} = 0$. Using Theorem 5.2.4 along with Lemma 5.4.8 $(m-2)$ more times, we have

$$I^m f(\gamma) = m!(-1)^m I^0 v_m(\gamma) = 0$$

where $v_m$ is 0-tensor i.e. a function. Now using [16, Theorem 1], we can conclude $v_m = 0$ on $\Omega \smallsetminus K$. And since $f = d^m v_m$ on an open connected set $\Omega \smallsetminus K$ therefore $f$ is also 0 on $\Omega \smallsetminus K$. □

## 5.5 Proof of Theorem 5.2.6

To prove Theorem 5.2.6, we will need the $s$-injectivity of the ray transform for symmetric $m$-tensor fields. The proof of $s$-injectivity for symmetric 2-tensor fields is given in [35]. The same proof will also work for a symmetric tensor field of any order. For details, we will refer the reader to [35, Sections 2,3,4]. Hence we have,

**Theorem 5.5.1** *[35, Theorem 1.4] Let $(\Omega, g)$ be a compact simple real analytic manifold with smooth boundary and $f$ be a symmetric $m$-tensor field with components in $L^2(\Omega)$. If $I^0 f(\gamma) = 0$ for all $\gamma$ which are geodesics in $\Omega$, then $f^s = 0$ in $\Omega$.*

**Theorem 5.5.2** *Let $\Omega$ be a compact simple Riemannian manifold with boundary. Let $m \geq 0$ and $p \geq m$ be integers. Then for any $f \in L^2(S^m(\Omega))$, there exist uniquely determined $v_0, \ldots, v_m$ with $v_i \in H^i(S^{m-i}\Omega)$ for $i = 0, 1, \ldots, m$ such that*

$$f = \sum_{i=0}^{m} d^i v_i, \qquad \text{with } v_i \text{ solenoidal for } 0 \leq i \leq m-1$$

$$\text{and for each } 0 \leq i \leq m-1, \quad \sum_{j=0}^{i} d^j v_{m-i+j} = 0 \text{ on } \partial\Omega.$$

*Proof:* This follows from a repeated application of [30, Theorem 3.3.2]. □

*Proof (Proof of Theorem 5.2.6):* We have from Theorem 5.5.2 that

$$f = \sum_{i=0}^{m} d^i v_i, \qquad \text{with } v_i \text{ solenoidal for } 0 \le i \le m-1$$

$$\text{and for each } 0 \le i \le m-1, \quad \sum_{j=0}^{i} d^j v_{m-i+j} = 0 \text{ on } \partial\Omega. \tag{5.19}$$

Using $s$-injectivity of $I$, we know that $v_0 = 0$, since it is solenoidal. Now consider

$$\begin{aligned} 0 = I^1 f(\gamma) &= I^1\left(\sum_{i=0}^{m} d^i v_i\right)(\gamma) \\ &= I^1\left(d\left(\sum_{i=1}^{m} d^{i-1} v_i\right)\right)(\gamma), \qquad \text{since } v_0 = 0 \\ &= -I^0\left(\sum_{i=1}^{m} d^{i-1} v_i\right)(\gamma) \qquad \text{(using Lemma 5.4.8)} \end{aligned}$$

From this, we can conclude $v_1$ is also 0 because it is solenoidal part of tensor field $\sum_{i=1}^{m} d^{i-1} v_i$.

Now suppose that $v_1, \cdots, v_k$ can be shown to be equal to 0 from the knowledge of $I^1 f, \cdots, I^k f$. Then

$$\begin{aligned} 0 = I^{k+1}\left(f - \sum_{0}^{k} d^i v_i\right) &= I^{k+1}\left(\sum_{i=k+1}^{m} d^i v_i\right) \\ \Rightarrow \qquad I^{k+1}\left(\sum_{i=k+1}^{m} d^i v_i\right) &= 0 \\ \Rightarrow (-1)^{k+1}(k+1)! I^0\left(\sum_{i=k+1}^{m} d^{i-k-1} v_i\right) &= 0, \quad \text{(using Lemma 5.4.8, } (k+1) \text{ times).} \end{aligned}$$

Therefore $v_{k+1} = 0$ because it is the solenoidal part of the tensor field $\left(\sum_{i=k+1}^{m} d^{i-k-1} v_i\right)$. By induction, the proof is now complete. □

## 5.6 Appendix

### 5.6.1 Proof of Lemma 5.3.3

*Proof:* First, let us recall for a $(m-1)$-tensor field $v$,

$$(dv)_{i_1\ldots i_m} = \sigma(i_1,\ldots,i_m)\left(\frac{\partial v_{i_1\ldots i_{m-1}}}{\partial x^{i_m}} - \sum_{l=1}^{m-1}\Gamma^p_{i_m i_l} v_{i_1,\ldots i_{l-1} p i_{l+1}\ldots i_{m-1}}\right).$$

We will prove this result for $k = 0,1,2$ and then for general $k \le m$.

$$\begin{aligned}
(dv)_{n\ldots n} &= \frac{\partial v_{n\ldots n}}{\partial x^n}, \quad \text{for } k = 0\\
(dv)_{n\ldots ni} &= \frac{m-1}{m}\frac{\partial v_{n\ldots ni}}{\partial x^n} - \frac{2(m-1)}{m}\Gamma^p_{in} v_{n\ldots np} + \frac{1}{m}\frac{\partial v_{n\ldots n}}{\partial x^i}, \quad \text{for } k = 1
\end{aligned}$$

And for $k = 2$, we have

$$\begin{aligned}
(dv)_{n\ldots nij} &= \sigma(n,\ldots,n,i,j)\left(\frac{\partial v_{n\ldots ni}}{\partial x^j} - \Gamma^p_{ij} v_{n\ldots np} - (m-2)\Gamma^p_{nj} v_{n\ldots nip}\right)\\
&= \frac{\sigma(n,\ldots,n,i)}{m}\left\{(m-1)\frac{\partial v_{n\ldots nij}}{\partial x^n} + \frac{\partial v_{n\ldots ni}}{\partial x^j} - 2\Gamma^p_{ij} v_{n\ldots np} - (m-2)\Gamma^p_{in} v_{n\ldots npj}\right.\\
&\quad \left. -2(m-2)\Gamma^p_{nj} v_{n\ldots nip} - (m-2)^2\Gamma^p_{in} v_{n\ldots npj}\right\}\\
&= \frac{\sigma(n,\ldots,n,i)}{m}\left\{(m-1)\frac{\partial v_{n\ldots nij}}{\partial x^n} + \frac{\partial v_{n\ldots ni}}{\partial x^j} - 2\Gamma^p_{ij} v_{n\ldots np} - 2(m-2)\Gamma^p_{nj} v_{n\ldots nip}\right.\\
&\quad \left. -(m-1)(m-2)\Gamma^p_{in} v_{n\ldots npj}\right\}\\
&= \frac{1}{m(m-1)}\left\{(m-1)\left((m-2)\frac{\partial v_{n\ldots nij}}{\partial x^n} + \frac{\partial v_{n\ldots nj}}{\partial x^i}\right) + (m-1)\frac{\partial v_{n\ldots ni}}{\partial x^j}\right.\\
&\quad - 2(\Gamma^p_{ij} v_{n\ldots np} + (m-2)\Gamma^p_{nj} v_{n\ldots npi}) - 2(m-2)\left(\Gamma^p_{ij} v_{n\ldots np} + (m-2)\Gamma^p_{nj} v_{n\ldots npi}\right)\\
&\quad \left. - 2(m-1)(m-2)\Gamma^p_{in} v_{n\ldots npj}\right\}\\
&= \frac{1}{m(m-1)}\left\{(m-1)(m-2)\frac{\partial v_{n\ldots nij}}{\partial x^n} + (m-1)\frac{\partial v_{n\ldots nj}}{\partial x^i} + (m-1)\frac{\partial v_{n\ldots ni}}{\partial x^j}\right.\\
&\quad \left. - 2(m-1)\Gamma^p_{ij} v_{n\ldots np} - 2(m-1)(m-2)\Gamma^p_{nj} v_{n\ldots npi} - 2(m-1)(m-2)\Gamma^p_{in} v_{n\ldots npj}\right\}\\
&= \frac{1}{m}\left\{(m-2)\frac{\partial v_{n\ldots nij}}{\partial x^n} + \frac{\partial v_{n\ldots nj}}{\partial x^i} + \frac{\partial v_{n\ldots ni}}{\partial x^j} - 2(m-2)\Gamma^p_{nj} v_{n\ldots npi} - 2(m-2)\Gamma^p_{in} v_{n\ldots npj}\right.\\
&\quad \left. -2\Gamma^p_{ij} v_{n\ldots np}\right\}\\
&= \frac{m-2}{m}\left\{\frac{\partial v_{n\ldots nij}}{\partial x^n} - 2\Gamma^p_{nj} v_{n\ldots npi} - 2\Gamma^p_{in} v_{n\ldots npj}\right\} + \frac{1}{m}\left\{\frac{\partial v_{n\ldots nj}}{\partial x^i} + \frac{\partial v_{n\ldots ni}}{\partial x^j} - 2\Gamma^p_{ij} v_{n\ldots np}\right\}.
\end{aligned}$$

From above, we see that the result is true for $k = 0, 1$ and 2. Now, we are going to prove that the result is also true for $k \leq m$. Consider

$$
\begin{aligned}
(dv)_{n\ldots ni_k\ldots i_1} &= \sigma(n,\ldots n, i_k,\ldots,i_1)\left(\frac{\partial v_{n\ldots ni_k\ldots i_2}}{\partial x^{i_1}} - \sum_{l=2}^{k}\Gamma^p_{i_l i_1} v_{n\ldots ni_k\ldots\hat{i_l}\ldots i_2 p} - (m-k)\Gamma^p_{ni_1} v_{n\ldots ni_k\ldots i_2 p}\right)\\
&= J + J_k^1 + (m-k)J_k^2.
\end{aligned}
$$

where

$$
\begin{aligned}
J &= \sigma(n,\ldots n, i_k,\ldots,i_1)\left(\frac{\partial v_{n\ldots ni_k\ldots i_2}}{\partial x^{i_1}}\right),\\
J_k^1 &= \sigma(n,\ldots n, i_k,\ldots,i_1)\left(\sum_{l=2}^{k}\Gamma^p_{i_l i_1} v_{n\ldots ni_k\ldots\hat{i_l}\ldots i_2 p}\right),\\
\text{and}\quad J_k^2 &= \sigma(n,\ldots n, i_k,\ldots,i_1)\left(\Gamma^p_{ni_1} v_{n\ldots ni_k\ldots i_2 p}\right).
\end{aligned}
$$

$$
\begin{aligned}
J &= \sigma(n,\ldots n, i_k,\ldots,i_1)\left(\frac{\partial v_{n\ldots ni_k\ldots i_2}}{\partial x^{i_1}}\right)\\
&= \frac{\sigma(n,\ldots n, i_k,\ldots,i_2)}{m}\left(\frac{\partial v_{n\ldots ni_k\ldots i_2}}{\partial x^{i_1}} + (m-1)\frac{\partial v_{n\ldots ni_k\ldots i_1}}{\partial x^n}\right)\\
&= \frac{1}{m}\frac{\partial v_{n\ldots ni_k\ldots i_2}}{\partial x^{i_1}} + \frac{\sigma(n,\ldots n, i_k,\ldots,i_3)}{m}\left(\frac{\partial v_{n\ldots ni_k\ldots i_3 i_1}}{\partial x^{i_2}} + (m-2)\frac{\partial v_{n\ldots ni_k\ldots i_1}}{\partial x^n}\right)\\
&= \frac{1}{m}\sum_{l=1}^{k}\frac{\partial v_{n\ldots ni_k\ldots i_{l-1}i_{l+1}\ldots i_1}}{\partial x^{i_l}} + \frac{m-k}{m}\frac{\partial v_{n\ldots ni_k\ldots i_1}}{\partial x^n}, \quad \text{repeating similar arguments .}
\end{aligned}
$$

$$
\begin{aligned}
J_k^2 &= \sigma(n,\ldots n, i_k,\ldots,i_1)\left(\Gamma^p_{ni_1} v_{n\ldots ni_k\ldots i_2 p}\right)\\
&= \frac{\sigma(n,\ldots n, i_k,\ldots,i_2)}{m}\left(2\Gamma^p_{ni_1} v_{n\ldots ni_k\ldots i_2 p} + (m-2)\Gamma^p_{ni_2} v_{n\ldots ni_k\ldots i_3 i_1 p}\right)\\
&= \frac{2\sigma(n,\ldots n, i_k,\ldots,i_3)}{m(m-1)}\left(\Gamma^p_{i_2 i_1} v_{n\ldots ni_k\ldots i_3 p} + (m-2)\Gamma^p_{ni_1} v_{n\ldots ni_k\ldots i_2 p}\right)\\
&\quad + \frac{(m-2)\sigma(n,\ldots n, i_k,\ldots,i_3)}{m(m-1)}\left(2\Gamma^p_{ni_2} v_{n\ldots ni_k\ldots i_3 i_1 p} + (m-3)\Gamma^p_{ni_3} v_{n\ldots ni_k\ldots i_4 i_2 i_1 p}\right)\\
&= \frac{2}{m(m-1)}\Gamma^p_{i_2 i_1} v_{n\ldots ni_k\ldots i_3 p} + \frac{2(m-2)\sigma(n,\ldots n, i_k,\ldots,i_3)}{m(m-1)}\sum_{q=1}^{2}\Gamma^p_{ni_q} v_{n\ldots ni_k\ldots i_3\hat{i_q}i_1 p}\\
&\quad + \frac{(m-3)(m-2)\sigma(n,\ldots n, i_k,\ldots,i_3)}{m(m-1)}\Gamma^p_{ni_3} v_{n\ldots ni_k\ldots i_2 i_1 p}\\
&= \frac{2}{m(m-1)}\sum_{q,r=1,q\neq r}^{3}\Gamma^p_{i_q i_r} v_{n\ldots ni_k\ldots i_4\hat{i_q}\hat{i_r}i_1 p} + \frac{2(m-3)\sigma(n,\ldots n, i_k,\ldots,i_4)}{m(m-1)}\sum_{q=1}^{3}\Gamma^p_{ni_q} v_{n\ldots ni_k\ldots i_4\hat{i_q}i_1 p}
\end{aligned}
$$

$$+\frac{(m-4)(m-3)\sigma(n,\dots n,i_k,\dots,i_4)}{m(m-1)}\Gamma^p_{ni_4}v_{n\dots ni_k\dots i_5i_3i_2i_1p},$$

repeating similar calculation $(k-3)$ times, we get

$$=\frac{2(k-1)}{m(m-1)}\sum_{q,r=1,q\neq r}^{k}\Gamma^p_{i_qi_r}v_{n\dots ni_k\dots\hat{i_q}\dots\hat{i_r}\dots i_1p}+\frac{2(m-k)}{m(m-1)}\sum_{q=1}^{k}\Gamma^p_{ni_q}v_{n\dots ni_k\dots\hat{i_q}\dots i_1p}.$$

$$J^1_k=\sigma(n,\dots n,i_k,\dots,i_1)\left(\sum_{l=2}^{k}\Gamma^p_{i_li_1}v_{n\dots ni_k\dots\hat{i_l}\dots i_2p}\right)$$

$$=\frac{\sigma(n,\dots n,i_k,\dots,i_2)}{m}\left(2\sum_{l=2}^{k}\Gamma^p_{i_li_1}v_{n\dots ni_k\dots\hat{i_l}\dots i_2p}+(m-2)\sum_{l=3}^{k}\Gamma^p_{i_li_2}v_{n\dots ni_k\dots\hat{i_l}\dots i_3i_1p}\right.$$

$$\left.+(m-2)\Gamma^p_{i_3i_2}v_{n\dots ni_k\dots i_4i_1p}\right)$$

$$=\frac{\sigma(n,\dots n,i_k,\dots,i_3)}{m(m-1)}\left\{2(k-1)\Gamma^p_{i_2i_1}v_{n\dots ni_k\dots i_3p}+(m-2)\left(2\sum_{l=3}^{k}\Gamma^p_{i_li_1}v_{n\dots ni_k\dots\hat{i_l}\dots i_2p}\right.\right.$$

$$+2\Gamma^p_{i_3i_1}v_{n\dots ni_k\dots i_4i_2p}+2\sum_{l=3}^{k}\Gamma^p_{i_li_2}v_{n\dots ni_k\dots\hat{i_l}\dots i_3i_1p}+(m-3)\sum_{l=4}^{k}\Gamma^p_{i_li_3}v_{n\dots ni_k\dots\hat{i_l}\dots i_4i_2i_1p}$$

$$\left.\left.+(m-3)\Gamma^p_{i_3i_4}v_{n\dots ni_k\dots i_5i_2i_1p}+2\Gamma^p_{i_3i_2}v_{n\dots ni_k\dots i_4i_1p}+(m-3)\Gamma^p_{i_3i_4}v_{n\dots ni_k\dots i_5i_2i_1p}\right)\right\}$$

$$=\frac{(m-2)\sigma(n,\dots n,i_k,\dots,i_3)}{m(m-1)}\left\{2\sum_{q=1}^{2}\left(\sum_{l=3}^{k}\Gamma^p_{i_li_q}v_{n\dots ni_k\dots\hat{i_l}\dots\hat{i_q}i_1p}+\Gamma^p_{i_3i_q}v_{n\dots ni_k\dots i_4\hat{i_q}i_1p}\right)\right.$$

$$\left.+(m-3)\left(\sum_{l=4}^{k}\Gamma^p_{i_li_3}v_{n\dots ni_k\dots\hat{i_l}\dots i_4i_2i_1p}+2\Gamma^p_{i_3i_4}v_{n\dots ni_k\dots i_5i_2i_1p}\right)\right\}+\frac{2(k-1)}{m(m-1)}\Gamma^p_{i_2i_1}v_{n\dots ni_k\dots i_3p}$$

$$=\frac{(m-3)\sigma(n,\dots n,i_k,\dots,i_4)}{m(m-1)}\left\{2\sum_{q=1}^{3}\left(\sum_{l=4}^{k}\Gamma^p_{i_li_q}v_{n\dots ni_k\dots\hat{i_l}\dots\hat{i_q}i_1p}\right)+4\sum_{q=1}^{3}\left(\Gamma^p_{i_4i_q}v_{n\dots ni_k\dots\hat{i_q}i_1p}\right)\right.$$

$$\left.+(m-4)\left(\sum_{l=5}^{k}\Gamma^p_{i_li_4}v_{n\dots ni_k\dots\hat{i_l}\dots i_1p}+3\Gamma^p_{i_5i_4}v_{n\dots ni_k\dots i_1p}\right)\right\}+\frac{2(k-1)}{m(m-1)}\sum_{q,r=1,q\neq r}^{3}\Gamma^p_{i_qi_r}v_{n\dots ni_k\dots\hat{i_q}\hat{i_r}i_1p}$$

Repeating this expansion for $(k-2)$ times more to get

$$J^1_k=\frac{(m-k+1)\sigma(n,\dots,n,i_k)}{m(m-1)}\left\{2\sum_{q=1}^{k-1}\Gamma^p_{i_ki_q}v_{n\dots ni_{k-1}\dots\hat{i_l}\dots i_1p}+2(k-2)\sum_{q=1}^{k-1}\Gamma^p_{i_ki_q}v_{n\dots ni_{k-1}\dots\hat{i_l}\dots i_1p}\right.$$

$$\left.+(m-k)(k-1)\Gamma^p_{ni_k}v_{n\dots ni_{k-1}\dots i_1p}\right\}+\frac{2(k-1)}{m(m-1)}\sum_{q,r=1,q\neq r}^{k-1}\Gamma^p_{i_qi_r}v_{n\dots ni_k\dots\hat{i_q}\hat{i_r}i_1p}$$

$$=\frac{(m-k+1)\sigma(n,\dots,n,i_k)}{m(m-1)}\left\{2(k-1)\sum_{q=1}^{k-1}\Gamma^p_{i_ki_q}v_{n\dots ni_{k-1}\dots\hat{i_q}\dots i_1p}\right.$$

$$\left.+(m-k)(k-1)\Gamma^p_{ni_k}v_{n\dots ni_{k-1}\dots i_1p}\right\}+\frac{2(k-1)}{m(m-1)}\sum_{q,r=1,q\neq r}^{k-1}\Gamma^p_{i_qi_r}v_{n\dots ni_k\dots\hat{i_q}\hat{i_r}i_1p}$$

$$= \frac{2(k-1)}{m(m-1)} \sum_{q,r=1,q\neq r}^{k} \Gamma^p_{i_q i_r} v_{n\ldots ni_k\ldots\hat{i_q}\ldots\hat{i_r}\ldots i_1 p} + \frac{2(k-1)(m-k)}{m(m-1)} \sum_{q=1}^{k} \Gamma^p_{ni_q} v_{n\ldots ni_k\ldots\hat{i_q}\ldots i_1 p}$$

After putting the values of $J, J^1_k$ and $J^2_k$ in $dv$, we get

$$\begin{aligned}(dv)_{n\ldots ni_k\ldots i_1} &= \frac{(m-k)}{m}\frac{\partial v_{n\ldots ni_k\ldots i_1}}{\partial x^n} - \frac{2(m-k)}{m}\sum_{l=1}^{k}\Gamma^p_{i_l n} v_{n\ldots ni_k\ldots\hat{i_l}\ldots i_1 p}\\ &+ \frac{1}{m}\sum_{l=1}^{k}\frac{\partial v_{n\ldots ni_k\ldots\hat{i_l}\ldots i_1}}{\partial x^{i_l}} - \frac{2}{m}\sum_{l,q=1,l\neq q}^{k}\Gamma^p_{i_l i_q} v_{n\ldots ni_k\ldots\hat{i_l}\ldots\hat{i_q}\ldots i_1 p}. \qquad \square\end{aligned}$$

### 5.6.2 Proof of estimates

*Proof (Proof of estimate* (5.11)*):* Let ${}^tL = \frac{\Phi_\xi.\partial_\xi}{i\lambda|\Phi_\xi|^2}$. Then as already noted

$${}^tL^N(e^{i\lambda\Phi(x,\xi)}) = e^{i\lambda\Phi(x,\xi)}.$$

Consider,

$$\begin{aligned}&\left|\iint_{|x-y|>\delta/C_0}({}^tL^N(e^{i\lambda\Phi(y,x,\xi,\eta)}))\tilde{a_N}(x,\xi)f_{i_1\ldots.i_m}(z)\tilde{b}^{i_1}(x,\xi)\ldots\tilde{b}^{i_m}(x,\xi)dxd\xi\right|\\ &\quad\leq\left|\iint_{|x-y|>\delta/C_0} e^{i\lambda\Phi(y,x,\xi,\eta)}L^N(\tilde{a_N}(x,\xi)f_{i_1\ldots.i_m}(z)\tilde{b}^{i_1}(x,\xi)\ldots\tilde{b}^{i_m}(x,\xi))dxd\xi\right|\\ &\qquad+N\int_{|x-y|>\delta/C_0} e^{-\lambda\delta^2/2}\left|f_{i_1\ldots i_m}(x)B^{i_1\ldots i_m}(x,\xi_{bdry})\right|dx.\end{aligned}$$

Using the fact that, $f$ is compactly supported and using (5.7), we get (5.11). $\square$

*Proof (Proof of the estimate* (5.13)*):* Consider

$$\left|\iint_{|x-y|<\delta/C_0}\left(e^{i\lambda\Phi(y,x,\xi,\eta)}\right)\tilde{a_N}(x,\xi)f_{i_1\ldots i_m}(z)\tilde{b}^{i_1}(x,\xi)\cdots\tilde{b}^{i_m}(x,\xi)dxd\xi\right|.$$

Rewrite the above as :

$$\left|\int_{|x-y|<\delta/C_0}(e^{i\lambda\Phi(y,x,\xi_c,\eta)})(e^{-i\lambda\Phi(y,x,\xi_c,\eta)})\int_{|\xi-\eta|<\delta/C_0}(e^{i\lambda\Phi(y,x,\xi,\eta)})\tilde{a_N}(x,\xi)f_{i_1\ldots i_m}(z)\tilde{b}^{i_1}(x,\xi)\cdots\tilde{b}^{i_m}(x,\xi)dxd\xi\right|$$

Using (2.10) of [33] to the above, we get

$$\Big|\int_{|x-y|<\delta/C_0}(e^{i\lambda\Phi(y,x,\xi_c,\eta)})f_{i_1\ldots i_m}(x)\sum_{0\le k\le\lambda/C}C_n\frac{1}{k!}\lambda^{-n/2-k}\frac{(\triangle^k)}{2}\big(\tilde{\tilde{a_N}}(x,\xi_c)\tilde{b}^{i_1}(x,\xi_c)\cdots\tilde{b}^{i_m}(x,\xi_c)\big)$$

$$+R(x,y,\eta,\lambda)dx\big|$$

where $|R(x,y,\eta;\lambda)|\le\Omega/Ce^{-\lambda/c}$, (See 2.10, [33]).

**Lemma 5.6.1**

$$\sum_{0\le k\le\lambda/C}C_n\frac{1}{k!}\lambda^{-n/2-k}\frac{(\triangle^k)}{2}\big(\tilde{\tilde{a_N}}(x,\xi_c)\tilde{b}^{i_1}(x,\xi_c)\cdots\tilde{b}^{i_m}(x,\xi_c)\big)$$

*is a formal analytic symbol.*

*Proof:* Let,

$$A_k=\frac{1}{k!}\frac{(\triangle^k)}{2}\big(\tilde{\tilde{a_N}}(x,\xi_c)\tilde{b}^{i_1}(x,\xi_c)\cdots\tilde{b}^{i_m}(x,\xi_c)\big)$$

Then from Cauchy integral formula [33, Section 2.4],

$$\begin{aligned}|A_k| &\le C_n(k+1)^{n/2}(k-1)!2^k\sup_{B(\xi_c)}\big(\tilde{\tilde{a_N}}(x,\xi_c)\tilde{b}^{i_1}(x,\xi_c)\ldots\tilde{b}^{i_m}(x,\xi_c)\big)\\ &\le C1_n(k+1)^{n/2}(k-1)!2^k\\ &\le C2_n(k+1)^{n/2}e^{2-k}(k-1)^{k-1/2}2^k \text{ (Using Stirling's approximation)}\\ &\le C2_n\left(\frac{2}{e}\right)^{k+1}(k+1)^{n/2+k}\\ &\le \tilde{C_n}^{k+n/2}(k+n/2)^{n/2+k}.\end{aligned}$$

Hence,

$$\sum_{0\le k\le\lambda/C}C_n\frac{1}{k!}\lambda^{-n/2-k}\frac{(\triangle^k)}{2}\big(\tilde{\tilde{a_N}}(x,\xi_c)\tilde{b}^{i_1}(x,\xi_c)\ldots\tilde{b}^{i_m}(x,\xi_c)\big)=\sum_{0\le k\le\lambda/C}\lambda^{-n/2-k}A_{k+n/2}$$

is a formal analytic symbol $B^{i_1\cdots i_m}(x,y,\eta;\lambda)$ by [33, Excercise 1.1]. □

Hence,

$$\begin{aligned}\int_{|x-y|<\delta/C_0} (e^{i\lambda\Phi(y,x,\xi,\eta)}) \tilde{\tilde{a}}_N(x,\xi) f_{i_1\ldots i_m}(z) \tilde{b}^{i_1}(x,\xi) \cdots \tilde{b}^{i_m}(x,\xi) dx d\xi \\ = \int_{|x-y|<\delta/C_0} (e^{i\lambda\Phi(y,x,\xi_c,\eta)}) f_{i_1\ldots i_m}(x) B^{i_1\ldots i_m}(x,y,\eta;\lambda) dx \\ + \int_{|x-y|<\delta/C_0} (e^{i\lambda\Phi(y,x,\xi_c,\eta)}) f_{i_1\ldots i_m}(x) R(x,y,\eta;\lambda) dx d\xi. \qquad \square\end{aligned}$$

But,

$$\left| \int_{|x-y|<\delta/C_0} (e^{i\lambda\Phi(y,x,\xi_c,\eta)}) f_{i_1\ldots i_m}(x) R(x,y,\eta;\lambda) dx \right| = \mathcal{O}(e^{-\lambda/c})$$

since

$$|R(x,y,\eta;\lambda)| \leq \Omega/Ce^{-\lambda/c}$$

(See 2.10, [33]). So, this along with (5.10) and (5.11), gives us:

$$\left| \int_{|x-y|<\delta/C_0} (e^{i\lambda\Phi(y,x,\xi_c,\eta)}) f_{i_1\ldots i_m}(x) B^{i_1\ldots i_m}(x,y,\eta;\lambda) dx \right| = \mathcal{O}(e^{-\lambda/c}).$$

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