REFLEXIVE FORMS, CONFIGURATIONS, AND HYPERGRAPHS: A DEEP DIVE

A thesis submitted by

Amanda V. Tran

in partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematics

TUFTS UNIVERSITY

May 2025

Advisor: George McNinch

Abstract

The world of bilinear forms and their relation to configuration geometry highlights interesting constructions and phenomena; in this thesis, we discuss the tools required to make constructions of polar spaces associated with sesquilinear (with a focus on bilinear) forms over finite fields. In particular, when we talk about totally singular of bilinear forms as hypergraphs, we often find that they live in highly regular configurations (where the dual of the "graph" formed by the totally singular lines is the same graph).

Here, we aim to explore various bilinear forms over finite fields as configurations, and tie it together the machinery native to graph coloring.

Acknowledgements

First and foremost, I would like to thank my parents and my fiance, Chris, for the support throughout this particularly treacherous journey we call a thesis. I wouldn't be able to do it without their patience for my shoddy time management and stress induced hysteria.

Second, I would like to thank all the professors and graduate students at Tufts I've had the joy of interacting with, especially as a student, for your care and dedication over the last two years. I would very specifically like to thank Flame for his love and dedication to all of us as our algebra TA — I don't know where I'd be if it weren't for him letting me camp out in his office for hours on end, well after office hours ended, just to let me workshop my homework. I hope to love my students as deeply one day.

In a very special way, I want to thank my advisor, George McNinch, for putting up with my awkward pacing over the last year. I know the last few months must have been a headache and a half, especially with my insistence on doing things in strange and unorthodox ways. Thank you, as well, for entertaining my second-wind aversion to matrices and any number that isn't prime!

Similarly, I would like to give a **massive shout-out** to Robert Lemke Oliver and David Smyth for agreeing to serve on my committee on short notice. Y'all are real ones for that. :)

Third, I would like to thank all my professors at UMass Boston who not only inspired me and fed my curiosity, but *significantly more importantly,* humored me. Namely, I wanted to thank Gabriel K. Cunningham for recruiting me to be his undergrad TA which set me off to be a permanent fixture in the UMB math department for the subsequent three years, Mirjana Vuletic for giving me my first taste of graphs, Steven Glen Jackson for my introduction to finite fields, and Joel Watson Fish for my obsession with set builder notation. Unfortunately, this unholy combination has set a plight upon mathematic academia!

But in particular, I wanted to thank Eric Grinberg, my undergraduate thesis advisor, for setting me down this spiral for my undergraduate thesis *to begin with,* then proceeded to let me go off the rails and throw graph stuff at the whole thing! I had decided to do a thesis last minute, and my academic advisor, on the first day of classes, had to scramble to find an adult willing to blend the aforementioned unholy combination together into something worth doing. The whimsy is not lost on me.

Finally, thank you to all my friends both at Tufts and UMass. You have witnessed me persist despite the horrors, and we're still friends, and I think that's worth something. Erjon, Jack, V, Ishaan, Chami, *Cat**, and MJ, yall have *seen horrors beyond your comprehension* and still stuck around, so shout-out to y'all! The late nights in the math library, making sure I ate, and insisting I beat the sunrise on my walk home are not lost on me. (And Logo, who will randomly drop random bits of comforting wisdom and reassurance while rushing off to this, that, or the other thing!)

Reflexive Forms, Configurations, and Hypergraphs: A Deep Dive

---

Contents

Reflexive Forms, Configurations, and Hypergraphs: A Deep Dive

Finite Geometries and Projective Spaces

We'll be discussing reflexive sesquilinear forms, with a focus on the finite field flavor. And to talk about them in this particular approach is to inherently talk about finite geometry and projective geometry.

Big picture wise, our first step is to classify, up to isometry, the relevant forms on our vector spaces with an emphasis on finite fields. From there, we want to classify the point-line geometries arising from sub-geometries of projective spaces over finite fields determined by reflexive sesqui-linear forms.

Sesquilinear forms, which will be defined a bit later on, work under the insistence that we don't take the zero vector as arguments. Given that, projective geometry, which also insist on working without the zero vector (as well as certain equivalence class properties), is a natural angle to approach this with.

A more generalized, and intuitive, way of discussing the geometries present also requires a bit of point-line geometry (and some hypergraph) machinery as well, especially given the finite field perspective.

Given this, we'll begin by front-loading some definitions.

Definition 1 (Point-Line Geometry)**.** A point-line geometry is a set P of points, a set L of lines, and a relation $R \subseteq P \times L$, where for a pair $(p, \ell) \in P \times L$, the condition $(p, \ell) \in R$ means that the point p is incident to the line ℓ.

Definition 2 (Incident)**.** A point and line in a point-line geometry are *incident* if the point belongs to the line.

More generally, two objects of distinct types are *incident* if one object belongs to, or lives on, the other.

The total incidence number of a point line-geometry is the sum of the degrees of all points in the geometry.

Definition 3 (Incidence Structure)**.** An incidence structure on a graph or point-line geometry is a matrix that encodes the relationship between an ordered pair, $(p, \ell) \in P \times L$. With the rows representing the lines, and the columns representing the points, the position corresponding to each (p, l) will be filled with a 1 if the two objects are incident, and a 0 otherwise.

Remark. Of course, incidence structures can encode the relationship between *any* two distinct classes of objects, and whether one object type is on the row or column is arbitrary as well.

Definition 4 (Adjacent)**.** Two points, p and $q \in P$ are adjacent (or "collinear") if there is a line $\ell \in L$ such that p and q are both incident to ℓ. In other words, there exists an ℓ such that (p, ℓ) and $(q, \ell) \in R$

Similarly, a pair of points, ℓ and $\ell' \in L$ are adjacent if there is a point $p \in P$ such that (p, ℓ) and $(p, \ell') \in R$

Definition 5 (Adjacency Structure)**.** An adjacency structure on a graph or point-line geometry is a matrix that encodes whether two objects of the same type, typically p and $q \in P$ or ℓ and $\ell' \in L$, are adjacent.

In particular, the adjacency matrix is a square matrix, where each row corresponds to a unique point, and each column corresponds to a unique point. If two points are adjacent, then the intersection of their respective rows and columns contain a one, and otherwise contain a zero.

Since every point is self adjacent, the row and column corresponding to the same point will have a one, and assuming the labeling order of rows and columns are the same, the matrix will be symmetric.

A similar convention holds for the adjacency matrix of lines (as well as any other object one would wish to talk about).

Definition 6 (Degree or Valency)**.** The degree, or valency, of a point p (or node/vertex) is the total number of lines that pass through point p. It's sometimes denoted as $deg(p)$.

Definition 7 (Point or Line Regular)**.** A point-line geometry is called *point regular* if every point has the same degree.

If all lines in a point-line geometry contain the same number of points, then it is called *line regular*.

Definition 8 (Projective Space)**.** For us, the projective space $P_{k-1}(\mathbb{F})$ of a vector space $V_k(\mathbb{F})$ is the point-line geometry whose points are the one-dimensional vector subspaces of $V_k(\mathbb{F})$ that pass through the origin, and the lines are the two-dimensional vector subspaces of $V_k(\mathbb{F})$ that pass the origin. The incidence relation between points and lines is just determined by containment. Specifically, a point p is incident to a line ℓ if the one-dimensional subspace p of $V_k(\mathbb{F})$ is contained in the two-dimensional subspace ℓ of $V_k(\mathbb{F})$.

Remark. The dimension of the projective space is one less than the vector space, and all subspaces of the projective space are one less than their corresponding subspace in the vector space.

For example, a projective line comes from a plane in the vector space, and the projective points on a projective line come from the vectors on the aforementioned plane.

Particularly, the joy of projective space is that we collapse our space down through the zero vector - which works out perfectly for our purposes because we insist our forms don't use the zero vector anyway! (More on that later!)

Example 1 (Projective Space Over Finite Field $\mathbb{F}_3$ - $V_3(\mathbb{F}_3)$)**.** To construct a finite projective space of k dimensions, we consider the number of coordinate positions - four in this case - and how many options we have for coordinate values - three in this case.

Thus, this gives us 3^3 coordinate points.

However, as we're working in projective spaces, we have to consider that the projection through the origin will remove the origin altogether and collapse scalar multiples of points (for example, 1000 and 2000). Because of this, we must divide by the number of nonzero scalars in the field, giving us $\frac{3^3-1}{2} = 13$, for a total of thirteen points in our projective plane. We also get a canonical and homogenized coordinate system out of this!

One important note to remember is that projective points are vectors in the original vector space, projective lines are planes in the vector space, projective planes are three dimensional spaces in the vector space, and so forth.

To determine how many projective points belong on a projective line, we consider that a projective line is really just the projection of a plane through the origin. So in a similar way as calculating the number of points in our overall projective space, we get $\frac{3^2-1}{2} = 4$ points to a line.

Now to calculate how many lines there are total, we note that two points determine a line, every line may be generated by two of its points, and thus we get $\frac{\binom{13}{2}}{\binom{4}{2}} = 13$.

In fact, because we're working in a projective space, we're guaranteed point regularity as well as line regularity! Meaning that the total incident is $4 \cdot 13$, and that should be evenly divided up between all thirteen, giving each point belonging to four distinct lines. This structure is also known as one of the (13_4) configurations.

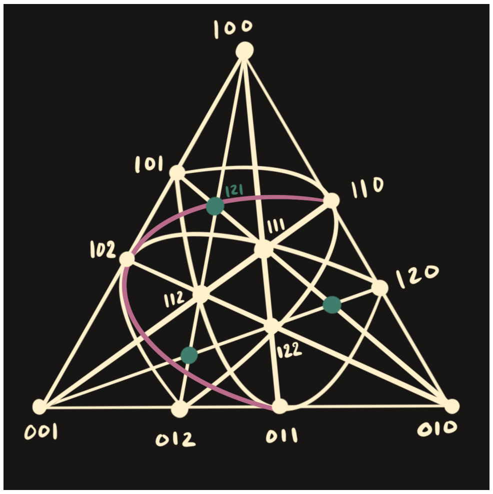


Figure 1: $\mathbb{P}_2(\mathbb{Z}_3)$

Definition 9 (Configuration)**.** A configuration is a point-line geometry with point regularity and line regularity. It's denoted as (p_i, l_k), where p is the total number of points, i is the number of lines through a point, l is the total number of lines, and k is the number of points through a line. These spaces are finite, as well as point and line regular.

Configurations satisfy the equation $pi = lk$ — which is called the number of flags. Furthermore, configurations with the same (p_i, l_k) don't necessarily have the same incidence structure, and certainly aren't obligated to have the same algebraic structure either! (More on this later.) Using the incidence structure is an efficient way of determining if two configurations are, in fact, isomorphic.

Lemma 1. All projective spaces over finite fields are (p_l) configurations - there are p points, p

lines, every point belongs to l lines, and every line contains l points.

This is easy enough to verify, especially by induction, and will actually be proven for a projection of two dimensions later on!

Remark. Since some of our examples come from $\mathbb{P}_3(\mathbb{F}_2)$, they'll be, typically, of form $((3j)_2, (2j)_3)$ such that $2 \leq j \leq 5$. The reason is that configurations with an odd number of lines typically behave poorly because of a smaller subconfiguration with an even number of lines. Thus, we'll typically only address configurations with an even number of vertices.

And due to the nature of our vector space, every plane has three vectors, thus, all projective lines contain three points, $k = 3$, hence the second subscript. The poor behavior (and evenness of the number of lines) tells us that the incidence of the j must be two. We of course will have other examples that behave poorly but for other reasons as well.

It's also worth noting that for any finite field $\mathbb{F}_q$, we'll find that some of the most interesting configurations will be of the form kv_q, ke_q for similar reasons (and more often than not, $e = v$).

Definition 10 (Chromatic Number and Chromatic Index)**.** The *chromatic number* of a point-line geometry (or graph!) is the minimum number of colors needed such that all points that are adjacent are colored in a different color.

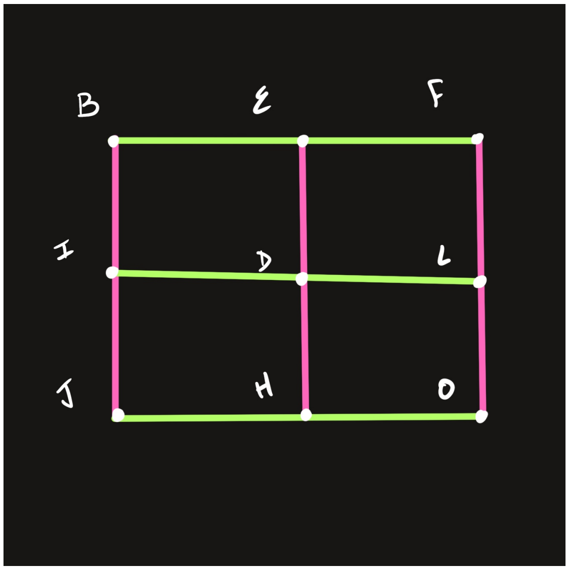
The *chromatic index* of a point-line geometry is the minimum number of colors needed such that all lines that are adjacent to each other are colored in a different color.

The following are some examples of point line geometries!

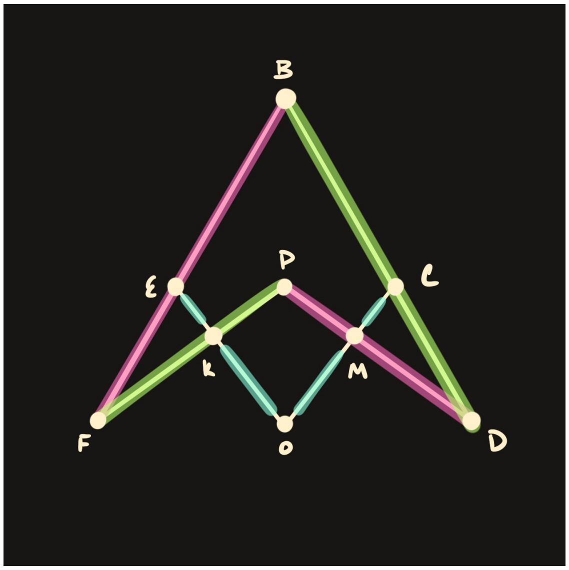
Remark. Observe that the chromatic indices between thef two $(9_2, 3_6)$ configurations are different.

Multi-linear Algebra

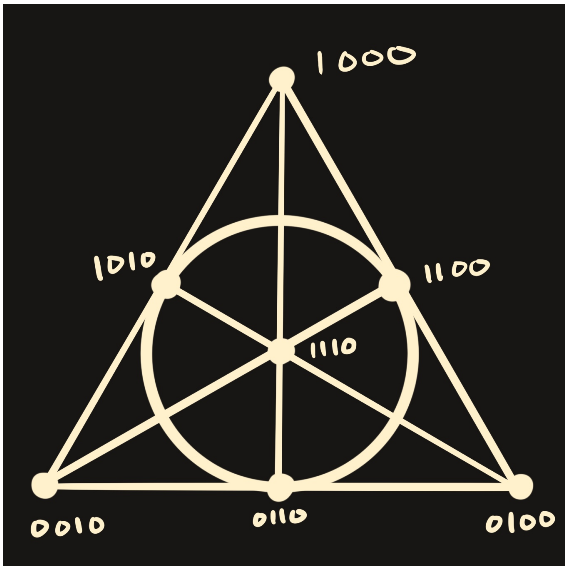
Let $\mathbb{F}$ be a field, $V_k(\mathbb{F})$ be a vector space over $\mathbb{F}$, k be the dimension of said vector space, and σ be an automorphism of $\mathbb{F}$.



(a) Points and lines of the quadric $xy = zw$ in $\mathbb{F}_2$ — a configuration denoted as $(9_2, 6_3)$



(b) Somehow another distinct $(9_2, 6_3)$ configuration — not $xy = zw$.



(a) The Fano Plane - a (7_3) configuration which is notably highly regular

(b) A point-line geometry with irregular number of lines through a point

Definition 11. A σ-sesquilinear is a map $b : V_k(\mathbb{F}) \times V_k(\mathbb{F}) \to \mathbb{F}$ such that

1. $b(u, v)$ is a linear form in the variable u for any fixed $v \in V_k(\mathbb{F})$
2. the map $b(u, v)$ is additive for any fixed $u \in V_k(\mathbb{F})$
3. $b(u, \lambda v) = \lambda^\sigma b(u, v) \forall v \in V_k(\mathbb{F})$ and $\lambda \in \mathbb{F}$

[Ball, pg 25]

Definition 12 (Bilinear Form)**.** A bilinear form is a σ-sesquilinear map $b : V_k(\mathbb{F}) \times V_k(\mathbb{F}) \to \mathbb{F}$ on a vector space V over a field $\mathbb{F}$ such that $b(u, v)$ is a linear form in the variable u for any fixed v, and is a linear form in the variable v for any fixed u.

An alternative representation, which is helpful for calculation purposes, guarantees that b satisfies the following three conditions for $u, v \in V$ and $k \in \mathbb{F}$:

1. $b(ku, v) = kb(u, v) = b(u, kv)$
2. $b(u_1 + u_2, v) = b(u_1, v) + b(u_2, v)$
3. $b(u, v_1 + v_2) = b(u, v_1) + b(u, v_2)$

where σ (as addressed in the σ-sesquilinear definition) is the identity.

Definition 13 (Isometric (equivalent))**.** Two σ-sesquilinear forms, b and b', are isometric if there is an isomorphism α of $V_k(\mathbb{F})$ such that $b(u, v) = b'(\alpha(u), \alpha(v)) \forall\, u, v \in V_k(\mathbb{F})$.

[Ball, pg 26]

Definition 14 (Reflexive)**.** A $\sigma-$sesquilinear form b is *reflexive* if $b(u, v) = 0 \iff b(v, u) = 0$.

[Ball, pg 26]

So now consider a reflexive $\sigma-$sesquilinear form b on some $V_k(\mathbb{F})$.

Definition 15 (Degenerate Form)**.** b is degenerate if there exists a vector $u \in V_k(\mathbb{F})\backslash\{0\}$ such that $b(u, v) = 0\ \forall v \in V_k(\mathbb{F})$.

A similar definition applies to degenerate quadratic forms, f, where f is degenerate if there is a nonzero vector $u \in V_k(\mathbb{F})$ such that $f(u) = 0$ and $b(u, v) = 0 \forall v \in V_k(\mathbb{F})$ (where b is the associated bilinear form of f).

We'll discuss quadratic forms a little bit later.

Definition 16 (Alternating Forms)**.** A bilinear form b is said to be an alternating form if $\forall u \in V_k(\mathbb{F}),\ b(u, u) = 0$.

[Ball, pg 28]

Definition 17 (Symmetric Form)**.** A bilinear form b is said to be symmetric if $\forall u, v \in V_k(\mathbb{F}),\ b(u,v) = b(v,u)$.

If $char(\mathbb{F}) \neq 2$, then $\frac{1}{2}b(u,u)$ is a quadratic form.

[Ball, pg 28]

Definition 18 (Quadratic Form)**.** A *quadratic form,* f, on $V_k(\mathbb{F})$ is a map from $V_k(\mathbb{F}) \to \mathbb{F}$ such that $f(\lambda u) = \lambda^2 f(u)\ \forall \lambda \in \mathbb{F}$ and $u \in V_k(\mathbb{F})$ and $b(u,v) = f(u+v) - f(u) - f(v)$ is a bilinear symmetric form on $V_k(\mathbb{F})$.

[Ball, pg 40]

Definition 19 (Hermitian Forms)**.** A Hermitian form is a non-degenerate reflexive form, b, on $V_k(\mathbb{F})$, up to scalar factor, such that $\forall u \in V_k(\mathbb{F}),\ b(u,v) = b(u,v)^{\sigma}$.

We insist, as well, that σ is a non-identity automorphism of order two such that $\sigma^2 = id$.

[Ball, pg 28]

Definition 20 (Radical of A Form)**.** The radical of the form b is defined as $rad_l(b) := \{u \in V_k(\mathbb{F}) |\ b(u,v) = 0\ \forall v \in V_k(\mathbb{F})\}$. It can also be regarded as $ker(b)$.

A similar setup is done for the right position argument, called $rad_r(b)$.

Lemma 2. If b is a reflexive σ-sesquilinear form, then $rad_l(b)$ and $rad_r(b)$ are linear subspaces of $V_k(\mathbb{F})$, and the form is non-degenerate if and only if $rad_l(b) = \{0\} = rad_r(b)$.

Proof.

Suppose b is a σ-sesquilinear form over $V_k(\mathbb{F})$.

Subproof.

First we'll show that $rad_l(b)$ and $rad_r(b)$ are linear subspaces of $V_k(\mathbb{F})$, starting with the left.

We'll start with associativity. Consider vectors $u,\ v,$ and w.

$$(u+v)+w = ([u_1, u_2, ...u_k] + [v_1, v_2, ...v_k]) + [w_1, w_2, ...w_k]$$

$$= [u_1 + v_1, u_2 + v_2, ...u_k + v_k] + [w_1, w_2, ...w_k]$$

$$= [u_1 + v_1 + w_1, u_2 + v_2 + w_2, ...u_k + v_k + w_k]$$

$$= [u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), ...u_k + (v_k + w_k)]$$

$$= [u_1, u_2, ...u_k] + [v_1 + w_1, v_2 + w_2, ...v_k + w_k]$$

$$= u + (v + w)$$

As for the identity, $\vec{0}$ is always in the left radical by definition.

Now we'll show that $\forall v \in rad_l(b)$, $\exists(-v) \in rad_l(b)$ such that $v + (-v) = 0$ and $b(-v, u) = 0 \forall u$.

Consider now some $v \in rad_l(b)$. Given this we know $b(v, u) = 0\ \forall u \in V_k(\mathbb{F})$. We'll use this fact to show that $b(-v, u) = 0$ and thus, $-v$ lives in the radical.

$$0 = b(v, u)$$

$$= b(v, u) + b(v + -v, u)$$

$$= b(v, u) + b(-v, u) + b(v, u) \qquad = b(-v, u) + 2b(v, u) = b(-v, u) + 2(0) = b(-v, u)$$

Finally we'll show closure under scalar multiplication!

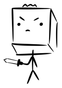
Pick an $\alpha \in \mathbb{F}$, and lest not forget that $b(v, u) = 0$! $b(\alpha v, u) = \alpha b(v, u) = 0$.

Subproof.

Now we'll proceed to show the form is non-degenerated if and only if $rad_l(b) = \{0\} = rad_r(b)$.

In the forward direction, if b is non-degenerate, then $\nexists\ v \in V_k(\mathbb{F})$ such that $b(v, u) = 0$. A similar argument is made for u on the righthand argument as well. But that means the left radical is trivial, and of course this means the right radical is trivial as well because the radicals are the same dimension.

In the backwards direction, if the radicals are trivial, then we can't procure a v such that $b(v, u) = 0$.



Remark. For notational purposes when working in $V_k\mathbb{F}_2$, where k is some low number like 4, I like to use $xyzw$ notation - that is, x corresponds to the first coordinate vector value, y corresponds to the second, and so forth. For example, 101 corresponds to the vector $[1, 0, 1]$, and polynomials using this notation would be something like $xy + zw = 0$.

If a form takes two arguments, x_1y_2 mean to take the first coordinate of the first point or vector, and the second coordinate of the second point or vector.

The reason is strictly for legibility purposes, but when we extend to arbitrary dimension k, we'll revert back to $u = [u_1, u_2, ...u_k]$ notation.

Example 2. What we're going to do is *construct* a degenerate form, b, by rigging what we want the (left) kernel, or $rad_l(b)$, to be spanned by!

So suppose that we're working in $V_3(\mathbb{F}_2)$ and we want $rad_l(b) = \{\langle 100\rangle\}$.

Now to construct $b(u, v)$, we'll want to write all possible second degree terms, so let α be a scalar in $\mathbb{F}_2$, and u and v are vectors in $V_3(\mathbb{F}_2)$.

$$b(u,v) = \sum_{i=1}^{3}\sum_{j=1}^{3}\alpha_{i,j}u_iv_j$$

$$= \alpha_{1,1}u_1v_1 + \alpha_{1,2}u_1v_2 + \alpha_{1,3}u_1v_3 + \alpha_{2,1}u_2v_1 + \alpha_{2,2}u_2v_2$$

$$+ \alpha_{2,3}u_2v_3 + \alpha_{3,1}u_3v_1 + \alpha_{3,2}u_3v_2 + \alpha_{2,2}u_3v_3$$

From here, we'll recall that 100 is the nontrivial left kernel element, and consider, in particular, $b(100, v)$, which returns

$$b(100,v) = \sum_{j=1}^{3}\alpha_{1,j}(1)v_j + \sum_{j=1}^{3}\alpha_{2,j}(0)v_j + \sum_{j=1}^{3}\alpha_{3,j}(0)v_j$$

$$= \sum_{j=1}^{3}\sum_{i=1}^{3}(\alpha_{i,j})v_j$$

$$= \alpha_{1,1}v_1 + \alpha_{1,2}v_2 + \alpha_{13}v_3$$

Now what we're going to need is to figure out what α_i has to equal in order for this equation to vanish for every point, and α certainly lives in $\mathbb{F}_2$. What we'll do is input the remaining points of $V_3(\mathbb{F}_2)$ into b, and solve from there!

$b(100, 010) = \alpha_2 v_2 = 0$

$b(100, 001) = \alpha_3 v_3 = 0$

$b(100, 110) = \alpha 1 v_1 + \alpha_2 v_2 = 0 \implies \alpha 1 v_1 = 0$ by equation 2

$b(100, 101) = \alpha_2 v_1 + \alpha_3 v_3 = 0$

$b(100, 011) = \alpha_2 v_2 + \alpha_3 v_3 = 0$

$b(100, 111) = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

At this point, by the third equation, we already have that $\alpha_1,\ \alpha_2,$ and α_3 are zero (known particularly because of that last equation!)

Of course, we'll also find that the other six coefficients are free to be whatever they please (certainly, we don't want them to all be zero because the radical will be the whole space.)

For our purposes, let's set all coefficients to be 1, giving us the bilinear form

$$b(u, v) = u_2 v_1 + u_2 v_2 + u_2 v_3 + u_3 v_1 + u_3 v_2 + u_3 v_3$$

Now we'll verify that this is, in fact, a bilinear form by testing the three bilinearity conditions:

Suppose $s, t, u, v, w \in V_3(\mathbb{F}_2)$

$$\begin{aligned}
b(s+t, v) &= (s_2+t_2)v_1 + (s_2+t_2)v_2 + (s_2+t_2)v_3 + (s_3+t_3)v_1 + (s_3+t_3)v_2 + (s_3+t_3)v_3 \\
&= s_2v_1 + t_2v_1 + s_2v_2 + t_2v_2 + s_2v_3 + t_2v_3 + s_3v_1 + t_3v_1 + s_3v_2 + t_3v_2 + s_3v_3 + t_3v_3 \\
&= (s_2v_1 + s_2v_2 + s_2v_3 + s_3v_1 + s_3v_2 + s_3v_3) + (t_2v_1 + t_2v_2 + t_2v_3 + t_3v_1 + t_3v_2 + t_3v_3) \\
&= b(s, v) + b(t, v)
\end{aligned}$$

$$\begin{aligned}
b(u, v+w) &= u_2(v_1+w_1) + u_2(v_2+w_2) + u_2(v_3+w_3) + u_3(v_1+w_3) + u_3(v_2+w_2) + u_3(v_3+w_3) \\
&= u_2v_1 + u_2w_1 + u_2v_2 + u_2w_2 + u_2v_3 + u_2w_3 + u_3v_1 + u_3w_3 + u_3v_2 + u_3w_2 + u_3v_3 + u_3w_3 \\
&= (u_2v_1 + u_2v_2 + u_2v_3 + u_3v_1 + u_3v_2 + u_3v_3) + (u_2w_1 + u_2w_2 + u_2w_3 + u_3w_3 + u_3w_2 + u_3w_3) \\
&= b(u, v) + b(u, w)
\end{aligned}$$

Now let's suppose $k \in \mathbb{F}_2$

$$\begin{aligned}
b(ku, v) &= (ku_2)v_1 + (ku_2)v_2 + (ku_2)v_3 + (ku_3)v_1 + (ku_3)v_2 + (ku_3)v_3 \\
&= k(u_2v_1 + u_2v_2 + u_2v_3 + u_3v_1 + u_3v_2 + u_3v_3) \\
&= kb(u, v) \\
&= ku_2v_1 + ku_2v_2 + ku_2v_3 + ku_3v_1 + ku_3v_2 + ku_3v_3 \\
&= u_2(kv_1) + u_2(kv_2) + u_2(kv_3) + u_3(kv_1) + u_3(kv_2) + u_3(kv_3) \\
&= b(u, kv)
\end{aligned}$$

Of course, there's no reason to limit ourselves to $\mathbb{F}_2$ — in fact, we'll do an example for any arbitrary

$\mathbb{F}_q$ over any finite dimensional vector space of dimension k! (It can also be generalized for any $\mathbb{F}$ for that matter.)

But before we do that, we're going to establish a tool that makes our job a *lot* easier.

Lemma 3. b is degenerate $\iff$ the matrix $A = [a_{i,j}]$ has determinant 0.

Proof.

In the forward direction, let's suppose b is degenerate in some $V_k(\mathbb{F})$, then there is some $u \in V_k(\mathbb{F})$ such that $b(u, v) = 0 \; \forall v \in V_k(\mathbb{F})$.

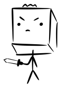
That is, we can evaluate b at u as $b(u, v) = u \cdot A \cdot v^T$.

Now what we know is that $u \cdot A \in V_k(\mathbb{F})$ satisfies $y \cdot A \cdot v^T = 0 \; \forall v \in V_k(\mathbb{F})$.

However, the only vector $w \in V_k(\mathbb{F})$ where $w \cdot v^T = 0 \; \forall v \in V$ is $w = \vec{0}$.

This of course shows that $u \cdot A = 0$, and since $u \neq \vec{0}$, we conclude that A has a non-trivial null space, and is thus, not invertible.

In the backwards direction, suppose A is not invertible. Well if that were true, then $Null(A) \neq \{0\}$, and thus, there exists some $u \in V_k(\mathbb{F})$ such that $u \cdot A = 0$. But if $u \cdot A = 0$, then $(u \cdot A) \cdot v = 0$ for every v, and thus, $b(u, v) = 0$ hence b is degenerate.



Example 3. Consider $b(u, v)$ in $V_k(\mathbb{F}_q)$ with unit vectors $e_1, e_2, ...e_k$. We're going to rig $rad_l(b)$ to be e_1. That is, $rad_l(b)$ will be the span of e_1 since the left radical is a subspace of $V_k(\mathbb{F}_q)$.

In the same way as before, we're going to evaluate $b(e_1, v)$ and set the image to be zero.

Let $A = [\alpha_{i,j}]$

$$0 = b(e_1, v)$$

$$= e_1 \cdot A \cdot v^T$$

$$= \sum_{j=1}^{k} \alpha_{1,j}(1)v_j + \sum_{j=1}^{k} \alpha_{2,j}(0)v_j ... \sum_{j=1}^{k} \alpha_{k,j}(0)v_j$$

$$= \alpha_{1,1}v_1 + \alpha_{1,2}v_2 ... + \alpha_{1,k}v_3$$

Lemma 4. If b is a bilinear form on $V_k(\mathbb{F})$, then $dim(rad_l(b)) = dim(rad_r(b))$

Proof.

Let b be a bilinear form on $V_k(\mathbb{F})$ and suppose $dim(rad_l(b)) = j$. Our goal is to show that $dim(rad_r(b)) = j$ as well. That is, $rad_r(b)$ has j basis elements.

First, let's note that $rad_l(b) := \{v \in V_k(\mathbb{F}) | \ \forall u \in V_k(\mathbb{F}), b(v, u) = 0\}$ *and* that it's a proper vector subspace of $V_k(\mathbb{F})$.

What *we're* going to do is we're going to take advantage of the rank-nullity theorem. That is:

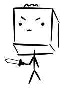
$$dim(ker_l(b)) = dim(V_k(\mathbb{F})) - dim(im(b))$$

$$= k - dim(im(b))$$

But in particular, right, $im(b) = M_b x$, or better known as $rank(M_b)$.

And to take the rank of M_b is to RREF M_b and count the rows, let's call that j.

In a similar way, to take the dimension of the right null space is to subtract the dimension of the right image from k. That is, we RREF M_b^T and count the rows for rank. But of course we know that the rank of the matrix and its transpose are the same, and thus, $rank(M_b^T) = j$ as well!



Definition 21 (Singular, Totally Singular, Maximum Totally Singular)**.** For some bilinear form b, v is said to be *isotropic* with respect to b if $b(v, v) = 0$. Isotropic is also known as singular, which is what we'll be using.

A subspace U is said to be *totally singular* if $\forall\ u, v \in U,\ b(u, v) = 0$

A subspace is said to be *maximally totally singular* if it isn't contained in a larger totally singular subspace.

The same naming convention is used for singular spaces relative to quadratic forms.

[Ball, pg 27]

Definition 22 (Non-Singular Subspaces)**.** A *nonsingular subspace* S with respect to a quadratic f is a subspace such that $\forall v \in S \setminus \{0\},\ f(v) \neq 0$. That is, $ker(f) = \{\vec{0}\}$ — or the kernel is trivial.

A (nonzero) vector v is singular if $f(v) = 0$.

[Ball, pg 41]

Definition 23 (Degenerate Quadratic Form)**.** A quadratic form f is degenerate if there is a singular vector $u \in V_k(\mathbb{F})$ such that $b(u, v) = 0 \forall v \in V_k(\mathbb{F})$. This bilinear form defined by a quadratic form is described above on page 18.

Remark. What's worth considering is that the bilinear form looks like $b(u, v) = f(u+v) - f(u) - f(v)$, and to cause degeneracy is to say this equation is equal to zero. But as we've established, u is singular, so $f(u) = 0$, so what we're really saying is that $f(u + v) = f(v)\ \forall\ v \in V_k(\mathbb{F})$.

Definition 24 (Hyperbolic Subspaces)**.** A *hyperbolic subspace* with respect to a quadratic form, f, is a two dimensional subspace $\langle u, v\rangle$ such that $f(u) = 0 = f(v)$ and $b(u, v) \neq 0$.

Or more particularly, it's a subspace spanned by two nonzero singular vectors but the space itself is not totally singular.

This notion also holds for hermitian, symmetric, and alternating forms.

[Ball, pg 41]

Corollary 1. Suppose $S \subseteq \mathbb{P}(V_k(\mathbb{F}_q))$. If S is a complete hypergraph that is also totally singular with respect to a quadratic form f, then S does not contain any hyperbolic subspaces.

Proof.

Suppose S a complete hypergraph that is also totally singular with respect to a quadratic form S.

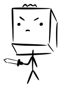
Then not only do we get that every pair of points determine a line, but every point on that line also vanishes on f and thus, the whole line does.

Or more particularly, because every projective point is associated with a vector, a pair of points are associated with a pair of vectors. So consider point u and v, associated with vectors $\vec{u}$ and $\vec{v}$.

Every point on the line $\overline{uv}$ is thus associated with a vector in $\langle u, v \rangle$.

But because S is totally singular, $\forall\ p \in \overline{uv},\ f(p) = 0$ and thus, $\forall \vec{p} \in \langle u, v \rangle,\ f(\vec{p}) = 0$.

But if *every* line in S behaves as such, we conclude that none of them are associated with hyperbolic subspaces because the whole associated plane vanishes on f.



Theorem 1. Let b be a non-degenerate σ-sesquilinear form defined on vector space $V_k(\mathbb{F})$, and M_b be the matrix associated with b.

$$dim(S) + dim(S^{\perp}) = rank(M_b) + null(M_b) = dim(im(b)) + dim(ker(b)) = k$$

Remark. Note that the first term of each equation are equivalent, and the second term of each equation are equivalent!

Theorem 2. A maximal, totally singular subspace with respect to a non-degenerate σ-sesquilinear form defined on $V_k(\mathbb{F})$ has dimension less than or equal to $\lfloor \frac{k}{2} \rfloor$. An alternative representation is if we let $k = 2r + c$ (where $c \in \{0, 1\}$), then the dimension is r.

[Ball, pg 27]

Proof.

We're going to prove this by binding the dimensions from above and below. That is to say, if S is the maximal, totally singular subspace of b, then $dim(S) \leq \lfloor \frac{k}{2} \rfloor$ and everything smaller *must* be a subspace of S.

Subproof.

Let's start with the (arguably) harder statement - that $dim(S)$ is bounded by $\lfloor \frac{k}{2} \rfloor$.

First, let's not forget that $dim(U) + dim(U^{\perp}) = k \implies dim(U^{\perp}) = k - dim(U) \leq k$

If S were totally isotropic, then $S \subseteq U^{\perp} \implies dim(S) \leq dim(U^{\perp})$. Thus we can do the following:

$$\begin{aligned} dim(S) + dim(U^{\perp} &\leq 2dim(U^{\perp}) \\ &\leq k \\ &\implies dim(U^{\perp}) \leq \frac{k}{2} \\ &\implies dim(S) \leq dim(U^{\perp} \leq k \end{aligned}$$

Subproof.

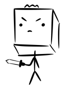
Now we're going to show that any smaller totally singular subspace has to belong to S itself.

We'll exploit the fact that b has an associated matrix, and the associated matrix has an orthogonal and non-orthogonal space, and the dimension sum of the two is k.

That is, $dim(U) + dim(U^{\perp}) = k = dim(im(b)) + dim(ker(b))$ where the orthogonal dimension is the same as the null space dimension which is thus the same as the kernel dimension (and thus, the rank is the same as the not orthogonal dimension which is the same as the image dimension).

With this logic, $U^{\perp} = S$ as $S = ker(b)$.

But here's the thing, right? $\forall T$ such that $v \in T \implies b(v, u) = 0,\ v \in ker(b) \implies v \in U^{\perp} \implies v \in S \implies T \subseteq S$.



Classification of Reflexive Forms

In particular, we'll prove here and now why there are *only* three reflexive σ-sesquilinear forms, namely, the alternating, symmetric, and Hermitian forms.

Theorem 3 (Three Sesquilinear Forms)**.** A non-degenerate reflexive $\sigma-$sesquilinear form on $V_k(\mathbb{F})$ is one of the following types:

- b is alternating — $b(u, u) = 0\ \forall u\ \in V_k(\mathbb{F})$
- b is symmetric — $b(u, v) = b(v, u)\ \forall u$ and $v\ \in V_k(\mathbb{F})$
- b is hermitian — $b(u, v) = b(v, u)^{\sigma}\ \forall u$ and $v\ \in V_k(\mathbb{F})$

We'll go into detail how each of these forms behave later!

Alternating Form

Let's recall what the definition of an alternating form is, first.

Definition 25 (Alternating Forms)**.** An alternating form is a non-degenerate reflexive form, b, on $V_k(\mathbb{F})$, up to scalar, such that $\forall u \in V_k(\mathbb{F}),\ b(u,u) = 0$

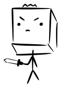
[Ball, pg 28]

Lemma 5. If b is an alternating form, we also get the property that $b(u,v) = -b(v,u)\ \forall u, v \in V_k(\mathbb{F})$

Proof.

Consider some $u, v \in V_k(\mathbb{F})$, and recall that $\forall w \in V_k(\mathbb{F}), b(w,w) = 0$.

$$
\begin{aligned}
0 &= b(u+v, u+v) \\
&= b(u+v, u) + b(u+v, v) \\
&= b(u,u) + b(v,u) + b(u+v, v) \\
&= b(u,u) + b(v,u) + b(u,v) + b(v,v) \\
&= 0 + b(v,u) + b(u,v) + 0 \\
&= b(v,u) + b(u,v)
\end{aligned}
$$



Theorem 4. A maximum totally singular subspace with respect to a non-degenerate alternating form on $V_k(\mathbb{F})$ has dimension $\frac{1}{2}k$.

In particular, if $V_k(\mathbb{F})$ has an alternating bilinear form, then $dim(V_k(\mathbb{F}))$ is even.

Proof.

Let's consider a maximum, totally singular subspace S with respect to a non-degenerate alternating form on $V_k(\mathbb{F})$.

We'll decompose down the properties of S. First, a totally singular space means that for all $u \in S \subseteq V_k(\mathbb{F})$, $b(u,u) = 0$. Second, there is no bigger $T \subset V_k(\mathbb{F})$ such that $S \subset T \subset V_k(\mathbb{F})$ and $S \neq T$ where T is also totally singular.

Now by the theorem described in the Multi-linear Algebra section, we know that the upper bound for $dim(S)$ is $\lfloor \frac{k}{2} \rfloor$, so all we need to do is show that first, k is even, and second, we actually *attain* that max.

As for the second part, certainly if there is some bigger $U \subset V_k(\mathbb{F})$ such that $S \subset V_k(\mathbb{F})$, then the dimension of U would still be bounded by $\frac{k}{2}$ and U would be the actual maximally singular subspace.

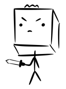
What we really want to show is that all totally singular subspaces must live inside a bigger totally singular subspace of dimension $\frac{k}{2}$ or itself be dimension $\frac{k}{2}$.

What we'll exploit is the fact that b has an associated matrix, and the associated matrix has an orthogonal and non-orthogonal space, and the dimension sum of the two is k.

That is, $dim(U) + dim(U^{\perp}) = k = dim(im(b)) + dim(ker(b))$ where the orthogonal dimension is the same as the null space dimension which is thus the same as the kernel dimension (and thus, the not null space dimension is the same as the not orthogonal dimension which is the same as the image dimension).

With this logic, $U^{\perp} = S$ as $S = ker(b)$.

But here's the thing, right? $\forall T$ such that $v \in T \implies b(v,u) = 0,\ v \in ker(b) \implies v \in U^{\perp} \implies v \in S \implies T \subseteq S$.



Corollary 2. A non-degenerate alternating form b on $V_k(\mathbb{F})$ is, with respect to a basis B, $b(u,v) = \sum_{i=1}^{r}(u_{2i-1}v_{2i} - u_{2i}v_{2i-1})$ where $k = 2r$

Now an interesting example of an alternating form is $W_3\mathbb{F}_2$, which is an alternating form in four variables over $\mathbb{F}_2$, given by $b(u,v) = (u_1v_2 - u_2 - v_1) + (u_3v_4 - u_4v_3)$. This also happens to be a

(15_3) configuration hiding some $(9_2, 6_3)$ and $(12_2, 8_3)$ configurations inside of it! (Remember that (15_3) means fifteen points, fifteen lines, three points per line, three lines per point, and $(9_2, 6_3)$ is nine points, each on two lines, and six lines, each with two points!)

Figure 4: An example of $W_3\mathbb{F}_2$

In fact, $W_3\mathbb{F}_2$ has a particularly fun property. If we choose three points that satisfy the form, not all collinear, and we place them down on the "unfilled" $W_3\mathbb{F}_2$ above, what we'll find is that there's exactly two ways to "fill in" the rest of the points on the form. Of course, this is given that we have to insist that this "filling in" — or "completion" — respects certain properties (you can't claim that points $a, b,$ and c live on the same line when they most definitely don't, for example).

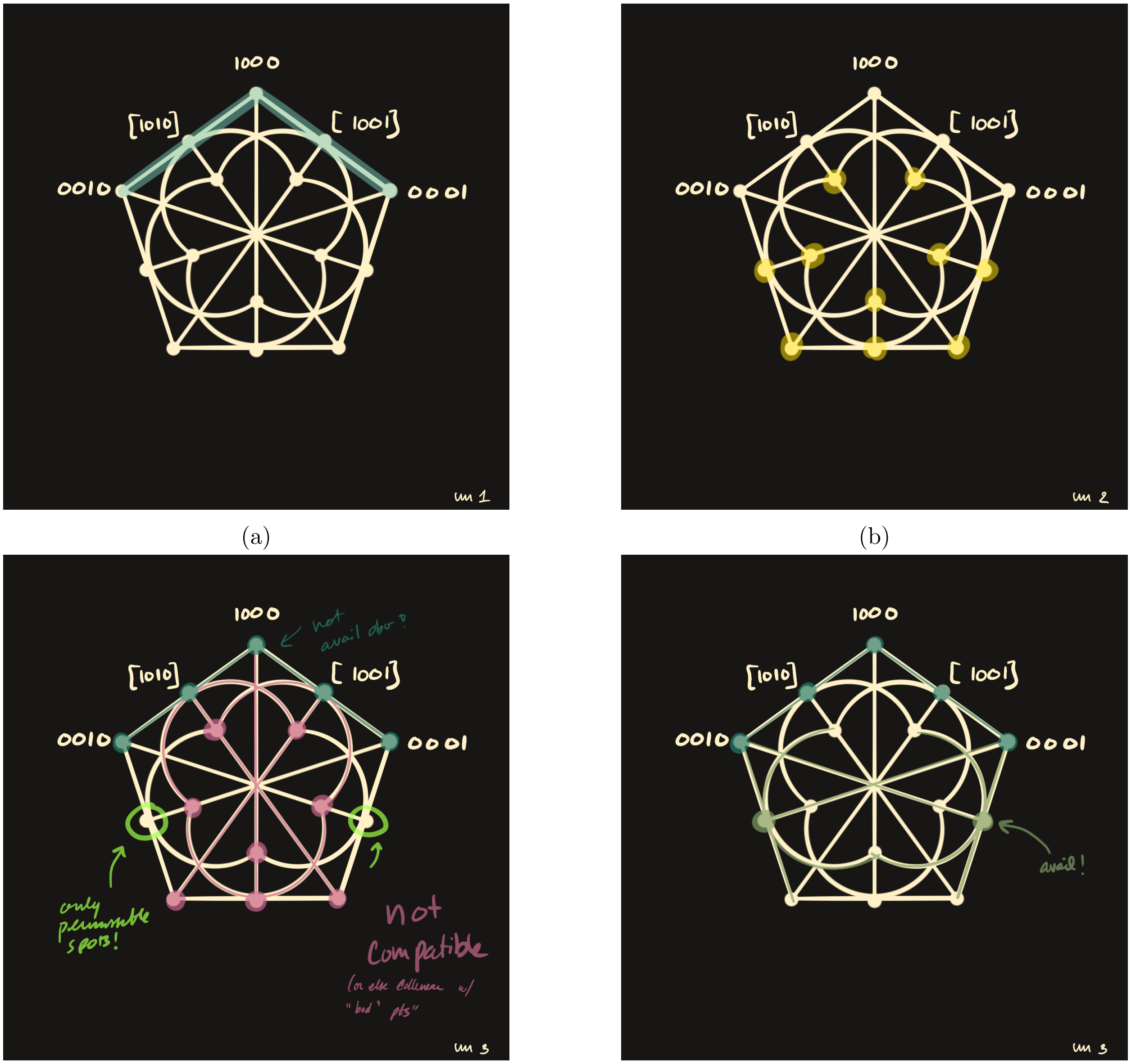
Before we do all this, however, we're going to prove that the fixing four (or more) points, no three collinear, fixes the entire completion of $W_3\mathbb{F}_2$.

Lemma 6. The fixing of four or more points, no three collinear, in $W_3\mathbb{F}_2$ fixes its completion.

Proof.

Consider the symmetric form $x_1y_2 + y_1x_2 + z_1w_2 + w_1z_2 = 0$, where for any p_1 and $p_2 \in L$ such that $L \in W_3\mathbb{F}_2$, $p_1 = [x_1, y_1, z_1, w_1]$, and p_2 is of a similar form.

Without loss of generality, we can fix W_3F_2 in the following way, *including* 0100 and get the same results up to a change of basis. Thus, any points that live on these lines must satisfy the following properties - strictly of course looking at our first four points.



(a) (b) (c) (d)

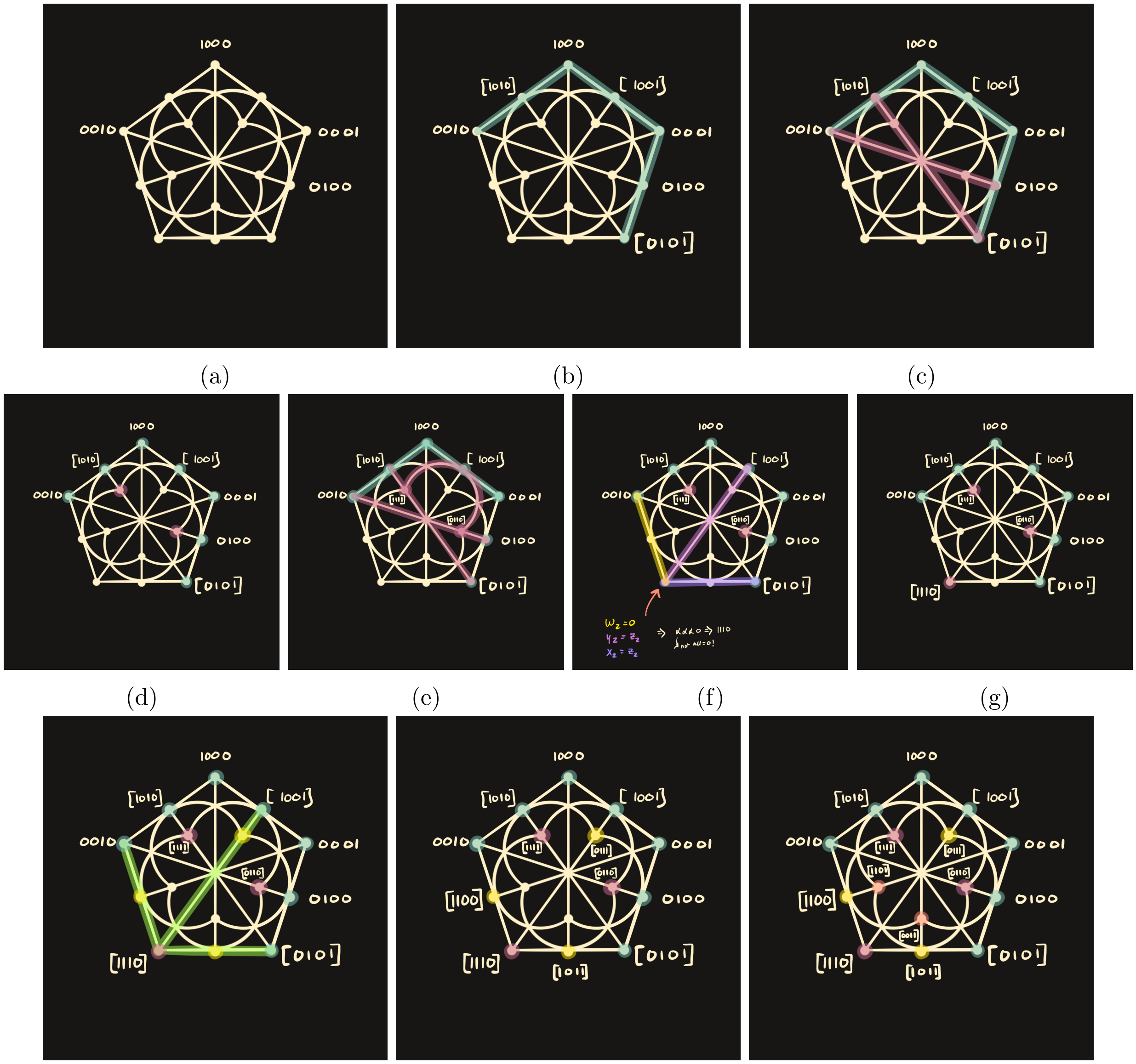
Figure 5: A filling of $W_3\mathbb{F}_2$ given three points, not all collinear

$$0 \cdot y_2 + 0 \cdot x_2 + 1 \cdot w_2 + 0 \cdot z_2 = 0$$

$$1 \cdot y_2 + 0 \cdot x_2 + 0 \cdot w_2 + 0 \cdot z_2 = 0$$

$$0 \cdot y_2 + 0 \cdot x_2 + 0 \cdot w_2 + 1 \cdot z_2 = 0$$

$$0 \cdot y_2 + 1 \cdot x_2 + 0 \cdot w_2 + 0 \cdot z_2 = 0$$



(a) (b) (c)

(d) (e) (f) (g)

(h) (i) (j)

Figure 6: A filling of $W_3\mathbb{F}_2$ given four points, no three collinear

Of course, this fixes the point between 0010 and 1000 as well as 1000 and 0001, and 0001 and 0100 to be 1010, 1001, and 0101 respectively.

Now up until this point, we haven't actually made use of our notation, nor the alternating form. ✨However, we can fix that ✨.

The verification that these previous five points satisfy the given symmetric form is straightforward and won't be addressed here.

To complete the remaining points, we know that any points on the same line as the first five must satisfy the following properties:

$$0 \cdot y_2 + 0 \cdot x_2 + 1 \cdot w_2 + 0 \cdot z_2 = 0$$

$$1 \cdot y_2 + 0 \cdot x_2 + 0 \cdot w_2 + 0 \cdot z_2 = 0$$

$$0 \cdot y_2 + 0 \cdot x_2 + 0 \cdot w_2 + 1 \cdot z_2 = 0$$

$$0 \cdot y_2 + 1 \cdot x_2 + 0 \cdot w_2 + 0 \cdot z_2 = 0$$

$$1 \cdot y_2 + 0 \cdot x_2 + 1 \cdot w_2 + 0 \cdot z_2 = 0$$

$$1 \cdot y_2 + 0 \cdot x_2 + 0 \cdot w_2 + 1 \cdot z_2 = 0$$

$$0 \cdot y_2 + 1 \cdot x_2 + 0 \cdot w_2 + 1 \cdot z_2 = 0$$

Remark. Observe that the first four equations are just the given points, and the remaining three correspond to points between/on the previous lines.

These equations are obtained by plugging in the values of each position of each point, going from left to right. The placement of each additional point is contingent on whether the new point can live on the intersection of a pair of lines, and thus, a pair of equations.

Let's address the lower left vertex now, which must satisfy the following three equations:

$$w_2 = 0$$

$$y_2 = z_2$$

$$x_2 = z_2$$

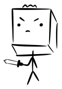
Remark. The equations are simplified down to make the combinatorics a bit more intuitive.

Which gives $x_2 = y_z = z_2$ and thus, our mystery point is of the form $jjj0$ ($j \in \mathbb{Z}_2$), and combinatorially, we have two options. But it's okay, because not all coordinates can be zero, and thus, returns to us 1110.

The addition of this point then determines three more points, noted in yellow, as well as the green lines being used to determine their identity. With that, we end up finding that the remaining two points are properly sandwiched between determined points, and thus completes our structure!

Remark. We can implicitly verify that these points satisfy our alternating form on the grounds that the two determining points satisfy a pair of equations derived from the alternating form, and the direct sum of the two points must return the third point. So all we need to do is add the two points together to get the missing point, then check to make sure it satisfies the sum of the equation pair. (It does)

What would be astronomically problematic would be if the missing point (which is fixed by the first two) does NOT satisfy the equation pair.



Now let us proceed to prove the fixture of three points, not all collinear, will give us a pair of completions of our alternating form.

Theorem 5. The fixing of three points, not all collinear, fixes a pair of completions for $W_3\mathbb{F}_2$

Proof.

Consider the symmetric form $x_1y_2 + y_1x_2 + z_1w_2 + w_1z_2 = 0$, where for any p_1 and $p_2 \in L \in W_3\mathbb{F}_2$, $p_1 = [x_1, y_1, z_1, w_1]$, and p_2 is of a similar form.

Without loss of generality, we can fix $W_3\mathbb{F}_2$ as described in 5a and get the same results up to a change of basis.

Of course, this fixes the point between 0010 and 1000 as well as 1000 and 0001 to be 1010 and 1001 respectively as highlighted in 5a.

The verification that these previous five points satisfy the given symmetric form is straightforward and won't be addressed here.

To complete the remaining points, we know that any points on the same line as the first five must satisfy the following properties:

$$0 \cdot y_2 + 0 \cdot x_2 + 1 \cdot w_2 + 0 \cdot z_2 = 0$$

$$1 \cdot y_2 + 0 \cdot x_2 + 1 \cdot w_2 + 0 \cdot z_2 = 0$$

$$1 \cdot y_2 + 0 \cdot x_2 + 0 \cdot w_2 + 0 \cdot z_2 = 0$$

$$1 \cdot y_2 + 0 \cdot x_2 + 0 \cdot w_2 + 1 \cdot z_2 = 0$$

$$0 \cdot y_2 + 0 \cdot x_2 + 0 \cdot w_2 + 1 \cdot z_2 = 0$$

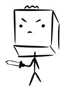
These equations are obtained by plugging in the values of each position of each point, going from left to right. The placement of each additional point is contingent off of whether the new point can live on the intersection of a pair of lines, and thus, a pair of equations. This is written, as of now, entirely clockwise from the leftmost vertex to the rightmost vertex.

Observe that this system of equations is the same as one from earlier with the exception of the last handful of lines because we have yet to fix the point 0100.

But where could this point live, anyway? Based off of our equations, it can either live on equation 1 or 5, that is, lines that 0010 or 0001 live on — in this case, it happens to be an *and*.

It's also worth noting that 0100 *cannot* be collinear with *any* of the previous points (other than the ones on lines 1 and 5), and if we were to block out the incompatible locations, we confirm that not only are there two lines that 0100 can live on, there are *exactly two* spots that it can occupy!

So now we have two choices - but of course, once you fix four points, no three collinear, you fix all of $W_3\mathbb{F}_2$ as established above, so we're done! There are only two ways to complete this space whilst still satisfying our alternating form!



Quadratic Forms

Let's recall the definition of the quadratic form for this next section!

Definition 26 (Quadratic Form)**.** A *quadratic form,* f, on $V_k(\mathbb{F})$ is a map from $V_k(\mathbb{F}) \to \mathbb{F}$ such that $f(\lambda u) = \lambda^2 f(u)\ \forall \lambda \in \mathbb{F}$ and $u \in V_k(\mathbb{F})$ and $b(u,v) = f(u+v) - f(u) - f(v)$ is a bilinear symmetric form on $V_k(\mathbb{F})$.

[Ball, pg 40]

Now the reason we're taking a quick detour from our previously listed forms is because quadratic forms are actually foundational to symmetric forms.

Theorem 6. A maximum totally singular subspace U has dimension $\frac{k-dim(x)}{2}$, where X is a non-singular subspace of maximum dimension.

[Ball, pg 43]

Lemma 7. If $char(\mathbb{F}) \neq 2$, and b is a symmetric bilinear form on $V_k(\mathbb{F})$, then $b(u,u)$ is a quadratic form on $V_k(\mathbb{F})$.

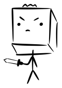
[Ball, pg 40]

Proof.

To prove these two are the same is to say that $b(\lambda u, \lambda u) = \lambda^2 f(u)$. That is, the factoring of any scalar on both arguments is squared on the outside of the function (which is similar to linear factors being pulled out having exponent 1).

Thus, suppose $f(v) = b(v, v)$

$$\begin{aligned}
\implies f(\lambda u) &= b(\lambda u, \lambda u) \\
&= \lambda b(u, \lambda u) \text{ By first property of bilinear forms on first argument} \\
&= \lambda(\lambda b(u, u)) \text{ By first property of bilinear forms on second argument} \\
&= \lambda^2 b(u, u) \\
&= \lambda^2 f(u).
\end{aligned}$$



Theorem 7. If S is a subspace of $V_k(\mathbb{F}_q)$ such that $k \geq 3$, then S contains a non-zero singular vector with respect to a quadratic form f.

[Ball, pg 45]

Proof.

Let $S \subseteq V_k(\mathbb{F}_q)$ such that $dim(S) \geq 3$ and consider some quadratic form f.

If we were to consider an arbitrary quadratic form f, we can write it to be of the following form:

$$f(v) = \sum_{i=1}^{k}\sum_{j=1}^{k}\alpha_{i,j}v_iv_j$$

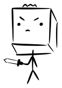
$$= v_1(\sum_{j=1}^{k}\alpha_{i,j}v_j) + v_2(\sum_{j=1}^{k}\alpha_{i,j}v_j) + v_3(\sum_{j=1}^{k}\alpha_{i,j}v_j) + ...v_k(\sum_{j=1}^{k}\alpha_{i,j}v_j)$$

Now let's choose the unit vector e_1. Evaluating f at e_1 we get

$$f(e_1) = 1(\sum_{j=1}^{k}\alpha_{i,j}0) + 0(\sum_{j=1}^{k}\alpha_{i,j}v_j) + 0(\sum_{j=1}^{k}\alpha_{i,j}v_j) + ...0(\sum_{j=1}^{k}\alpha_{i,j}v_j)$$

$$= 0$$

And if I'm not mistaken, all unit vectors will behave in such a way.



Corollary 3. If X is a non-singular subspace, then $dim(x) \leq 2$

[Ball, pg 45]

Theorem 8. Let f be a non-singular quadratic form on $V_k(\mathbb{F}_q)$. Then $k = 2r,\ 2r+1,$ or $2r+2$, and there is a basis B with respect to which, respective to k,

$$f(u) = u_1u_2 + ... + u_{2r-1}u_{2r} = \sum_{i=1}^{r} u_{2i-1}u_{2i}$$

$$f(u) = u_1u_2 + ... + u_{2r-1}u_{2r} + au_{2r+1}^2 = (\sum_{i=1}^{r} u_{2i-1}u_{2i}) + au_{2r+1}^2$$

where $a = 1$ if q is even, and $a = 1$ or some other non-square if q is odd

$$f(u) = u_1u_2 + ... + u_{2r-1}u_{2r} + u_{2r+1}^2 + au_{2r+1}u_{2r+2} + bu_{2r+2}^2 = (\sum_{i=1}^{r} u_{2i-1}u_{2i}) + au_{2r+1}u_{2r+2} + bu_{2r+2}^2$$

$b = 1$ and the trace of a^{-1} from $\mathbb{F}_q$ to $\mathbb{F}_2$ is 1 if q is even,

and $a = 0$ and $-b$ is a chosen non-square if q is odd

That, and r is the dimension of a maximum totally singular subspace.

We also have the property that if $char(\mathbb{F}) \neq 2$, classification of non-degenerate quadratic forms is the same as classifying non-degenerate symmetric forms!

[Ball, pg 45]

Definition 27 (Hyperbolic, Parabolic, and Elliptic Quadratic Forms)**.** If $k = 2r$ then f is *hyperbolic*, if $k = 2r + 1$ then f is *parabolic*, and if $k = 2r + 2$ then f is *elliptic*.

That is:

Hyperbolic: $f(u) = \sum_{i=1}^{r} u_{2i-1}u_{2i}$

Parabolic: $f(u) = (\sum_{i=1}^{r} u_{2i-1}u_{2i}) + au_{2r+1}^2$

Elliptic: $f(u) = (\sum_{i=1}^{r} u_{2i-1}u_{2i}) + au_{2r+1}u_{2r+2} + bu_{2r+2}^2$

[Ball, pg 45]

Remark. An important note is that an even dimensional space is both $2r$ and $2r+2$ - the reason why we notate it in two different ways is because when we generate our quadratic forms, there are two distinct, non-isometric quadratics (hyperbolic and elliptic) that show up. We'll come back to this when we address polar spaces!

Example 4 (Quadratic Examples)**. Parabolic** For the parabolic example, we'll be looking at $Q_4\mathbb{F}_2$ — although parabolic subspaces exist outside of $\mathbb{F}_2$, this example will be useful again later! Particularly, this gives us $r = 2$.

$Q_4\mathbb{F}_2 := u_1u_2 + u_3u_4 + u_5^2 = 0.$

What we'll do is give each term a name, that is, the first term will be A, the second will be B, and the third will be C.

This gives us a *very* nice and *intuitive* set builder notation setup.

$$A_0 := \{u_1,\ u_2 | u_1u_2 = 0\} \text{ and } A_1 := \{[u_1,\ u_2] | u_1u_2 = 1\}$$

$$B_0 := \{u_3,\ u_4 | u_3u_4 = 0\} \text{ and } B_1 := \{u_3,\ u_4 | u_3u_4 = 1\}$$

$$C_0 := \{u_5 | u_5^2 = 0\} \text{ and } C_1 := \{u_5 | u_5^2 = 1\}$$

Or, more specifically,

$$A_0 := \{u_1,\ u_2 | u_1 \text{ or } u_2 = 0\} \text{ and } A_1 := \{u_1,\ u_2 | u_1 \text{ and } u_2 = 1\}$$

$$B_0 := \{u_3,\ u_4 | u_3 \text{ or } u_4 = 0\} \text{ and } B_1 := \{u_3,\ u_4 | u_3 \text{ and } u_4 = 1\}$$

$$C_0 := \{u_5 = 0\} \text{ and } C_1 := \{u_5 = 1\}$$

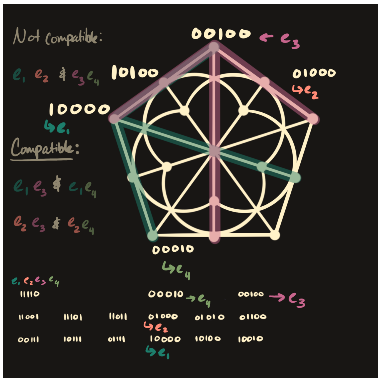
This all of a sudden becomes *really nice* because we can use some combinatorics to work it out.

First is when all terms are equal to zero with the exception of $\vec{0}$, which is just $A_0 \times B_0 \times C_0 \setminus \{\vec{0}\}$ with size $|A_0| \cdot |B_0| \cdot |C_0| - 1 = 8$. We use cross product notation to keep it consistent with the set builder notation, though if we were so inclined, we could create each set to be a set of points that satisfy some equation, then intersect them. Fundamentally, we get to the same place.

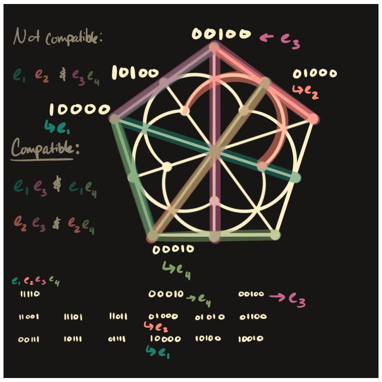
Similarly, we want when two out of three terms equal one with the insistence that the third one equals zero, giving us $(A_1 \times B_1 \times C_0) \cup (A_1 \times B_0 \times C_1) \cup (A_0 \times B_1 \times C_1)$, with a total cardinality of eight.

For the sake of brevity, we won't list all of fifteen of these points out here. An important note is that this is *not* the whole space, because the whole space (with the exception of the zero vector) has a total of $2^5 - 1 = 31$ points (this will be useful for later!)

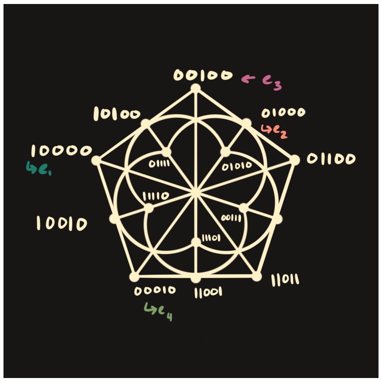
For some intuition, the following construction is working under the premise that $\langle e_1,\ e_2\rangle$ and $\langle e_3,\ e_4\rangle$ are not on our list of points and is thus incompatible with our quadric. This gives us a nice syllogism puzzle to work with and, with some elbow grease, returns to us the following:



(a)



(b)



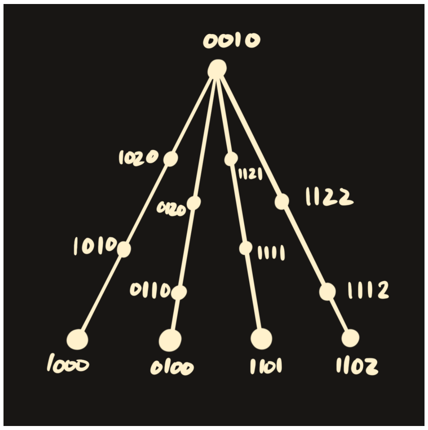
(c)

Parabolic and Hyperbolic

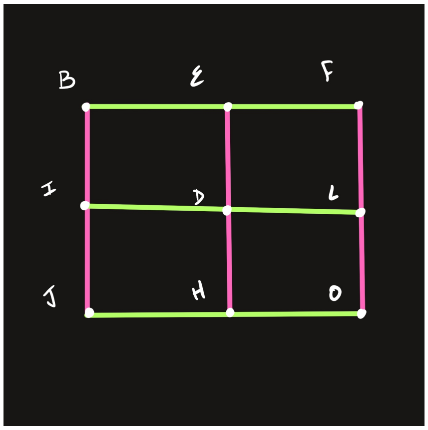
Symmetric Form

Definition 28 (Symmetric Form)**.** A symmetric form is a non-degenerate symmetric form, b, on $V_k(\mathbb{F})$, up to scalar, such that $\forall u, v \in V_k(\mathbb{F}),\ b(u, v) = b(v, u)$.

If $char(\mathbb{F}) \neq 2$, then $\frac{1}{2}b(u, u)$ is a quadratic form.



(a) An ellipse in $\mathbb{F}_3$ where $r = 1$



(b) An ellipse in $\mathbb{F}_2$ where $r = 1$

[Ball, pg 28]

Theorem 9. If b is a symmetric bilinear form on $V_k(\mathbb{F})$ and the characteristic of $\mathbb{F}$ is *is* two, then $V_k(\mathbb{F}) = E \oplus F$ (where E of b is an alternating form and F is either a non-singular one dimensional subspace or $\{0\}$).

[Ball, pg 38]

Corollary 4. If $char(\mathbb{F}) = 2$, then there's a basis such that a non-degenerate symmetric form b on $V_k(\mathbb{F})$ is

$$b(u, v) = (\sum_{i=1}^{r} u_{2i-1}v_{2i} + u_{2i}v_{2i-1}) + u_{2r+1} + v_{2r+1} \text{ for } k = 2r + 1$$

or

$$b(u, v) = (\sum_{i=1}^{r} u_{2i-1}v_{2i} + u_{2i}v_{2i-1}) \text{ for } k = 2r$$

[Ball, pg 39]

Hermitian Form

Definition 29 (Hermitian Forms)**.** A Hermitian form is a non-degenerate reflexive form, b, on $V_k(\mathbb{F})$, up to scalar, such that $\forall u \in V_k(\mathbb{F})$, $b(u, v) = b(u, v)^{\sigma}$. Particularly, σ is a non-identity automorphism, and $\sigma^2 = id$.

[Ball, pg 28]

Theorem 10. A maximum, totally singular subspace with respect to a non-degenerate Hermitian form defined on $V_k(\mathbb{F}_q)$ has dimension $\lfloor \frac{k}{2} \rfloor$. An alternative representation is if we let $k = 2r + c$ (where $c \in \{0, 1\}$), then the dimension is r. :)

[Ball, pg 35]

Now the question we have to ask is: How do we go about procuring a Hermitian form?

And the answer is graciously given to us by Simeon Ball in the following corollary.

Corollary 5. A non-degenerate Hermitian form, b, on $V_k(\mathbb{F}_q)$ is, with respect to a basis B, defined as follows:

$$b(u, v) = \sum_{i=1}^{r} (u_{2i-1} v_{2i}^{\sigma} + u_{2i} v_{2i-1}^{\sigma})$$

$$= u_1 v_2^{\sigma} + u_2 v_1^{\sigma} + ... + u_{2r-1} v_{2r}^{\sigma} + u_{2r} v_{2r-1}^{\sigma} \text{ if } k = 2r$$

$$b(u, v) = (\sum_{i=1}^{r} u_{2i} v_{2i-1}^{\sigma} + u_{2i-1} v_{2i}^{\sigma}) + u_{2r+1} v_{2r+1}^{\sigma}$$

$$= u_1 v_2^{\sigma} + u_2 v_1^{\sigma} + ... + u_{2r-1} v_{2r}^{\sigma} + u_{2r} v_{2r-1}^{\sigma} + u_{2r+1} v_{2r+1}^{\sigma} \text{ if } k = 2r + 1$$

[Ball, pg 36]

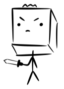
Now before we proceed, we're going to show that if $b(u, u) = 0$ (that is: u is singular), we also get that αu is singular! This is for the strict purpose of making our calculations more streamlined.

Lemma 8. For $u \in V_k(\mathbb{F}_q)$, if $b(u, u) = 0 \implies b(\alpha u, \alpha u) = 0$. That is, if u is singular, then all of its scalar multiples are, as well.

Proof.

$$b(\alpha u, \alpha u) = \alpha b(u, \alpha u)$$

$$= \alpha \alpha^{\sigma} b(u, u) = 0$$



Our instructive example would be the (more on "the" later) Hermitian form of size $\frac{4}{2}$ for $\mathbb{P}_3(\mathbb{F}_2)$.

Example 5. Now because we're insisting everything is in $\mathbb{P}_3(\mathbb{F}_2)$, we get that $k = 4$ and thus, $r = 2$, so we'll use the first equation.

$$b(u, v) = u_1 v_2^\sigma + u_2 v_1^\sigma + ... + u_{2r-1} v_{2r}^\sigma + u_{2r} v_{2r-1}^\sigma$$

$$\implies b(u, v) = u_1 v_2^\sigma + u_2 v_1^\sigma + u_3 v_4^\sigma + u_4 v_3^\sigma$$

Which isn't actually all that exciting to be honest!

Example 6. Another example to consider is $V^3(\mathbb{F}_3)$.

Combinatorially, we have $\frac{9^3-1}{8} = 81$ unique points (up to scalar). And it's worth noting, as well, that $\mathbb{F}_9$ is really just $\mathbb{F}_3[\sqrt{2}]$ (because in $\mathbb{F}_3$, 2 is not a square number!).

Now for $\sigma^2 = id$ purposes, we'll observe that $\sigma : a + b\sqrt{2} \mapsto a - b\sqrt{2}$.

So to put a Hermitian form on this is to say

$b(u, v) = x_1 y_2^\sigma + y_1 x_2^\sigma + z_1 z_2^\sigma$

where σ takes each coordinate of the second vector to its conjugate.

Based off of the dimension counting corollary above, we get that $\forall u, v \in V^3(\mathbb{F}_3),\ dim(b(u, v)) = r$.

However, $k = 3$ which means $r = 1$, and that means things that satisfy the Hermitian form will just be points - no lines here! We'll show what these points look like nonetheless.

First we'll look at parts where $z = 0$ for both vectors, that is, the form is $xy0$ Particularly, this gives us a form that only takes *one* argument since it isn't bilinear!

Thus, the form we need to satisfy is

$$b(u,u) = xy^\sigma + yx^\sigma \text{ and if } x \text{ and } y \in \mathbb{F}_3, \text{ then we get that } u = u^\sigma \forall u$$

$$= xy + xy = 2xy$$

Now because this should equal zero, and we're working in a field so we have no zero divisors, either x or y (but not both!) are equal to zero - that is, $k00$ or $0k0$, where $k \in \mathbb{F}_3^\times$. Now of course, since all coordinates except one are zero, and we're looking at homogeneous points, this actually gives us exactly two solutions, 100 and 010!

For the remaining points, we have a very particular property we want to use:

$$\begin{aligned}
0 &= xy^\sigma + yx^\sigma \\
&= (a+b\sqrt{2})(a'+b'\sqrt{2})^\sigma + (a'+b'\sqrt{2})(a+b\sqrt{2})^\sigma \\
&= (a+b\sqrt{2})(a'-b'\sqrt{2}) + (a'+b'\sqrt{2})(a-b\sqrt{2}) \\
&= (aa' - ab'\sqrt{2} + a'b\sqrt{2} - 2bb') + (aa' + ab'\sqrt{2} - a'b\sqrt{2} - 2bb') \\
&= 2(aa' - 2bb') \\
\implies 0 &= 2(aa' - 2bb') \\
&= aa' - 2bb' \\
\implies aa' &= 2bb'
\end{aligned}$$

From here, we'll take a combinatorial approach, where we insist that not both b and b' are zero (or else we capture the points we've already counted above). This then gives us the following options:

$$a = 0 = a' \implies 2bb' = 0 \implies b = 0 \text{ (exclusive) or } b' = 0 \implies 0 + 0\sqrt{2} \text{ and } 0 + 0\sqrt{2} \text{ and } 0 + b'$$

$$b = 0 = b' \implies aa' = 0 \implies a = 0 \text{ (exclusive) or } a' = 0 \implies 0 + 0\sqrt{2} \text{ and } a' + 0\sqrt{2}$$

$$a = 0 = b' \implies a' \text{ and } b \text{ are free (a symmetric argument holds for the other (redundant) pairing}$$

$$\implies 0 + b\sqrt{2} = b(\sqrt{2}) \text{ and } a' + 0\sqrt{2} = a'(1)$$

$$aa' = 1 \text{ and } bb' = 2 \implies 1 + 1\sqrt{2}, \text{ and } 1 + 2\sqrt{2}$$

$$aa' = 2 \text{ and } bb' = 1 \implies 2 + 1\sqrt{2} = 2(1 + 2\sqrt{2}), \text{ and } 2 + 2\sqrt{2} = 2(1 + 1\sqrt{2})$$

Of course, the first two solutions aren't satisfying, because this gives one of our inputs as the zero vector which isn't in use, and thus, only the remaining three are viable options.

This gives us two solution pairs up to scalar: $0 + \sqrt{2}$ and $1 + 0\sqrt{2}, 1 + 1\sqrt{2}$ and $1 + 2\sqrt{2}$

A Brief Detour Into Finite Geometry and (Hyper)Graph Theory

To address the next section, polar spaces, we need to address some finite geometry and hypergraph theory notions.

Definition 30 (Near Linear Space)**.** A space, S is near linear if:

1. $\forall l \in S$, l contains at least two points
2. All pairs of points are on at most one line. That is, two points determine at most one line.

[Batten, pg 4]

Definition 31 (Linear Space)**.** A space, S is linear it satisfies all conditions of near-linearity, and in particular, *every* pair of points determine exactly one line.

[Batten]

Definition 32 (Complete Graph)**.** A *complete graph* is a graph, G, such that for all v_1 and $v_2 \in G$, $\exists e_{1,2}$ such that v_1 and $v_2 \in e_{1,2}]$. That is, every pair of nodes have an edge connecting them.

Definition 33 (Path On A Graph)**.** A path on a graph G is a sequence of nodes such that each subsequent node is connected to the previous one by an edge.

Definition 34 (Graph Distance)**.** The distance between a pair of nodes, denoted $d(v_1, v_2)$, is the minimal number of nodes needed to form a path between v_1 and v_2.

Definition 35 (Hypergraph)**.** A *hypergraph* is a generalized notion of a graph - that is, edges may connect *at least* two vertices, but may contain more. These edges are also called *hyperedges*.

Definition 36 (Simple Hypergraph)**.** A *simple* hypergraph is a hypergraph where *no* edge has a sub-edge.

For our purposes, we'll always assume our graphs are simple!

Definition 37 (r-uniform Hypergraph)**.** A *r-uniform hypergraph* is a hypergraph in which every node is degree r.

Example 7. A stellar example of this is $xy = zw$ in $\mathbb{P}_3(\mathbb{F}_2)$! This is a 2-uniform hypergraph!

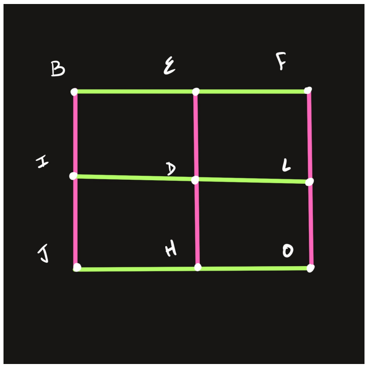
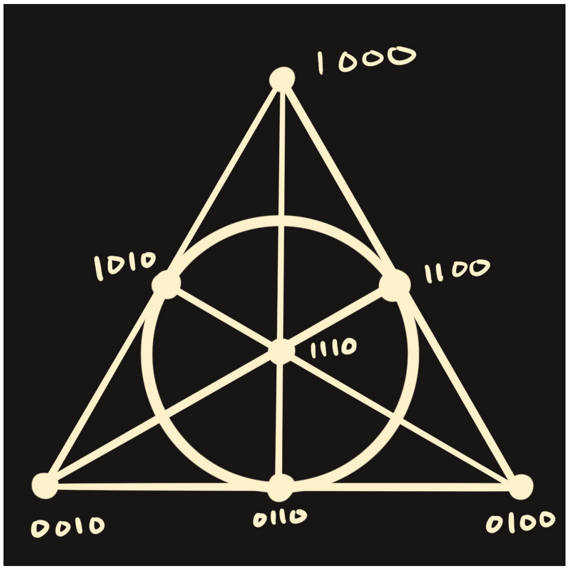


Figure 9: The quadratic $xy = zw$ in $\mathbb{P}_3(\mathbb{F}_2)$

Definition 38 (Complete k-uniform Hypergraph)**.** A *complete k-uniform hypergraph* is an r-uniform hypergraph where for any set of k nodes, there is a hyper-edge connecting them.

Example 8. A fantastic example of a complete 3-uniform hypergraph, which also happens to be a (7_3) configuration, is the Fano plane!

Another example is $W_3\mathbb{F}_2$, a (3)-uniform hypergraph and (15_3) configuration!



(a) Behold, a Fano Plane! (7_3)

(b) $W_3\mathbb{F}_2$

Lemma 9. All finite field projective planes are symmetric configurations and linear spaces.

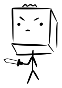
Proof.

Consider some finite field $\mathbb{F}_q$. Its three dimensional vector space will have $q^3 - 1$ nonzero vectors, and each plane will have $q^2 - 1$ nonzero vectors.

Already, this gives us that the projective plane will have $q^3 - 1$ projective points, and each projective line will have $\frac{q^2-1}{q-1} = q + 1$ projective points.

All we need now is to show that there are $q^3 - 1$ projective lines, and every projective point lives on $q + 1$ lines. A symmetric argument is made and we find that our plane will have exactly $q^3 - 1$ lines, with $q + 1$ lines to a point.

By the Pigeon Hole Lemma, we'll quickly see that in order to have a $(q^3 - 1_{q+1})$ configuration, every point must be collinear to every other point, and every line must intersect every other line. Thus, our plane is linear!



As we can see, the discussion of finite field reflexive forms would be incomplete if we don't collaborate closely with the configuration geometry and hypergraph machinery. All of a sudden, the discussion of uniformity and regularity and graph coloring that was front-loaded makes perfect sense.

Definition 39 (Path On A Hypergraph)**.** A path between v_1 and v_2 on a hypergraph is an alternating sequence of nodes and edges starting with v_1 and ending with v_2. It's formed by choosing an edge containing v_1, followed by choosing another node on said edge, repeating until the last node chosen is v_2.

Definition 40 (Hypergraph Distance)**.** The distance between two nodes on a hypergraph, denoted $d_h(v_1, v_2)$, is the minimal number of edges needed to form a path between v_1 and v_2.

For the purpose of metric space compliancy, we insist that $d(v_1, v_1) = 0$ because you don't need an edge to get from a node back to itself, and the distance from v_1 to any other v_2 adjacent to v_1 is one.

Definition 41 (Connection number)**.** The connection number between point v and line e, denoted as $c(v, e)$, is the number of points of e adjacent to v, which is the same as the number of lines containing v that intersect e. If $v \in e$, we can either use the first condition (points of e different from v) or unique lines through v that intersect e (which is just e itself). We use the second one and thus, $c(v, e) = 1$ for $v \in e$.

Polar Spaces

Now here's the issue, you see. As algebraists, we want to use Ball's definition for polar spaces as follows:

Definition 42 (Polar Space)**.** A *polar space* is a near linear space defined from $V_k(\mathbb{F})$ equipped with a non-degenerate σ-sesquilinear form or non-degenerate quadratic form. We do not consider every non-trivial subspace of $V_k(\mathbb{F})$ but only those that are totally singular with respect to the form.

In particular, the points of the polar spaces are the points of $P_{k-1}(\mathbb{F})$ which are singular for the form, the lines are lines of $P_{k-1}(\mathbb{F})$ that is totally singular for the form, and the incidence relation between these points and lines is inherited from the projective space.

[Ball, pg 54]

More generally speaking, we can look at polar spaces as the image of a non-degenerate form on some $V_k(\mathbb{F})$. What is *very important to note* is that this space is under no obligation to be linear - which means the hypergraph interpretation isn't necessarily complete, and we're allowed to have hyperbolic spaces. Consequentially, the other vectors spanned by the singular ones are under no obligation to be, themselves, singular either.

The upshot here is that polar spaces aren't spaces in the traditional "vector" sense — there's no need for them to hold the identity (they don't, anyway, because we don't have $\vec{0}$!) nor are they obligated to hold the direct sum of two other elements. More generally, they can be viewed as "sub-geometries", or subsets of points and lines in the projection of the vector space.

But as geometers, this is all nonsense! In fact, we actually might want Batten's definition, which requires some finite geometry machinery — namely the connection number defined in the previous section.

Definition 43 (Polar Space - equivalent definition)**.** Let S be a near linear space, e be a line in S, and v be a point in S.

A *polar space* is a near linear space, S, such that for every point $v \notin e,\ c(v, e) = 1$.

Now in particular, there's a special class of properties using the finite geometry machinery that tie *very nicely* with the algebra half of the conversation.

Definition 44 (Degenerate Polar Space)**.** A polar space, S, is called *degenerate* if there $\exists\ v_1 \in S$ such that $\forall v_2 \in S,\ \exists e_{1,2}$ such that v_1 and $v_2 \ \in e_{1,2}$. That is, there is some point v_1 that is collinear with every other point in S.

Why is this interesting to us?

This definition is *very* ***very*** useful, because geometrically, it tells us that this problematic point is in the radical/ which will make all lines from it totally singular!

When we talk about polar spaces, we don't want degenerate spaces or else there's some point that's a distance one from every other point (excluding itself) — your graph coloring gets all wonky.

More interestingly, complete k-uniform hypergraphs are degenerate polar spaces by virtue of being complete.

What does this mean for our study of polar spaces then?

When talking projectively, *we don't want to talk about linear spaces*, which are inherently complete in every sense of the word, and thus degenerate. In fact, this means we will *never* address the whole projective space. We actually *insist* that our space is a **hyperbolic subspace** (not all lines vanish on our form b although all points will).

Or in other words? Our space is strictly *not linear* — but only *near* linear.

Theorem 11. Up to isomorphism of point-line geometry, every polar space is isomorphic one of the following six families. In particular, for every $r \in \mathbb{Z}_{\geq 2}$, there are exactly six polar families. They are as follows

$W_{2r-1}\mathbb{F}_q,\ H_{2r-1}\mathbb{F}_q,\ H_{2r}\mathbb{F}_q,\ Q^+_{2r-1}\mathbb{F}_q,\ Q_{2r}\mathbb{F}_q,$ and $Q^-_{2r+2}\mathbb{F}_q$.

And the families are described by the following table:

Form	k	Name	Polar Space	Equation	ϵ
Alternating	$2r$	Symplectic	$W_{2r-1}(\mathbb{F}_q)$	$b(u,v) = \sum_{i=1}^{r}(u_{2i-1}v_{2i} - u_{2i}v_{2i-1})$	0
Hermitian	$2r$	Hermitian	$H_{2r-1}(\mathbb{F}_q)$	$b(u,v) = \sum_{i=1}^{r}(u_{2i-1}v^{\sigma}_{2i} + u_{2i}v^{\sigma}_{2i-1})$	$-\frac{1}{2}$
Hermitian	$2r+1$	Hermitian	$H_{2r}(\mathbb{F}_q)$	$b(u,v) = (\sum_{i=1}^{r}u_{2i}v^{\sigma}_{2i-1} + u_{2i-1}v^{\sigma}_{2i}) + u_{2r+1}v^{\sigma}_{2r+1}$	$\frac{1}{2}$
Quadratic	$2r$	Hyperbolic	$Q^+_{2r-1}(\mathbb{F}_q)$	$f(u) = \sum_{i=1}^{r}u_{2i-1}u_{2i}$	-1
Quadratic	$2r+1$	Parabolic	$Q_{2r}(\mathbb{F}_q)$	$f(u) = (\sum_{i=1}^{r}u_{2i-1}u_{2i}) + au^2_{2r+1}$	0
Quadratic	$2r+2$	Elliptic	$Q^-_{2r+1}(\mathbb{F}_q)$	$f(u) = (\sum_{i=1}^{r}u_{2i-1}u_{2i}) + au_{2r+1}u_{2r+2} + bu^2_{2r+2}$	1

[Ball, pg 55]

Proof.

By Theorem 8 and Theorem 3, we can guarantee that we either get a hermitian, alternating, or one of three quadratics (as a symmetric).

For the hermitian form, we get exactly two unique, non isomorphic, and non degenerate forms by $threftwo_hermitians$.

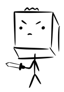
Similarly, by $threfalternating_basis$, we get that there's only one alternating form up to a change of basis.

So if we're working with a quadratic form, we'll consider the cases where $k = 2r,\ k = 2r{+}1,$ and $k = 2r + 2$.

By $threfquad_basis$, we know these are the only options, and all non-degenerate quadric forms for when $k = 2r$ are isometric, so there's only one!

As for the $k = 2r + 1$ case, what we'll find is that for all non-equivalent $k = 2r + 1$ forms, call them g_1 and g_2 there's an isomorphism between the singular vectors of g_1 to the totally singular vectors of g_2. Thus, the polar forms associated with g_1 and g_2 are isometric! Thus, there's only one $k = 2r + 1$ polar space up to isomorphism.

What we'll also find is that $k = 2r + 2$ behaves in the exact same way, and so altogether, we have three quadratic forms.



Theorem 12. The number of points of a finite polar space $\mathscr{P}$ of rank r is $\frac{(q^r-1)(q^{r+\epsilon+1})}{(q-1)}$

[Ball, pg 63]

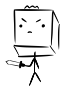
Definition 45 (Dual)**.** The dual of some space S is a mapping from S to S^* in which points are mapped to dual lines, and lines are mapped to dual points, where adjacency and incidence structures are preserved. It is most intuitive in the point-line geometry context.

Lemma 10. A symmetric configuration, (p_l), is self dual.

Proof.

Consider a configuration C. Let the incidence matrix of C be M, with each line ℓ_i for rows, and each point, p_j for columns. If $p_j \in \ell_i$, then the position (i, j) will have a 1, and if not, it will have a 0.

Now let's consider the dual configuration C^*, and its corresponding matrix M^*. This configuration is constructed by mapping each ℓ to a dual point, and each p to a dual line. What we'll find is that this configuration has the same matrix as the original, is denoted as (p_l), and thus, C is self dual.



Theorem 13. $Q_4(\mathbb{F})$ is isomorphic to the dual of $W_3\mathbb{F}$.

[Ball, pg 73]

Theorem 14 (A Series Of Equivalent Statements)**.** If $char(\mathbb{F}) = 2$ then...

- $W_3\mathbb{F}$ is self dual
- $Q_4\mathbb{F}$ is self dual
- $Q_4\mathbb{F}$ is isomorphic to $W_3\mathbb{F}$

[Ball, pg 73]

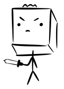
Proof.

We'll do this in three parts!

First we'll show $W_3\mathbb{F}$ is a symmetric (p_i) configuration and thus self dual. The best part is that we already know what $W_3\mathbb{F}$ looks like, and it already *is* clearly a (15_3) configuration and thus self dual.

Now by the previous theorem, we establish that $Q_4\mathbb{F}$ is isomorphic to $W_3\mathbb{F}^*$, and is thus a (15_3) configuration and therefore self dual.

For the last part, since $W_3\mathbb{F}^* \cong Q_4\mathbb{F}$, then we get $W_3\mathbb{F} \cong W_3\mathbb{F}^* \cong Q_4\mathbb{F}$.



Example 9. A particularly good example of the previous theorem is over $\mathbb{F}_2$, which is the $W_3\mathbb{F}_2$ configuration we addressed way back!

Example 10. What I find particularly instructive is given some space of dimension k, what are all the possible polar forms living in there. Let's consider then $\mathbb{P}_3(\mathbb{F}_2)$ — where $k = 4$. Although characteristic two has, as we've previously discussed, some rather unique properties with some isomorphisms, it's still instructive to be able to see all the possible polar spaces hidden within it.

For $r = 1$, we'll have $W_1\mathbb{F}_2,\ Q_1^+\mathbb{F}_2, Q_2\mathbb{F}_2,$ and $Q_3^-\mathbb{F}_2$.

For $r = 2$, we'll have only one polar spaces, $W_3\mathbb{F}_2$ — that is, the symplectic, and hyperbolic. The reason is because the other three requires a larger vector space than we have, and it's also worth noting that $Q_4\mathbb{F}_2 \cong W_3\mathbb{F}_2$, so we actually only end up with *two* unique forms.

We also don't have hermitians because $\mathbb{F}_2$ doesn't have any order two automorphisms!

Example 11 (A Foray Into $\mathbb{F}_3$)**.** For the sake of variety, we'll also address some interesting geometries in $\mathbb{F}_3$.

The problem is that all other iterations get *astronomically larger* than I'm willing to draw (booooo), so we will simply get the SAGE code to show us what these points look like.

A Revisitation of ✨ Shapes and Colors ✨

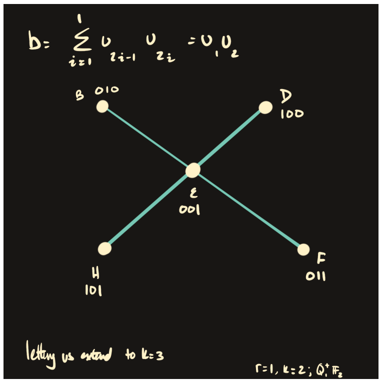
Definition 46. A polarity, π, of an incidence structure is an incidence preserving bijection from the points to the lines. That is, $x \in \ell \iff \pi^{-1}(\ell) \in \pi(x)$. In particular, this means the incidence structure is self dual and a symmetric incidence structure.

[Ball, pg 74]

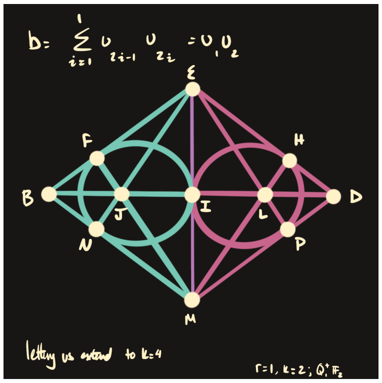
Theorem 15. If q is an odd power of two, then $W_3\mathbb{F}_q$ has a polarity.

[Ball, pg 74]

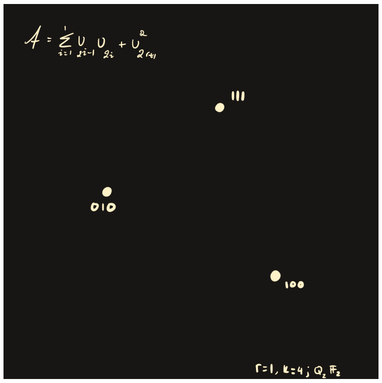
(a) $Q_1^+\mathbb{F}_2$



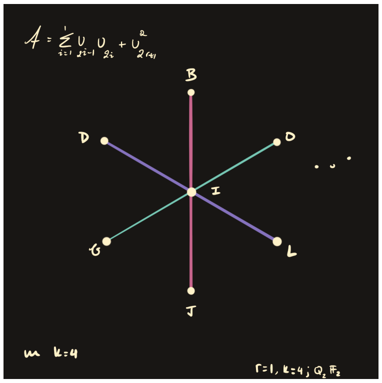
(b) $Q_1^+\mathbb{F}_2$ if you let it take on three coordinates



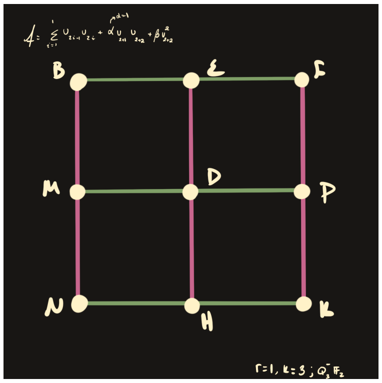
(c) $Q_1^+\mathbb{F}_2$ if you let it take on four coordinates



(d) $Q_2\mathbb{F}_2$

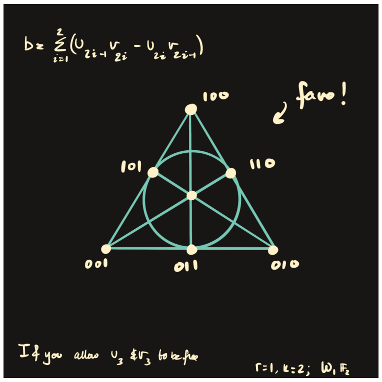


(e) $Q_2\mathbb{F}_2$ if we give it four coordinates

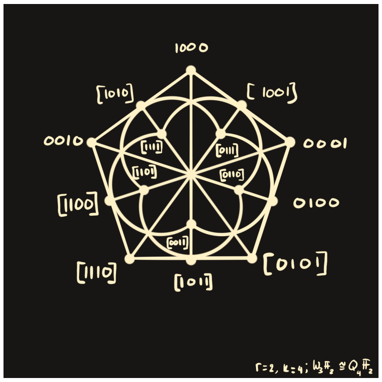


(f) $Q_3^-\mathbb{F}_2$

(g) $W_1\mathbb{F}_2$



(h) $W_1\mathbb{F}_2$ if we let it live in four dimensions (Fano Plane!)

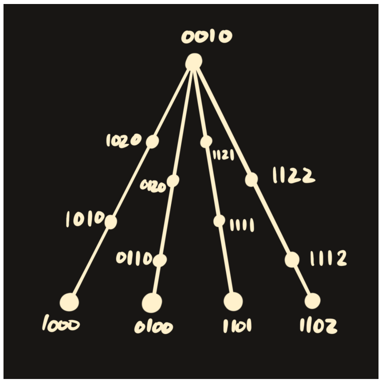


(i) $W_3\mathbb{F}_2 \cong Q_4\mathbb{F}_2$

Incidentally, this means $W_3\mathbb{F}_2$ has a polarity.

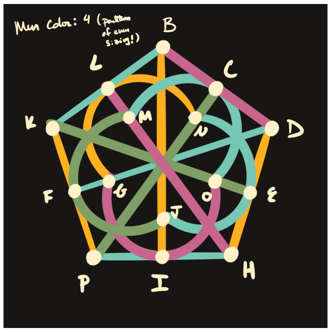
Now let's revisit some of the interesting shapes we came across in the previous section!

First, we'll begin by addressing the minimal hypergraph coloring — which behaves similarly to standard graph coloring.



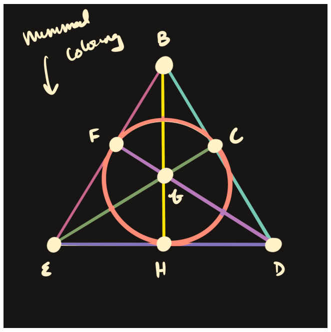
(a) $Q_3^- \mathbb{F}_3$

Example 12. This is the coloring of $W_3\mathbb{F}_2$!



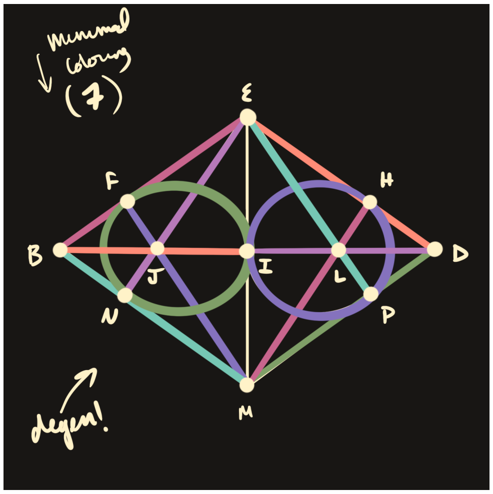
Any space that is linear will require a new color for each line.

Example 13. This is the coloring of the Fano plane!

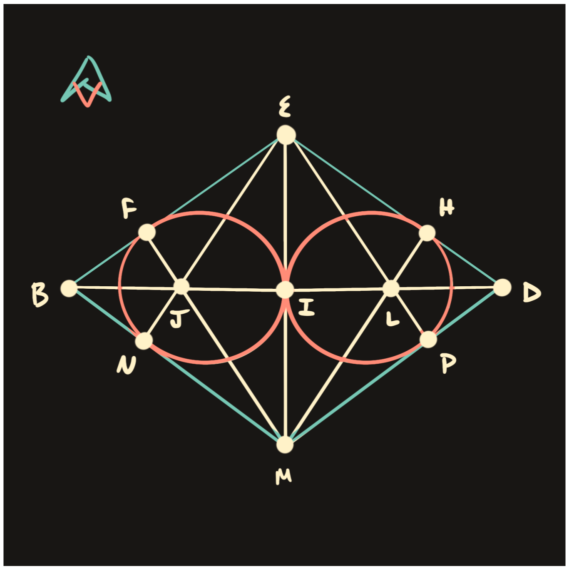


We can also look at spaces with nontrivial kernels, also known as degenerate polar spaces with an emphasis on the Batten perspective on degeneracy.

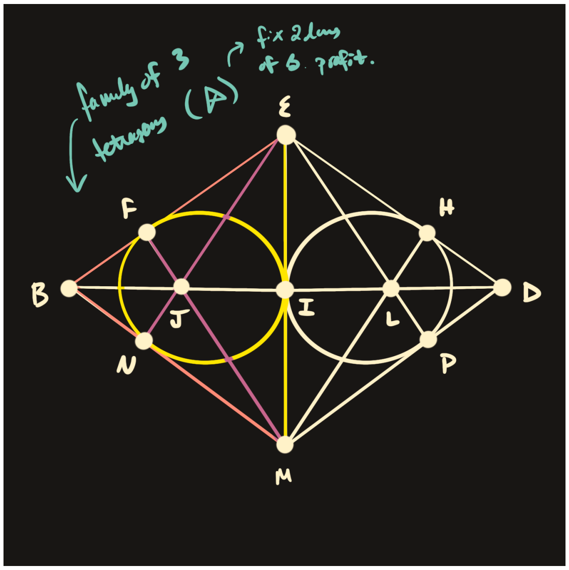
Example 14. Another interesting example is one of the degenerate and totally isotropic subspaces of $\mathbb{P}_3(\mathbb{F}_2)$ — this particular one is a pair of Fano's glued at the base.



Example 15. Or we can look at subconfigurations!

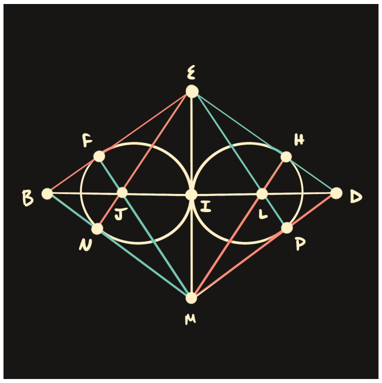


(a) $W_3\mathbb{F}_2$ with non degenerate hexagon

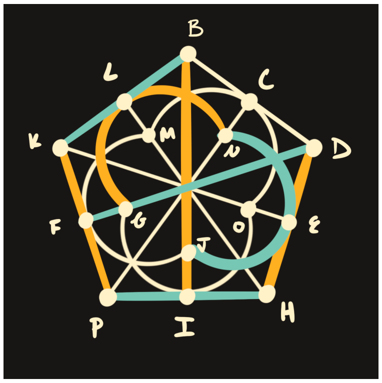


(b) Double Fano with tetragons - $(6_2, 4_3)$

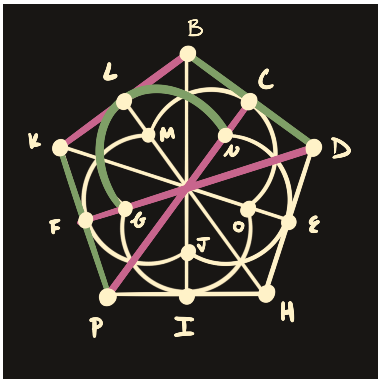
Example 16. If we're feeling fancy, we can look at totally singular subspaces with totally singular lines that have an even color (which ultimately degenerates the incidence matrix!)



(a) $W_3\mathbb{F}_2$ with 4-4 octagon



(b) $W_3\mathbb{F}_2$ with octagon



(c) $W_3\mathbb{F}_2$ with hexagon ($xy = zw$)

Bibliography

References

[Ball] Simeon Ball (2015) *Finite Geometry and Combinatorial Applications*

[Batten] Lynn Margaret Batten (1997) *Combinatorics of Finite Geometry*, 2nd ed.

[Zheng] MJ Zheng *Horrors-Persisting* available at https://github.com/WenJunMZheng