

TIME-FREQUENCY ANALYSIS ON
NON-EUCLIDEAN DOMAINS AND
SPECTRAL PROPERTIES OF THE GRAM
MATRIX FOR GABOR SYSTEMS
GENERATED BY B-SPLINES

A dissertation

submitted by

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in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

TUFTS UNIVERSITY

August 2025

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Abstract

The short-time Fourier transform (STFT) and Gabor frames are central tools in time-frequency analysis providing local Fourier decompositions of functions. While well-studied classically, few attempts have been made to extend these tools to non-Euclidean spaces. Here, we present a definition of a STFT on such spaces and investigate their properties. A STFT on graphs is one realization of this definition that is built on the graph Laplacian and graph heat kernel. We prove it has desirable spanning properties, an explicit inversion formula, and is energy-preserving on two large classes of algebraic graphs. A STFT on the Sierpinski gasket (SG) fractal is another realization defined likewise using a fractal Laplacian and heat kernel. We show that it shares properties with the classical STFT such as being continuous, satisfying orthogonality relations, and invertibility. This definition is then extended to the post-critical finite class of fractals. We investigate a related problem in time-frequency analysis on sampling the STFT on $\mathbb{R}^d$. Specifically, we show that the Gram matrix corresponding to time-frequency shifts of a B-spline has remarkable Toeplitz block structure amenable to spectral analysis and provide explicit formulas for the spectrum of its Toeplitz blocks.

Acknowledgements

An enormous thanks to my advisor Kasso, whose passion for mathematics is palpable and contagious. He has been overwhelmingly supportive and generous with his time and I would not have made it through graduate school without him. I also wouldn't be in graduate school without my parents who nurtured a sense of curiosity and value in science, math, learning, and education that I hold dear. They have been incredibly supportive and understanding in my not having a salary for five years.

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Chapter 1

Introduction

Fourier analysis studies how to decompose, represent, and reconstruct a function from the basic building blocks of trigonometric functions. Introduced by Joseph Fourier in the 18th and 19th century while constructing solutions to the heat equation, Fourier analysis developed into its own branch of mathematics that touches a vast array of fields including complex analysis, partial differential equations, approximation theory, and representation theory. In application it is indispensable in signal processing, quantum physics, modern data science, and machine learning. Classically, Fourier analysis revolved around the study of Fourier series and the Fourier transform formalizing this representation in terms of the trigonometric functions. These trigonometric functions provide in a sense a *global* representation; their support and energy are spread across the domain. However, if one begins to ask for a local representation in terms of these trigonometric functions, the Fourier transform and Fourier series fall short. The *short-time Fourier transform* (STFT) is a tool in time-frequency analysis to capture *local* frequency information of a function. This is accomplished via a window that localizes a function to a particular portion of its domain before performing the Fourier transform. The function class that the window belongs to, the trade-off in localization between domains, and the existence of *Gabor frames* tied to the sampling of the STFT are delicate problems in time-frequency analysis. The ingredients for the STFT on $\mathbb{R}^d$ are (1) a window that localizes a function to a region of its support (2) an underlying (algebraic) structure that allows one to translate the window across the domain and localize and (3) the complex exponentials that one integrates against to study the function in the frequency domain.

A natural question to ask is to what extent can Fourier analysis and the STFT be extended to less-structured (non-Euclidean) domains that still retains its essential

properties? Here, we present a definition of a STFT on the non-Euclidean domains of graphs and fractals and investigate its properties. Directly importing the tools from classical time-frequency analysis and STFT theory is challenging as the three ingredients listed above do not have direct analogues on these domains. We prove that using a heat kernel as a window and integrating against the eigenfunctions of a Laplacian allows for a well-defined STFT and leads to desirable properties of this STFT. In the graph case, this leads to an explicit form of a frame operator, its eigenvalues, and inverse that provides a decomposition of a function in terms of building blocks created from this STFT. In the fractal case, we show that using a weighted measure allows for isometries between the domain and co-domain and an explicit inverse (reconstruction) formula. Finally, we investigate a related problem within time-frequency analysis of sampling the STFT and the associated Gabor frame theory. Motivated by the *frame set problem*, we study the spectrum of the Gram matrix and Gram operator constructed from time-frequency shifts of a B-spline and provide new results on the structure of these operators and their spectrum including asymptotic formulas for their eigenvalues and estimates on the frame bounds. The layout of this thesis is as follows:

1. Chapter 2 contains the mathematical background in Fourier analysis, time-frequency analysis, and fractal analysis.
2. Chapter 3 presents the definition of a (G)STFT on graphs. This definition (Equation 3.1) is built entirely from the spectrum of the graph Laplacian by integrating a function against its eigenfunctions and a heat kernel. The key results include conditions on when the graph Gabor system forms a frame for $\mathbb{C}^N$, explicit expressions for eigenvalues of the time-varying frame operator, a formula for the left inverse of the STFT, and conditions on the (algebraic) structure of the underlying graph that are sufficient for the Gabor frame to form a tight frame. These are contained in Theorems 3.2.1, 3.2.3, 3.3.2, and 3.3.3.
3. Chapter 4 presents results on a (F)STFT on the Sierpinski gasket (SG) and

related post-critically finite fractals. As in the graph case it is built on the spectrum of a Laplacian and integrates against its heat kernel and eigenfunctions. The main theorems are that the STFT admits a set of orthogonality relations, is an isometry into its range, and invertible with an explicit form for its left-inverse. These results are extended to the larger class of post-critically finite fractals whose spectral dimension is not too large, see Proposition 4.2.4 and Theorems 4.2.2, 4.4.3.

4. Chapter 5 looks in detail at a numerical approximation technique, finite elements, for solving differential equations on fractals studied previously by Gibbons [37]. In-depth comparison to classical finite elements is made and the results of numerical experiments for solving the Helmholtz equation on SG are presented. This provides numerical solutions and visualizations to the eigenvalue problem for the Laplacian used to build the short-time Fourier transform on SG .
5. Chapter 6 investigates the structure of a Gram matrix related to a sampling of the short-time Fourier transform of the 2nd-order B-spline. The main results show that the Gram matrix contains Toeplitz and block structure making it amenable to spectral analysis. We fully characterize the asymptotic behavior of the eigenvalues of blocks within the finite Gram matrix which provide estimates on the finite frame bounds. The frame bounds are shown to be tied to the extrema of a finite Fourier series built from the Toeplitz patterns in the Gram matrix. These results are Theorems 6.2.2, 6.3.1 and Corollary 6.3.2.

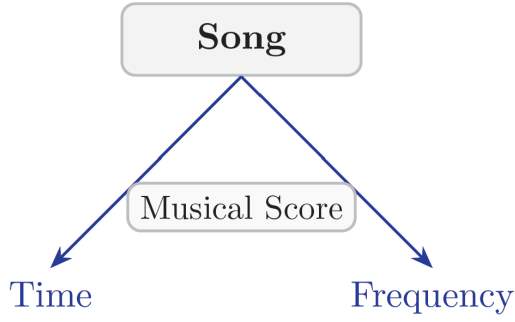


Figure 1.1: A musical score is a time-frequency analysis of a song. The score decomposes the song into building blocks that are intervals of time and intervals of frequency or pitch.

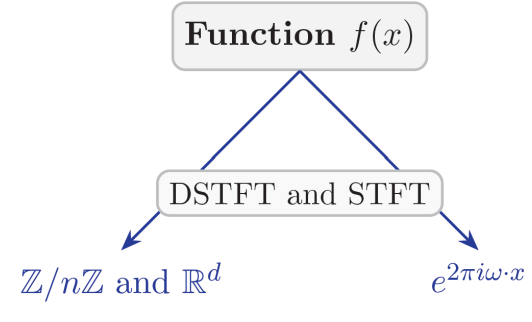


Figure 1.2: The short-time Fourier transform (STFT) and discrete short-time Fourier transform (DSTFT) in Chapter 2 are a time-frequency analysis of a function on Euclidean $\mathbb{R}^d$ and group $\mathbb{Z}/n\mathbb{Z}$.

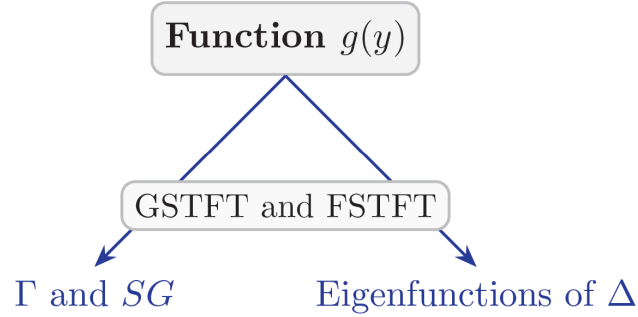


Figure 1.3: The graph short-time Fourier transform (GSTFT) in Chapter 3 and fractal short-time Fourier transform (FSTFT) in Chapter 4 defined and studied in this thesis can be viewed as an extension of time-frequency analysis to the non-Euclidean spaces of a graph Γ and fractal SG .

Chapter 2

Background

2.1 The Heat Equation, Fourier Series, and Function Decompositions

We start with a classic example that Joseph Fourier studied in the 18th and 19th century regarding heat flow. It will introduce Fourier analysis and motivate a number of the central ideas, questions, and mathematical characters that are in this thesis including function representation, bases, the Laplacian, and heat kernels. This leads naturally to the question of what other building blocks can one represent a function by beyond a basis? Is it required that these building blocks even form a basis? Denis Gabor and J. Von Neumann asked these questions with particular interest in representing a function via certain time-frequency shifts of the Gaussian. These are not a basis for $L^2(\mathbb{R}^d)$ but instead form an overcomplete representation system called a *frame*. We focus on a special type of frame, *Gabor frames*, and their continuum cousin the *short-time Fourier transform*. This thesis is largely an investigation into cases where the spaces the Gabor frames and short-time Fourier transforms live and act on is no longer required to be Euclidean $\mathbb{R}^d$.

2.1.1 The Heat Equation and a Fourier Series Representation

In his treatise *The Analytic Theory of Heat*, Fourier studied a particular form of the partial differential equation modeling heat flow, the *heat equation*. We sketch his study here. Fix a thin metal rod of length L that is completely insulated at the ends so that no heat flows in or out and the temperature there is fixed at zero. If the initial temperature distribution across the rod is given by the function $f(x)$, then the heat $u(x, t)$ at position x and time t within the rod is a solution to the following

partial differential equation,

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = 0, & (x, t) \in (0, L) \times (0, \infty) \\ u(x, t) = 0, & (x, t) \in \{0, L\} \times (0, \infty) \\ u(x, t) = f(x), & (x, t) \in (0, L) \times \{t = 0\}, \end{cases}$$

where $\Delta := \partial_{xx}$ is the *Laplacian* operator. Via a technique called separation of variables, where one assumes the solution has the form $u(x, t) = \varphi(x)\psi(t)$, we arrive at an ordinary differential equation for the spatial variable, the *Helmholtz equation*:

$$\begin{cases} -\Delta \varphi(x) = \lambda \varphi(x), & x \in (0, L) \\ \varphi(x) = 0, & x \in \{0, L\}. \end{cases}$$

The Helmholtz equation is an eigenvalue problem for the Laplacian Δ whose solutions in this case are the countable collection of sinusoids,

$$\{(\lambda_\ell, \varphi_\ell)\}_{\ell=1}^\infty = \left\{ \left(\frac{\ell^2 \pi^2}{L^2}, \sqrt{\frac{2}{L}} \sin \frac{\ell \pi x}{L} \right) \right\}_{\ell=1}^\infty.$$

We also arrive at an ordinary differential equation for the time variable, $-\psi'(t) = \lambda \psi(t)$, with solutions,

$$\{(\lambda_\ell, \psi_\ell)\}_{\ell=1}^\infty = \left\{ \left(\frac{\ell^2 \pi^2}{L^2}, e^{-\frac{\ell^2 \pi^2 t}{L^2}} \right) \right\}_{\ell=1}^\infty.$$

Since the PDE is linear, we hypothesize a sum of solutions is again a solution, and construct the general solution by adding each solution in a series $u_\ell(x, t) := \varphi_\ell(x)\psi_\ell(t) = \sqrt{\frac{2}{L}} \sin \frac{\ell \pi x}{L} e^{-\frac{\ell^2 \pi^2 t}{L^2}}$,

$$u(x, t) = \sum_{\ell=1}^\infty \alpha_\ell \varphi_\ell(x) \psi_\ell(t) = \sum_{\ell=1}^\infty \alpha_\ell \sin \frac{\ell \pi x}{L} e^{-\frac{\ell^2 \pi^2 t}{L^2}}$$

$$\alpha_\ell = \frac{2}{L} \int_0^L f(x) \varphi_\ell(x) dx = \frac{2}{L} \int_0^L f(x) \sin \frac{\ell \pi x}{L} dx.$$

If we allow the interchange of summation and integration, we arrive at another form of the solution in terms of an integral kernel called the *heat kernel* $H_t(x, y)$ expressed

entirely in terms of the eigenvalues and eigenfunctions of the Laplacian:

$$\begin{aligned} u(x, t) &= \sum_{\ell=1}^{\infty} \left[\frac{2}{L} \int_0^L f(y) \varphi_{\ell}(y) dy \right] \varphi_{\ell}(x) e^{-\lambda_{\ell} t} \\ &= \frac{2}{L} \int_0^L \left[\sum_{\ell=1}^{\infty} e^{-\lambda_{\ell} t} \varphi_{\ell}(x) \varphi_{\ell}(y) \right] f(y) dy \\ &= \frac{2}{L} \int_0^L H_t(x, y) f(y) dy. \end{aligned}$$

This procedure produces the general solution to the heat equation on the compact domain $[0, L]$. It also produces a decomposition of the initial temperature distribution $f(x)$ in terms of these sinusoids, called a *Fourier series*,

$$f(x) = \sum_{\ell=1}^{\infty} \alpha_{\ell} \sin \frac{\ell \pi x}{L}, \quad \alpha_{\ell} = \frac{2}{L} \int_0^L f(x) \sin \frac{\ell \pi x}{L} dx.$$

Formally, we decompose (represent or reconstruct) $f \in L^2([-\pi, \pi])$ as orthogonal projections onto a countable orthonormal basis which consist of eigenfunctions of the Laplacian Δ . For $f \in L^2([-\pi, \pi])$ (see Section 2.4.2 for formal introductions to Hilbert spaces and operator theory), its *Fourier series* is the representation in terms of the orthonormal eigenbasis of the Laplacian $\{\varphi_{\ell}\}_{\ell \in \mathbb{Z}} = \{e^{i\ell x}\}_{\ell \in \mathbb{Z}}$ of $L^2(\mathbb{R})$:

$$f(x) = \sum_{\ell \in \mathbb{Z}} \alpha_{\ell} e^{i\ell x}, \quad \alpha_{\ell} = \langle f, \varphi_{\ell} \rangle_{L^2([-\pi, \pi])}. \quad (2.1)$$

If $f \in L^2([-\pi, \pi])$, then

1. we have the isometry (called *Parseval's relation*, see Section 2.2): $\sum_{\ell \in \mathbb{Z}} |\alpha_{\ell}|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$,
2. the mapping $C : f \mapsto \{\alpha_{\ell}\}$ is a unitary correspondence (an invertible isometry, see Section 2.4.2) between $L^2([-\pi, \pi])$ and $\ell^2(\mathbb{Z})$,
3. the Fourier series of f converges to f in the $L^2([-\pi, \pi])$ norm.

In the 20th century, Denis Gabor and John Von Neumann continued to explore this idea of function decomposition beyond Fourier series. Rather than decomposing a function into complex exponentials, they looked to time-frequency shifts of

the Gaussian. The Gaussian is central to time-frequency analysis for a number of reasons including the beautiful fact that its time-frequency shifts minimize a classic uncertainty principles providing optimal time-frequency resolution [47, Chapter 2].

2.1.2 Function Decompositions

Gabor investigated sequences of functions of the form $\{M_\ell T_k e^{-x^2/2}\}_{\ell,k \in \mathbb{Z}}$ where M_ℓ and T_k are the modulation and translation operators (Section 2.2.2), so that one (hopefully) has the representation

$$f(x) = \sum_{\ell,k \in \mathbb{Z}^2} c_{\ell k} e^{2\pi i \ell b x} e^{-(x-ka)^2/2},$$

with convergence in $L^2(\mathbb{R})$. Although the support of the Gaussian is the entire real line, one can motivate this type of function decomposition with another function of fast decay that incorporates a Fourier series decomposition: the indicator function $\chi_{[0,1)}(x)$. Multiplying by $\chi_{[0,1)}(x)$ *windows* the original function to have compact support like the initial heat distribution $f(x)$ in Section 2.1.1. Let $g(x) = \chi_{[0,1)}(x)$ and window (localize) $f \in L^2(\mathbb{R})$ via $f(x)g(x-k)$. Then, f can be recovered via summing over all these localizations with convergence in $L^2(\mathbb{R})$:

$$f(x) = \sum_{k \in \mathbb{Z}} f(x)g(x-k).$$

Each $f(x)g(x-k)$ is $L^2([k, k+1))$ and so has a Fourier series decomposition:

$$f(x)g(x-k) = \sum_{\ell \in \mathbb{Z}} c_{\ell k} e^{2\pi i x \ell}$$

where the coefficients have the form,

$$c_{\ell k} = \langle f(\cdot)g(\cdot-k), e^{2\pi i \ell \cdot} \rangle_{L^2[k, k+1)} = \langle f, g_{\ell k} \rangle_{L^2[k, k+1)},$$

and $g_{\ell k} := g(\cdot - k)e^{2\pi i \ell \cdot}$. This leads to the following decomposition of $f \in L^2(\mathbb{R})$:

$$\begin{aligned} f(x) &= \sum_{k \in \mathbb{Z}} f(x)g(x - k) \\ &= \sum_{\ell, k \in \mathbb{Z}^2} c_{\ell k} e^{2\pi i \ell x} \\ &= \sum_{\ell, k \in \mathbb{Z}^2} \langle f, g_{\ell k} \rangle_{L^2[k, k+1]} g_{\ell k}(x). \end{aligned}$$

Indeed, the system of time-frequency shifts of the characteristic function $\{M_\ell T_k \mathbb{1}_{[0,1]}\}_{\ell, k \in \mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$. However, with the Gaussian one *cannot* represent every $f \in L^2(\mathbb{R})$ using the system $\{M_{\ell b} T_{ka} e^{-x^2/2}\}_{\ell, k \in \mathbb{Z}}$ for arbitrary $a, b > 0$. Janssen [59, 61] and Heller and David [24] showed that indeed $\{M_{\ell b} T_{ka} e^{-x^2/2}\}_{\ell, k \in \mathbb{Z}}$ does not form what is called a *frame* for $L^2(\mathbb{R})$ and hence one does not have the representation $f(x) = \sum_{\ell, k \in \mathbb{Z}^2} c_{\ell k} e^{2\pi i \ell b x} e^{-(x-ka)^2/2}$ when $a = b = 1$. If instead the *lattice parameters* have product $ab < 1$, then this system of time-frequency shifts of the Gaussian form a frame for $L^2(\mathbb{R})$ and one can represent every $f \in L^2(\mathbb{R})$ in this way.

2.2 The Fourier Transform

Fourier analysis began as an investigation via the heat equation into the decomposition of a function into fundamental building blocks of trigonometric functions and complex exponentials. This is the action of the Fourier transform and Fourier series. The Fourier transform integrates across the entire domain of the complex exponentials. The Fourier series in Equation (2.1) uses the integer samples of the complex exponential for functions with compact support or are periodic. Here, we present the definition and fundamental properties of the Fourier transform. We also discuss its limitations. This leads naturally to the introduction of a local Fourier analysis and local Fourier transform studied in the next section. This local Fourier analysis is referred to as *time-frequency analysis* and the local Fourier transform as the *short-time Fourier transform*. The short-time Fourier transform is closely related to *Gabor systems* $\{M_{\ell b}T_{ka}g\}_{\ell,k \in \mathbb{Z}}$ used in a discretization process to recover a function from its short-time Fourier transform. This is discussed at the end of Section 2.1 and in detail in Section 2.4.

2.2.1 Definition, Riemann-Lebesgue, and Parseval-Plancharel

The *Fourier transform* of a function $f \in L^1(\mathbb{R}^d)$ is initially defined pointwise as the integral

$$\hat{f}(\omega) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \omega} dx,$$

where $x \cdot \omega = \sum_{i=1}^d x_i \omega_i$ is the inner product on $\mathbb{R}^d$ and the integral is taken with respect to the Lebesgue measure. Ideally, we'd like to define the Fourier transform of a function $f \in L^2(\mathbb{R}^d)$, which has no reason a priori to be finite as $L^2(\mathbb{R}^d) \not\subset L^1(\mathbb{R}^d)$. Defining the Fourier transform on $L^2(\mathbb{R}^d)$ would allow full use of its Hilbert space structure and geometry. Furthermore, in applications $L^2(\mathbb{R}^d)$ is a natural setting as one often wants to measure the *energy* of the signal,

$$\mathcal{E}(f) := \int_{\mathbb{R}^d} |f(x)|^2 dx = \int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 d\omega < \infty.$$

Defining the Fourier transform for $f \in L^2(\mathbb{R}^d)$ is accomplished via a limit in a space that is dense in $L^2(\mathbb{R}^d)$ where the Fourier transform is already well-defined: at the intersection of $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ or in the Schwartz space $S(\mathbb{R}^d)$. We first state two classic results in Fourier analysis, the *Riemann-Lebesgue* lemma and the *Parseval-Plancharel* theorem. These state that the Fourier transform of a function is continuous, decays to zero, and preserves energy.

Lemma 2.2.1 (Riemann-Lebesgue) *If $f \in L^1(\mathbb{R}^d)$, then $\hat{f}$ is uniformly continuous and $\lim_{|\omega| \rightarrow \infty} |\hat{f}(\omega)| = 0$.*

Another way to state the Riemann-Lebesgue lemma is that the Fourier transform operator

$$\mathcal{F}\{f\}(\omega) := \hat{f}(\omega)$$

maps into the Banach space of continuous functions that vanish at infinity,

$$\mathcal{F} : L^1(\mathbb{R}^d) \rightarrow C_0(\mathbb{R}^d).$$

Theorem 2.2.2 (Parseval-Plancharel) *If $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then*

$$\|f\|_{L^2(\mathbb{R}^d)} = \|\hat{f}\|_{L^2(\mathbb{R}^d)}.$$

The Fourier transform operator $\mathcal{F}$ extends to a unitary operator on $L^2(\mathbb{R}^d)$ and satisfies Parseval's formula,

$$\langle f, g \rangle_{L^2(\mathbb{R}^d)} = \langle \hat{f}, \hat{g} \rangle_{L^2(\mathbb{R}^d)}, \quad \forall f, g \in L^2(\mathbb{R}^d).$$

That $\mathcal{F}$ is unitary when extended to $L^2(\mathbb{R}^d)$ and hence preserves the inner product is an often exploited property in Fourier analysis and used in the proofs throughout this thesis. Proofs of the Riemann-Lebesgue lemma and Plancharel-Parseval can be found in any standard Fourier analysis text such as [47, Chapter 1].

Defining the Fourier transform on $L^2(\mathbb{R}^d)$ uses Parseval: identify a dense subspace of $X \subset L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, choose a sequence of functions that converge to f , and

show the sequence of Fourier transforms converge. For example, $X = L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ or the Schwartz space,

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} |D^\alpha X^\beta f(x)| < \infty, \forall \alpha, \beta \in \mathbb{Z}_+^d\},$$

where D^α and X^β are the derivative and multiplication operators respectively for multi-indices $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\beta = (\beta_1, \dots, \beta_d)$. If $X = L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, one then chooses a sequence $\{f_n\}_{n=1}^\infty \subset X$ such that $\lim_{n \rightarrow \infty} \|f_n - f\|_{L^2} = 0$. The Fourier transform is well-defined on $X \subset L^1(\mathbb{R}^d)$ and hence by Parseval, $\|f_n - f_m\|_{L^2} = \|\hat{f}_n - \hat{f}_m\|_{L^2}$, and $\{\hat{f}_n\}_{n=1}^\infty$ is a Cauchy sequence. The Fourier transform for $f \in L^2(\mathbb{R}^d)$ is then *defined* as the unique limit

$$\hat{f} := \lim_{n \rightarrow \infty} \hat{f}_n.$$

2.2.2 Properties of the Fourier Transform

The Fourier transform is linear,

$$\mathcal{F}(\alpha f + \beta g) = \alpha \mathcal{F}(f) + \beta \mathcal{F}(g), \quad \alpha \text{ and } \beta \in \mathbb{C}$$

due to linearity of the Lebesgue integral. The Fourier transform *transforms* a set operations in the x -domain or t -domain to a set of operations in the ω -domain. A convolution of two functions is transformed into pointwise multiplication. Derivation is transformed into multiplication by monomials. Translation is transformed into modulation by the complex exponentials. For $x, \omega \in \mathbb{R}^d$ define the *translation* operator $T_x : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$ as

$$T_x f = f(t - x).$$

Define the *modulation* operator $M_\omega : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$ as

$$M_\omega f = e^{2\pi i \omega \cdot t} f(t).$$

Both operations are isometries on $L^p(\mathbb{R}^d)$ due to the translation invariance of the Lebesgue integral and the fact that $|e^{2\pi i \omega \cdot t} f(t)|^p = |f(t)|^p$.

Lemma 2.2.3 (Fourier Transform of a Translation, Chapter 1 [47]) *The Fourier transform $\mathcal{F} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ takes translation in the t -domain to modulation in the ω -domain and vice-versa,*

$$(\widehat{T_x f}) = M_{-x} \hat{f}, \quad (\widehat{M_\omega f}) = T_\omega \hat{f}.$$

Lemma 2.2.3 can be seen from a simple change-of-variables,

$$(\widehat{T_x f}) = \int_{\mathbb{R}^d} f(t-x) e^{-2\pi i t \cdot \xi} dt = \int_{\mathbb{R}^d} f(t) e^{-2\pi i (t+x) \cdot \xi} dt = M_{-x} \hat{f}$$

and

$$(\widehat{M_\omega f}) = \int_{\mathbb{R}^d} (e^{2\pi i t \cdot \omega} f(t)) e^{-2\pi i t \cdot \xi} dt = \int_{\mathbb{R}^d} f(t) e^{-2\pi i t \cdot (\xi - \omega)} dt = T_\omega \hat{f}.$$

Not only are these operators related to one another via the Fourier transform, their composition $T_x M_\omega$ and $M_\omega T_x$ (called *time-frequency shifts*), are central to time-frequency analysis and the properties of the short-time Fourier transform seen in Section 2.3.

Given two functions $f, g \in L^1(\mathbb{R}^d)$, the *convolution* $f * g$ is the integral,

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y) g(x-y) dy.$$

One immediately has the estimate $\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}$ by a simple application of Fubini's theorem. The following lemma describes how convolution is transformed under the Fourier transform.

Lemma 2.2.4 (Fourier Transform of a Convolution, Chapter 1 [47]) *The Fourier transform $\mathcal{F} : L^1(\mathbb{R}^d) \rightarrow C_0(\mathbb{R}^d)$ takes convolution to point-wise multiplication,*

$$(\widehat{f * g}) = \hat{f} \cdot \hat{g}.$$

Lemma 2.2.4 follows from Fubini's theorem:

$$\begin{aligned}
 (\widehat{f * g})(\omega) &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(y)g(x-y)dy \right) e^{-2\pi i y \cdot \omega} e^{-2\pi i (x-y) \cdot \omega} dx \\
 &= \int_{\mathbb{R}^d} f(y) e^{-2\pi i y \cdot \omega} \left(\int_{\mathbb{R}^d} g(x-y) e^{-2\pi i (x-y) \cdot \omega} dx \right) dy \\
 &= \hat{f}(\omega) \hat{g}(\omega).
 \end{aligned}$$

For a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{Z}_+^d$, the order of the multi-index is given by the sum $|\alpha| = \sum_{j=1}^d \alpha_j$. Define the *derivative operator* $D^\alpha : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ as $D^\alpha f(x) := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} f(x)$ and the *multiplication operator* $X^\alpha : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ as $X^\alpha f(x) := x^\alpha f(x)$. The Fourier transform is classically treated as an operator on an $L^p(\mathbb{R}^d)$ space. There is no reason a priori to expect the function to be differentiable. As such, a proper treatment of the relationship between the Fourier transform, the derivation operator, and the multiplication operator requires the space of Schwartz functions and its dual space of tempered distributions. For an in-depth treatment, see [44]. We state the relationship between these operators and provide a brief justification.

Lemma 2.2.5 (Fourier Transform of a Derivation, Chapter 1 [47]) *The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ takes derivation to multiplication by monomials and vice-versa:*

$$(\widehat{D^\alpha f})(\omega) = (2\pi i \omega)^\alpha \hat{f}(\omega), \quad ((-2\pi i x)^\alpha f)(\omega) = D^\alpha \hat{f}(\omega).$$

Using the operators D^α and X^α , the relationship is

$$\mathcal{F} D^\alpha = (2\pi i)^{|\alpha|} X^\alpha \mathcal{F}, \quad \mathcal{F} X^\alpha = \left(\frac{i}{2\pi}\right)^{|\alpha|} D^\alpha \mathcal{F}.$$

Lemma 2.2.5 follows from the fact that Schwartz functions are smooth $C^\infty(\mathbb{R}^d)$ functions and we can relentlessly interchange limits and integral,

$$D^\alpha \hat{f}(\omega) = \int_{\mathbb{R}^d} f(x) D^\alpha (e^{-2\pi i x \cdot \omega}) dx = \int_{\mathbb{R}^d} (-2\pi i x)^\alpha f(x) e^{-2\pi i x \cdot \omega} dx,$$

and integrate-by-parts,

$$(\widehat{D^\alpha f})(\omega) = \int_{\mathbb{R}^d} D^\alpha f(x) e^{-2\pi i x \cdot \omega} dx = \int_{\mathbb{R}^d} f(x) D^\alpha e^{-2\pi i x \cdot \omega} dx.$$

The last crucial property of the Fourier transform is that it is invertible. We can reconstruct (represent) a function in terms of a continuous superposition of the complex exponentials $e^{2\pi i x \cdot \omega}$. The inversion formula below is stated for functions $f \in L^1(\mathbb{R}^d)$ and is extended to other function spaces such as $L^2(\mathbb{R}^d)$ by approximation in a dense subspace as described below Theorem 2.2.2.

Theorem 2.2.6 (Fourier Transform Inversion, Theorem 1.2.2 [47]) *If $f \in L^1(\mathbb{R}^d)$ and $\hat{f} \in L^1(\mathbb{R}^d)$, then*

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\omega) e^{2\pi i x \cdot \omega} d\omega, \quad x \in \mathbb{R}^d.$$

The inversion formula and Plancherel's theorem are indeed equivalent. If Plancherel's theorem is true, then $\mathcal{F}$ is a unitary operator and hence its inverse is the adjoint, $\mathcal{F}^{-1} = \mathcal{F}^*$. The adjoint of the integral operator $\mathcal{F}f(\omega) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \omega} dx$ is the integral operator $\mathcal{F}^*g(x) = \int_{\mathbb{R}^d} g(\omega) e^{2\pi i x \cdot \omega} d\omega$, and hence the inversion formula holds. Now suppose, the inversion formula holds for all f such that $f, \hat{f} \in L^1(\mathbb{R}^d)$. Then we have,

$$f * f^*(0) = \int_{\mathbb{R}^d} f(x) f(0 - x) dx = \|f\|_{L^2(\mathbb{R}^d)}^2$$

where $f^*(x) = \bar{f}(-x)$ is an involution, and

$$(\widehat{f * f^*})(\omega) = \hat{f}(\omega) \hat{f}^*(\omega) = |\hat{f}(\omega)|^2 \in L^1(\mathbb{R}^d).$$

Hence,

$$\|f\|_{L^2(\mathbb{R}^d)}^2 = f * f^*(0) = \int_{\mathbb{R}^d} (\widehat{f * f^*})(\omega) e^{2\pi i 0 \cdot \omega} d\omega = \int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 d\omega = \|\hat{f}\|_{L^2(\mathbb{R}^d)}^2.$$

Before proceeding into joint time-frequency representations in Section 2.3, we mention three trade-offs that exist between representing a function in the x or t domains and the ω domain: (1) scaling, (2) smoothness and rate of decay at infinity,

and (3) the uncertainty principle.

From a change-of-variables one sees that scaling the independent variable in one domain by $a > 0$ inverse-scales the independent and dependent variable in the other:

$$\mathcal{F}\{f(ax)\}(\omega) = \int_{\mathbb{R}^d} f(ax) e^{-2\pi i x \cdot \omega} dx = \frac{1}{a} \int_{\mathbb{R}^d} f(u) e^{-2\pi i \frac{u}{a} \cdot \omega} du = \frac{1}{a} \mathcal{F}\{f\}\left(\frac{\omega}{a}\right).$$

Loosely, we flatten and stretch the Fourier transform when contracting the original function and vice-versa. The second trade-off is between the order of its smoothness and how fast its Fourier transform approaches zero as $|\omega| \rightarrow \infty$. This is summarized in Lemma 2.2.7.

Lemma 2.2.7 (Bessel Potential Spaces, Theorem 1.2.3 [47]) $D^\alpha f \in L^2(\mathbb{R}^d)$ for all $|\alpha| < n$, if and only if $\int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 (1 + |\omega|^2)^n d\omega < \infty$.

This is a refinement of the Riemann-Lebesgue Lemma 2.2.1. The Fourier transform must decay to zero and Lemma 2.2.7 states that it must decay like $\frac{1}{|\omega|^{nd}}$. The famous last set of trade-offs between a function and its Fourier transform, and a natural segue into time-frequency analysis, are called *uncertainty principles*. The uncertainty principles state that one cannot concentrate both a function and its Fourier transform simultaneously. Concentration can be measured multiple ways. For example, deviation of the function about its center of mass or the size of its support. The classic uncertainty principle from quantum physics is stated below and is derived from a more general statement involving the commutators of two self-adjoint operators on a Hilbert space, see Lemma 2.2.2 in [47].

Theorem 2.2.8 (Heisenburg-Pauli-Weyl Uncertainty Principle) If $f \in L^2(\mathbb{R}^d)$ and $a, b \in \mathbb{R}$ are arbitrary, then

$$\left(\int_{\mathbb{R}} (x - a)^2 |f(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} (\omega - b)^2 |\hat{f}(\omega)|^2 d\omega \right)^{1/2} \geq \frac{1}{4\pi} \|f\|_{L^2(\mathbb{R})}^2.$$

Each quantity on the left-hand side is a measure of concentration of f and $\hat{f}$ about a and b , respectively. If f is normalized so that $\|f\|_{L^2(\mathbb{R})}^2 = 1$, these concentration

measures are minimized about the center of masses, $a = \bar{x} = \int x|f(x)|^2 dx$ and $b = \bar{\omega} = \int \omega|f(\omega)|^2 d\omega$. Let

$$\Delta_f x := \int_{\mathbb{R}} (x - \bar{x})^2 |f(x)|^2 dx$$

and

$$\Delta_f \omega := \int_{\mathbb{R}} (\omega - \bar{\omega})^2 |\hat{f}(\omega)|^2 d\omega.$$

In signal processing and in music, $\Delta_f x$ is a measure of duration, "essential support" of f , or concentration of f about its center of mass; $\Delta_f \omega$ is a measure of the "essential bandwidth" of f about its average frequency. In probability, $\Delta_f x$ is the standard deviation of a random variable x with distribution $|f|^2$; $\Delta_f \omega$ measures the essential support of its characteristic function. In quantum physics, $\Delta_f x$ measures the precision of the position of a particle with distribution $|f|^2$; $\Delta_f \omega$ measures the precision of the momentum of a particle with distribution $|\hat{f}|^2$. After normalizing the Fourier transform by a factor of $\frac{1}{2\pi}$, the Heisenburg uncertainty principle reads in its simplest and recognizable form,

$$\Delta_f x \Delta_f \omega \geq 1. \tag{2.2}$$

This makes explicit the trade-off in concentration between the two domains. Gröchenig [47] states this as *a realizable signal occupies a region of area at least one in the time-frequency plane*. Analyzing a signal simultaneously in both these domains in the face of the uncertainty principle is a core piece of time-frequency analysis. The short-time Fourier transform seen in Equation (4.1) is a central tool in time-frequency analysis that performs this simultaneous analysis.

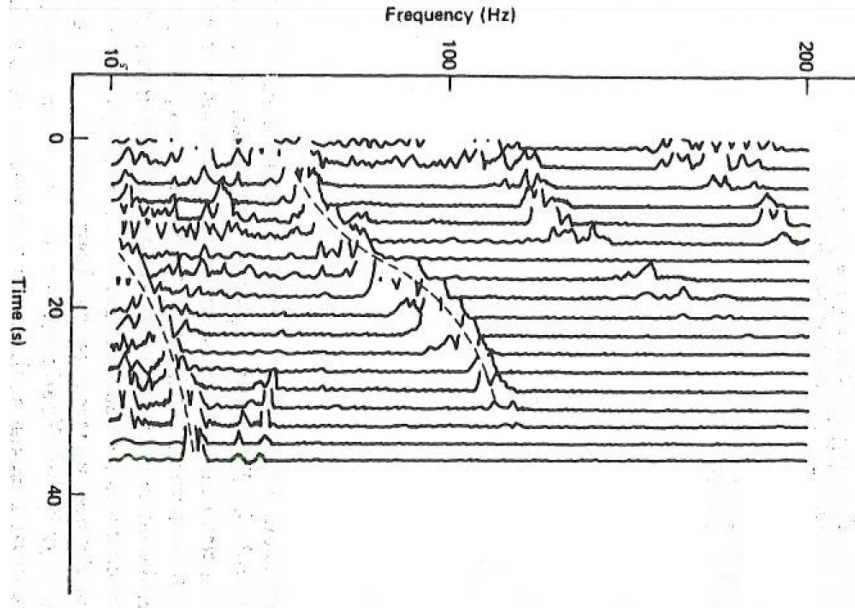


Figure 2.1: A spectrogram visualizing the STFT of a recording of an arriving helicopter [78]. The STFT is a tool in time-frequency analysis that analyzes a function simultaneously in time or space and frequency. The shift towards higher frequencies is the Doppler effect.

2.2.3 Examples: The Gaussian and B-spline

The Gaussian,

$$f(x) := e^{-\pi\|x\|^2/a}, \quad x \in \mathbb{R}^d$$

plays an essential role in Fourier analysis, especially in time-frequency analysis. It minimizes the classic Heisenberg uncertainty principle, is its own Fourier transform, is infinitely smooth and decays to zero rapidly. These latter properties make it a mathematically attractive choice of window to study for the short-time Fourier transform. It also happens to be the functional form of the heat kernel on $\mathbb{R}^d$. In Chapters 3 and 4, we use Gaussian-like heat kernels as a part of a definition (functionally, the window) of a short-time Fourier transform on the non-Euclidean domains of graphs and fractals.

The 2nd-order B-spline

$$g(x) := \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x) * \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x) = \begin{cases} 1 - |x|, & x \in [-1, 1] \\ 0, & x \notin [-1, 1]. \end{cases}$$

(the *hat function*) is a convolution of an indicator function with itself with properties such as continuity and compact support that also make it an attractive window in the short-time Fourier transform. We study the time-frequency shifts of the B-spline and spectral properties of a matrix built from these shifts in Chapter 6 as part of an investigation into sets of lattice parameters that allow reconstruction of a function. Remarkably, we can explicitly compute the Fourier transform of both the Gaussian and the B-spline.

Example 2.2.9 (Fourier Transform of the Gaussian) The Gaussian f is in the Schwartz space $\mathcal{S}(\mathbb{R})$, so Lemma 2.2.5 applies. In dimension $d = 1$,

$$f'(x) = (-2\pi x a)^{-1} f(x).$$

Thus,

$$\begin{aligned} \frac{d}{d\omega} \hat{f}(\omega) &= \widehat{(-2\pi i x f)}(\omega) \\ &= \widehat{ia \frac{d}{dx} f}(\omega) \\ &= ia(2\pi i \omega) \hat{f}(\omega). \end{aligned}$$

This is an ordinary differential equation for the Fourier transform $\hat{f}$, which one can check directly has solution

$$\hat{f}(\omega) = \sqrt{a} e^{-a\pi\omega^2}.$$

When $d > 1$, the Fourier transform factors as $\mathcal{F}(\prod_{i=1}^d f_i(x_i)) = \prod_{i=1}^d \hat{f}_i(\omega_i)$. Therefore, since $f(x) = \prod_{i=1}^d e^{-\pi x_i^2/a}$, $\hat{f}(\omega) = a^{d/2} f(a\omega)$. In particular if $a = 1$, we have the relation

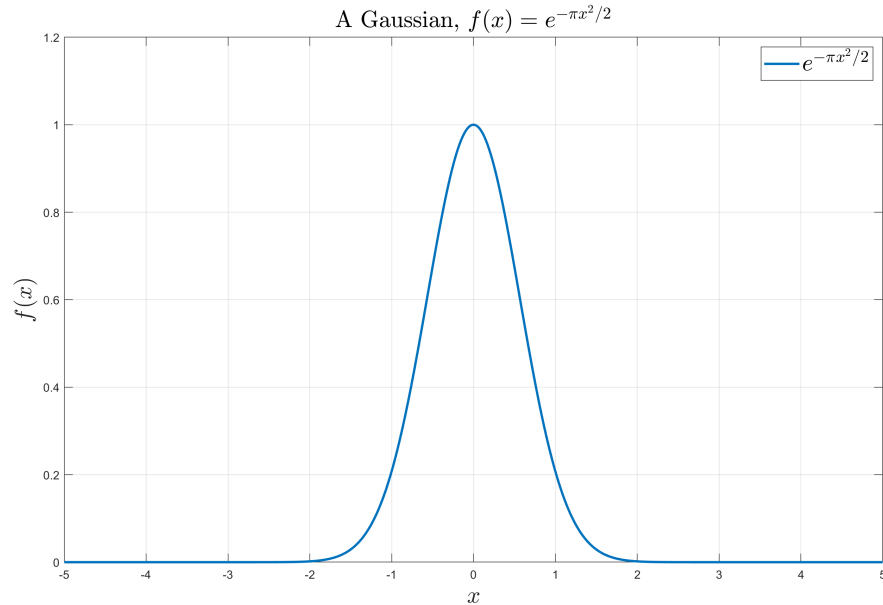
$$\hat{f} = f. \quad \diamond$$

Example 2.2.10 (Fourier Transform of the B-spline) We first show that the Fourier transform of the indicator function $\chi_{[-\frac{1}{2}, \frac{1}{2}]}(x)$ is the $\text{sinc}(\omega) := \frac{\sin(\pi\omega)}{\pi\omega}$. The integral here can be Riemann or Lebesgue (though, it is interesting $\text{sinc}(\omega)$ is not Lebesgue integrable and when integrating it must be viewed as an improper Riemann integral):

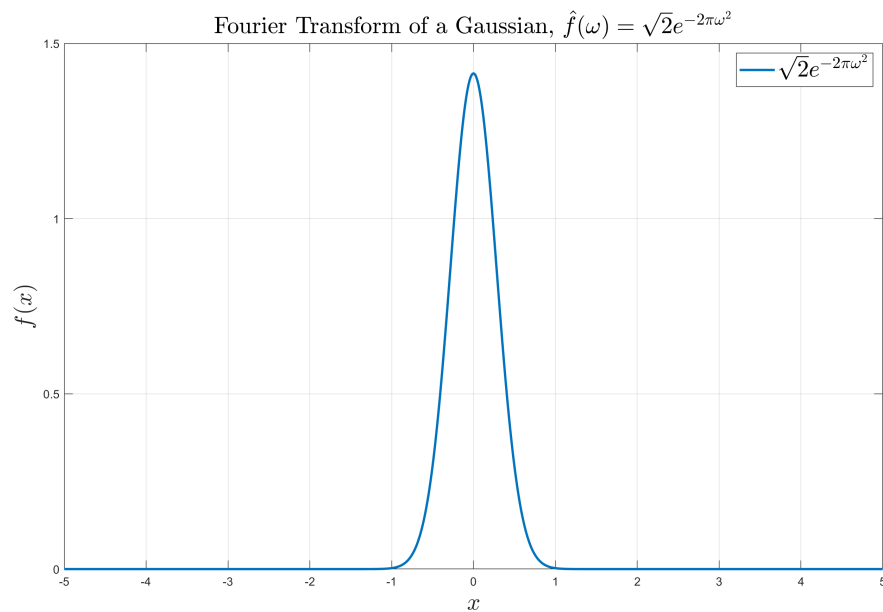
$$\begin{aligned}
 \mathcal{F}\{\chi_{[-\frac{1}{2}, \frac{1}{2}]}(\omega) &= \int_{\mathbb{R}} \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x) e^{-2\pi i x \omega} dx \\
 &= \int_{-1/2}^{1/2} e^{-2\pi i x \omega} dx \\
 &= -\frac{1}{2\pi i \omega} e^{-2\pi i x \omega} \Big|_{x=-1/2}^{x=1/2} \\
 &= -\frac{1}{2\pi i \omega} \left(\cos(-\pi\omega) + i \sin(-\pi\omega) - \cos(\pi\omega) - i \sin(\pi\omega) \right) \\
 &= \frac{\sin(\pi\omega)}{\pi\omega}.
 \end{aligned}$$

The indicator function does not offer good frequency resolution as a window in time-frequency analysis precisely because of the slow decay of $\text{sinc}(\omega)$. The 2nd-order B-spline is the convolution $g(x) := \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x) * \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x)$. Since it and its convolution are $L^1(\mathbb{R})$ functions, we apply Lemma 2.2.4:

$$\mathcal{F}\{\chi_{[-\frac{1}{2}, \frac{1}{2}]} * \chi_{[-\frac{1}{2}, \frac{1}{2}]}(\omega) = \mathcal{F}\{\chi_{[-\frac{1}{2}, \frac{1}{2}]}(\omega) \cdot \mathcal{F}\{\chi_{[-\frac{1}{2}, \frac{1}{2}]}(\omega) = \text{sinc}^2(\omega). \quad \diamond$$

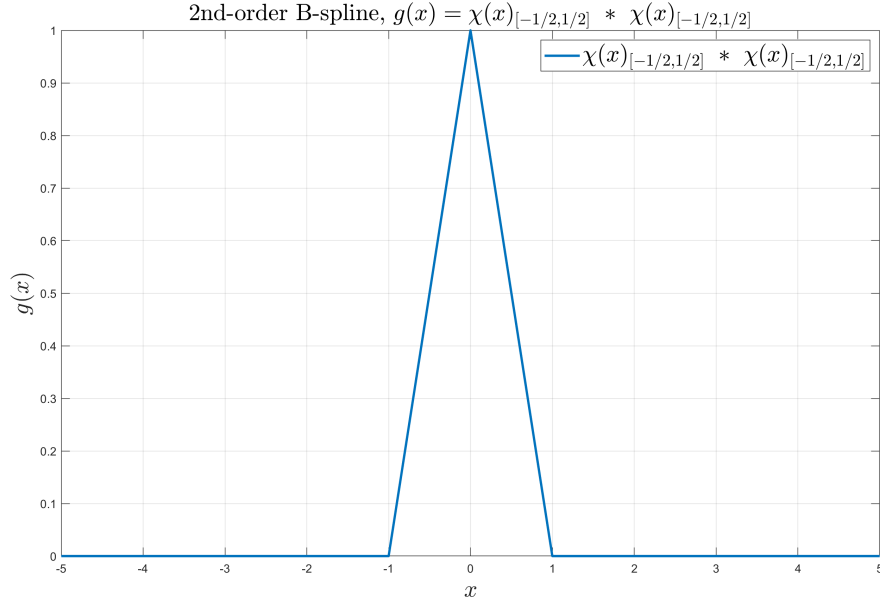


(a) A famous function in mathematics, the Gaussian. It played an essential role in the development of time-frequency analysis. The Gaussian is a classic choice of window in the STFT, minimizes the Heisenberg uncertainty principle, and was explored as an alternative building block to the complex exponentials in a series representation of a function.

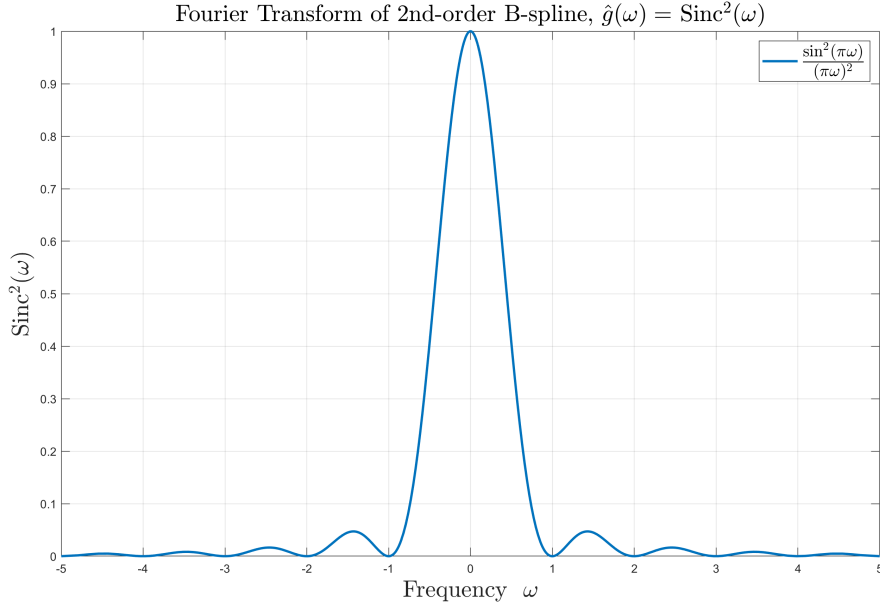


(b) The Fourier transform of a Gaussian is another Gaussian! It is a stationary point of the Fourier transform when scaled appropriately ($a = 1$). Gaussian-like heat-kernels are used as a window for a STFT on non-Euclidean domains in Chapters 3 and 4.

Figure 2.2: Fourier Transform of a Gaussian



(a) The 2nd-order B-spline $g(x)$ (or hat function) is a convolution of two indicator functions. The B-splines are a class of windows for the STFT in $C_c(\mathbb{R})$ whose frame set has not been fully characterized [3]. Chapter 6 investigates spectral properties of a Gram matrix constructed from the time-frequency shifts $\{M_{\ell b}T_{ka}g\}_{\ell, k \in \mathbb{Z}}$.



(b) The Fourier transform of the 2nd-order B-spline is a $\text{sinc}^2(\omega)$. The $\text{sinc}(\omega)$ is a fundamental function in signal processing. Its integer translates form a basis for the Paley-Wiener space of band-limited functions that allow reconstruction of a function from its samples [16].

Figure 2.3: Fourier Transform of a 2nd-order B-spline

2.3 Time-Frequency Analysis

2.3.1 A Musical Score Analogy

A simple and illuminating analogy for time-frequency analysis is the musical score, laid out beautifully in *Uncertainty Principles in Fourier Analysis* [26] and the beginnings of *Foundations of Time-Frequency Analysis* [47]. Let f be a piece of music; $f(x)$, for example, measures the amplitude of the vibrations of a speaker membrane as the music is played through it. Observing the ups-and-downs and vibrational patterns of the membrane lets us discern the rhythm or volume of the music. However, the key or melody is absent from this representation of the music. If the music is represented via its Fourier transform $\hat{f}$, we can study the melody and key. Neither representation captures all the information that make up the music as we hear it and as is processed by our ears and brain. The ear-brain system performs a joint and simultaneous analysis and experience of the time and frequency information of the music. When a composer writes a piece of music via the musical score (Figure 2.4) they are performing a time-frequency analysis: the horizontal axis is time and the vertical axis is frequency. In Figure 2.5 we see two pure tones (cosines of two different frequencies) played back-to-back. In the temporal representation in Figure 2.5a we can discern the volume (amplitude) and exactly when a change in tone occurs. In the frequency representation in Figure 2.5b this information is hard to discern but we can clearly see which two frequencies compose the original signal. In Section 2.3.2, the short-time Fourier transform is introduced to provide a joint time-frequency representation combining aspects of the function and its Fourier transform.

Crumbling Castle
as perf. by King Gizzard & The Wizard Lizard Transcribed by Andre Sauzier

The musical score is a time-frequency representation of the song "Crumbling Castle" by King Gizzard & The Wizard Lizard. It is transcribed by Andre Sauzier. The score is divided into several sections: Intro (measures 1-4), Bass & Drums (measures 5-8), Gtr. & Vox (measures 9-12), Gtr. Only (measures 13-16), Band In (measures 17-20), Drum Breaks (measures 21-24), and Verse (measures 25-27). The lyrics are written below the vocal lines. The score includes various musical notations such as notes, rests, and dynamic markings (e.g., *tr*, *dr*, *etc.*). The horizontal axis represents time (measures) and the vertical axis represents frequency (pitch).

Figure 2.4: The musical score for *King Gizzard and the Lizard Wizard's Crumbling Castle* [79]. The musical score is a time-frequency representation of a piece of music: the horizontal axis represents time and the vertical axis represents frequency (pitch).

2.3.2 Definition and Properties of the Short-time Fourier Transform

The short-time Fourier transform provides a joint time-frequency information of a function and can be viewed as an attempt to capture local frequency information in a way that the classic Fourier transform cannot, illustrated in Figures 2.6a and 2.6b. The uncertainty principle prevents the ideal time-frequency representation which would provide the frequency spectrum at each instant or position x . The short-time Fourier transform $V_g f(x, \omega)$ approximates the strength of frequency ω at instant or position x and jointly analyzes a function in these domains in the face of the uncertainty principle. This is accomplished via the window g that restricts f to an area of its support before performing a Fourier transform. For a non-zero window $g \in L^2(\mathbb{R}^d)$ the *short-time Fourier transform (STFT)* of a function $f \in L^2(\mathbb{R}^d)$ with respect to g is defined as

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i t \cdot \omega} dt, \text{ for } x \text{ and } \omega \in \mathbb{R}^d.$$

The choice of window, especially its function class, smoothness, and decay is a delicate problem that has tremendous consequences on the resolution in the ω -domain and how f is represented by its STFT. When $g \in L^2(\mathbb{R}^d)$, the STFT is defined point-wise, is uniformly continuous on $\mathbb{R}^{2d}$, and obeys the *orthogonality relations* on inner products. These properties and a host of expressions of the STFT are seen in Lemma 2.3.1 and Theorem 2.3.2 .

Lemma 2.3.1 (Faces of the STFT, Chapter 3 [47]) *If $f, g \in L^2(\mathbb{R}^d)$, then $V_g f$ is uniformly continuous on $\mathbb{R}^{2d}$ and*

$$V_g f(x, \omega) = (\widehat{f \cdot T_x g})(\omega) \quad (2.3)$$

$$= \langle f, M_\omega T_x \rangle \quad (2.4)$$

$$= \langle \hat{f}, T_\omega M_{-x} \hat{g} \rangle \quad (2.5)$$

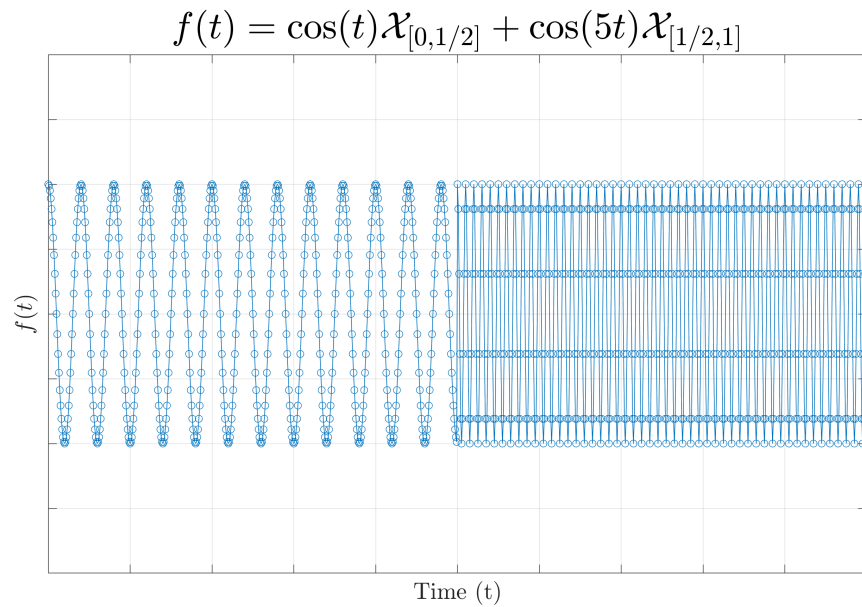
$$= e^{-2\pi i x \cdot \omega} V_{\hat{g}} \hat{f}(\omega, -x) \quad (2.6)$$

$$= e^{-2\pi i x \cdot \omega} (f * M_\omega g^*)(x) \quad (2.7)$$

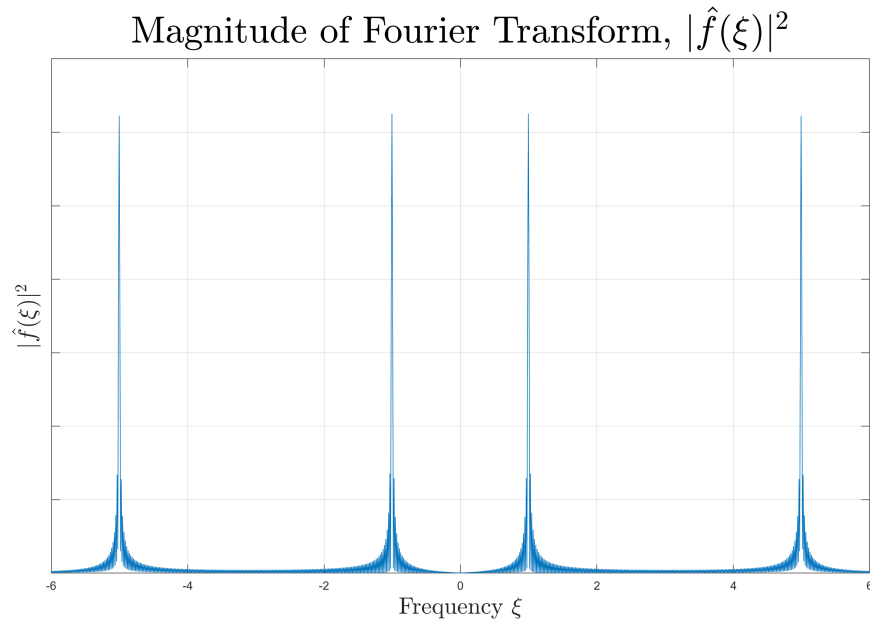
$$= (\hat{f} * M_{-x} \hat{g}^*)(\omega) \quad (2.8)$$

$$= e^{-\pi i x \cdot \omega} \int_{\mathbb{R}^d} f(t + \frac{x}{2}) \overline{g(t - \frac{x}{2})} e^{-2\pi i t \cdot \omega} dt. \quad (2.9)$$

Equation (2.3) expresses the STFT as the Fourier transform of a product of f and a translated window g . Equation (2.4) expresses it as a simple $L^2(\mathbb{R}^d)$ inner product of f with a time-frequency shifted window g . The STFT on graphs defined in Chapter 3 (see Equation (3.1)) is likewise viewed as an inner product between a function f and a "space-frequency" shifted window g . Sometimes referred to as the *fundamental identity of time-frequency analysis*, Equation (2.6) relates the STFT of f with respect to g to the STFT of the Fourier transform $\hat{f}$ with respect to $\hat{g}$. Equation (2.6) states the Fourier transform changes the coordinate system of the time-frequency plane by a rotation of $-\pi/2$.

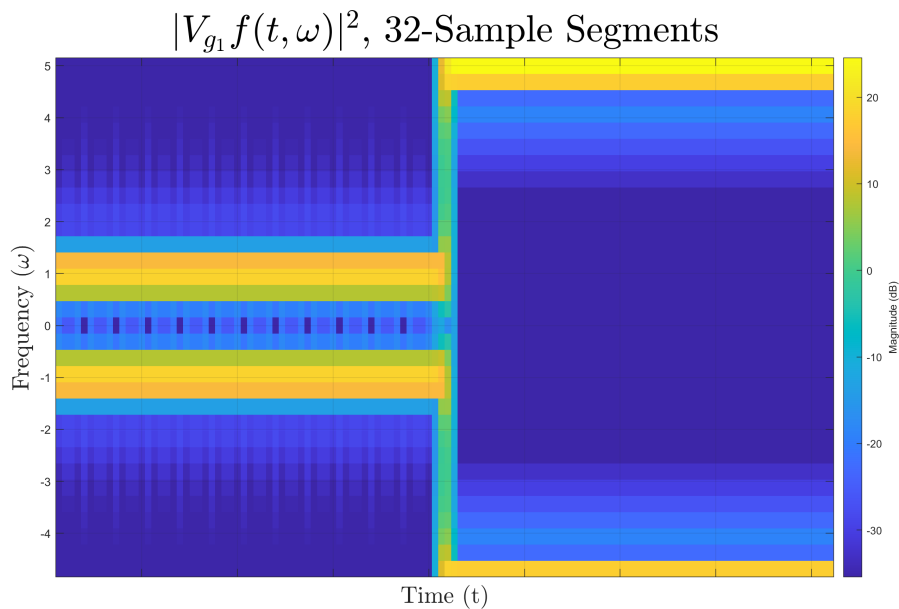


(a) $f(t) = \cos(\alpha_1 t)\chi_{[0,1/2]}(t) + \cos(\alpha_2 t)\chi_{[1/2,1]}(t)$ is a function of two pure tones (frequencies).

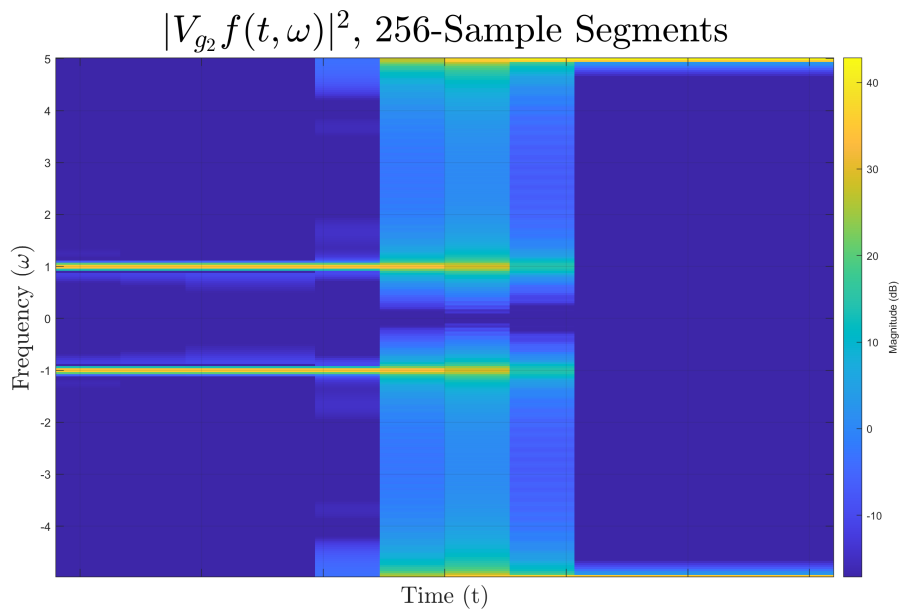


(b) Magnitude-squared of Fourier transform $|\hat{f}(\omega)|^2$.

Figure 2.5: A piecewise cosine $f(t)$ and the magnitude of the Fourier transform $|\hat{f}(\omega)|^2$. The magnitude $|\hat{f}(\omega)|^2$ localizes around the modulus of the constituent frequencies but provides no information on when the frequency change occurs.



(a) Spectrogram $|V_{g_1}f(t, \omega)|^2$. A narrow window has high resolution in the t -domain but lower resolution in the ω -domain.



(b) Spectrogram $|V_{g_2}f(t, \omega)|^2$. A wider window has high resolution in the ω -domain but lower resolution in the t -domain.

Figure 2.6: Spectrograms visualize the magnitude of the STFT $|V_g f(t, \omega)|^2$ and provide joint time-frequency information of $f(t)$. The windowing allows the STFT to determine when the frequency change occurs in $f(t)$ as well as the constituent frequencies in Figure 2.5a. It is still subject to uncertainty-type principles.

Theorem 2.3.2 (Orthogonality Relations for STFT, Chapter 3 [47]) *Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$; then $V_{g_j} f_j \in L^2(\mathbb{R}^{2d})$ for $j = 1, 2$ and*

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \langle g_1, g_2 \rangle_{L^2(\mathbb{R}^d)}.$$

The orthogonality relations express that the STFT is unitary on its range and as a consequence it is similar to Parseval's formula for the Fourier transform. Indeed, taking $g_1 = g_2$ and $f_1 = f_2$ with $\|g_1\| = \|g_2\| = 1$ one sees that the STFT is an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$. In Chapter 4, we construct a STFT on the Sierpinski gasket fractal that is likewise an isometry when using the appropriate measure. We provide a proof of Theorem 2.3.2. It is a simple proof but demonstrates techniques such as Fubini's theorem which we will use relentlessly in Chapter 4 as well as translation invariance of the Lebesgue integral on $\mathbb{R}^d$, which we emphatically do not have in the non-Euclidean cases studied in Chapters 3 and 4.

Proof (Proof of Theorem 2.3.2 [47]): First, assume that the windows $g_j \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, so that $f_j \cdot T_x \overline{g_j} \in L^2(\mathbb{R}^d)$ for all $x \in \mathbb{R}^d$. Apply Parseval to the ω -integral,

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} V_{g_1} f_1(x, \omega) \overline{V_{g_2} f_2(x, \omega)} d\omega dx \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} (f_1 \cdot T_x \overline{g_1})(\omega) \overline{(f_2 \cdot T_x \overline{g_2})(\omega)} d\omega \right) dx \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f_1(t) \overline{f_2(t)} g_1(t-x) \overline{g_2(t-x)} dt \right) dx. \end{aligned}$$

Since $f_1 \overline{f_2} \in L^1(\mathbb{R}^d, dt)$ and $\overline{g_1} g_2 \in L^1(\mathbb{R}^d, dx)$, we apply Fubini to interchange the order of integration:

$$\begin{aligned} \langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^{2d})} &= \int_{\mathbb{R}^d} f_1(t) \overline{f_2(t)} \left(\int_{\mathbb{R}^d} \overline{g_1(t-x)} g_2(t-x) dx \right) dt \\ &= \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \overline{\langle g_1, g_2 \rangle_{L^2(\mathbb{R}^d)}}. \end{aligned}$$

The extension to $g_j \in L^2(\mathbb{R}^d)$ is done by a standard density argument of $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ within $L^2(\mathbb{R}^d)$, □

2.3.3 Inversion

Because V_g is an isometry, we have $V_g f = 0 \implies f = 0$ almost everywhere. This in turn implies that $\{M_\omega T_x g : x, \omega \in \mathbb{R}^d\}$ spans a dense subspace of $L^2(\mathbb{R}^d)$. We'd like to recover f from its STFT V_g in the same spirit that we can invert the ordinary Fourier transform that uses $\hat{f}$ as a weighting function in the continuous superposition of the complex exponentials $e^{2\pi i x \cdot \omega}$ seen in Equation (2.2.6). The invertibility formula for the STFT is derived from the orthogonality relations.

Theorem 2.3.3 (STFT Inversion, Chapter 3 [47]) *If $g, \gamma \in L^2(\mathbb{R}^d)$ and $\langle \gamma, g \rangle_{L^2(\mathbb{R}^d)} \neq 0$, then for all $f \in L^2(\mathbb{R}^d)$*

$$f = \frac{1}{\langle \gamma, g \rangle_{L^2(\mathbb{R}^d)}} \iint_{\mathbb{R}^{2d}} V_g f(x, \omega) M_\omega T_x \gamma d\omega dx. \quad (2.10)$$

The time-frequency shifts $M_\omega T_x \gamma$ are not necessarily in $L^2(\mathbb{R}^{2d})$; however the inner product $\langle M_\omega T_x \gamma, k \rangle$ is just $V_\gamma k(x, \omega) \in L^2(\mathbb{R}^{2d})$. Thus, this inversion formula is *weak* in the sense that one integrates the left- and right-hand sides against an arbitrary $k \in L^2(\mathbb{R}^d)$ as

$$\langle f, k \rangle_{L^2(\mathbb{R}^d)} = \frac{1}{\langle \gamma, g \rangle_{L^2(\mathbb{R}^d)}} \iint_{\mathbb{R}^{2d}} V_g f(x, \omega) \langle M_\omega T_x \gamma, k \rangle_{L^2(\mathbb{R}^d)} d\omega dx = \frac{\langle V_g f, V_\gamma k \rangle_{L^2(\mathbb{R}^{2d})}}{\langle \gamma, g \rangle_{L^2(\mathbb{R}^d)}}.$$

Equation (2.10) uses the STFT $V_g f(x, \omega)$ as a weight for the time-frequency shifts $M_\omega T_x \gamma$ to recover f in the same spirit as the inversion formula for the Fourier transform in Equation (2.2.6).

Proof (Proof of Inversion Formula [47]): Since $V_g f \in L^2(\mathbb{R}^{2d})$ by the orthogonality relations in Theorem 2.3.2, the integral

$$\tilde{f} = \frac{1}{\langle \gamma, g \rangle_{L^2(\mathbb{R}^d)}} \iint_{\mathbb{R}^{2d}} V_g f(x, \omega) M_\omega T_x \gamma dx d\omega$$

is a well-defined function in $L^2(\mathbb{R}^d)$. Using the orthogonality relations again,

$$\langle \tilde{f}, k \rangle_{L^2(\mathbb{R}^d)} = \frac{1}{\langle \gamma, g \rangle_{L^2(\mathbb{R}^d)}} \iint_{\mathbb{R}^{2d}} V_g f(x, \omega) \overline{\langle k, M_\omega T_x \gamma \rangle} dx d\omega$$

$$= \frac{1}{\langle \gamma, g \rangle_{L^2(\mathbb{R}^d)}} \langle V_g f, V_\gamma k \rangle_{L^2(\mathbb{R}^{2d})} = \langle f, k \rangle_{L^2(\mathbb{R}^d)}.$$

Hence, $\tilde{f} = f$ almost everywhere. $\square$

Because the STFT provides joint time-frequency information of a function it is subject to uncertainty principles. Figure 2.6 visualizes this trade-off in concentration via the STFT. In Equation (2.2), the tradeoff in concentration in this uncertainty principle is stated in terms of standard deviations of a function and its Fourier transform. If one were instead to measure the concentration of a function in terms of sets that contain the support, we arrive at *Donoho and Stark's uncertainty principle*: if $f \in L^2(\mathbb{R}^d)$, $\text{supp } f \subset T$ and $\text{supp } \hat{f} \subset \Omega$, then $\mu(T)\mu(\Omega) \geq 1$. Likewise, there exists a simple uncertainty principle for the STFT that describes a trade-off in the size of a subset of its support and how much energy is concentrated in this subset.

Lemma 2.3.4 (Uncertainty Principle for the STFT, Chapter 3 [47]) *Suppose*

$\|f\|_{L^2(\mathbb{R}^d)}^2 = \|g\|_{L^2(\mathbb{R}^d)}^2 = 1$, and $U \subset \mathbb{R}^{2d}$ and $\epsilon > 0$ are such that

$$\iint_U |V_g f(x, \omega)|^2 dx d\omega \geq 1 - \epsilon.$$

Then, $|U| \geq 1 - \epsilon$.

Since $\|V_g f\|_{L^2(\mathbb{R}^{2d})} = 1$, as $\epsilon \rightarrow 0$, U must contain more energy of $V_g f$ and as a consequence U must increase in measure. The lower bound estimate on $|U|$ can be improved to $|U| \geq (1 - \epsilon)^{\frac{p}{p-2}} \left(\frac{p}{2}\right)^{\frac{2d}{p-2}}$ using $L^p(\mathbb{R}^{2d})$ -norm bounds on $V_g f$, see Theorem 3.3.3 in Grochenig [47]. One cannot have an uncertainty principle like this on the Sierpinski gasket in Chapter 4 due to the existence of localized eigenfunctions [86] that concentrate on arbitrarily small sets. However, there exists a *weak uncertainty principle* on the Sierpinski gasket that provides a lower bound on the product of the energy of a function $\mathcal{E}$ and a measure of its variance Var_d weighted by the resistance metric d_R one can place on the Sierpinski gasket [72]. See Section 2.5 for the background on fractals. While Lemma 4.3.1 is not a true uncertainty principle in that the Sierpinski gasket is compact and the lower bound depends on the window

parameter $t > 0$, it nonetheless describes a trade-off between where the STFT V_t concentrates its energy, its support, and the size of the window t . Similarly, in the graph case it is possible for the eigenfunctions of the graph Laplacian to be highly localized making meaningful uncertainty principle statements difficult, see [75].

2.4 Frame Theory

2.4.1 Bases and Frames in Finite Dimensions

In finite dimensions, frames can be viewed as the last step in a sequence of relaxing the constraints of an orthonormal basis,

$$\begin{array}{ccccc} \text{Orthonormal Basis} & \xrightarrow{\quad \underbrace{\quad} \quad} & \text{Basis} & \xrightarrow{\quad \underbrace{\quad} \quad} & \text{Frame.} \\ & \text{Drop Orthonormality} & & \text{Drop Linear Independence} & \end{array}$$

A basis is first introduced in the finite dimensional Hilbert spaces of $\mathbb{R}^N$ or $\mathbb{C}^N$. A *basis* is a finite collection of vectors $\{\varphi_i\}_{i=1}^N$ that are linearly independent and span the space. With a basis there is a unique expression for every vector, $x = \sum_{i=1}^N c_i \varphi_i$. The coefficients can be explicitly computed and tell how much each basis vector contribute to x . Questions about how a transformation (a matrix) act on x are reduced to simpler questions of how a transformation acts on the basis. Furthermore, different bases reveal different characteristics of x . For example, the canonical basis versus the Fourier basis seen in Equation (2.11). In an *orthonormal basis* (ONB) the collection $\{\varphi_i\}_{i=1}^N$ has the additional properties of orthogonality $\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$ and unit norm $\|\varphi_i\| = 1$.

If we stack $\{\varphi_i\}_{i=1}^N$ as columns in a matrix $\varphi \in \mathbb{R}^{N \times N}$, we have the compact representation

$$x = \sum_{i=1}^N \langle x, \varphi_i \rangle \varphi_i = \varphi \varphi^* x = \varphi^* \varphi x,$$

where $\varphi^* \in \mathbb{R}^{N \times N}$ is the (conjugate) transpose. The coefficients for an ONB can be explicitly computed as the inner products $c_i = \langle x, \varphi_i \rangle$. This representation is simply a sum of orthogonal projections of f onto the set $\{\varphi_i\}_{i=1}^N$. An ONB also enjoys properties such as *least squares approximations* when considering subsets of $\{\varphi_i\}_{i=1}^N$ and Parseval, $\|x\|^2 = \sum_{i=1}^N |\langle x, \varphi_i \rangle|^2$. Of course, the standard basis is an ONB for $\mathbb{C}^N$. Another ONB, the *Fourier basis* $\{f_i\}_{i=1}^N$, consists of powers of primitive roots of unity $\omega = e^{2\pi i/N}$,

$$f_i = [1, \omega^i, \omega^{2i}, \dots, \omega^{(N-1)i}]. \quad (2.11)$$

If we stack the f_i^* as rows of a matrix $W_N \in \mathbb{C}^{N \times N}$ and normalize by $\frac{1}{\sqrt{N}}$ we get the unitary change-of-basis matrix, the *discrete Fourier transform (DFT)*, see Equation (2.13). The DFT is indispensable in digital signal processing. This can be seen if we view its action on a sampled function $\mathbf{g}(x_i)$ at $x_i = -a + \frac{i(a+b)}{N}$ for $i = 1, 2, \dots, N$ as a Riemann sum approximation of the Fourier transform integral,

$$\int_{-a}^b g(x) e^{-2\pi i \xi x} dx \approx \frac{a+b}{N} \sum_{i=1}^N g(x_i) e^{-2\pi i \xi x_i} = \frac{1}{\sqrt{N}} W_N^\xi \mathbf{g},$$

where W_N^ξ is the DFT matrix with entries pointwise raised to the ξ -power. The DFT can be studied from an algebraic point-of-view as its columns are the eigenvectors of a representation of the Cayley graph $\mathbb{Z}/N\mathbb{Z}$, [89, Chapter 2]. The relationship between a graph short-time Fourier transform and the algebraic properties of the underlying graph are explored in Chapter 3.

For general bases, orthogonality is relaxed and there is freedom in choosing basis vectors. However, we lose the properties of Parseval and the compact representation of x using only basis vectors. We require another collection, a *dual basis* $\tilde{\varphi} = \{\tilde{e}_i\}_{i=1}^N$, and instead have the representation

$$f = \sum_{i=1}^N \langle \tilde{\varphi}_i, f \rangle \varphi_i = \sum_{i=1}^N \langle \varphi_i, f \rangle \tilde{\varphi}_i.$$

The pair φ and $\tilde{\varphi}$ are *biorthogonal bases* that have the property

$$\langle \varphi_i, \tilde{\varphi}_j \rangle = \delta_{ij}.$$

Indeed, $\tilde{\varphi}^* = \varphi^{-1}$ by the above bi-orthogonality relation, so one can explicitly compute the dual basis once a basis is fixed. We then have the representation,

$$x = \varphi \tilde{\varphi}^* x = \tilde{\varphi} \varphi^* x.$$

Bases have their limitations. There is zero flexibility in the choice of coefficients

$\{c_i\}_{i=1}^N$ and the representation: once the basis is chosen, the coefficients are determined. Bases are fragile: there is zero redundancy built into the system in that adding or subtracting a basis vector destroys the basis. Important operators and systems such as the discrete short-time Fourier transform and wavelets consist of spanning sets that are emphatically not bases (they are not linearly independent) but instead are *frames* for a particular Hilbert space [14].

Example 2.4.1 Start with the standard ONB for $\mathbb{R}^2$ and add a vector: $\varphi = \{\varphi_1, \varphi_2, \varphi_3\} = \{(1, 0)^T, (0, 1)^T, (1, -1)^T\}$. Note $\varphi_3 = \varphi_1 - \varphi_2$, and this set is no longer linearly independent. It spans the space as we can simply exclude the third vector,

$$x = \langle x, \varphi_1 \rangle \varphi_1 + \langle x, \varphi_2 \rangle \varphi_2.$$

Now, add the zero vector:

$$x = \langle x, \varphi_1 \rangle \varphi_1 + \langle x, \varphi_2 \rangle \varphi_2 + (\langle \varphi_1, x \rangle - \langle \varphi_1, x \rangle)(\varphi_1 - \varphi_2)$$

And rearrange as,

$$x = \langle 2\varphi_1, x \rangle \varphi_1 + \langle (-\varphi_1 + \varphi_2), x \rangle \varphi_2 + \langle -\varphi_1, x \rangle (\varphi_1 - \varphi_2).$$

Now, rename the vectors in the inner products:

$$\tilde{\varphi}_1 = 2\varphi_1, \quad \tilde{\varphi}_2 = -\varphi_1 + \varphi_2, \quad \tilde{\varphi}_3 = -\varphi_1.$$

Re-writing x , we can express it as

$$x = \sum_{i=1}^3 \langle \tilde{\varphi}_i, x \rangle \varphi_i,$$

If we collect the vectors in a matrix, we have the two rectangular matrices

$$\varphi = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad \tilde{\varphi} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then, $x = \varphi \tilde{\varphi}^* x = \tilde{\varphi} \varphi^* x$, like the reconstruction for biorthogonal bases except the matrices are rectangular and the columns are linearly dependent. We also see that x has multiple representations in this system. The collection φ is a *frame* for $\mathbb{R}^2$ and $\tilde{\varphi}$ is a corresponding *dual frame* for $\mathbb{R}^2$. $\diamond$

A family $\varphi = \{\varphi_i\}_{i \in I}$ in a finite-dimensional Hilbert space $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$, such that for all $x \in \mathcal{H}$,

$$A\|x\|^2 \leq \sum_{i \in I} |\langle \varphi_i, x \rangle|^2 \leq B\|x\|^2, \quad (2.12)$$

where I is a finite index set. The constants A, B are called *frame bounds* (or the *optimal frame bounds* if they are tightest possible bounds). In finite-dimensional Hilbert spaces, the upper frame inequality is always satisfied. Apply Cauchy-Schwartz,

$$\sum_{i \in I} |\langle \varphi_i, x \rangle|^2 \leq \underbrace{\left(\sum_{i \in I} \|\varphi_i\|^2 \right)}_B \|x\|^2.$$

The lower frame inequality is equivalent to the statement that the family spans the space: if the lower frame bound is equal to zero, then each of the terms in the sum must be zero. Therefore, there exists a $x \in \mathcal{H}$ that is orthogonal to all of the frame vectors $\{\varphi_i\}_{i \in I}$ and therefore not in their span. The lower frame inequality can also be violated if for each integer $m \in \mathbb{Z}$, there is an element $x_m \in \mathcal{H}$ such that $\|x_m\| = 1$ and

$$\sum_{i \in I} |\langle \varphi_i, x_m \rangle|^2 < \frac{1}{m}.$$

Since the sequence is bounded, apply Bolzano-Weierstrass to identify a subsequence $\{x_{m_j}\}_{j=1}^\infty$ that converges to a $x \in \mathcal{H}$. Then,

$$0 = \lim_{j \rightarrow \infty} \sum_{i \in I} |\langle x_{m_j}, \varphi_i \rangle|^2 = \sum_{i \in I} |\langle x, \varphi_i \rangle|^2.$$

Therefore, x is orthogonal to the frame φ , and hence does not span.

Frames can be analyzed through the study of their frame operators. Without loss

of generality, fix the finite-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^N$ and index set $|I| = M$ for some $M \geq N$. To define these operators, let the *analysis operator* $C : \mathbb{C}^N \rightarrow \mathbb{C}^M$ take an element to its sequence of frame coefficients, $Cx = \{\langle x, \varphi_i \rangle\}_{i=0}^{M-1}$. Its adjoint $C^* : \mathbb{C}^M \rightarrow \mathbb{C}^N$ takes a finite sequence to an element of $\mathbb{C}^N$ via $C^* \{c_i\}_{i=0}^{M-1} = \sum_{i=0}^{M-1} c_i \varphi_i$. The *frame operator* $S : \mathbb{C}^N \rightarrow \mathbb{C}^N$ is defined as the composition C^*C , so that

$$Sx = \sum_{i=0}^{M-1} \langle x, \varphi_i \rangle \varphi_i.$$

The frame condition (2.12) can then be re-written in terms of the frame operator as

$$A\|x\|^2 \leq \langle Sx, x \rangle \leq B\|x\|^2.$$

Therefore, the frame bounds $A = \lambda_{\min}$, $B = \lambda_{\max}$ are indeed the extremal eigenvalues of the frame operator S as the quadratic form $\langle Sx, x \rangle$ is extremized by the eigenvectors corresponding to $\lambda_{\min}$ and $\lambda_{\max}$. The *Gram matrix* $G : \mathbb{C}^M \rightarrow \mathbb{C}^M$ is the composition $G = CC^*$ and stores the inner products between the frame elements.

The non-zero eigenvalues of $S \in \mathbb{C}^{N \times N}$ and $G \in \mathbb{C}^{M \times M}$ are identical. If the frame is linearly dependent, the Gram matrix G will have zero as an eigenvalue whereas S is always a positive matrix. If the frame is linearly independent (a basis) then G will be invertible and will not have zero as an eigenvalue. Its spectrum up to multiplicity will be identical to S . In this case, the frame bounds can be computed using the minimum eigenvalue of G [65]. In Chapter 6, we investigate spectral properties of a Gram matrix constructed from frame elements that are time-frequency shifts of the B-spline. The spectrum correspondence between S and G is part of the motivation of the investigation in Chapter 6.

An element $x \in \mathcal{H}$ can have more than one representation or expansion in terms of the frame elements. One choice, the *canonical dual frame* $\tilde{\varphi} := \{\tilde{\varphi}_i\}_{i \in I}$, is defined via the inverse of the frame operator,

$$\tilde{\varphi}_i := S^{-1} \varphi_i, \quad i \in I.$$

It can be shown that indeed $\{\tilde{\varphi}_i\}_{i \in I}$ is a frame for $\mathcal{H}$. A frame is referred to as *tight* if the frame bounds are equal, $A = B$. In this case, the frame operator $S = AI$ is a multiple of the identity. A frame is a *Parseval tight frame* if $A = B = 1$. A frame is called *equiangular* if the frame vector norms are equal to one and $|\langle \varphi_i, \varphi_j \rangle| = C$, for $i \neq j$ and some $C \geq 0$. Tight frames share properties with ONB and have the following attractive features:

1. Parseval tight frames are self-dual:

$$\tilde{\varphi} = S^{-1}\varphi = \frac{1}{A}I\varphi = \varphi.$$

Thus, we have the compact representation, $x = \sum_{i \in I} \langle \varphi_i, x \rangle \varphi_i$.

2. Tight frames preserve the norm (Parseval-Plancharel) by a simple re-scaling of the frame elements:

$$\begin{aligned} \|x\|^2 &= \frac{1}{A} \sum_{i \in I} |\langle x, \varphi_i \rangle|^2 \\ &= \sum_{i \in I} \left| \left\langle \frac{1}{\sqrt{A}} \varphi_i, x \right\rangle \right|^2. \end{aligned}$$

This is equivalent to saying the analysis operator is a scalar multiple of an isometry $\|\varphi^* x\|^2 = A\|x\|^2$, which is equivalent to the matrix $\frac{1}{\sqrt{A}}\varphi$ having orthonormal rows.

3. An optimal condition number in numerical reconstruction: $x = \sum_{i \in I} \langle x, \varphi_i \rangle S^{-1} \varphi_i$ requires computing or estimating the inverse of the frame operator S^{-1} or solving an associated system, the quality and speed of which depends heavily on the condition number $\text{cond}(S) = \lambda_{\max}/\lambda_{\min} = A/B = 1$.
4. *Naimark's Theorem*: A set $\varphi = \{\varphi_i\}_{i \in I}$ in a Hilbert space $\mathcal{H}$ is a Parseval tight frame if and only if there is a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ and an ONB $\{e_i\}_{i \in I}$ for $\mathcal{K}$ so that the orthogonal projection P of $\mathcal{K}$ onto $\mathcal{H}$ satisfies $Pe_i = \varphi_i$, $i \in I$. In fact, any frame can be obtained by projecting a biorthogonal basis from a larger space.

5. The frame bound $A = B$ records the redundancy of the system. Assuming $|I| = M$, then normalizing the tight frame to unit-norm:

$$M = \sum_{i \in I} \|\varphi_i\|^2 = \text{Trace}(G) = \text{Trace}(S) = NA,$$

hence $A = \frac{M}{N}$. As such, one way to quantify tightness is the *Frobenius norm* of the operator $\|S - \frac{M}{N}I\|_F$:

$$\begin{aligned} 0 &\leq \left\| \varphi\varphi^* - \frac{M}{N}I_{N \times N} \right\|_F^2 := \text{Trace}\left((S - \frac{M}{N}I_{N \times N})^2\right) \\ &= \text{Trace}(S^2) - \frac{2M}{N}\text{Trace}(S) + \frac{M^2}{N^2}\text{Trace}(I_{N \times N}) \\ &= \text{Trace}(G^2) - \frac{2M}{N}\text{Trace}(G) + \frac{M^2}{N^2} \\ &= \sum_{i,j \in I} |\langle \varphi_i, \varphi_j \rangle|^2 - \frac{M^2}{N^2} \\ &= \sum_{j \neq i} |\langle \varphi_i, \varphi_j \rangle|^2 - M\left(\frac{M}{N} - 1\right) \\ &\leq M(M-1) \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2 - M\left(\frac{M}{N} - 1\right). \end{aligned}$$

This shows that measuring tightness and the equiangularity of a frame corresponds to the *coherence* of system, $\max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2$. This is a problem of *line packing*: $|\langle \varphi_i, \varphi_j \rangle| = \cos \theta_{ij}$ and typically we seek to minimize this quantity so that the minimum pairwise angle between two lines is as large as possible. The coherence is minimized when the frame is an equiangular tight frame so that $\|\varphi\varphi^* - \frac{M}{N}I\|_F^2 = 0$ and $|\langle \varphi_i, \varphi_j \rangle|^2$ is constant for all $i \neq j$.

In Section 3.3 we consider when the underlying graph structure determines if a system of graph Gabor frames are tight frames. For further work on the fundamentals of frame theory and tight frames, see [14, 55].

Example 2.4.2 (Peres-Wooter or Mercedes-Benz Frame) This frame is a equiangular tight frame in $\mathbb{R}^2$ consisting of three vectors,

$$\varphi = \{\varphi_1, \varphi_2, \varphi_3\} = \{(0, 1)^T, (-\frac{\sqrt{3}}{2}, -\frac{1}{2})^T, (\frac{\sqrt{3}}{2}, -\frac{1}{2})^T\}.$$

Stacking the frame vectors into a matrix,

$$\varphi = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

It is a frame for $\mathbb{R}^2$ since it is a spanning set. We calculate the frame operator S ,

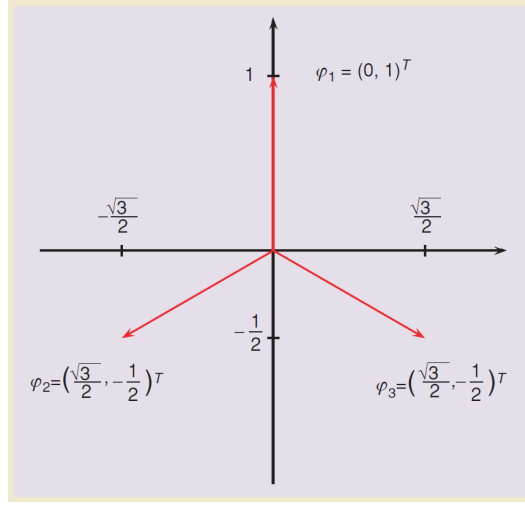


Figure 2.7: The Peres-Wootter (Mercedes-Benz) frame is a tight equiangular frame for $\mathbb{R}^2$ [65].

the Gram matrix G , and the canonical dual frame via S^{-1} :

$$S = \varphi\varphi^* = \frac{3}{2}I, \quad \tilde{\varphi} = S^{-1}\varphi = \frac{2}{3}\varphi, \quad G = \varphi^*\varphi = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}.$$

The frame operator is a scalar multiple of the identity as the frame is tight. The scalar $\frac{3}{2}$ encodes the redundancy of the frame as there are three frame vectors in a two-dimensional space. The off-diagonal entries $G_{ij} = \langle \varphi_i, \varphi_j \rangle$ for $i \neq j$ all have the same magnitude, hence it is an equiangular frame. If we rescale the frame vectors by $\frac{1}{\sqrt{A}} = \sqrt{\frac{2}{3}}$, the frame is now Parseval:

$$\varphi = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}.$$

We have $x = \varphi\varphi^*x$ and any element can be expressed as the linear combination

$$x = \langle (0, \sqrt{\frac{2}{3}}), x \rangle (0, \sqrt{\frac{2}{3}}) + \langle (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}), x \rangle (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}) + \langle (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}), x \rangle (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}).$$

We see Naimark's theorem in action as this is the projection of the following ONB for $\mathbb{R}^3$,

$$\Psi = \begin{bmatrix} 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

with projection matrix

$$P = \frac{1}{\sqrt{3}} \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{2} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

In the presence of additive noise (corrupt coefficients $\langle \varphi_i, x \rangle$ with Gaussian white noise), the mean-squared error (MSE) is reduced using the Mercedes-Benz frame [65]. With an ONB if the MSE is σ^2 , the MSE with the Mercedes-Benz frame is $\frac{2}{3}\sigma^2$. Further, if one of the system vectors is erased, say φ_2 , we still have construction (a representation) of x which would not be possible with an ONB:

$$x = \langle \varphi_1, x \rangle \tilde{\varphi} + \langle \varphi_3, x \rangle \tilde{\varphi}, \text{ where } \tilde{\varphi}_1 = (\frac{1}{\sqrt{3}}, 1), \tilde{\varphi}_3 = (\frac{2}{\sqrt{3}}, 0).$$

The MSE averaged over all possible erasures is minimized when the original frame is tight [65]. This is a crucial property in error-correction and communications. $\diamond$

We recall key definitions and concepts from finite-dimensional Fourier analysis focusing on the discrete Fourier transform, the discrete short-time Fourier transform (DSTFT), and associated Gabor frames for $\mathbb{C}^N$. This provides the background for Chapter 3. The corresponding operators defined on graphs in Section 3.2 will coincide with those defined here when the underlying graph is the Cayley graph of $\mathbb{Z}/N\mathbb{Z}$, the *circle graph* or *ring graph* on N vertices. This is due to the eigenvector structure of the graph Laplacian on the circle graph whose entries are powers of primitive roots

of unity [89, Chapter 2]. The graph short-time Fourier transform and associated graph Gabor frame in Chapter 3 can therefore be viewed as generalizing the DSTFT and Gabor frames for $\mathbb{C}^N$.

Given a vector $f \in \mathbb{C}^N$, the *discrete Fourier transform* $DFT : \mathbb{C}^N \rightarrow \mathbb{C}^N$ is defined pointwise as

$$DFT\{f\}(m) = \hat{f}(m) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n) e^{-2\pi i n m / N}, \quad m = 0, \dots, N-1. \quad (2.13)$$

The DFT can also be expressed in matrix notation. Let the *DFT matrix* $W_N \in \mathbb{C}^{N \times N}$ be defined as

$$W_N = \left(\frac{1}{\sqrt{N}} \omega^{-rs} \right)_{r,s=0}^{N-1}, \quad \omega = e^{2\pi i / N}.$$

Then, the DFT can be expressed as the matrix-vector product $\hat{f} = W_N f$. The DFT shares fundamental properties with the classical Fourier transform defined on functions in $L^2(\mathbb{R}^d)$ such as an inversion (reconstruction) formula, Parseval-Plancherel, Poisson summation formulas, and uncertainty principles [89, Chapters 12, 14]. These can be derived from the fact that the normalized harmonics $\frac{1}{\sqrt{N}} e^{2\pi i n \cdot / N}$ for $n = 0, \dots, N-1$ form an orthonormal basis for $\mathbb{C}^N$:

$$f(m) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \hat{f}(n) e^{2\pi i n m / N}.$$

The inversion formula uses the DFT as coefficients or weights on the basis elements in reconstructing f . The DSTFT localizes the Fourier transform and provides information on constituent frequencies that make up a function and where those frequencies occur. This is accomplished via the composition of a translation operator and a modulation operator acting on a window function. The *cyclic shift operator* $T : \mathbb{C}^N \rightarrow \mathbb{C}^N$ is given by

$$Tf = T(f(0), f(1), \dots, f(N-1))^T = (f(N-1), f(0), f(1), \dots, f(N-2))^T.$$

Then, for $k \in \{0, 1, \dots, N-1\}$ the *translation operator* T_k is given by

$$T_k f(n) := T^k f(n) = f(n-k), \quad n = 0, 1, \dots, N-1.$$

The *modulation operator* $M_\ell : \mathbb{C}^N \rightarrow \mathbb{C}^N$ for $\ell = 0, 1, \dots, N-1$ is given by pointwise multiplication with harmonics $e^{2\pi i \ell \cdot / N}$,

$$M_\ell f = (e^{2\pi i \ell 0 / N} f(0), e^{2\pi i \ell 1 / N} f(1), \dots, e^{2\pi i \ell (N-1) / N} f(N-1)).$$

Translation operators are also called time-shift operators. Likewise, modulation operators are called frequency-shift operators and much of time-frequency analysis is based on the interplay between these two operators, related by

$$DFT M_\ell = T_\ell DFT.$$

The *time-frequency operator* $\pi(k, \ell) : \mathbb{C}^N \rightarrow \mathbb{C}^N$ takes $f \mapsto \pi(k, \ell)f = M_\ell T_k f$. The *discrete short-time Fourier transform (DSTFT)* $V_g : \mathbb{C}^N \rightarrow \mathbb{C}^{N \times N}$ with respect to a *window* $g \in \mathbb{C}^N \setminus \{0\}$ is then given by

$$V_g f(k, \ell) = \langle f, \pi(k, \ell)g \rangle = \sum_{n=0}^{N-1} f(n) \overline{g(n-k)} e^{-2\pi i \ell n / N}. \quad (2.14)$$

The set $(g, \Gamma) = \{\pi(k, \ell)g\}_{(k, \ell) \in \Gamma}$ for $\Gamma = \{0, 1, \dots, N-1\}^2$ is called the *full Gabor system* generated by the window g and the set Γ . The DSTFT is thus an inner product between f and an element of a Gabor system. The following reconstruction (inversion) formula holds for each $f \in \mathbb{C}^N$:

$$f(n) = \frac{1}{N \|g\|^2} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} V_g f(k, \ell) g(n-k) e^{-2\pi i \ell n / N}.$$

In fact, the Gabor system (g, Γ) is a tight frame [14]. Like inversion with the DFT, the inversion formula here uses the DSTFT as weights on the Gabor frame elements when reconstructing f .

We take a brief detour into a connection between algebraic graphs and finite

frame theory. *Equiangular tight frames* (ETFs) for $\mathbb{R}^N$ are in 1-1 correspondence with a subclass of graphs from algebraic graph theory called *strongly regular graphs* defined below and in Section 3.3. A set of lines in $\mathbb{R}^N$ can be represented by a set of unit vectors $\varphi = \{\varphi_1, \dots, \varphi_M\}$. If φ contains an equiangular set of lines, then

$$\varphi_i^T \varphi_j = \pm \cos \theta, \quad i \neq j.$$

Recall $G = \varphi^* \varphi \in \mathbb{R}^{M \times M}$ is the symmetric positive semi-definite Gram matrix of inner products between the vectors $\varphi_i \in \varphi$. It will have the same rank N as φ . Conversely, given a symmetric positive semi-definite matrix of specified rank we can find a matrix φ such that $G = \varphi^* \varphi$. For a set of equiangular lines, G has the special form

$$G = I + \alpha S$$

where S (a *Seidel matrix*) is a symmetric matrix with entries in $0, \pm 1$ [41, Chapter 11]. If we interpret 1 as adjacent and -1 as non-adjacent, we can build an adjacency matrix for a graph as

$$A = (J - I - S)/2$$

where J is the all 1's matrix. If we start with a graph on M vertices and form the Seidel matrix, then since the trace is zero and $S \neq 0$, the smallest eigenvalue of S is negative, $\lambda_{\min} = -\alpha$. Thus,

$$I + \frac{1}{\alpha} S$$

is a symmetric positive semi-definite matrix. If its rank is N , then it is the Gram matrix of a set of equiangular lines in $\mathbb{R}^N$ with $\pm \cos \theta = \frac{1}{\alpha}$ [41, Chapter 11]. Thus, the geometric problem of finding the smallest N such that there are M equiangular lines in $\mathbb{R}^N$ is equivalent to the graph-theoretic problem of finding graphs Γ on M vertices such that the multiplicity of $\lambda_{\min}$ is as large as possible.

As we saw above, the Gram matrix G of an ETF can be converted to the adjacency matrix of a graph. Also since $G^2 = \alpha G$, it is reasonable to believe A also satisfies a quadratic relationship such as the one defining a strongly regular graph.

This implies the graph possesses certain symmetries. A graph $\Gamma = (V, E)$ is *strongly regular* with non-negative parameters (v, k, λ, μ) if any two adjacent vertices have exactly λ neighbors in common while any two non-adjacent vertices have μ neighbors in common. A strongly regular graph can be characterized algebraically via the relation

$$A^2 = (\lambda - \mu)A + (k - \mu)I + \mu J.$$

That is, it suffices to show there exist integers x, y, z such that $A^2 = xA + yI + zJ$. This is due to the fact that A^2 counts the number of two-step paths from vertex i to vertex j :

$$A^2(i, j) = \begin{cases} k, & i = j \\ \lambda, & i \neq j, A_{ij} = 1 \\ \mu, & i \neq j, A_{ij} = 0. \end{cases}$$

A SRG can also be characterized by its spectrum and in fact any connected regular graph with exactly three eigenvalues is a strongly regular graph [41, Chapter 10]. The eigenvalues and their multiplicities are determined completely by the parameters of the SRG.

Establishing the existence of a specified number of equiangular lines is much more difficult in $\mathbb{C}^N$. It is known that for a set of equiangular lines to exist in $\mathbb{C}^N$, one must have $M \leq N^2$. It is of interest in quantum computing [7, 20] where the following conjecture was stated about a maximal packing of N^2 lines in $\mathbb{C}^N$.

Theorem 2.4.3 (The SIC-POVM/Zauner Conjecture, [76]) *For each dimension N there exists N^2 equiangular lines in $\mathbb{C}^N$. That is, there exists an equiangular tight frame with N^2 frame vectors.*

2.4.2 Hilbert Spaces and Related Operators

We review basic definitions, operators, and results in Hilbert spaces. Recall that a *Banach space* $\mathcal{X}$ is a complete normed space and a *Hilbert space* $\mathcal{H}$ is a Banach space whose norm is derived from an inner product. For $1 \leq p < \infty$, the Banach space

$L^p(\mathbb{R}^d)$ is defined as

$$L^p(\mathbb{R}^d) := \{f : \mathbb{R}^d \rightarrow \mathbb{C} \mid f \text{ Lebesgue measurable and } \int_{\mathbb{R}^d} |f(x)|^p dx < \infty\}.$$

For exposition on classical measure theory and the Lebesgue measure, see Folland [31] or Stein and Shakarchi [84]. The norm on $L^p(\mathbb{R}^d)$ is

$$\|f\|_{L^p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{1/p}.$$

We will be working with the Hilbert space $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d, \lambda)$ equipped with the Lebesgue measure λ and $L^2(SG) = L^2(SG, \mu)$ where SG is the Sierpinski gasket equipped with the self-similar measure μ , see Section 2.5. The inner product on $L^2(\mathbb{R}^d)$ is the integral,

$$\langle f, g \rangle_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x) \overline{g(x)} dx.$$

On SG , the inner product is defined likewise using the self-similar measure μ . The discrete counterpart to the $L^p(\mathbb{R}^d)$ spaces are the spaces of p -summable sequences, denoted $\ell^p(I)$, where I is a countable index set:

$$\ell^p(I) := \left\{ \{x_i\}_{i \in I} \mid x_i \in \mathbb{C}, \sum_I |x_i|^p < \infty \right\}.$$

When $1 \leq p < \infty$, these are Banach spaces with norm

$$\|\{x_i\}_{i \in I}\|_{\ell^p} := \left(\sum_{i \in I} |x_i|^p \right)^{1/p}.$$

When $p = 2$, it is a Hilbert space with inner product

$$\langle \{x_i\}_{i \in I}, \{y_i\}_{i \in I} \rangle := \sum_{i \in I} x_i \overline{y_i}.$$

The $L^\infty(\mathbb{R}^d)$ and $\ell^\infty(I)$ are the spaces of (essentially) bounded functions, equipped with the supremum norms. The Hilbert spaces $L^2(\mathbb{R}^d)$ and $L^2(SG)$ in this thesis

are *separable*: there exists a sequence $\{\varphi_i\}_{i=1}^{\infty}$ such that for each $f \in \mathcal{H}$ and each $\epsilon > 0$, there is a $\varphi_{i'}$ such that

$$\|f - \varphi_{i'}\| < \epsilon.$$

Roughly, we can approximate f arbitrarily well with the sequence $\{\varphi_i\}_{i=1}^{\infty}$. The infinite series $\sum_{i=1}^{\infty} c_i \varphi_i$ is *convergent* with sum $f \in \mathcal{H}$ if

$$\|f - \sum_{i=1}^n c_i \varphi_i\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

The *range* of an operator $U : \mathcal{H} \rightarrow \mathcal{H}$ is the subspace of the co-domain that is hit by the outputs,

$$\text{Range}(U) := \{y \in \mathcal{H} : y = U(x), x \in \mathcal{H}\}.$$

The *nullspace* or *kernel* of $U : \mathcal{H} \rightarrow \mathcal{H}$ is the subspace of the domain that is mapped to zero,

$$\text{Ker}(U) := \{x \in \mathcal{H} : U(x) = 0\}.$$

A subspace $V \subset \mathcal{H}$ is *closed* if it contains all its limit points. The *closure* $\overline{V}$ is the subspace V unioned with its limit points. The *orthogonal complement* $V^{\perp}$ is the space

$$V^{\perp} := \{w \in \mathcal{H} : \langle w, v \rangle = 0, \forall v \in V\}.$$

If $V \subset \mathcal{H}$ is closed, we can decompose $\mathcal{H}$ as

$$\mathcal{H} = V \oplus V^{\perp},$$

meaning every element $x \in \mathcal{H}$ can be written uniquely as $x = v + w$, where $v \in V$ and $w \in V^{\perp}$. The *adjoint* of a bounded operator $U : \mathcal{H} \rightarrow \mathcal{H}$ is the unique operator $U^* : \mathcal{H} \rightarrow \mathcal{H}$, such that

$$\langle Ux, y \rangle = \langle x, U^*y \rangle, \forall x, y \in \mathcal{H}.$$

A bounded operator $U : \mathcal{H} \rightarrow \mathcal{H}$ is called *unitary* if $UU^* = U^*U = I$. When U is unitary it preserves inner products

$$\langle Ux, Uy \rangle = \langle x, y \rangle, \quad \forall x, y \in \mathcal{H}$$

and the adjoint is its inverse, $U^{-1} = U^*$. U is a *self-adjoint* operator if $U = U^*$. In this case the inner product $\langle Ux, x \rangle$ is always real and the operator norm can be calculated explicitly as

$$\|U\| = \sup_{\|x\|=1} |\langle Ux, x \rangle|.$$

A Hilbert-Schmidt operator $U : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded operator that has finite Hilbert-Schmidt norm,

$$\|U\|_{HS} := \sum_{i \in I} \|Ue_i\|_{\mathcal{H}}^2,$$

where $\{e_i\}_{i \in I}$ is any orthonormal basis for the (separable) Hilbert space, see Section 2.4.3. Hilbert-Schmidt operators are a type of compact operator. Recall if X and Y are Banach spaces and $T : X \rightarrow Y$ is linear, then T is *compact* if whenever $\{x_n\}_{n=1}^\infty$ is a bounded sequence in X , then $\{Tx_n\}_{n=1}^\infty$ has a convergent subsequence in Y . A great property of compact operators is that they can be approximated by finite-rank operators.

Theorem 2.4.4 (Compact Operator Approximation, Theorem 4.11 [67]) *If a bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is a compact operator on Hilbert space $\mathcal{H}$ having a countable orthonormal basis, there exists finite rank operators T_n with $\|T - T_n\| \rightarrow 0$; that is, every compact operator is a limit of finite rank operators.*

The finite rank operators are projections $T_n = P_n T$ onto the span of the first n basis vectors $\{e_1, e_2, \dots\}$. These types of projection operators are used in the *finite section method* discussed in Chapter 6 to try and approximate an infinite matrix with a finite one. This method appeals to pointwise convergence of operators, a weaker notion than operator (uniform) convergence. A sequence of operators $\{U_n\}_{n=1}^\infty$ on $\mathcal{H}$

is said to converge *uniformly* to $U : \mathcal{H} \rightarrow \mathcal{H}$ if

$$\lim_{n \rightarrow \infty} \|U_n - U\| = 0.$$

The sequence is said to converge *pointwise* or strongly if

$$\lim_{n \rightarrow \infty} U_n x = Ux, \quad \forall x \in \mathcal{H}.$$

The sequence converges *weakly* if

$$\lim_{n \rightarrow \infty} \langle y, U_n x \rangle = \langle y, Ux \rangle, \quad \forall y \in \mathcal{H}.$$

If the compact operator is self-adjoint then the spectrum is characterized via a spectral theorem. If $U : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded linear operator, the set of $\lambda \in \mathbb{C}$ for which $U - \lambda I$ is not invertible is the *spectrum* of U ,

$$\text{Spec}(U) := \{\lambda \in \mathbb{C} : U - \lambda I \text{ is singular}\}.$$

The spectrum can be divided up in many ways. $U - \lambda I$ fails to be invertible if $U - \lambda I$ is not injective or not surjective. If it is not injective, then there is a $x \neq 0$ in $\text{Ker}(U)$ and

$$Ux = \lambda x, \quad \text{for some } \lambda \in \mathbb{C}.$$

The λ is an *eigenvalue* of U and x is the associated *eigenvector*. If $U - \lambda I$ fails to be surjective, then the *continuous spectrum* is

$$\{\lambda \in \mathbb{C} : \overline{\text{Range}(U - \lambda I)} = \mathcal{H}\}$$

and the *residual spectrum* is

$$\{\lambda \in \mathbb{C} : \overline{\text{Range}(U - \lambda I)} \subsetneq \mathcal{H}\}.$$

There is a common division among the eigenvalues into the *point spectrum* consisting of all eigenvalues and the discrete spectrum consisting of the eigenvalues that are isolated and of finite multiplicity. The complement of the discrete spectrum is often referred to as the *essential spectrum*. The following theorems characterize the spectrum of a self-adjoint operator and a compact self-adjoint operator.

Theorem 2.4.5 (Spectrum of Self-Adjoint Operators, Theorem 4.20 [67])

If $U : \mathcal{H} \rightarrow \mathcal{H}$ is a self-adjoint, then every eigenvalue of U is real and the eigenvectors for distinct eigenvalues are orthogonal. Furthermore, the residual spectrum is empty.

Theorem 2.4.6 (Spectral Theorem, Theorem 4.24, [67]) *Let $U \neq 0$ be a compact linear self-adjoint operator on $\mathcal{H}$. There exists a finite or countably finite orthonormal set $\{g_n\}$ of eigenvectors of U , with corresponding real eigenvalues $\{\lambda_n\}$, such that*

$$Ux = \sum_n \lambda_n \langle x, g_n \rangle g_n.$$

If $\{\lambda_n\}$ is infinite, then $\lambda_n \rightarrow 0$.

Two fundamental inequalities in Hilbert spaces are the *triangle inequality* and the *Cauchy-Schwartz inequality*. The triangle inequality bounds the norm of a sum of two elements,

$$\|f + g\| \leq \|f\| + \|g\|. \quad (2.15)$$

Geometrically, the longest side of a triangle in $\mathcal{H}$ cannot be longer than the sum of the two short sides. Equality is achieved via the Pythagorean theorem when the triangle is a right triangle, $\langle f, g \rangle = 0$. The Cauchy-Schwartz inequality bounds the magnitude of the inner product as

$$|\langle f, g \rangle| \leq \|f\| \|g\|. \quad (2.16)$$

If we normalize $\|g\| = 1$ and view the inner product as the magnitude of the projection of one element onto another, Cauchy-Schwartz states the projection cannot be larger than the length of f . Equality is achieved when f and g are parallel, $f = \alpha g$, $\alpha \in \mathbb{C}$.

A classic result in analysis is the *Fubini-Tonelli* theorem. This allows one to calculate an integral on a product space in terms of integrals over factors of the product space regardless of the order of integration. We use Fubini-Tonelli in Chapter 4 when deriving an explicit left-inverse for the STFT on the Sierpinski gasket. Recall that if X is a set and $\mathcal{M}$ is a subset of the power set, then $(X, \mathcal{M})$ is called a *measurable space*. If $\mu : \mathcal{M} \rightarrow [0, \infty]$ is a measure on $(X, \mathcal{M})$, then the triple $(X, \mathcal{M}, \mu)$ is called a *measure space*. If $X = \cup_{i=1}^{\infty} E_i$ where $E_i \in \mathcal{M}$ and each $\mu(E_i) < \infty$, $(X, \mathcal{M}, \mu)$ is a *σ -finite measure space*. The proof of Fubini-Tonelli and additional measure theory background can be found in Folland [31].

Theorem 2.4.7 (Fubini-Tonelli) *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ -finite measure spaces.*

1. (Tonelli) *If $f \in L^+(X \times Y)$, then the functions $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$ are in $L^+(X)$ and $L^+(Y)$, respectively, and*

$$\int f d(\mu \times \nu) = \int \left[\int f(x, y) d\nu(y) \right] d\mu(x) \quad (2.17)$$

$$= \int \left[\int f(x, y) d\mu(x) \right] d\nu(y) \quad (2.18)$$

2. (Fubini) *If $f \in L^1(\mu \times \nu)$, then $f_x \in L^1(\nu)$ for almost every $x \in X$, $f^y \in L^1(\mu)$ for almost every $y \in Y$, the almost everywhere defined functions $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$ are in $L^1(\mu)$ and $L^1(\nu)$, respectively, and Equation (2.17) holds.*

2.4.3 Frame Theory in Infinite Dimensions

The inversion formula in Equation (2.10) expresses a function f as a continuum superposition of time-frequency shifts of a window γ using the STFT $V_g f$ as a weighting function. However, $L^2(\mathbb{R}^d)$ is a separable Hilbert space. Therefore, we hope that a series expansion in terms of a countable number of time-frequency shifts is sufficient to represent or reconstruct a function from its STFT $V_g f$. In practice one needs to discretize, sample, and approximate these function representations regardless. One

approach would be to discretize the integral and approximate via a Riemann-type sum,

$$f \approx \sum_{k, \ell \in \mathbb{Z}^d} \langle f, M_{\ell b} T_{ka} \gamma \rangle M_{\ell b} T_{ka} g. \quad (2.19)$$

Alternatively, interpreting $|V_g f(x, \omega)|^2$ as the energy in a time-frequency cell located around $(x, \omega) \in \mathbb{R}^{2d}$, we hope to capture all the energy (recall Theorem 2.3.2) by sampling finely enough in this domain. This asks sampling of the STFT $V_g f$ to satisfy the two following inequalities,

$$A \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \sum_{\ell, k \in \mathbb{Z}^d} |V_g f(\ell b, ka)|^2 \leq B \|f\|_{L^2(\mathbb{R}^d)}^2 \quad (2.20)$$

Roughly, the sampling operator is bounded above therefore continuous if the upper inequality is satisfied. It is bounded below, therefore injective, and f can be recovered from the samples if the lower inequality is satisfied. If the *time-frequency lattice* parameters $a, b > 0$ that control the resolution are chosen so that the inequalities in Equation (2.20) are satisfied for all $f \in L^2(\mathbb{R}^d)$, the sequence $\{M_{\ell b} T_{ka} g\}_{\ell, k \in \mathbb{Z}^d}$ is said to form a *frame* for $L^2(\mathbb{R}^d)$. As a consequence, one has the representation of type seen in Equation (2.19). Frames constructed from time-frequency shifts of a window g are called *Gabor frames*.

The concepts of linear independence, span, basis, and vector (function) representation are delicate in the infinite-dimensional setting. Like in the finite dimensional setting, we'd like a representation of $f \in \mathcal{H}$ of the form,

$$f = \sum_{i=1}^{\infty} c_i \varphi_i. \quad (2.21)$$

We will not worry about conditional convergence and all sums will converge unconditionally independent of the ordering of terms in the sum. For a sequence $\{\varphi_i\}_{i=1}^{\infty} \subset \mathcal{H}$, its *span* is the vector space consisting of all finite linear combinations,

$$\text{span}\{\varphi_i\}_{i=1}^{\infty} = \{a_1 \varphi_1 + a_2 \varphi_2 + \dots + a_N \varphi_N : N \in \mathbb{N}, a_1, a_2, \dots, a_N \in \mathbb{C}\}.$$

If each $f \in \mathcal{H}$ has representation in Equation (2.21), then the $\text{span}\{\varphi_i\}_{i=1}^\infty$ is dense in $\mathcal{H}$. Hence its closure is the entire space, $\overline{\text{span}}\{\varphi_i\}_{i=1}^\infty = \mathcal{H}$. A sequence $\{\varphi_i\}_{i=1}^\infty$ whose closed span is the entire space is called *complete*. Note that a priori having a representation in Equation (2.21) is not equivalent to separability: there exist Banach spaces that are separable but do not have this basis-like representation [16, Chapters 3, 4]. Luckily in a Hilbert space, separability and the existence of a basis are equivalent [16, Chapters 3, 4].

A sequence $\{e_i\}_{i=1}^\infty \subset \mathcal{X}$ is a (Schauder) *basis* for $\mathcal{X}$ if for each $f \in \mathcal{X}$, there exist unique scalar coefficients $\{c_i(f)\}_{i=1}^\infty$ such that f has the representation,

$$f = \sum_{i=1}^{\infty} c_i(f) e_i.$$

The definition of a basis requires both the existence of coefficients to represent f by and uniqueness of these coefficients. Uniqueness is correlated to notions of independence, similar to the finite case, except that there exist three notions instead of one [16, Lemma 3.1.3]. We list these notions in order of strength. A sequence $\{e_i\}_{i=1}^\infty$ is *linearly independent* if every finite subsequence is linearly independent. The $\{e_i\}_{i=1}^\infty$ is ω -*independent* if whenever the series $\sum_{i=1}^\infty c_i e_i$ is convergent and equal to zero for some coefficient sequence $\{c_i\}_{i=1}^\infty$, then $c_i = 0$ for all $i \in \mathbb{Z}_+$. The $\{e_i\}_{i=1}^\infty$ is *minimal* if $e_i \notin \text{span}\{e_j\}_{j \neq i}$.

A sequence $\{e_i\}_{i=1}^\infty \subset \mathcal{H}$ is a *Bessel sequence* if there is a constant $B > 0$ such that

$$\sum_{i=1}^{\infty} |\langle f, e_i \rangle|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{H}$$

One of the conditions that define a frame in a Hilbert space is the Bessel inequality. Every basis in a Hilbert space $\mathcal{H}$ leads to an expansion with coefficients determined by inner products with a *dual basis*.

Theorem 2.4.8 (Dual Basis and Biorthogonality, Theorem 3.3.3 [16]) *Assume $\{e_i\}_{i=1}^\infty$ is a basis for the Hilbert space $\mathcal{H}$. Then there exists a unique family, the dual*

basis $\{f_i\}_{i=1}^\infty$, such that

$$f = \sum_{i=1}^{\infty} \langle f, f_i \rangle e_i, \quad \forall f \in \mathcal{H}.$$

The sequence $\{f_i\}_{i=1}^\infty$ is a basis for $\mathcal{H}$ and the two sequences are biorthogonal,

$$\langle e_i, f_j \rangle = \delta_{ij}, \quad \forall i, j \in \mathbb{Z}_+$$

A sequence $\{e_i\}_{i=1}^\infty$ in $\mathcal{H}$ is called an *orthonormal system* if

$$\langle e_i, e_j \rangle = \delta_{ij}, \quad \forall i, j \in \mathbb{Z}_+.$$

An *orthonormal basis (ONB)* is an orthonormal system $\{e_i\}_{i=1}^\infty$ that is a basis for $\mathcal{H}$.

We have the following equivalent characterizations of an ONB including Parseval's formula and self-duality.

Theorem 2.4.9 (Characterizations of an ONB, Theorem 3.4.2 [16]) *For an orthonormal system $\{e_i\}_{i=1}^\infty$ in a Hilbert space $\mathcal{H}$ the following are equivalent:*

1. $\{e_i\}_{i=1}^\infty$ is an ONB
2. $f = \sum_{i=1}^\infty \langle f, e_i \rangle e_i, \quad \forall f \in \mathcal{H}$
3. $\sum_{i=1}^\infty |\langle f, e_i \rangle|^2 = \|f\|^2, \quad \forall f \in \mathcal{H}$
4. $\overline{\text{span}}\{e_i\}_{i=1}^\infty = \mathcal{H}$
5. If $\langle f, e_i \rangle = 0, \quad \forall i \in \mathbb{N}$, then $f = 0$
6. $\langle f, g \rangle = \sum_{i=1}^\infty \langle f, e_i \rangle \langle e_i, g \rangle, \quad \forall f, g \in \mathcal{H}.$

Given an ONB $\{e_i\}_{i=1}^\infty$ in $\mathcal{H}$, all other ONB can be characterized as the sequences $\{Ue_i\}_{i=1}^\infty$, where $U : \mathcal{H} \rightarrow \mathcal{H}$ is a unitary operator. If we relax the constraint that U is unitary, we arrive at the notion of a Riesz basis.

A *Riesz basis* for $\mathcal{H}$ is a family of the form $\{Ue_i\}_{i=1}^\infty$, where $\{e_i\}_{i=1}^\infty$ is an orthonormal basis and $U : \mathcal{H} \rightarrow \mathcal{H}$ is an invertible operator. One can show that the dual basis of a Riesz basis is another Riesz basis, similar to Theorem 2.4.8.

Theorem 2.4.10 (Representation and Riesz Inequalities, Theorem 3.6.3 [16])

Let $\{e_i\}_{i=1}^{\infty}$ be a Riesz basis for $\mathcal{H}$. Then there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, e_i \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

The largest possible value for the constant A is $\frac{1}{\|U^{-1}\|^2}$ and the smallest possible value for B is $\|U\|^2$. Furthermore, if $\{f_i\}_{i=1}^{\infty}$ is the dual basis, then any $f \in \mathcal{H}$ can be expressed as

$$f = \sum_{i=1}^{\infty} \langle f, e_i \rangle f_i = \sum_{i=1}^{\infty} \langle f, f_i \rangle e_i.$$

The upper inequality states that the series $\sum_{i=1}^{\infty} |\langle f, e_i \rangle|^2$ is convergent for any $f \in \mathcal{H}$ and is in fact a Bessel sequence. The lower inequality implies that we have a basis-like representation (Equation (2.21)) of each $f \in \mathcal{H}$. It also implies that $\{e_i\}_{i=1}^{\infty}$ is a complete sequence, $\overline{\text{span}}\{e_i\}_{i=1}^{\infty} = \mathcal{H}$. By a similar argument given in Section 2.4.1, if it weren't complete there would be a non-zero element in the orthogonal complement and the lower inequality would be zero. Note, a basis-like representation of the form in Equation (2.21) implies the existence of a complete sequence but *not* vice-versa, see Example 5.4.6 in [16]. These upper and lower inequalities are the defining features of a frame.

For the Gabor systems introduced in , one has to abandon the use of ONB and Riesz bases to achieve good time-frequency resolution of a function via the STFT. There is no localized window $g \in L^2(\mathbb{R})$ in time or frequency when using an ONB or a Riesz basis: *the uncertainty principle is maximized*. This is stated formally in the *Balian-Low* theorem.

Theorem 2.4.11 (Balian-Low, Theorem 4.2.1 [16]) *If $\{M_{\ell}T_k g\}_{\ell, k \in \mathbb{Z}}$ is a Riesz basis for $L^2(\mathbb{R})$, then*

$$\left(\int_{\mathbb{R}} |xg(x)|^2 dx \right) \left(\int_{\mathbb{R}} |\omega \hat{g}(\omega)|^2 d\omega \right) = \infty.$$

Thus, with Gabor systems one is naturally led to the overcomplete systems of frames. Frames in a Hilbert space are a natural relaxation of the constraints of a

basis. They arise naturally as both mathematical curiosity and in application:

1. Like in the finite-dimensional setting (Section 2.4.1), frames provide basis-like representations of an $f \in \mathcal{H}$ as $f = \sum_{i=1}^{\infty} c_i \varphi_i$ with added flexibility in choosing the coefficients. Frames answer the question: how can we generalize a basis and relax its defining features but still retain the representation $f = \sum_{i=1}^{\infty} c_i \varphi_i$?
2. All the ONB in $\mathcal{H}$ are of the form $\{Ue_i\}_{i=1}^{\infty}$, where $U : \mathcal{H} \rightarrow \mathcal{H}$ is unitary and $\{e_i\}_{i=1}^{\infty}$ is an ONB; all the Riesz bases are of the form $\{Ue_i\}_{i=1}^{\infty}$, where $U : \mathcal{H} \rightarrow \mathcal{H}$ is invertible; all the frames are of the form $\{Ue_i\}_{i=1}^{\infty}$, where $U : \mathcal{H} \rightarrow \mathcal{H}$ is bounded and surjective.
3. If one considers the non-harmonic system of exponentials $\{e^{2\pi i \ell b x}\}_{\ell=1}^{\infty}$ as Duffin and Schaeffer [28] did in their seminal work on frames, then:
 - When $b > 1$, the system is incomplete and we cannot have a basis-like representation of $f \in L^2([0, 1))$.
 - When $b = 1$, we recover the traditional harmonic exponentials used in a Fourier series and the system is a ONB for $L^2([0, 1))$.
 - When $b < 1$, we still have a basis-like representation of $f \in L^2([0, 1))$ but the system is *redundant* or *overcomplete* and emphatically not a basis but a frame for $L^2([0, 1))$.
4. Consider the ONB for $L^2([0, 1))$ consisting of the exponentials $\{e^{2\pi i \ell x}\}_{\ell=1}^{\infty}$. Their restriction to a proper open sub-interval $I \subset [0, 1)$ is no longer a basis and each $f \in L^2(I)$ can be expressed using non-identical sequences of coefficients,

$$f(x) = \sum_{\ell=1}^{\infty} c_{\ell} e^{2\pi i \ell x} = \sum_{\ell=1}^{\infty} c'_{\ell} e^{2\pi i \ell x}, \quad \{c_{\ell}\}_{\ell=1}^{\infty} \neq \{c'_{\ell}\}_{\ell=1}^{\infty}.$$

Indeed this is a realization of a general property of projecting an ONB of a Hilbert space $\mathcal{H}$ onto a proper subspace $V \subset \mathcal{H}$: there are always choices for coefficients in expansions of $f \in V$ when using the original ONB by adding an element of $V^{\perp}$, see Proposition 4.1.2 in [16].

5. When using local Fourier analysis tools like the STFT one is led to Gabor frames, a discrete cousin of the STFT. Indeed via Balian-Low (Theorem 2.4.11), one has to abandon using a ONB for $L^2(\mathbb{R}^d)$ when sampling or discretizing the STFT to achieve good time-frequency resolution in the face of the uncertainty principles.
6. Certain families of wavelets and the DSTFT are cornerstones of modern signal processing that are built on properties of frames [66] that offer flexibility, redundancy, and sparsity in their representations [66] [14, Chapter 9].

A sequence $\{\varphi_i\}_{i=1}^\infty$ of elements in $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i=1}^\infty |\langle f, \varphi_i \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \quad (2.22)$$

This defining pair of inequalities is referred to as a *pseudo-Parseval* or *pseudo-Plancharel* as it loosens the equality (isometry) stated in classic Parseval-Plancharel theorems as in Theorem 2.2.2. The bounds A, B are *frame bounds* and called *optimal frame bounds* if they are the tightest bounds possible. If $A = B$, the frame is a *tight frame*. Similar to Riesz bases, the upper inequality states that the sequence $\{\varphi_i\}_{i=1}^\infty$ is Bessel with bound B and ensures the coefficients $\{\langle f, \varphi_i \rangle\}_{i=1}^\infty$ we'd like to use in a basis-like expansion of f is a convergent sequence. The lower inequality implies $\{\varphi_i\}_{i=1}^\infty$ is complete in $\mathcal{H}$ and that we have the basis-like expansion,

$$f = \sum_{i=1}^\infty \langle f, \tilde{\varphi}_i \rangle \varphi_i.$$

This can be seen with the introduction of the operators below. It can also be understood via the surjectivity of the synthesis operator $T : \ell^2(\mathbb{N}) \rightarrow \mathcal{H}$ defined below, see Theorem 5.5.1 in [16].

Because $\{\varphi_i\}_{i=1}^\infty$ is a Bessel sequence, the *synthesis operator* $T : \ell^2(\mathbb{N}) \rightarrow \mathcal{H}$,

$$T\{c_i\}_{i=1}^\infty := \sum_{i=1}^\infty c_i \varphi_i$$

is well-defined (bounded). Its adjoint $T^* : \mathcal{H} \rightarrow \ell^2(\mathbb{N})$ the *analysis operator*, is

$$T^* f := \{\langle f, \varphi_i \rangle\}_{i=1}^\infty.$$

Composing T and T^* , define the *frame operator* $S : \mathcal{H} \rightarrow \mathcal{H}$ that takes an $f \in \mathcal{H}$ to its basis-like expansion,

$$Sf := TT^* f = \sum_{i=1}^\infty \langle f, \varphi_i \rangle \varphi_i.$$

Composing T^* and T , define the *Gram operator* $G : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ as

$$G\{c_i\}_{i=1}^\infty := T^*T\{c_i\}_{i=1}^\infty = \{\langle \sum_{i=1}^\infty c_i \varphi_i, \varphi_i \rangle\}.$$

The matrix representation of G in the canonical orthonormal basis for $\ell^2(\mathbb{N})$ has entries given by the inner products of frame elements, $G_{ij} = \langle \varphi_i, \varphi_j \rangle$, for $i, j \in \mathbb{N} \times \mathbb{N}$.

The frame condition implies that S is a positive invertible operator. This leads to two canonical representations of f that can be computed explicitly.

Lemma 2.4.12 (S is a Positive Invertible Operator, Lemma 5.1.5 [16]) *Let $\{\varphi_i\}_{i=1}^\infty$ be a frame with frame operator S and frame bounds A, B . Then,*

1. S is bounded, positive, invertible, and self-adjoint and
2. $\{S^{-1}\varphi_i\}_{i=1}^\infty$ is a frame called the *canonical dual frame* with frame bounds A^{-1}, B^{-1} and frame operator S^{-1} .

Theorem 2.4.13 (The Canonical Dual Frame, Theorem 5.1.6 [16]) *Let $\{\varphi_i\}_{i=1}^\infty$ be a frame with frame operator S . Then,*

$$f = \sum_{i=1}^\infty \langle f, S^{-1}\varphi_i \rangle \varphi_i, \quad \forall f \in \mathcal{H}$$

and

$$f = \sum_{i=1}^\infty \langle f, \varphi_i \rangle S^{-1}\varphi_i, \quad \forall f \in \mathcal{H}.$$

We also have the following characterization of Riesz bases in terms of the Gram matrix, Bessel sequences, and completeness.

Theorem 2.4.14 (Gram Matrix Invertibility, Theorem 3.6.6 [16]) *For a sequence $\{\varphi_i\}_{i=1}^\infty$ in $\mathcal{H}$, the following conditions are equivalent:*

1. $\{\varphi_i\}_{i=1}^\infty$ is a Riesz basis for $\mathcal{H}$
2. $\{\varphi_i\}_{i=1}^\infty$ is complete in $\mathcal{H}$ and there exist constants $A, B > 0$ such that for every finite scalar sequence $\{c_i\}$

$$A \sum_{i=1}^\infty |c_i|^2 \leq \left\| \sum_{i=1}^\infty c_i \varphi_i \right\|^2 \leq B \sum_{i=1}^\infty |c_i|^2,$$

3. $\{\varphi_i\}_{i=1}^\infty$ is complete and its Gram matrix $\{\langle \varphi_i, \varphi_j \rangle\}_{i,j=1}^\infty$ defines a bounded invertible operator on $\ell^2(\mathbb{N})$
4. $\{\varphi_i\}_{i=1}^\infty$ is a complete Bessel sequence and it has a complete biorthogonal sequence $\{\psi_i\}_{i=1}^\infty$ which is also a Bessel sequence.

A special type of frame in time-frequency analysis are Gabor frames. Define the time-frequency lattice,

$$\Lambda := \{(\ell b, ka) : a, b > 0 \text{ and } \ell, k \in \mathbb{Z}^d\}.$$

Given a window $g \in L^2(\mathbb{R}^d)$, a *Gabor system* $\mathcal{G}(g, a, b)$ is the set of time-frequency shifts

$$\mathcal{G}(g, a, b) := \{M_{\ell b} T_{ka} g : (\ell b, ka) \in \Lambda\},$$

where $M_{\ell b}$ and T_{ka} are the modulation and translation operators defined in Section 2.2. A Gabor system is a *Gabor frame* for $L^2(\mathbb{R}^d)$ when

$$A \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \sum_{\ell, k \in \mathbb{Z}^{2d}} |\langle f, M_{\ell b} T_{ka} g \rangle|^2 \leq B \|f\|_{L^2(\mathbb{R}^d)}^2. \quad (2.23)$$

Gabor frames are a discretization or sampling of the STFT. Note, the inner product above is the STFT evaluated at $(\ell b, ka) \in \Lambda$,

$$V_g f(\ell b, ka) = \int_{\mathbb{R}^d} f(t) g(t - ka) e^{-2\pi i t \cdot \ell b} dt = \langle f, M_{\ell b} T_{ka} g \rangle_{L^2(\mathbb{R}^d)}.$$

Further, with a Gabor frame we have the representations,

$$\begin{aligned} f &= \sum_{\ell, k \in \mathbb{Z}^{2d}} \langle f, M_{\ell b} T_{ka} \tilde{g} \rangle M_{\ell b} T_{ka} g \\ &= \sum_{\ell, k \in \mathbb{Z}^{2d}} \langle f, M_{\ell b} T_{ka} g \rangle M_{\ell b} T_{ka} \tilde{g}, \end{aligned}$$

where $\tilde{g} := S^{-1}g \in L^2(\mathbb{R}^d)$ is the *canonical dual window*. That is, sample the STFT and use the samples $\{\langle f, M_{\ell b} T_{ka} g \rangle\}_{(\ell b, ka) \in \Lambda}$ or $\{\langle f, M_{\ell b} T_{ka} \tilde{g} \rangle\}_{(\ell b, ka) \in \Lambda}$ as coefficients in a frame reconstruction of f . This raises the question about how exactly one must sample or discretize the STFT $V_g f$ on $\mathbb{R}^{2d}$. This is answered partially in Proposition 2.4.15.

Proposition 2.4.15 (Lattice Density, Proposition 4.2.2 [16]) *Let g be a continuous function of compact support and $a, b > 0$. Then:*

1. *If $\mathcal{G}(g, a, b)$ is a Gabor frame for $L^2(\mathbb{R}^d)$, then $ab \leq 1$.*
2. *If $ab > 1$, then $\mathcal{G}(g, a, b)$ is not complete in $L^2(\mathbb{R}^d)$.*
3. *$\mathcal{G}(g, a, b)$ is an ONB for $L^2(\mathbb{R}^d)$ if and only if it is a tight frame, $ab = 1$, and $\|g\|_{L^2(\mathbb{R}^d)} = 1$.*

Via Theorem 2.4.15 we can interpret the frame inequalities in Equation (2.23) as placing requirements on how the energy of the STFT is captured via sampling on the time-frequency lattice. Finally, we state an important conjecture regarding Gabor systems, the HRT conjecture. Let $\mathcal{F}$ be a finite index set. The *HRT conjecture* states that any finite collection of distinct points $\{(x_n, \omega_n)\}_{n \in \mathcal{F}} \subset \mathbb{R}^2$ and a non-zero function g , the Gabor system $\{M_{\omega_n} T_{x_n} g\}_{n \in \mathcal{F}}$ is linearly independent. The conjecture is unsolved but is true for arbitrary $g \in L^2(\mathbb{R})$ in the lattice case, $\{(x_n, \omega_n)\}_{n \in \mathcal{F}} = \{(\ell b, ka)\}_{\ell, k=1}^{L, K}$ [71]. The conjecture is also true for arbitrary points $\{(x_n, \omega_n)\}_{n \in \mathcal{F}}$ if $g \in L^2(\mathbb{R})$ is compactly supported [71]. A consequence is that the lower frame bound for finite Gabor frames is forced to converge to zero, see Theorem 13.4.4 in [16]. Results in Chapter 6 on finite frame bounds refine this result and provide

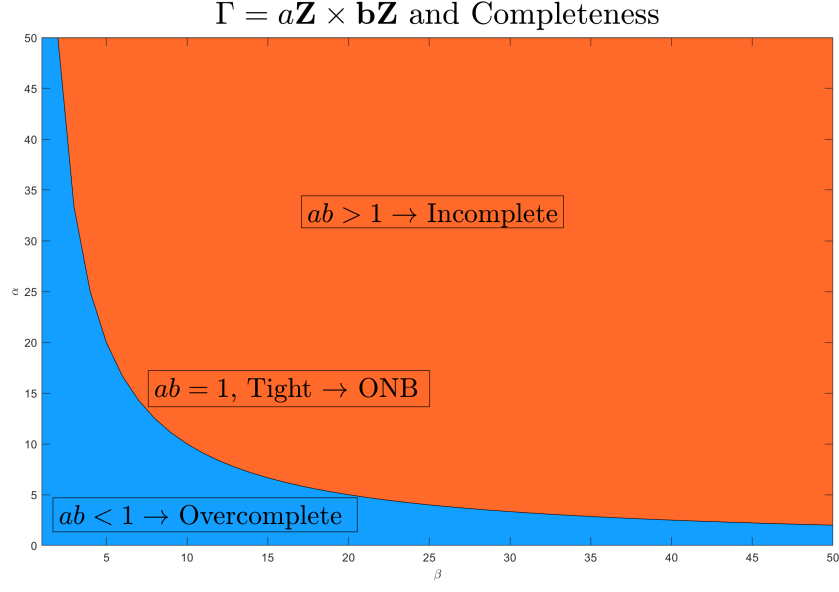


Figure 2.8: Theorem 2.4.15 provides conditions under which the time-frequency lattice $\Lambda = \{(\ell b, ka) : a, b > 0 \text{ and } \ell, k \in \mathbb{Z}^d\}$ is dense enough to allow reconstruction of a function $f \in L^2(\mathbb{R}^d)$ from samples of its STFT $V_g f \in L^2(\mathbb{R}^{2d})$.

both estimates and exact values of the frame bounds for a particular Gabor frame constructed from time-frequency shifts of the B-spline.

2.5 Fractals and the Sierpinski Gasket (SG)

2.5.1 Definition and Graph Approximations

Let (X, d) be a complete metric space. If the map $f_i : X \rightarrow X$ is a contraction with respect to the metric d for $i \in \{1, 2, \dots, N\}$, then there exists a unique non-empty compact subset K of X that satisfies a *self-similar identity*:

$$K = f_1(K) \cup \dots \cup f_N(K).$$

The *fractal* K is this unique fixed point of the contraction maps $\{f_i\}_{i=1}^N$. We write $f_w = f_{w_1} \circ f_{w_2} \circ \dots \circ f_{w_m}$ for a *word* $w = (w_1, \dots, w_m)$ of length m , where each $w_i \in \{1, 2, \dots, N\}$. By iteration, one can then write the self-similar identity and decompose K in terms of the *cells* $f_w(K)$,

$$K = \bigcup_{|w|=m} f_w K.$$

Classic examples of fractals include the unit interval, the Cantor set, the Sierpinski gasket, Hata's tree-like set, and the Koch curve.

Example 2.5.1 (Unit Interval) The unit interval: let $I = [0, 1]$ and define the contraction maps $f_1(x) = \frac{x}{2}$ and $f_2(x) = \frac{x}{2} + \frac{1}{2}$ that send the unit interval to its left and right halves. Then,

$$I = f_1 I \cup f_2 I = \bigcup_{|w|=m} f_w I. \quad \diamond$$

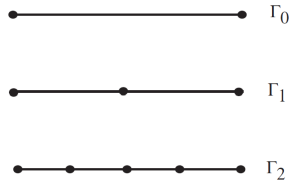


Figure 2.9: The unit interval $I = \Gamma_0 = [0, 1]$ and the first two images of an iteration of the contraction maps f_1, f_2 : $\Gamma_1 = f_1(\Gamma_0) \cup f_2(\Gamma_0)$ and $\Gamma_2 = f_1 f_1 \Gamma_0 \cup f_0 f_1 \Gamma_0 \cup f_1 f_0 \Gamma_0 \cup f_0 f_0 \Gamma_0$.

Example 2.5.2 (The Koch Curve) Let $X = \mathbb{C}$. Suppose $\alpha \in \{z : z \in \mathbb{C}, |z|^2 + |1 - z|^2 < 1\}$, $f_1(z) = \alpha\bar{z}$, and $f_2(z) = (1 - \alpha)(\bar{z} - 1) + 1$. Then the fractal

$$K_\alpha = f_1(K_\alpha) \cup f_2(K_\alpha)$$

defined by these contraction maps gives the Koch curve seen in Figure 2.10. $\diamond$

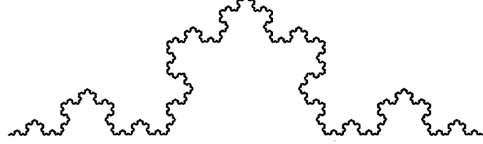
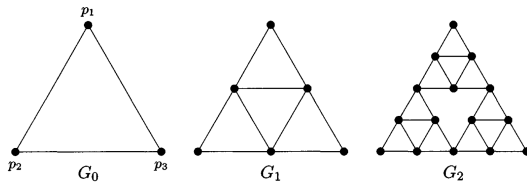


Figure 2.10: The Koch curve fractal K_α , $\alpha = \frac{1}{2} + \frac{1}{2\sqrt{3}}i$.

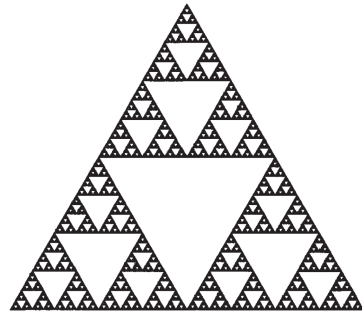
Example 2.5.3 (The Sierpinski Gasket (SG)) Let $X = \mathbb{C}$ and set $\{p_1, p_2, p_3\}$ to be the vertices of an equilateral triangle. Define $f_j(z) = (z - p_j)/2 + p_j$ for $j = 1, 2, 3$. The resulting fractal K defined by these contraction maps is the Sierpinski gasket SG (see Figure 2.11),

$$K = f_1K \cup f_2K \cup f_3K = \bigcup_{|w|=m} f_w K.$$

$\diamond$



(a) Graph approximations G_0, G_1, G_2 to the Sierpinski gasket SG . The closure of union $G^* := \bigcup_{m=1}^{\infty} G_m$ with respect to a graph *resistance metric* can be identified with SG .



(b) The Sierpinski gasket (SG) is a prototypical pcf fractal with a well-developed theory of analysis.

Figure 2.11: Analysis SG can be accomplished via a discrete analysis on sequence of graphs $G_0, G_1, G_2, \dots$ that approximate SG .

One can define fractals and proceed with a theory of analysis without any metric structure and can be studied as a purely topological object [62]. However, a beautiful

theory of analysis on SG and in general for pcf fractals proceeds through an analysis on graphs. The central operator in this analysis is the *Laplacian*, which we define below as pointwise limit of graph Laplacians. Let $V_0 = \{p_1, p_2, p_3\}$ be the vertices of an equilateral triangle as seen in Figure 2.11a and define a sequence of finite sets $\{V_m\}_{m \geq 0}$ inductively by $V_{m+1} = f_1(V_m) \cup f_2(V_m) \cup f_3(V_m)$. We then have a natural graph whose set of vertices is V_m . The graph G_m has edges defined by $x \sim_m y$ if there is a cell of level m that contains both x and y :

$$x \sim_m y \iff \exists w, |w| = m \text{ and } i, j \in \{0, 1, 2\} \text{ s.t. } x = f_w p_i \text{ and } y = f_w p_j.$$

If $l(V_m) = \{f : f \text{ maps } V_m \text{ to } \mathbb{R}\}$, then the *graph Laplacian* $\Delta_m : l(V_m) \rightarrow l(V_m)$ is the linear operator

$$(\Delta_m u)(x) = \sum_{x \sim_m y} u(x) - u(y)$$

for any $u \in l(V_m)$ and any $x \in V_m$. The *Laplacian* on SG is the renormalized limit of these finite difference operators,

$$\Delta u(x) = \lim_{m \rightarrow \infty} 5^m (\Delta_m u)(x).$$

In analysis on SG , in order to integrate functions one introduces a *self-similar measure*. Self-similar measures on the interval $[0, 1]$ are the *Bernoulli measures*. The interpretation of a self-similar measure on SG is similar: assign probabilities $\{\mu_0, \mu_1, \mu_2\}$ such that $\mu_0 + \mu_1 + \mu_2 = 1$ to index set $\{0, 1, 2\}$ which dictates the probability that a cell $F_w K$ of level m is chosen by the rule

$$\mu(F_w K) = \prod_{j=1}^m \mu_{w_j}, \text{ for } |w| = m.$$

Equivalently, a self-similar measure on SG is the unique regular probability measure μ that satisfies for every finite union of cells within SG , $A = \cup_j C_j$ where $C_j = F_{w_j} K$

for some word w_j , the identity

$$\mu(A) = \sum_{i=1} \mu_i \mu(F_i^{-1}A). \quad (2.24)$$

We take μ to be the *standard self-similar measure* where $\mu_0 = \mu_1 = \mu_2 = 1/3$. Intuitively, this probability measure simply assigns weight 3^{-m} to a cell of level m . A more general definition and properties of the self-similar measures can be found in [62, Chapter 1].

A crucial property of the Laplacian Δ on SG is that it admits an orthonormal basis of eigenfunctions $\{\varphi_{\lambda_n}\}_{n=1}^\infty$ for $L^2(SG, \mu)$ in the same spirit as the complex exponentials forming an orthonormal basis for $L^2(\Omega, \lambda)$ where Ω is a compact subset of $\mathbb{R}^d$ and λ is the Lebesgue measure. One can associate both Neumann and Dirichlet boundary conditions with the Laplacian Δ on SG with corresponding families of eigenfunctions. Proving these statements about the Laplacian on SG is delicate and relies on theories of normal derivatives, Gauss-Green formulas, weak formulations of Δ , and *Dirichlet forms*: a theory of normal derivatives is required to establish Gauss-Green formulas by which one integrates-by-parts to define a weak form of Δ , and to determine that the eigenfunctions form an orthonormal basis of $L^2(SG, \mu)$ one proves that the non-negative self-adjoint operator (a Laplacian) corresponding to a Dirichlet form on $L^2(SG, \mu)$ has compact resolvent. The key results about the spectrum of Δ is summarized below.

Theorem 2.5.4 (Dirichlet and Neumann Spectrum, Theorem 3.4.6 [62]) *The Dirichlet and Neumann eigenvalues $\Lambda := \{\lambda_n\}_{n=1}^\infty$ are non-negative, have finite multiplicity, and the only accumulation point is ∞ . The associated system $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ form a complete orthonormal system for $L^2(SG, \mu)$.*

Part of the utility of the $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ is we can explicitly construct a *heat kernel* on SG . Define the heat kernel $p_b(t, x, y)$ for $(t, x, y) \in (0, \infty) \times SG \times SG$ as

$$p_b(t, x, y) = \sum_{\lambda \in \Lambda} e^{-\lambda t} \varphi_\lambda(x) \varphi_\lambda(y), \quad (2.25)$$

where $b = N$ uses the Neumann Laplacian and $b = D$ uses the Dirichlet Laplacian. The heat kernel $p_b(t, x, y)$ is non-negative and continuous on $(0, \infty) \times SG \times SG$ and satisfies the partial differential equation

$$\frac{\partial p_b(t, x, y)}{\partial t} = \Delta p_b(t, x, y). \quad (2.26)$$

with appropriate boundary conditions. We leverage known estimates on the eigenfunctions $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ and the heat kernel $p_b(t, x, y)$ in order to prove the STFT on SG is an isometry is invertible. Specifically, we require the following uniform bound on the heat kernel of SG where for $b = D, N$ there exists $c_1 > 0$ such that

$$\sup_{x, y \in SG} p_b(t, x, y) \leq c_1 t^{-d_S \bar{\delta}/2} \quad (2.27)$$

for any $t \in (0, 1]$, where $c_1 > 0$ is a constant, $\bar{\delta}$ is a fixed quantity computed from the self-similar measure μ , and d_S is the spectral dimension of SG . Furthermore, the diagonal of the Neumann heat kernel $p_N(t, x, x)$ is known to have decay estimates from below,

$$c_2 t^{-d_S/2} \leq p_N(t, x, x) \leq c_1 t^{-d_S/2} \quad (2.28)$$

for $t \in (0, 1]$, positive constants $c_1, c_2 > 0$, and any $x \in SG$. The formal statements of these bounds from Kigami [62] are below in Section 2.5.2.

2.5.2 Analysis on Fractals

We recall the notions of Dirichlet forms and Dirichlet energies that are used to bound integrals in the proof of a formula for the left-inverse of the STFT on SG in Chapter 4. Let X be a locally compact metric space and $\mathcal{E}$ be a non-negative quadratic closed form on $L^2(X, \mu)$, see [62, Definition B.1.6]. Set $\mathcal{F} = \text{dom}(\mathcal{E})$. Then, the *unit contraction* operates on $\mathcal{E}$ if and only if $\bar{u} \in \mathcal{F}$ and $\mathcal{E}(\bar{u}, \bar{u}) \leq \mathcal{E}(u, u)$ for any $u \in \mathcal{F}$,

where:

$$\bar{u} = \begin{cases} 1, & u(p) \geq 1 \\ u(p), & 0 < u(p) < 1 \\ 0, & u(p) \leq 0 \end{cases}.$$

The pair $(\mathcal{E}, \mathcal{F})$ is said to have the *Markov property* if and only if the unit contraction operates on $\mathcal{E}$. A subspace $\mathcal{C}$ of $\mathcal{F} \cap C_0$ is called a *core* of $(\mathcal{E}, \mathcal{F})$ if and only if $\mathcal{C}$ is dense in $\mathcal{F}$ with respect to the $\mathcal{E}^*$ -norm

$$\mathcal{E}^*(u, v) := \mathcal{E}(u, v) + \langle u, v \rangle_{L^2(X, \mu)}$$

and in $C_0(X)$ with respect to the supremum norm $\|\cdot\|_\infty$, where

$$C_0(X) := \{f : X \rightarrow \mathbb{R}, f \text{ continuous and } \text{supp}(f) \text{ is compact}\}.$$

Because $C_0(X)$ is dense in $L^2(X, \mu)$, the core $\mathcal{C}$ is also dense in $L^2(X, \mu)$. Let $\mathcal{E}$ be a closed form on $L^2(X, \mu)$ and $\mathcal{F} = \text{dom}\mathcal{E}$. Then the pair $(\mathcal{E}, \mathcal{F})$ is called a *Dirichlet form* on $L^2(X, \mu)$ if and only if it has the Markov property. It is called *regular* if and only if it posses a core.

The *eigenvalue counting function* is defined as

$$\rho_b(x, \mu) := \sum_{\lambda \leq x} \dim E_b(\lambda) = \max\{k, \lambda_k^b \leq x\}$$

where $E_b(\lambda)$ is the eigenspace of eigenfunction φ_λ associated with eigenvalue λ , and $d_S > 0$ is the *spectral dimension* (or spectral exponent) of the fractal K . The spectral dimension governs the asymptotic behavior of $\rho_b(x, \mu)$. Theorem 2.5.5 is the analogue of Weyl's theorem on self-similar structures. Below, the factors $\{\mu_i\}_{i=1}^N$ and $\{r_i\}_{i=1}^N$ are the weights defining the self-similar measure μ and the renormalization factors defining a self-similar energy on a sequence of compatible resistance networks, see Chapter 3 in Kigami [62]. The triple $(\mathcal{E}, \mathcal{F}, \mu)$ is the Dirichlet form associated to the Laplacian Δ .

Theorem 2.5.5 (Eigenvalue Counting Function, Theorem 4.1.5 [62]) *Let μ be a self-similar measure with weight $(\mu_i)_{i=1,2,\dots,N}$. Assume that $\mu_i r_i < 1$ for all $i \in \{1, 2, \dots, N\}$. Let d_S be the unique real number d that satisfies*

$$\sum_{i=1}^N \gamma_i^d = 1,$$

where $\gamma_i = \sqrt{r_i \mu_i}$. Then

$$0 < \liminf_{x \rightarrow \infty} \rho_b(x, \mu) / x^{d_S/2} \leq \limsup_{x \rightarrow \infty} \rho_b(x, \mu) / x^{d_S/2} < \infty$$

for $b = D$ and N . d_S is called the spectral exponent of $(\mathcal{E}, \mathcal{F}, \mu)$. Moreover,

1. *Non-lattice case: If $\sum_{i=1}^N \mathbb{Z} \log \gamma_i$ is a dense subgroup of $\mathbb{R}$, then $\lim_{x \rightarrow \infty} \rho_b(x, \mu) / x^{d_S/2}$ exists and is independent of the boundary conditions.*
2. *Lattice case: If $\sum_{i=1}^N \mathbb{Z} \log \gamma_i$ is a discrete subgroup of $\mathbb{R}$, let $T > 0$ be its generator. Then, as $x \rightarrow \infty$,*

$$\rho_b(x, \mu) = \left(G\left(\log \frac{x}{2}\right) + o(1) \right) x^{d_S/2},$$

where G is a right-continuous, T -periodic function satisfying $0 < \inf G(x) \leq \sup G(x) < \infty$. Moreover, G is independent of the boundary conditions.

If μ is a self-similar measure on SG with weights $\{\mu_i\}_{i=1}^N$, then $\sum_{i=1}^N (\mu_i r_i)^{d_S/2} = 1$ by Theorem 2.5.5. Define $\nu_i = (\mu_i r_i)^{d_S/2}$ and ν to be a new self-similar measure on K with weights $\{\nu_i\}_{i=1}^N$. Then, we have the following theorem bounding an eigenfunction of Δ in terms of its L^2 -norm.

Theorem 2.5.6 (Eigenfunction Bounds, Theorem 4.5.3 [62]) *There exist positive constants c_1 and c_2 such that*

$$\|u\|_\infty \leq c_1 |\lambda|^{d_S \bar{\delta}(\nu, \mu)/4} \|u\|_2$$

for any $u \in \mathcal{D}_\mu$ with $\Delta_\mu u = -\lambda u$ and $\lambda \geq c_2$, where $\mathcal{D}_\mu = \text{dom} \Delta$ is the domain of Δ

and

$$\bar{\delta}(\nu, \mu) = \max_i \frac{\log \mu_i}{\log \nu_i}, \quad \underline{\delta}(\nu, \mu) = \min_i \frac{\log \mu_i}{\log \nu_i}.$$

We use bounds on the heat kernel $p_b(t, x, y)$ in Equation (2.25) that is defined in terms of the $\{\varphi_\lambda\}_{\lambda \in \Lambda}$. These bounds and the fundamental properties of heat kernels on a connected pcf fractal K are summarized below. Property (4) is a convolution-type property used for example in the proof of Theorem 4.2.2 to invert the STFT on SG .

Proposition 2.5.7 (Heat Kernel Properties, Proposition 5.1.2 [62]) *For $b = D, N$, the heat kernel $p_b(t, x, y)$ has the properties:*

1. *it is non-negative and continuous on $(0, \infty) \times K \times K$,*
2. *for any $(x, y) \in K \times K$, $p_b(\cdot, x, y) \in C^1((0, \infty))$,*
3. *for any $(t, x, y) \in (0, \infty) \times K \times K$*

$$\frac{\partial p_b(t, x, y)}{\partial t} = (\Delta p_b^{t, x})(y),$$

4. *for any $s, t > 0$ and any $x, y \in K$,*

$$p_b(t + s, x, y) = \int_K p_b(t, x, x') p_b(x, x', y) \mu(dx').$$

Theorem 2.5.8 (Heat Kernel Bounds, Theorem 3.5.1 [62]) *For $b = D, N$, there exists $c_1 > 0$ such that*

$$\sup_{x, y \in K} p_b(t, x, y) \leq c_1 t^{-d_S \bar{\delta}/2}$$

for any $t \in (0, 1]$. There exists $c_2 > 0$ such that

$$c_2 t^{-d_S \underline{\delta}/2} \leq p_N(t, x, x)$$

for any $t \in (0, 1]$ and any $x \in K$.

Chapter 3

Tight Frames Generated by a Graph

Short-time Fourier Transform

3.1 Introduction

The short-time Fourier transform (STFT) is an essential and widely used tool in Fourier analysis, quantum mechanics, and signal processing. Pioneered by J. von Neumann and Nobel prize winner Dennis Gabor [33] in the 20th century, the STFT provides joint time-frequency or space-frequency information of a function defined on $\mathbb{R}^d$. This is accomplished by localizing the standard Fourier transform via a *window* function. As a result, the STFT combines features of both the function and its Fourier transform in a way that provides important local frequency information that traditional Fourier transform methods cannot provide. For example, in speech analysis and audio signal processing the STFT is used to detect pitch changes and transient events in the underlying signal [45, 83]. In image processing and optics, the STFT and associated Gabor system has been applied to denoising image patches, feature extraction, and coherent optical setups [5, 25]. Windowed Fourier methods like the STFT and the related wavelet transforms are fundamental to communication algorithms and to compression algorithms such as JPEG [4, 34, 68]. In addition to its utility in application, the STFT draws from diverse theories in mathematics including complex analysis, partial differential equations, the Heisenberg group, representation theory, and operator algebras [30, 46].

The classical STFT and related Gabor transformation are defined when the signal of interest f and the window g belongs to dual function spaces, e.g., $f, g \in L^2(\mathbb{R}^d)$, or f is a tempered distribution and g is a Schwartz function defined on $\mathbb{R}^d$ [46]. More generally, these transformations can be extended to signals defined in non-Euclidean spaces. For example, see [32, 52] for the case of signals defined on a sphere. One can

therefore ask how to generalize the STFT for signals defined on less regular spaces. In this paper, we revisit this question for signals defined on graphs by defining a *graph short-time Fourier transform* that is built on

1. The eigenvectors $\{\varphi_{\lambda_\ell}\}_{\lambda_\ell \in \Lambda}$ of the graph Laplacian Δ to perform a graph Fourier transform
2. The graph heat kernel $H_t := e^{-t\Delta}$ as a window to localize and translate the graph Fourier transform.

Our approach is related to the vertex-frequency analysis introduced in [82] and will lead to the construction of a time-dependent family of finite Gabor-like frames for $\mathbb{C}^N$. One of the major difficulties in defining windowed Fourier transform methods on irregular spaces such as graphs is in defining meaningful notions of graph harmonics and graph translation [35, 81]. For functions on $\mathbb{R}^d$, the eigenfunctions of the Laplacian are the complex exponentials $e^{2\pi i x \xi}$ used in the Fourier transform and translation relies on an underlying group structure. For signals defined on graphs, the approach developed by Shuman et al. [82] is based on a notion of a graph Fourier transform through the spectral analysis of a graph Laplacian, and has found success in graph signal processing and data science applications [81]. The graph heat kernel H_t is a natural window to consider in defining a graph short-time Fourier transform. Since the graph heat kernel is a fundamental solution to the heat equation on a graph, it can be viewed as encoding a notion of translation via a convolution-type operator and relating one vertex to another via a random walker. For example, for two vertices (v_i, v_j) on a graph with underlying group structure such as the ring graph, $e^{-t\Delta}(v_i, v_j) = k_t(v_i - v_j)$, where $k_t(\cdot) = \sum_{\ell=1}^N e^{-\lambda_\ell t} \varphi_{\lambda_\ell}(\cdot)$ and $(\lambda_\ell, \varphi_{\lambda_\ell})$ are an eigenvalue and eigenvector of Δ [89, Chapter 7]. These two graph operators, Δ and H_t , are well-studied and fundamental tools in spectral graph theory, graph signal processing, and data science. Indeed, spectral graph theory is based on the study of the spectrum of various graph Laplacians which reveal a dazzling array of properties of the geometry of the graph and dynamic processes defined on the graph. This includes isoperimetric and Cheeger inequalities, mixing time of Markov chains, and

minimizing energies of Hamiltonian systems [19, 57]. In graph signal processing, a notion of graph Fourier transform is defined via a change-of-basis matrix consisting of eigenvectors of a graph Laplacian. This allows an efficient basis representation of signals defined on the graph that is tailored to the structure of the underlying graph. In data science and computer vision, the heat kernel is used to characterize the shape of a mesh describing an underlying manifold or a graph [88]. Furthermore, a plethora of clustering techniques, graph neural networks, and PageRanks are based on the spectrum of the graph Laplacian and graph heat kernels [64, 90].

Here, we use the heat kernel H_t to define the graph short-time Fourier transform on finite graphs. As a consequence, we obtain a time-dependent family of finitely many signals whose spanning properties we investigate. We note that Shuman et al. [82] were the first to propose this definition, but their main results were stated for general window functions. In particular, they prove the resulting Gabor-like system forms a frame for $\mathbb{R}^N$ [82, Theorem 3]. By contrast, our paper zeros in on the heat kernel H_t as the window of the graph-short-time-Fourier transform and obtains the full description of the spectra of the frame operators associated to these Gabor-like systems.

Our contributions are summarized as follows. In Section 3.2 we introduce our definition of a short-time Fourier transform on a general graph in Equation (3.1) using the graph Laplacian Δ and graph heat kernel H_t . Subsequently, an explicit form for the inverse transform, the associated frame operator, and its spectrum are provided in Theorems 3.2.1, 3.2.3, and 3.2.2. In Section 3.3, on two classes of algebraic graphs we show in Theorems 3.3.2 and 3.3.3 that the associated Gabor frame is tight and independent of the window parameter $t \in \mathbb{R}_{t \geq 0}$.

3.2 A Short-time Fourier Transform and Gabor Frames on Graphs

Let $V = \{v_1, \dots, v_N\}$ and $E \subset \{(v_i, v_j) : v_i, v_j \in V, i \neq j\}$ be the finite vertex and edge set of the graph $G = (V, E)$. We consider an undirected and connected graph with uniform edge weights equal to one. Let $\mathcal{C}(V)$ be the complex vector space of functions $f : V \rightarrow \mathbb{C}$ on the vertices of the graph. A standard measure to associate with G is the counting measure. For a subset of vertices $V' \subset V$, the measure $\mu(V')$ is simply its cardinality:

$$\mu(V') = |V'|.$$

The space $\mathcal{C}(V)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \sum_{v \in V} f(v) \overline{g(v)} \mu(v) = \sum_{v \in V} f(v) \overline{g(v)}.$$

Recall the *graph Laplacian* $\Delta : \mathcal{C}(V) \rightarrow \mathcal{C}(V)$ is defined pointwise as

$$\Delta f(v) = \sum_{u \sim v} (f(v) - f(u)).$$

In matrix notation, $\Delta = D - A$, where $A \in \mathbb{R}^{N \times N}$ and $D \in \mathbb{R}^{N \times N}$ are the adjacency matrix and diagonal degree matrix associated with G . Recall the semigroup associated with the graph Laplacian is the matrix exponential, called the *graph heat kernel* $H_t \in \mathbb{R}^{N \times N}$:

$$H_t := e^{-t\Delta} = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \Delta^k.$$

By the *spectral theorem*, Δ has orthonormal set of eigenvectors $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_N\}$ and real eigenvalues $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_N\} \subset [0, \infty)$. A *graph Fourier transform* $\hat{f} : \Lambda \rightarrow \mathbb{C}$ [81] of a function $f : V \rightarrow \mathbb{C}$ is defined via inner products with this orthonormal basis of eigenvectors,

$$\hat{f}(\lambda_\ell) := \langle f, \varphi_\ell \rangle = \sum_{i=1}^N f(v_i) \overline{\varphi_{\lambda_\ell}(v_i)}.$$

This can also be expressed as the matrix multiplication $\hat{f} = \Phi^* f$, where now $\Phi \in \mathbb{C}^{N \times N}$ is the matrix with columns $\varphi_1, \varphi_2, \dots, \varphi_N$. Equivalently, Φ_{ij} are the function values $\varphi_j(v_i)$. We therefore have the inversion formula and function representation

$$f(v_i) = \sum_{\ell=1}^N \hat{f}(\lambda_\ell) \varphi_{\lambda_\ell}(v_i) = \Phi \hat{f}.$$

This graph Fourier transform shares key properties with the classical Fourier transform. It can be shown that the graph Fourier transform obeys the *Parseval-Plancharel Identity* $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$ simply due to Φ being unitary:

$$\langle \hat{f}, \hat{g} \rangle = \langle \Phi^* f, \Phi^* g \rangle = \langle \Phi \Phi^* f, g \rangle = \langle f, g \rangle.$$

Furthermore, the eigenvalues correspond to *graph energies* in the sense that the quadratic form

$$\overline{\varphi_{\lambda_\ell}} \Delta \varphi_{\lambda_\ell} = \sum_{(u,v) \in E} (\varphi_{\lambda_\ell}(u) - \varphi_{\lambda_\ell}(v))^2$$

is small when there is small variation of eigenvector φ_{λ_ℓ} across the graph. Although computing a closed form expression for the eigenvectors of the Laplacian is an unsolved problem for an arbitrary graph, when the underlying graph is sufficiently structured such as the path graph or ring graph, the entries can be computed explicitly and chosen to be roots of unity or samples of a cosine [81]. In fact, when the graph is a ring graph the graph Fourier transform is the *Discrete Fourier Transform (DFT)* [14, Chapter 6]. This is due in part to the underlying group structure that is naturally equipped with the ring graph when viewed as the Cayley graph of the group $\mathbb{Z}/n\mathbb{Z}$ and the corresponding graph Laplacian being circulant [89, Chapter 2]. The group $\mathbb{Z}/n\mathbb{Z}$ also provides a natural notion of translation via adding the group generator and its inverse, the equivalence classes of ± 1 .

Frames on graphs have previously been studied within the graph signal processing community. A notion of wavelets on graphs were introduced in [51] via a wavelet generating kernel parameterized by a scaling parameter that acts directly on the

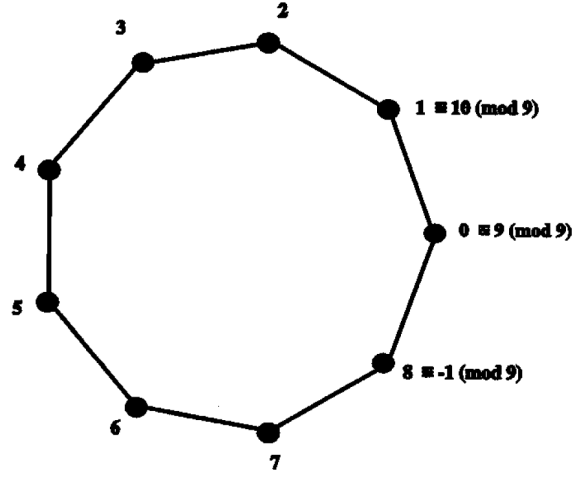


Figure 3.1: The Cayley graph $\Gamma(\mathbb{Z}/9\mathbb{Z}, \pm 1)$ [89, Chapter 1]. The DFT and DSTFT are Fourier analysis tools that can be viewed as being built on the Cayley graph $\Gamma(\mathbb{Z}/n\mathbb{Z}, \pm 1)$.

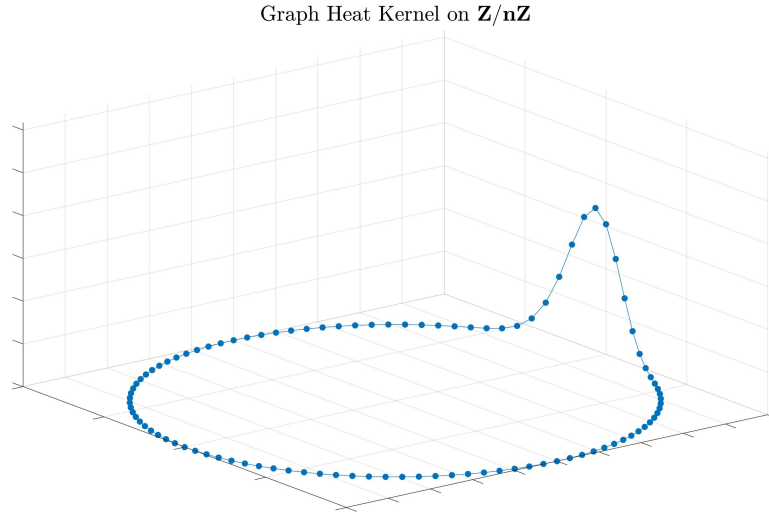


Figure 3.2: A column of the graph heat kernel $H_t(\cdot, v_i)$ on the Cayley graph $\Gamma(\mathbb{Z}/51\mathbb{Z}, \pm 1)$. The columns $H_t(\cdot, v_i)$ are used as a window in a definition of a graph short-time Fourier transform seen in Equation (3.1) and are viewed as solutions to a heat equation on the graph.

spectrum of the graph Laplacian. In similar spirit, a joint vertex-frequency representation of a graph signal was constructed by Shuman et al. [82] by defining and composing operators of translation and modulation in the graph Fourier domain. Frames on graphs have also found application in sparse representations of graph signals and de-noising [27, 38]. Recently, the choice of eigenbasis and underlying group structure of Cayley graphs have been used to define and study classes of translation operators for Gabor-type frames on graphs [6, 36]. Our construction of a graph short-time Fourier transform and the resulting graph Gabor frame is most similar to Shuman et al. [82]. However, as far as the authors know this is the first work to investigate the graph heat kernel specifically as the window function and relate the spectrum of the graph Laplacian to tight frames, particularly on algebraic graphs.

3.2.1 Definition and Frame Properties

For the graph short-time Fourier transform, we set the window to be the columns of the graph heat kernel $H_t(\cdot, v_i)$. Each entry of the window $H_t(\cdot, v_i) \geq 0$ is tunably localized around the vertex v_i . We have underlying geometry of the graph encoded in H_t via decay estimates on the trace in large time or recovering geodesics (shortest-path distances) in short time [19, Chapter 10]. Because the graph Laplacian Δ is diagonalizable, H_t has the compact form $H_t = \Phi e^{-t\Delta} \Phi^*$, which reads entry-wise as

$$H_t(v_i, v_j) = \sum_{\ell=1}^N e^{-\lambda_\ell t} \varphi_{\lambda_\ell}(v_i) \overline{\varphi_{\lambda_\ell}(v_j)}.$$

It is not difficult to show that the entries of H_t are positive for $t > 0$ and to compute an explicit form for the norms of its columns. We show in Theorem 3.2.2 that the spectrum of the frame operator corresponds to these column norms. Define the *graph short-time Fourier transform (GSTFT)* $V_t : \mathbb{C}^N \rightarrow \mathbb{C}^{N \times N}$ of a function $f : V \rightarrow \mathbb{C}$ on the vertices of the graph G :

$$(V_t f)(v_i, \lambda_j) := \sum_{k=1}^N f(v_k) H_t(v_i, v_k) \overline{\varphi_{\lambda_j}(v_k)}. \quad (3.1)$$

This definition can be viewed as an extension of the DSTFT on $\mathbb{C}^N$ in the same sense that the graph Fourier transform extends the discrete Fourier transform [14, Chapter 6]. See Chapter 2 for background on finite frames and the DSTFT. The GSTFT is a representation of a function on the the graph with entries indexed by vertex v_i and eigenvalue λ_j for $i, j \in \{1, 2, \dots, N\}$. We get a simpler representation of this

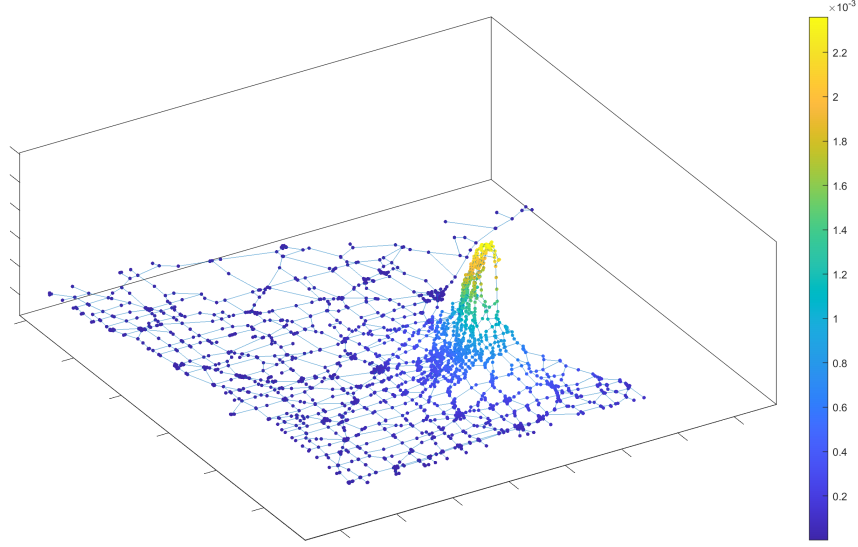


Figure 3.3: The graph heat kernel H_t on the Minnesota road graph dataset [82]. Chapter 3 is a time-frequency analysis on graphs when the underlying graph is not the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$.

GSTFT in terms of an inner product by defining an associated graph Gabor system. Let $D_i(t) \in \mathbb{R}^{N \times N}$ be the diagonal matrix containing the i -th column of the graph heat kernel matrix:

$$D_i(t) = \text{diag}(H_t(\cdot, v_i)).$$

Then, the *graph Gabor system* associated with the GSTFT is

$$\{\psi_{ij}(t)\}_{i,j=1}^N := \{D_i(t)\varphi_{\lambda_j}\}_{i,j=1}^N.$$

Thus, we can write each entry of the GSTFT $V_t f(v_i, \lambda_j)$ as the inner product

$$(V_t f)(v_i, \lambda_j) = \langle f, \psi_{ij}(t) \rangle.$$

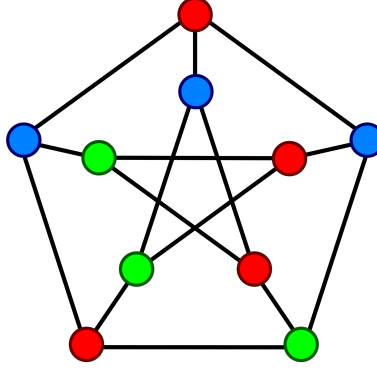


Figure 3.4: The Petersen graph is the smallest vertex-transitive graph that is not a Cayley graph of an underlying group. The GSTFT in Equation 3.1 is applied to the vertex-transitive graphs in Section 3.3.

With the classical short-time Fourier transform and associated Gabor system, the function can be reconstructed from its sampling on the time-frequency lattice when the Gabor system forms a frame. To show this graph Gabor system forms a frame for $\mathbb{C}^N$, we introduce the associated operators. The *analysis operator* $A(t) \in \mathbb{C}^{N^2 \times N}$ is the matrix with rows that are the conjugate transpose graph Gabor elements $\{\overline{\psi(t)_{ij}}\}_{i,j=1}^N$. The corresponding *frame operator* $S(t) \in \mathbb{C}^{N \times N}$ is the composition $S(t) := A(t)^* A(t)$ and the *Gram matrix* $G(t) \in \mathbb{C}^{N^2 \times N^2}$ is the composition $G(t) := A(t) A(t)^*$. We show that the frame operator $S(t)$ is a diagonal matrix, explicitly compute its form, and show the set of graph Gabor system always forms a frame.

Theorem 3.2.1 (Graph Gabor Frame) *The set of graph Gabor elements $\{\psi_{ij}(t)\}_{i,j=1}^N$ forms a frame for $\mathbb{C}^N$ for all $t \geq 0$. Furthermore, the frame operator $S(t)$ is a diagonal matrix and has the following form:*

$$S(t) = \sum_{i=1}^N D_i^2(t).$$

Proof: Fix $i \in \{1, 2, \dots, N\}$. Then consider the $N \times N$ matrix $D_i(t)\Phi$. This matrix has N columns that are a subset of the N^2 set of Gabor elements. This matrix is the composition of a full-rank diagonal matrix and a unitary matrix and hence spans $\mathbb{C}^N$. Now, a column of A_t^* is $D_i(t)\varphi_{\lambda_j}$ and a row of A_t is $\overline{\varphi_{\lambda_j}} D_i(t)$ for $i, j \in \{1, 2, \dots, N\}$.

We write the frame operator as a sum of N^2 outer products:

$$\begin{aligned} S(t) &= \sum_{i=1}^N \sum_{j=1}^N D_i(t) \varphi_{\lambda_j} \overline{\varphi_{\lambda_j}} D_i(t) = \sum_{i=1}^N D_i(t) \left(\sum_{j=1}^N \varphi_{\lambda_j} \overline{\varphi_{\lambda_j}} \right) D_i(t) \\ &= \sum_{i=1}^N D_i(t) I_{N \times N} D_i(t) = \sum_{i=1}^N D_i^2(t). \end{aligned} \quad \square$$

We leverage this explicit form of the frame operator to explicitly characterize its spectrum and describe when the frame formed by the graph Gabor elements are a tight frame.

Theorem 3.2.2 (Eigenvalues of $S(t)$) *The eigenvalues $\gamma_j(t)$ for $j \in \{1, \dots, N\}$ of the frame operator $S(t)$ are the norm-squared of the corresponding columns of the graph heat kernel,*

$$\gamma_j(t) = \|H_t(\cdot, v_j)\|^2 = \sum_{\ell=1}^N e^{-2\lambda_\ell t} |\varphi_{\lambda_\ell}(v_j)|^2.$$

In particular, the eigenvalues are positive and the frame for $\mathbb{C}^N$ formed by the graph Gabor system $\{\psi_{ij}(t)\}_{i,j=1}^N$ is tight for times $t \geq 0$ when the columns of the graph heat kernel all have the same norm.

Proof: By Theorem 3.2.1, we know that the frame operator is a diagonal matrix and thus has eigenvalues given by the diagonal entries with eigenvectors as the standard basis vectors. By direct computation, these eigenvalues $\gamma_j(t)$ for $j \in \{1, \dots, N\}$ have the form

$$\gamma_j(t) = S_{jj}(t) = \sum_{i=1}^N H_t(v_i, v_j)^2 = \|H_t(\cdot, v_j)\|^2.$$

The frame $\{\psi_{ij}(t)\}_{i,j=1}^N$ is tight when the eigenvalues of $S(t)$ are equal; that is, for times $t \in \mathbb{R}_{\geq 0}$ when $\|H_t(\cdot, v_i)\|^2 = \|H_t(\cdot, v_j)\|^2$ for all $i, j \in \{1, 2, \dots, N\}$. To obtain an

explicit formula for these eigenvalues, we compute:

$$\begin{aligned}
\langle H_t(\cdot, v_i), H_t(\cdot, v_i) \rangle &= \sum_{k=1}^N H_t(v_k, v_i) \overline{H_t(v_k, v_i)} \\
&= \sum_{k=1}^N \left[\sum_{\ell=1}^N e^{-\lambda_\ell t} \varphi_{\lambda_\ell}(v_k) \overline{\varphi_{\lambda_\ell}(v_i)} \right] \left[\sum_{\ell'=1}^N e^{-\lambda_{\ell'} t} \overline{\varphi_{\lambda_{\ell'}}(v_k)} \varphi_{\lambda_{\ell'}}(v_i) \right] \\
&= \sum_{k=1}^N \left[\sum_{\ell=1}^N \sum_{\ell'=1}^N e^{-(\lambda_\ell + \lambda_{\ell'}) t} \varphi_{\lambda_\ell}(v_k) \overline{\varphi_{\lambda_{\ell'}}(v_k)} \varphi_{\lambda_{\ell'}}(v_i) \overline{\varphi_{\lambda_\ell}(v_i)} \right] \\
&= \sum_{\ell=1}^N \sum_{\ell'=1}^N e^{-(\lambda_\ell + \lambda_{\ell'}) t} \varphi_{\lambda_{\ell'}}(v_i) \overline{\varphi_{\lambda_\ell}(v_i)} \left[\sum_{k=1}^N \varphi_{\lambda_\ell}(v_k) \overline{\varphi_{\lambda_{\ell'}}(v_k)} \right] \\
&= \sum_{\ell=1}^N \sum_{\ell'=1}^N e^{-(\lambda_\ell + \lambda_{\ell'}) t} \varphi_{\lambda_{\ell'}}(v_i) \overline{\varphi_{\lambda_\ell}(v_i)} \delta_{\ell\ell'} \\
&= \sum_{\ell=1}^N e^{-2\lambda_\ell t} |\varphi_{\lambda_\ell}(v_i)|^2.
\end{aligned}$$

If $|\varphi_{\lambda_\ell}(v_i)|^2 = 0$, $\forall \ell \in \{1, 2, \dots, N\}$, then $\{\varphi_{\lambda_\ell}\}_{\ell=1}^N$ would not be a basis for $\mathbb{C}^N$. Therefore, $|\varphi_{\lambda_\ell}(v_i)|^2 > 0$ for some $\ell \in \{1, 2, \dots, N\}$ and so $\sum_{\ell=1}^N e^{-2\lambda_\ell t} |\varphi_{\lambda_\ell}(v_i)|^2 > 0$. $\square$

In the limit as $t \rightarrow \infty$, the graph Gabor frame evolves towards a tight frame at a rate which is dominated by size of second smallest eigenvalue or *Fiedler value* of the graph Laplacian. This is due to the norm-squared of the columns of the graph heat kernel decaying at a rate that depends on the quantity $e^{-2\lambda_\ell t}$ for $\ell = 1, \dots, N$ as seen in the above theorem. Since the graphs we are considering are connected, $\lambda_1 = 0$ with unit multiplicity and thus $\lim_{t \rightarrow \infty} \|H_t(\cdot, v_i)\|^2 = 1/\sqrt{N}$, for all $v_i \in V$. In Figure 3.5, a collection of random regular graphs are generated using the *pairing model* (see [9] and [69]) and the eigenvalue gap $|\gamma_{\max} - \gamma_{\min}|$ of the frame operator $S(t)$ is plotted as a function of t . We observe that the graph Gabor frame becomes tight at a faster rate for graphs with a larger Fiedler value.

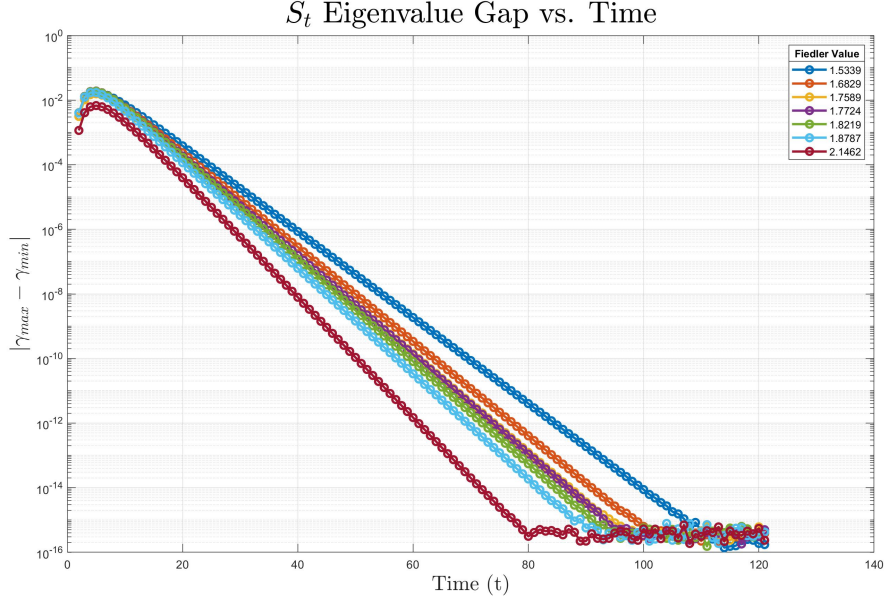


Figure 3.5: Difference between the maximum and minimum eigenvalue of the graph Gabor frame operator $S(t)$ as a function of time. Regular graphs with larger Fiedler values see faster decay towards a tight frame.

3.2.2 Inversion

Since the graph Gabor system forms a frame, we can reconstruct a function on the graph from the frame elements using the GSTFT as weights.

Theorem 3.2.3 (Inversion) *The operator $W_t : \mathbb{C}^{N \times N} \rightarrow \mathbb{C}^N$ defined as*

$$(W_t F)(v_i) = \frac{1}{\|H_t(\cdot, v_i)\|^2} \sum_{j=1}^N \sum_{k=1}^N F(v_k, \lambda_j) \psi_{jk}(t)(v_i)$$

is a left inverse for the GSTFT V_t .

Proof: By direct computation:

$$\begin{aligned}
(W_t V_t f)(v_i) &= \frac{1}{\|H_t(\cdot, v_i)\|^2} \sum_{j=1}^N \varphi_j(v_i) \left[\sum_{k=1}^N V_t f(v_k, \lambda_j) H_t(v_k, v_i) \right] \\
&= \frac{1}{\|H_t(\cdot, v_i)\|^2} \sum_{j=1}^N \varphi_j(v_i) \left[\sum_{k=1}^N \left[\sum_{\ell=1}^N f(v_\ell) \overline{\varphi_j(v_\ell)} H_t(v_\ell, v_k) \right] H_t(v_k, v_i) \right] \\
&= \frac{1}{\|H_t(\cdot, v_i)\|^2} \sum_{j=1}^N \sum_{k=1}^N \sum_{\ell=1}^N f(v_\ell) \varphi_j(v_i) \overline{\varphi_j(v_\ell)} H_t(v_\ell, v_k) H_t(v_k, v_i) \\
&= \frac{1}{\|H_t(\cdot, v_i)\|^2} \sum_{k=1}^N \sum_{\ell=1}^N f(v_\ell) H_t(v_\ell, v_k) H_t(v_k, v_i) \underbrace{\left[\sum_{j=1}^N \varphi_j(v_i) \overline{\varphi_j(v_\ell)} \right]}_{\delta_{i\ell}} \quad \square \\
&= \frac{1}{\|H_t(\cdot, v_i)\|^2} \sum_{k=1}^N f(v_i) H_t(v_i, v_k) H_t(v_i, v_k) \\
&= f(v_i) \frac{1}{\|H_t(\cdot, v_i)\|^2} \underbrace{\sum_{k=1}^N H_t(v_i, v_k)^2}_{\|H_t(\cdot, v_i)\|^2} \\
&= f(v_i).
\end{aligned}$$

Note that this definition based on the interplay between the graph Laplacian Δ and the graph heat kernel H_t is remarkably tied to a notion of graph short-time Fourier transforms defined via a *graph modulation* and *graph translation* operator in *Vertex-Frequency Analysis on Graphs* by Shuman et al. [82]. Recall that translation of a function on the graph $T_i f(v_k)$ from vertex v_k to vertex v_i for $i, k \in \{1, \dots, N\}$ was a kernel operation (a convolution) defined in the graph Fourier domain as

$$(T_i f)(v_k) := \sqrt{N} (f * \delta_i)(v_k) = \sqrt{N} \sum_{j=1}^N \hat{f}(\lambda_j) \overline{\varphi_{\lambda_j}(v_i)}.$$

Further, modulation $(M_j f)(v_i)$ for $j \in \{1, \dots, N\}$ was defined via pointwise multiplication with a eigenvector φ_j ,

$$(M_j f)(v_k) := \sqrt{N} f(v_k) \varphi_{\lambda_j}(v_k).$$

Their *windowed Fourier transform* is the composition of these operators as in the classical case of the STFT. For a window function $g \in \mathbb{R}^N$, they define a *windowed*

graph Fourier atom by:

$$g_{ij}(v_k) := (M_j T_i g)(v_k) = \underbrace{\sqrt{N} \varphi_{\lambda_j}(v_k)}_{\text{Modulation}} \underbrace{\sqrt{N} \sum_{\ell=1}^N \hat{g}(\lambda_\ell) \overline{\varphi_{\lambda_\ell}(v_i)} \varphi_{\lambda_\ell}(v_k)}_{\text{Translation}}.$$

Define a window in the spectral domain as $\hat{g}(\lambda_\ell) = C e^{-\tau \lambda_\ell}$ for a fixed $\tau > 0$, where the constant C is chosen so that $\|g\| = 1$. This is an example that is studied in [82] on the highly regular *Minnesota road graph* dataset [77]. The window is defined in the graph Fourier domain since the translation operator is a kernel operator given by integrating against $\hat{g}$. The (i, j) entry of this windowed Fourier transform of a function $f : V \rightarrow \mathbb{R}$ is then given by:

$$Sf(v_i, \lambda_j) := \langle f, g_{ij} \rangle = N \sum_{k=1}^N f(v_k) \varphi_{\lambda_j}(v_k) \left(\sum_{\ell=1}^N \hat{g}(\lambda_\ell) \overline{\varphi_{\lambda_\ell}(v_i)} \varphi_{\lambda_\ell}(v_k) \right). \quad (3.2)$$

If the window is chosen to be this window $\hat{g}(\lambda_\ell) = C e^{-\tau \lambda_\ell}$, then the above is written as:

$$Sf(v_i, \lambda_j) := \langle f, g_{ij} \rangle = N \sum_{k=1}^N f(v_k) \left[\sum_{\ell=1}^N C e^{-\tau \lambda_\ell} \varphi_{\lambda_\ell}(v_i) \overline{\varphi_{\lambda_\ell}(v_k)} \right] \overline{\varphi_{\lambda_j}(v_k)}.$$

Comparing with the GSTFT defined in Equation 3.1, we can see the two definitions coincide for functions $f : V \rightarrow \mathbb{R}$ by unpacking the definition of the graph heat kernel H_t :

$$\begin{aligned} (V_t f)(v_i, \lambda_j) &:= \sum_{k=1}^N f(v_k) H_t(v_i, v_k) \overline{\varphi_{\lambda_j}(v_k)} \\ &= \sum_{k=1}^N f(v_k) \left(\sum_{\ell=1}^N e^{-\lambda_\ell t} \varphi_{\lambda_\ell}(v_i) \overline{\varphi_{\lambda_\ell}(v_k)} \right) \overline{\varphi_{\lambda_j}(v_k)}. \end{aligned}$$

3.3 Tight Frames on Vertex-Transitive and Strongly Regular Graphs

Given that the graph Gabor system is a frame and parametrized by $t \in \mathbb{R}_{\geq 0}$, a natural question to ask is for what values of t is the frame tight? As discussed in Section 4.2, tight frames have remarkable reconstruction properties and closely resemble orthonormal bases. Here we show that on two classes of algebraic graphs, the corresponding graph Gabor frame is tight $\forall t \in \mathbb{R}_{\geq 0}$. We recall the definition of a graph automorphism which define *vertex-transitive graphs* and *strongly regular graphs*. As in the previous section, the graph $G = (V, E)$ is finite, undirected and unweighted, and assumed to be connected.

An *automorphism* of a graph G is a permutation π of the vertices V with the property that $(u, v) \in E$ if and only if $(\pi(u), \pi(v)) \in E$. The set of all automorphisms Π of a graph together with the operation of composition $\text{Aut}(G) = (\Pi, \circ)$ is a group called the *automorphism group*. A graph G is called *vertex-transitive* if $\text{Aut}(G)$ acts transitively on V . That is, for all pairs of vertices $u, v \in V$ there exists a permutation $\pi \in \text{Aut}(G)$ such that $\pi(u) = v$. This can also be understood in terms of group actions of $\text{Aut}(G)$. Essentially, each vertex is structurally identical and only distinguished via labeling under these permutations. A graph G on $|V| = n$ vertices that is neither complete nor empty is said to be *strongly-regular* with parameters (n, k, a, c) if:

1. it is k -regular,
2. every pair of adjacent vertices has a common neighbors,
3. every pair of non-adjacent vertices has c common neighbors.

3.3.1 Vertex-Transitive Graphs

A consequence of a graph being vertex-transitive is the adjacency matrix commutes with permutation matrices representing the maps $\pi \in \text{Aut}(G)$. A permutation π of the vertex set V can always be represented by a permutation matrix $P \in \mathbb{R}^{N \times N}$, where $P_{ij} = 1$ if $v_i = \pi(v_j)$ and $P_{ij} = 0$ otherwise.

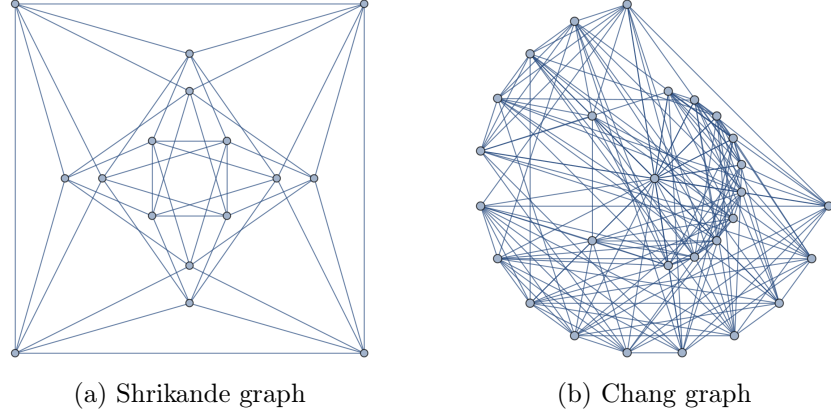


Figure 3.6: The *Shrikande graph* is a vertex-transitive and strongly regular graph with parameters $(16, 6, 2, 2)$; it is the Cayley graph of $\mathbb{Z}_4 \times \mathbb{Z}_4$. The *Chang graphs* are a family of strongly regular graphs with parameters $(28, 12, 6, 4)$.

Lemma 3.3.1 (PA=AP, Chapter 15 [8]) *Let A be the adjacency matrix of a graph G and π a permutation of vertices V . Then π is an automorphism of G if and only if $PA = AP$, where P is the permutation matrix representing π .*

A column of the graph heat kernel $H_t(\cdot, v_i)$ can be interpreted as solutions to the graph heat equation with initial conditions of a Dirac mass at the corresponding vertex:

$$\begin{aligned} \frac{d}{dt} H_t(\cdot, v_i) &= \Delta H_t(\cdot, v_i) \\ H_0(v_j, v_i) &= \delta_{ij}. \end{aligned}$$

This, together with Proposition 3.3.1 and Theorem 3.2.2 leads to Theorem 3.3.2. The intuition is the following: one can map every vertex to another via a permutation such that the resulting graph is isomorphic to the original. Therefore, heat flow out of a source vertex can be mapped to heat flow out of another source vertex. Since the columns of the graph heat kernel describe such heat flow, each column will be a permutation of another and therefore have the same norm. Note, that the following proof works for any operator on the graph that commutes with the permutation matrix P and in particular matrices of the form $f(tA)$, for analytic f . We focus on the case where the operator is the graph heat kernel, $f(tA) = \exp(-t\Delta)$. Using

the graph heat kernel has a number of advantages, including that the exponential guarantees 1) the graph Gabor system 3.2.1 is a frame for all times $t \in \mathbb{R}_{\geq 0}$ due to exponential being positive and analytic and 2) simple expressions for the eigenvalues of the frame operator $S(t)$ seen in Theorem 3.2.2 making the determination of a tight frame amenable to analysis. If one were to choose for example $f(tA) = \sin(tA)$ instead of the exponential, we could not guarantee that the existence of a frame due to its zeroes which would require careful analysis of the set of diagonal matrices $D_i(t)$ that appear in Theorem 3.2.1. Investigating alternative analytic functions f and therefore alternative windows besides the graph heat kernel is a direction of future work.

Theorem 3.3.2 (Tightness Conditions for $\{\psi_{ij}(t)\}_{i,j=1}^N$ I) *The graph Gabor frame $\{\psi_{ij}(t)\}_{i,j=1}^N$ forms a tight frame for $\forall t \in [0, \infty)$ when the underlying graph is vertex-transitive.*

Proof: Let $\pi \in \text{Aut}(G)$ be the automorphism such that $\pi(v_i) = v_j$. Then the associated permutation matrix P has the property that $Pe_i = e_j$. Thus:

$$h_j := H_t e_j = H_t (Pe_i) = (H_t P) e_i.$$

Now, if the graph heat kernel H_t and the permutation matrix P commute then we are done as $h_j = (H_t P) e_i = (P H_t) e_i = P h_i$. The graph G is assumed to be vertex-transitive and hence $PA = AP$. Therefore, since $H_t = \sum_{\ell=0}^{\infty} \frac{(-t)^\ell}{\ell!} \Delta^\ell$, we can determine if P commutes with H by checking if P commutes with Δ . Let $k \in \mathbb{Z}^+$ be the degree of each vertex in G as vertex-transitive graphs are regular [8, Chapter 3]. We have:

$$\begin{aligned} PL &= P(D - A) = P(kI - A) \\ &= kPI - PA \\ &= kIP - AP \\ &= (kI - A)P \\ &= (D - A)P \end{aligned}$$

$$= LP.$$

Thus, $LP = PL$ and hence $H_t P = P H_t$. Therefore, $h_j = P h_i$. We conclude that each column of the graph heat kernel is a permutation of every other column of the graph heat kernel, thus preserving column norm. $\square$

The vertex transitive graphs are a large class of graphs in algebraic graph theory and contain Cayley graphs, distance transitive graphs, and symmetric graphs [40]. For example, the *Shrikande graph* seen in Figure 3.6a is a Cayley graph on the group $\mathbb{Z}_4 \times \mathbb{Z}_4$ and therefore is also vertex-transitive [40].

3.3.2 Strongly Regular Graphs

Another large class of graphs in algebraic graph theory are the strongly regular graphs. They are an important class of graphs in frame theory and quantum computing due in part to the one-to-one correspondence between *equiangular tight frames* for $\mathbb{R}^N$ that minimize the *coherence* $\max_{i,i'} |\langle \psi_i, \psi_{i'} \rangle|^2$ and a subset of strongly regular graphs [91]. In quantum computing, the equiangular tight frames for $\mathbb{C}^N$ are called SIC-POVMs (*Symmetric Informationally Complete Positive Operator Valued Measurements*) and conditions on their existence are part of the *Zauner conjecture* and relevant for quantum tomography and cryptography [76]. Further, strongly regular graphs are good candidates for state transfer in continuous-time quantum walks [20, 39].

Building on well-understood properties of the spectrum of strongly regular graphs, we show that the graph Gabor frame $\{\psi_{ij}(t)\}_{i,j=1}^N$ is tight and independent of $t \in \mathbb{R}_{\geq 0}$. The spectrum of a strongly regular graph (its adjacency matrix $A \in \mathbb{R}^{N \times N}$) is well-understood in that the eigenvalues and their multiplicities can be explicitly computed in terms of the parameters (n, k, a, c) [8]. There are exactly three eigenvalues $\nu_1, \nu_2, \nu_3 \in \mathbb{R}$ of the adjacency matrix A :

$$\nu_1 = k, \quad \nu_2, \nu_3 = \frac{a - c \pm \sqrt{(a - c)^2 + 4(k - c)}}{2}.$$

Since strongly regular graphs are indeed regular, the spectrum of the graph Laplacian $\Delta = kI - A$ is simply $\{\lambda_i = k - \nu_i\}_{i=1}^3$.

Let m_1, m_2, m_3 denote the multiplicities of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of Δ . The graph Laplacian Δ in our setting has eigenvalue $\lambda_1 = 0$ with the corresponding normalized eigenvector $\mathbf{1}/\sqrt{N}$ [19]. Since an eigenvalue of the frame operator is the norm-squared of a column of the graph heat kernel by Theorem 3.2.2, we have the following expression for the eigenvalues of $S(t)$ for strongly regular graphs by decomposing the sum in Theorem 3.2.2 according to the eigenvalue multiplicities:

$$\begin{aligned}\gamma_i(t) &= \sum_{k=1}^N e^{-2\lambda_k t} |\varphi_k(v_i)|^2 \\ &= \frac{1}{N} + e^{-2\lambda_2 t} \sum_{k=2}^{m_2+1} |\varphi_k(v_i)|^2 + e^{-2\lambda_3 t} \sum_{k=m_2+2}^N |\varphi_k(v_i)|^2.\end{aligned}\tag{3.4}$$

We leverage this expression along with a simple counting argument to establish that $\{\psi_{ij}(t)\}_{i,j=1}^N$ is tight and independent of $t \in \mathbb{R}_{\geq 0}$.

Theorem 3.3.3 (Tightness Conditions for $\{\psi_{ij}(t)\}_{i,j=1}^N$ II) *The graph Gabor frame $\{\psi_{ij}(t)\}_{i,j=1}^N$ forms a tight frame for every time $t \in [0, \infty)$ when the underlying graph is strongly regular.*

Proof: Since the smallest eigenvalue of Δ is zero with normalized eigenvector $\mathbf{1}/\sqrt{N}$, we have for $i \in \{1, \dots, N\}$:

$$1 = \frac{1}{N} + \sum_{k=2}^{m_2+1} |\varphi_k(v_i)|^2 + \sum_{k=m_2+2}^N |\varphi_k(v_i)|^2.\tag{3.5}$$

From Equation (3.5) we get

$$\sum_{k=m_2+2}^N |\varphi_k(v_i)|^2 = \frac{N-1}{N} - \sum_{k=2}^{m_2+1} |\varphi_k(v_i)|^2.\tag{3.6}$$

Substituting Equation (3.6) into Equation (3.4) we see that for any $i \neq j$

$$\gamma_i(t) - \gamma_j(t) = (e^{-2\lambda_2 t} - e^{-2\lambda_3 t}) \left(\sum_{k=2}^{m_2+1} |\varphi_k(v_i)|^2 - |\varphi_k(v_j)|^2 \right).\tag{3.7}$$

By a simple count, $\|Le_i\|^2 = d^2 + d$ for all $i \in \{1, \dots, N\}$ since strongly regular graphs are regular. Further, via the spectral theorem the eigenvectors form an orthonormal basis and so $\Phi\Phi^* = I$. In particular,

$$d^2 + d = \|Le_i\|^2 = \|\Phi\Lambda\Phi^*e_i\|^2 = \left\| \sum_{k=1}^N \lambda_k \varphi_k \overline{\varphi_k} e_i \right\|^2 = \lambda_2^2 \sum_{k=2}^{m_2+1} |\varphi_k(v_i)|^2 + \lambda_3^2 \sum_{k=m_2+2}^N |\varphi_k(v_i)|^2.$$

It follows that

$$\sum_{k=m_2+2}^N |\varphi_k(v_i)|^2 = \frac{(d^2 + d) - \lambda_3^2 \sum_{k=2}^{m_2+1} |\varphi_k(v_i)|^2}{\lambda_2^2}. \quad (3.8)$$

Using Equation (3.6) and Equation (3.8) we have

$$\sum_{k=2}^{m_2+1} |\varphi_k(v_i)|^2 = \frac{(\frac{N-1}{N})\lambda_3^2 - (d^2 + d)}{\lambda_3^2 - \lambda_2^2}. \quad (3.9)$$

Because Equation (3.9) is independent of i and j we conclude that the term $\sum_{k=2}^{m_2+1} |\varphi_k(v_i)|^2 - |\varphi_k(v_j)|^2 = 0$ in Equation (3.7) is zero. Consequently, $\gamma_i(t) - \gamma_j(t) = 0$ for all $i \neq j$. $\square$

Chapter 4

A Short-time Fourier Transform on the Sierpinski Gasket and Post-Critically Finite Fractals

4.1 Introduction

Time-frequency analysis is a local Fourier analysis that uses the operations of translations and modulations to analyze functions of a particular class such as $L^2(\mathbb{R}^d)$. The *short-time Fourier transform* (STFT) is a central tool within time-frequency analysis and widely used across applications in signal processing and quantum mechanics. Pioneered by J. von Neumann and Nobel prize winner Dennis Gabor [33] in the 20th century, the STFT provides joint time-frequency or space-frequency information of a function defined on $\mathbb{R}^d$. It is local Fourier analysis in the sense that it modifies the classical Fourier transform to focus on specific regions of support via *windowing*. For example, in speech analysis and audio signal processing the STFT is used to detect pitch changes and transient events in the underlying signal [45, 83]. In image processing and optics, the STFT and associated Gabor transform has been applied to denoising image patches, feature extraction, and coherent optical setups [5, 25]. Windowed Fourier methods like the STFT and wavelet transforms are fundamental to communication algorithms and to compression algorithms such as JPEG [4, 68]. The mathematical theory is rich and draws from diverse fields including complex analysis, partial differential equations, the Heisenberg group, representation theory, and operator algebras [30, 47].

A natural question to ask is what other spaces can one start to construct a theory of time-frequency analysis via the STFT? Besides $L^2(\mathbb{R}^d)$, one can work with $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ for conjugate p, q or if f is a tempered distribution and g

is a Schwartz function defined on $\mathbb{R}^d$ [47]. Theories of the STFT also exist on non-Euclidean domains such as spheres and graphs [13, 32, 52, 82]. We introduce a notion of a STFT on the Sierpinski gasket (SG) fractal and show the construction can be extended to the larger class of *post-critically finite fractals* (pcf). Two example differences between Fourier analysis on fractals and Fourier analysis on Euclidean $\mathbb{R}^d$ are the existence of localized eigenfunctions [72] of a Laplacian and the absence of a group structure such as $(\mathbb{R}, +)$ that allows for translation. Further, to derive an inversion formula for the STFT on $\mathbb{R}^d$ one uses the translation invariance of the Lebesgue measure. While these differences cannot be rectified, we show one can still window with a Gaussian-like kernel as in the classical case and recover desirable properties of the STFT such as being an isometry into its range and provide an explicit inversion formula. The construction relies on the eigenfunctions of a Laplacian Δ on SG and the corresponding heat kernel $p_b(t, x, y)$ constructed from the eigenvalues and eigenfunctions of Δ . We summarize our contributions as follows:

1. We present a definition of a STFT V_t on SG in Section 4.2 and show it is well-defined, finite, and continuous for functions in $L^2(SG, \mu)$.
2. In Section 4.2.2, V_t is shown to be an isometry into its codomain $L^2(SG \times \Lambda, \mu \times \sigma)$ when using a measure weighted by a heat kernel. This leads to an explicit expression for the adjoint V_t^* as its left inverse and the main theorem of invertibility.
3. Section 4.3 details additional properties such as trade-offs in the support of the energy of V_t and the window size $t > 0$, example computations, and shows V_t is a solution to a heat equation on SG .
4. The definition of V_t and its properties from Section 4.2 are extended to the larger class of pcf fractals whose spectral dimension $d_S < 2$ in Section 4.4.

4.2 A STFT on the Sierpinski Gasket

4.2.1 Definition of the STFT $V_t f(x, \lambda)$

The heat kernel $p_b(t, x, y)$ is used as the window function to define a STFT on SG . The heat kernel on $\mathbb{R}^d$ is a Gaussian that minimizes a number of uncertainty principles and historically a natural choice of window in the STFT. On SG the heat kernel $p_b(t, x, y)$ (1) is bounded, smooth, integrable with unit integral in either space coordinate (2) has a tunable time parameter $t \geq 0$ that allows one to consider a *family* of windows by letting heat diffuse across the fractal (3) it is the integral kernel for the semigroup $e^{-t\Delta}$ associated to a Laplacian Δ on SG and has an explicit representation in terms of the eigenvalues and eigenfunctions and (5) has well-understood and well-behaved decay estimates that are leveraged to prove boundedness and invertibility of the STFT on SG .

We state the definition of the STFT V_t on SG and the main theorem of invertibility. Recall that $L^2(SG, \mu)$ is the space of square-integrable functions on SG with self-similar measure μ . Let Λ be the countable spectrum of the Dirichlet or Neumann Laplacian Δ and σ the counting measure. We denote $\|\cdot\|_{L^2(SG, \mu)} = \|\cdot\|_\mu$ for the $L^2(SG, \mu)$ -norm and $\|\cdot\|_{L^2(\Lambda, \sigma)} = \|\cdot\|_\sigma$ for the $L^2(\Lambda, \sigma)$ -norm. For background on analysis on fractals, see Section 2.5 and Kigami [62]. For $f \in L^2(SG, \mu)$ and fixed $t > 0$, the *short-time Fourier transform* $V_t : L^2(SG, \mu) \rightarrow L^2(SG \times \Lambda, \mu \times \sigma)$ of f is defined as the integral

$$V_t f(x, \lambda) = \int_{SG} f(y) p_b(t, x, y) \overline{\phi_\lambda(y)} d\mu(y). \quad (4.1)$$

For fixed $t > 0$ and λ , $V_t f(x, \lambda_n)$ is finite for any $f \in L^2(SG, \mu)$ and is (uniformly) continuous, as is the case for the classical STFT on $\mathbb{R}^d$.

Proposition 4.2.1 (Properties of $V_t f(x, \lambda)$) *Fix $t > 0$. For $f \in L^2(SG, \mu)$, the STFT $V_t f(x, \lambda)$ on SG is defined pointwise and finite with bound*

$$|V_t f(x, \lambda)| \leq c_1 t^{-d_S \bar{\delta}/2} \|f\|_\mu,$$

for some $c_1 > 0$. For fixed $\lambda \in \Lambda$ it is (uniformly) continuous in x .

Proof: Applying Cauchy-Schwartz, the uniform bound on the heat kernel $p_b(t, x, y)$ seen in Equation (2.27), and noting $\|\phi_\lambda\|_\mu = 1$, we have

$$\begin{aligned} |V_t f(x, \lambda)| &= |\langle f, p_b(t, x, \cdot) \phi_\lambda \rangle_\mu| \\ &\leq \|f\|_\mu \|p_b(t, x, \cdot) \phi_\lambda\|_\mu \\ &\leq c_1 t^{-d_S \bar{\delta}/2} \|f\|_\mu \|\phi_\lambda\|_\mu. \end{aligned}$$

For continuity, fix $\lambda \in \Lambda$, and apply Cauchy-Schwartz:

$$\begin{aligned} |V_t f(x, \lambda) - V_t f(x', \lambda)|^2 &= |\langle f, p_b(t, x, \cdot) \phi_\lambda \rangle_\mu - \langle f, p_b(t, x', \cdot) \phi_\lambda \rangle_\mu|^2 \\ &= |\langle f, p_b(t, x, \cdot) \phi_\lambda - p_b(t, x', \cdot) \phi_\lambda \rangle_\mu|^2 \\ &\leq \|f\|_\mu^2 \|(p_b(t, x, \cdot) - p_b(t, x', \cdot)) \phi_\lambda\|_\mu^2. \end{aligned}$$

Since $p_b(t, x, y)$ is bounded and continuous in both x and y , apply dominated convergence:

$$\begin{aligned} \lim_{x \rightarrow x'} |V_t f(x, \lambda) - V_t f(x', \lambda)|^2 &\leq \|f\|_\mu^2 \lim_{x \rightarrow x'} \int_{SG} |p_b(t, x, \cdot) - p_b(t, x', \cdot) \phi_\lambda|^2 d\mu(y) \\ &= \|f\|_\mu^2 \int_{SG} \lim_{x \rightarrow x'} |p_b(t, x, y) - p_b(t, x', y) \phi_\lambda(y)|^2 d\mu(y) \\ &= 0. \end{aligned}$$

Since SG is compact in either the topology induced by the Euclidean metric or the resistance metric [62, Chapter 2], $V_t f(\cdot, \lambda)$ is uniformly continuous. $\square$

The main result is Theorem 4.2.2 stating that the STFT V_t on SG has a left inverse and admits an explicit inversion formula. Hence $f \in L^2(SG, \mu)$ can be expressed (recovered) from its STFT V_t . Indeed, V_t is bounded from below (a consequence of Equation (2.28)) and is an isometry when using a measure weighted by the heat kernel, which is the statement of Proposition 4.2.4. The explicit formula for the adjoint V_t^* and left inverse requires a number of technical results in Lemmas 4.2.5

and 4.2.7.

Theorem 4.2.2 (V_t is Invertible) *Fix $t > 0$. When using the Neumann heat kernel ($b = N$) the operator $W_t = V_t^* : L^2(SG \times \Lambda, \mu \times \sigma) \rightarrow L^2(SG, \mu)$ is a left-inverse for the STFT V_t on SG and has the form*

$$(W_t F)(x) = \int_{\Lambda} \int_{SG} F(y, \lambda) \frac{p_N(t, x, y)}{p_N(2t, y, y)} \overline{\phi_{\lambda}(x)} d\mu(y) d\sigma(\lambda), \text{ for a.e. } x \in SG. \quad (4.2)$$

4.2.2 Orthogonality Relations and Invertibility

To justify the existence of the inverse provided in Theorem 4.2.2, we first show the STFT V_t satisfies an orthogonality relation. The classical STFT has the key property of being an isometry into $L^2(\mathbb{R}^{2d})$ which relies on the algebraic structure of $(\mathbb{R}^d, +)$ and shift invariance of the Lebesgue integral. This is seen from the orthogonality relations for the STFT where Fubini-Tonelli is applied to show

$$\begin{aligned} \langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^{2d})} &= \int_{\mathbb{R}^d} f_1(t) \overline{f_2(t)} \left(\int_{\mathbb{R}^d} \overline{g_1(t-x)} g_2(t-x) dx \right) dt \\ &= \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \langle g_1, g_2 \rangle_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Unfortunately no such structure exists to construct a translation invariant measure. The STFT V_t does not satisfy an orthogonality relation from $L^2(SG, \mu)$ into $L^2(SG \times \Lambda, \mu \times \sigma)$. However, when using the weighted measure

$$\mu_b^{t_1+t_2} := p_b(t_1 + t_2, y, y) \mu, \quad t_1, t_2 > 0$$

V_t admits an orthogonality relation and is an isometry from $L^2(SG, \mu_b^{t_1+t_2})$ into $L^2(SG \times \Lambda, \mu \times \sigma)$. Using the weighted measure $\mu_b^{t_1+t_2}$ we have the following Hilbert space inclusions.

Proposition 4.2.3 (Weighted Space Inclusions) *Fix $t > 0$. Using the Neumann ($p = N$) or Dirichlet heat kernel ($p = D$) we have the inclusion*

$$L^2(SG, \mu) \subset L^2(SG, \mu_b^{2t}).$$

With the Neumann heat kernel we have the inclusion

$$L^2(SG, \mu_N^{2t}) \subset L^2(SG, \mu).$$

Therefore with the Neumann heat kernel we have the norm equivalence $\|\cdot\|_\mu \sim \|\cdot\|_{\mu_N^{2t}}$ and identify the spaces $L^2(SG, \mu) \cong L^2(SG, \mu_N^{2t})$.

Proof: When $b = D, N$, apply the bound on $p_b(t, x, y)$ in Equation (2.27):

$$\begin{aligned} \|f\|_{\mu_b^{2t}} &= \int_{SG} |f(y)|^2 p_b(2t, y, y) d\mu(y) \\ &\leq c_1 (2t)^{-\frac{d_S \bar{\delta}}{2}} \int_{SG} |f(y)|^2 d\mu(y) \\ &= c_1 (2t)^{-\frac{d_S \bar{\delta}}{2}} \|f\|_\mu. \end{aligned}$$

When $b = N$, the diagonal $p_N(t, y, y)$ has positive lower bound seen in Equation (2.28). Therefore,

$$\begin{aligned} \|f\|_\mu &\leq c_2 (2t)^{-\frac{d_S}{2}} \int_{SG} f(y)^2 d\mu(y) \\ &\leq \int_{SG} f(y)^2 p_N(2t, y, y) d\mu(y) \\ &= \|f\|_{\mu_N^{2t}}. \end{aligned} \quad \square$$

Proposition 4.2.4 (V_t Orthogonality Relations) Fix $t_1, t_2 > 0$. The STFT $V_t : L^2(SG, \mu) \rightarrow L^2(SG \times \Lambda, \mu \times \sigma)$ satisfies the orthogonality relation

$$\langle V_{t_1} f, V_{t_2} g \rangle_{\mu \times \sigma} = \langle f, g \rangle_{\mu_b^{t_1+t_2}}, \quad \forall f, g \in L^2(SG, \mu),$$

where $\mu_b^{t_1+t_2}$ is a weighted measure of μ with density $p_b(t_1 + t_2, y, y)$. The STFT V_t is therefore an isometry into $L^2(SG \times \Lambda, \mu \times \sigma)$ when using the $\mu_b^{t_1+t_2}$ measure and consequently

1. is bounded below and has closed range,
2. is injective from $L^2(SG, \mu)$ into $L^2(SG \times \Lambda, \mu \times \sigma)$ and therefore invertible on

its range $V_t(L^2(SG, \mu)) \subset L^2(SG \times \Lambda, \mu \times \sigma)$,

3. the adjoint $V_t^* : L^2(SG \times \Lambda, \mu \times \sigma) \rightarrow L^2(SG, \mu)$ is a left inverse, $V_t^* V_t = I_{\mu_b^{2t}}$,

4. has operator norm $\|V_t\| = 1$.

Proof: Let $f \in L^2(SG, \mu)$ and recall that the heat kernel $p_b(t, x, y)$ for $t \in [0, 1)$ satisfies the supremum bound given by Equation (2.27). Thus for any $x \in SG$ and $t_1, t_2 > 0$, the product $f(\cdot)p_b(t_1, x, \cdot) \in L^2(SG, \mu)$ and $g(\cdot)p_b(t_2, x, \cdot) \in L^2(SG, \mu)$. Since V_t is an integral against the orthonormal basis of eigenfunctions $\{\phi_\lambda\}_{\lambda \in \Lambda}$, apply Parseval:

$$\begin{aligned} \langle V_{t_1} f, V_{t_2} g \rangle_{\mu \times \sigma} &= \int_{SG} \int_{\Lambda} V_{t_1} f(x, \lambda) \overline{V_{t_2} g(x, \lambda)} d\sigma(\lambda) d\mu(x) \\ &= \int_{SG} \langle f p_b(t_1, x, \cdot), g p_b(t_2, x, \cdot) \rangle_{L^2(\Lambda, \sigma)} d\mu(x) \\ &= \int_{SG} \langle f p_b(t_1, x, \cdot), g p(t_2, x, \cdot) \rangle_{L^2(SG, \mu)} d\mu(x) \\ &= \int_{SG} \int_{SG} f(y) p_b(t_1, x, y) \overline{g(y) p_b(t_2, x, y)} d\mu(y) d\mu(x) \end{aligned}$$

Applying Fubini-Tonelli, we have:

$$\begin{aligned} &\int_{SG} \int_{SG} f(y) p_b(t_1, x, y) \overline{g(y) p_b(t_2, x, y)} d\mu(y) d\mu(x) \\ &= \int_{SG} f(y) \overline{g(y)} \int_{SG} p_b(t_1, x, y) \overline{p_b(t_2, x, y)} d\mu(x) d\mu(y) \\ &= \int_{SG} f(y) \overline{g(y)} p_b(t_1 + t_2, y, y) d\mu(y) \\ &= \langle f, g \rangle_{\mu_b^{t_1+t_2}} \end{aligned}$$

Taking $t_1 = t_2$ and $f = g$,

$$\|V_t f\|_{\mu \times \sigma}^2 = \|f\|_{\mu_b^{2t}}^2.$$

□

Proposition 4.2.4 is almost a statement of invertibility of V_t as invertibility is equivalent to boundedness from below and having dense range. Since V_t is an isometry it is bounded below and has closed range. However, there is no reason a priori

to expect the range to be dense in $L^2(SG \times \Lambda, \mu \times \sigma)$. A function $f \in L^2(SG, \mu)$ can be recovered from its STFT $V_t f(x, \lambda)$ up to a set of measure zero akin to the *weak invertibility* of the classical STFT. To prove this, we require the notion of Dirichlet forms, properties of *regular Dirichlet forms* and their cores $\mathcal{C}$, see Section 2.5. We also use the Weyl-type result in Proposition 4.2.6 on the distribution of eigenvalues of the Laplacian on SG and the form of the Green's function $G(x, x)$.

Lemma 4.2.5 (Dirichlet Energy Bounds) *Fix $t > 0$. If $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form and $f \in \mathcal{C}$, the Dirichlet energy $\mathcal{E}(fp_b(t, x, \cdot))$ is uniformly bounded in x as*

$$\mathcal{E}(fp_b(t, x, \cdot)) \leq C \left(\frac{\|f\|_\infty^2}{t} + \frac{\mathcal{E}(f)}{t^{d_S \delta}} \right),$$

for some $C > 0$.

Proof: First, the Dirichlet energy of a product is bounded as

$$\mathcal{E}(fp_b) \leq \mathcal{E}(p_b(t, x, \cdot)) \|f\|_\infty^2 + \mathcal{E}(f) \|p_b(t, \cdot, \cdot)\|_\infty^2.$$

Now, $\mathcal{E}(p_b(t, x, \cdot))$ can be expressed in integral form [85][Theorem 3.7.d]:

$$\mathcal{E}(p_b(t, x, \cdot)) = \int_\Lambda \lambda |\widehat{p_b(t, x, \cdot)}(\lambda)|^2 d\sigma(\lambda).$$

The $\widehat{p_b(t, x, \cdot)}(\lambda)$ can be computed as its a linear combination of eigenfunctions of the Laplacian:

$$\begin{aligned} \widehat{p_b(t, x, \cdot)}(\lambda) &= \int_{SG} \left(\sum_{\lambda' \geq 1} e^{-\lambda' t} \phi_{\lambda'}(x) \phi_{\lambda'}(y) \right) \overline{\phi_\lambda(y)} d\mu(y) \\ &= \sum_{\lambda' \geq 1} e^{-\lambda' t} \phi_{\lambda'}(x) \int_{SG} \phi_{\lambda'}(y) \overline{\phi_\lambda(y)} d\mu(y) \\ &= e^{-\lambda t} \phi_\lambda(x). \end{aligned}$$

Since $e^{-t} \leq \frac{1}{t}$,

$$\mathcal{E}(p_b(t, x, \cdot)) = \int_\Lambda \lambda |\widehat{p_b(t, x, \cdot)}(\lambda)|^2 d\sigma(\lambda)$$

$$\begin{aligned}
&= \int_{\Lambda} \lambda |e^{-\lambda t} \phi_{\lambda}(x)|^2 d\sigma(\lambda) \\
&\leq \frac{1}{t^2} \int_{\Lambda} \frac{1}{\lambda} |\phi_{\lambda}(x)|^2 d\sigma(\lambda) \\
&= \frac{1}{t^2} G(x, x) \\
&\leq \frac{C}{t^2},
\end{aligned}$$

for some $C > 0$ since the Green's function $G(x, x)$ is bounded on SG . Therefore,

$$\begin{aligned}
\mathcal{E}(fp_b(t, x, \cdot)) &\leq \mathcal{E}(p_b(t, x, \cdot)) \|f\|_{\infty}^2 + \mathcal{E}(f) \|p_b(t, \cdot, \cdot)\|_{\infty}^2 \\
&\leq C \left(\frac{\|f\|_{\infty}^2}{t^2} + \frac{\mathcal{E}(f)}{t^{d_S \bar{\delta}}} \right). \quad \square
\end{aligned}$$

Lemma 4.2.7 allows us to interchange the order of integration in order to prove that $W_t = V_t^*$ is a left-inverse for the STFT V_t on SG . This requires Weyl-type estimates on the distribution of eigenvalues of the Laplacian Δ .

Proposition 4.2.6 (Estimates on $\rho_b(x, \mu)$, Example 4.1.6 [62]) *The eigenvalue counting function $\rho_b(x, \mu)$ for $b = D, N$ of the Laplacian Δ on SG has the form*

$$\rho_b(x, \mu) = G(\log \frac{x}{2}) x^{\frac{d_S}{2}} + o(1)$$

as $x \rightarrow \infty$, where G is a bounded and $\log \frac{5}{2}$ -periodic positive function.

We use knowledge of the distribution of eigenvalues via $\rho_b(x, \mu)$ to bound the integral $\int_{\Lambda} \frac{1}{1+\lambda} d\sigma(\lambda)$ by dividing the domain into dyadic intervals and estimating the number of eigenvalues within each interval. To show V_t is invertible with left inverse $W_t : L^2(SG \times \Lambda, \mu \times \sigma) \rightarrow L^2(SG, \mu)$ one needs to apply Fubini-Tonelli to the inner product $\langle W_t V_t f, k \rangle_{\mu}$; this is justified with Lemma 4.2.7.

Theorem 4.2.7 (Integral Bounds) *Fix $t > 0$. Then $\forall f \in \mathcal{C}$ and $\forall k \in L^2(SG, \mu)$ with $\|k\|_{\mu} = 1$ there exists a constant $C > 0$ such that*

$$\langle W_t V_t f, k \rangle_{\mu} = \int_{SG} \int_{SG} \int_{\Lambda} V_t f(x, \lambda) \frac{p_b(t, x, y)}{p_b(2t, y, y)} \overline{\phi_{\lambda}(x) k(x)} d\sigma(\lambda) d\mu(y) d\mu(x)$$

has bound

$$|\langle W_t V_t f, k \rangle_\mu| \leq C \left(\frac{\|f\|_\infty^2}{t^2} + \frac{\|f\|_\mu^2 + \mathcal{E}(f)}{t^{d_S \bar{\delta}}} \right)^{1/2} \leq C \sqrt{\mathcal{E}_*(f)} \left(\frac{1}{t^2} + \frac{1}{t^{d_S \bar{\delta}}} \right)^{1/2},$$

where $\mathcal{E}_*(\cdot) := \mathcal{E}(\cdot) + \|\cdot\|_\mu^2$. Hence, $\|W_t\|_\mu$ is bounded on $V_t(\mathcal{C})$.

Proof: Since $V_t f(x, \lambda) \in L^2(SG \times \Lambda, \mu \times \sigma)$, we have by Fubini-Tonelli for fixed $x \in SG$, $V_t f(x, \lambda_n) \in L^2(\Lambda, \sigma)$. By Cauchy-Schwartz and Lemma 4.2.5,

$$\begin{aligned} & \int_\Lambda |V_t f(x, \lambda)| |\phi_\lambda(x)| d\sigma(\lambda) \\ &= \int_\Lambda \sqrt{1+\lambda} |V_t f(x, \lambda)| \frac{|\phi_\lambda(x)|}{\sqrt{1+\lambda}} d\sigma(\lambda) \\ &\leq \left(\int_\Lambda (1+\lambda) |V_t f(x, \lambda)|^2 d\sigma(\lambda) \right)^{1/2} \left(\int_\Lambda \frac{|\phi_\lambda(x)|^2}{(1+\lambda)} d\sigma(\lambda) \right)^{1/2} \\ &= \left(\|f p_b(t, x, \cdot)\|_\mu^2 + \mathcal{E}(f p_b) \right)^{1/2} \left(\int_\Lambda \frac{|\phi_\lambda(x)|^2}{(1+\lambda)} d\sigma(\lambda) \right)^{1/2} \\ &\leq C \left(\frac{\|f\|_\mu^2}{t^{d_S \bar{\delta}}} + \frac{\|f\|_\infty^2}{t^2} + \frac{\mathcal{E}(f)}{t^{d_S \bar{\delta}}} \right)^{1/2} \left(\int_\Lambda \frac{|\phi_\lambda(x)|^2}{(1+\lambda)} d\sigma(\lambda) \right)^{1/2}. \end{aligned}$$

Define

$$C(f, t) := C \left(\frac{\|f\|_\infty^2}{t^2} + \frac{\|f\|_\mu^2 + \mathcal{E}(f)}{t^{d_S \bar{\delta}}} \right)^{1/2},$$

so the inner product is bounded as

$$\langle W_t V_t f, k \rangle_\mu \leq C(f, t) \int_{SG} |\overline{k(x)}| \int_{SG} \left| \frac{p_b(t, x, y)}{p_b(2t, y, y)} \right| \left(\int_\Lambda \frac{|\phi_\lambda(x)|^2}{(1+\lambda)} d\sigma(\lambda) \right)^{1/2} d\mu(y) d\mu(x).$$

By Proposition 4.2.6 we can estimate the spectral mass of the Laplacian Δ on dyadic intervals as

$$\begin{aligned} |\rho_b(2^{n+1}, \mu) - \rho_b(2^n, \mu)| &= \left| \left(G \left(\log \frac{2^{n+1}}{2} \right) 2^{(n+1) \frac{d_S}{2}} + o(1) \right) - \left(G \left(\frac{\log 2^n}{2} \right) 2^{n \frac{d_S}{2}} + o(1) \right) \right| \\ &\leq C 2^{\frac{nd_S}{2}} \left| 2^{\frac{d_S}{2}} M - m \right| \\ &= C 2^{\frac{nd_S}{2}}, \end{aligned}$$

where $M, m > 0$ are the upper and lower bounds respectively on the $\log \frac{5}{2}$ -periodic

function G . Therefore, we can bound the integral of $\frac{1}{1+\lambda}$ because the spectral dimension of SG has $d_S < 2$,

$$\begin{aligned} \int_{\Lambda} \frac{1}{(1+\lambda)} d\sigma(\lambda) &\leq C \sum_{n=0}^{\infty} 2^{-n} 2^{\frac{nd_S}{2}} \\ &= \frac{C}{1-2^r}, \end{aligned}$$

where $r = \frac{d_S}{2} - 1 = \frac{\log 9 / \log 5}{2} - 1$. WLOG let $\|k\|_{\mu} = 1$. Since $p_b(t, x, y)$ is $L^1(SG, \mu)$ -normalized in x , apply Cauchy-Schwartz:

$$\begin{aligned} \langle W_t V_t f, k \rangle_{\mu} &\leq C(f, t) \int_{SG} |\overline{k(x)}| \int_{SG} \left| \frac{p_b(t, x, y)}{p_b(2t, y, y)} \right| \left(\int_{\Lambda} \frac{|\phi_{\lambda}(x)|^2}{1+\lambda} d\sigma(\lambda) \right)^{1/2} d\mu(y) d\mu(x) \\ &= C(f, t) \int_{SG} |\overline{k(x)}| \left(\int_{\Lambda} \frac{|\phi_{\lambda}(x)|^2}{1+\lambda} d\sigma(\lambda) \right)^{1/2} d\mu(x) \\ &\leq C(f, t) \left(\int_{SG} \int_{\Lambda} \frac{|\phi_{\lambda}(x)|^2}{1+\lambda} d\sigma(\lambda) d\mu(x) \right)^{1/2}, \end{aligned}$$

where we absorbed the upper bound on $p_b(2t, y, y)$ into $C(f, t)$. Combining Fubini-Tonelli, Equation (12), and $\|\phi_{\lambda}\|_{\mu} = 1$, we have the inner product is bounded:

$$\begin{aligned} \langle W_t V_t f, k \rangle_{\mu} &\leq C(f, t) \left(\int_{SG} \int_{\Lambda} \frac{|\phi_{\lambda}(x)|^2}{1+\lambda} d\sigma(\lambda) d\mu(x) \right)^{1/2} \\ &= C(f, t) \left(\int_{\Lambda} \frac{1}{1+\lambda} d\sigma(\lambda) \right)^{1/2} \\ &= C(f, t) \left(\sum_{n=0}^{\infty} 2^{n(\frac{d_S}{2}-1)} \right)^{1/2} \\ &= \frac{C(f, t)}{\sqrt{1-2^r}}. \end{aligned}$$

The $\|\cdot\|_{\infty}$ -norm is bounded by the $\|\cdot\|_{\mu}$ and $\mathcal{E}(\cdot)$ -norms. Therefore,

$$\begin{aligned} C(f, t) &\leq C \left(\frac{(\sqrt{\mathcal{E}(f)} + \|f\|_{\mu})^2}{t^2} + \frac{\mathcal{E}_*(f)}{t^{d_S \bar{\delta}}} \right)^{1/2} \\ &\leq C \left(\frac{\mathcal{E}(f) + 2\sqrt{\mathcal{E}(f)}\|f\|_{\mu} + \|f\|_{\mu}^2}{t^2} + \frac{\mathcal{E}_*(f)}{t^{d_S \bar{\delta}}} \right)^{1/2} \\ &\leq C \left(\frac{\mathcal{E}(f) + \|f\|_{\mu}^2}{t^2} + \frac{\mathcal{E}_*(f)}{t^{d_S \bar{\delta}}} \right)^{1/2} \\ &= C \sqrt{\mathcal{E}_*(f)} \left(\frac{1}{t^2} + \frac{1}{t^{d_S \bar{\delta}}} \right)^{1/2}. \end{aligned}$$

□

We provide an explicit form of the left inverse W_t and show it is the adjoint $W_t = V_t^*$. This is the proof of Theorem 4.2.2 which follows from applying Fubini-Tonelli justified by Lemma 4.2.7.

Proof (Theorem 4.2.2): Let $f \in \mathcal{C}$ and $k \in L^2(SG, \mu)$. The inner product $\langle W_t V_t f, k \rangle_\mu$ has the form,

$$\begin{aligned}
\langle W_t V_t f, k \rangle_\mu &= \int_{SG} \left(\int_\Lambda \int_{SG} V_t f(y, \lambda) \frac{p_N(t, x, y)}{p_N(2t, y, y)} \overline{\phi_\lambda(x)} d\mu(y) d\sigma(\lambda) \right) \overline{k(x)} d\mu(x) \\
&= \int_{SG} \int_\Lambda \frac{V_t f(y, \lambda)}{p_N(2t, y, y)} \int_{SG} \overline{k(x)} p_N(t, x, y) \overline{\phi_\lambda(x)} d\mu(x) d\sigma(\lambda) d\mu(y) \\
&= \left\langle \frac{V_t f}{p_N^{2t}}, V_t k \right\rangle_{\mu \times \sigma} \\
&= \left\langle \frac{V_t f}{p_N^{2t}}, V_t k \right\rangle_{\mu_N^{2t}} \\
&= \langle V_t f, V_t k \rangle_\mu \\
&= \langle V_t^* V_t f, k \rangle_\mu.
\end{aligned}$$

We compute $\langle W_t V_t f, k \rangle_\mu$ for $k \in L^2(SG, \mu)$ and $f \in \mathcal{C}$:

$$\begin{aligned}
\langle W_t V_t f, k \rangle_\mu &= \int_{SG} \overline{k(x)} \int_\Lambda \overline{\phi_\lambda(x)} \int_{SG} V_t f(y, \lambda) \frac{p_N(t, x, y)}{p_N(2t, y, y)} d\mu(y) d\sigma(\lambda) d\mu(x) \\
&= \int_{SG} \int_\Lambda V_t f(y, \lambda) \left(\int_{SG} p_N(t, x, y) \overline{k(x)} \overline{\phi_\lambda(x)} d\mu(x) \right) d\sigma(\lambda) \frac{1}{p_N(2t, y, y)} d\mu(y) \\
&= \int_{SG} \int_\Lambda V_t f(y, \lambda) \overline{V_t k(y, \lambda)} d\sigma(\lambda) \frac{1}{p_N(2t, y, y)} d\mu(y) \\
&= \int_{SG} \left\langle p_N(t, \cdot, y) f, p_N(t, \cdot, y) k \right\rangle_\sigma \frac{1}{p_N(2t, y, y)} d\mu(y) \\
&= \int_{SG} \left\langle p_N(t, \cdot, y) f, p_N(t, \cdot, y) k \right\rangle_\mu \frac{1}{p_N(2t, y, y)} d\mu(y) \\
&= \int_{SG} \frac{f(y) \overline{k(y)}}{p_N(2t, y, y)} \int_{SG} p_N(t, x, y) p_N(t, x, y) d\mu(x) d\mu(y) \\
&= \langle f, k \rangle_\mu.
\end{aligned}$$

Therefore $W_t V_t = I_\mu$ when $f \in \mathcal{C}$. Also,

$$\sup_{\|f\|_\mu=1} \|W_t V_t f\|_\mu = \sup_{\|f\|_\mu=1} \|f\|_\mu = 1.$$

Now, let $f \in L^2(SG, \mu)$. The core $\mathcal{C}$ is dense in $L^2(SG, \mu)$ so there exists a sequence $\{f_n\}_{n=1}^\infty \subset \mathcal{C}$ such that $\lim_{n \rightarrow \infty} \|f_n - f\|_\mu = 0$. A subsequence $\{\langle W_t V_t f_{n_\ell}, k \rangle_\mu\}_{\ell=1}^\infty$ has a limit:

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \langle W_t V_t f_{n_\ell}, k \rangle_\mu &= \lim_{\ell \rightarrow \infty} \langle f_{n_\ell}, k \rangle_\mu \\ &= \langle \lim_{\ell \rightarrow \infty} f_{n_\ell}, k \rangle_\mu \\ &= \langle f, k \rangle_\mu. \end{aligned}$$

Since W_t is continuous on $V_t(\mathcal{C})$ and V_t is continuous on $\mathcal{C}$,

$$\lim_{\ell \rightarrow \infty} \langle W_t V_t f_{n_\ell}, k \rangle_\mu = \langle W_t V_t \lim_{\ell \rightarrow \infty} f_{n_\ell}, k \rangle_\mu = \langle W_t V_t f, k \rangle_\mu.$$

Therefore, $\langle W_t V_t f, k \rangle_\mu = \langle f, k \rangle_\mu$ and $W_t V_t = I_\mu$ when $f \in L^2(SG, \mu)$. □

4.3 Energy Tradeoffs, A Solution to the Heat Equation, and Further Examples

The argument in the proof of Proposition 4.2.1 implies a tradeoff between the amount that V_t concentrates its energy in a subset $U \subset SG \times \Lambda$ and the size of U . This depends on the window size $t > 0$. While not explicitly an uncertainty principle as this tradeoff depends on the window size, the statement is in the spirit of Donoho and Stark's uncertainty principle. See [72] for a weak uncertainty principle based on the energy $\mathcal{E}$ and a measure of variance Var_d weighted by the resistance metric d_R .

Proposition 4.3.1 (V_t Energy Bound) *Let $\|f\|_\mu = 1$. Fix $U = U_1 \times U_2 \subset SG \times \Lambda$ and $\epsilon > 0$ so that*

$$\iint_U |V_t f(x, \lambda_n)|^2 d\mu(x) d\sigma(\lambda) \geq 1 - \epsilon.$$

Then there exists a $c_1 > 0$ such that,

$$(\mu \times \sigma)(U) \geq \frac{(1 - \epsilon)t^{d_S \bar{\delta}}}{c_1^2}.$$

In particular, $\exists t^* > 0$ such that $(\mu \times \sigma)(U) \geq 1 - \epsilon$.

Proof: By Proposition 4.2.1, we have that $|V_t f(x, \lambda)| \leq c_1 t^{-d_S \bar{\delta}/2}$. Hence,

$$1 - \epsilon \leq \iint_U |V_t f(x, \lambda)|^2 d\mu(x) d\sigma(\lambda) \leq c_1^2 t^{-d_S \bar{\delta}} (\mu \times \sigma)(U).$$

and therefore $(\mu \times \sigma)(U) \geq \frac{(1 - \epsilon)t^{d_S \bar{\delta}}}{c_1^2}$. $\square$

Since $(\mu \times \sigma)(U)$ factors as $\mu(U_1)\sigma(U_2) = \sum_{\lambda \in U_2} \mu(U_1)$, we can write the trade-off in concentration in terms of the measure of U_1 , the measure of U_2 , and window size as

$$\mu(U_1) \geq \frac{(1 - \epsilon)t^{d_S \bar{\delta}}}{c_1^2 \sum_{\lambda \in U_2}} \quad \text{or} \quad \sum_{\lambda \in U_2} \geq \frac{(1 - \epsilon)t^{d_S \bar{\delta}}}{\mu(U_1)c_1^2} \quad \text{or} \quad \log t \leq \frac{1}{d_S \bar{\delta}} \log \frac{c_1^2 \mu(U_1) \sum_{\lambda \in U_2}}{1 - \epsilon}.$$

In general, explicitly evaluating the STFT V_t is difficult. However, we can compute it for the constant functions and estimate it for the ONB of eigenfunctions

$\{\varphi_\lambda\}_{\lambda \in \Lambda}$. This requires computing the integral of triples of eigenfunctions which we estimate using known bounds. Furthermore, given $f \in L^2(SG, \mu)$ the STFT $V_t f(x, \lambda)$ can be interpreted as a solution to a heat equation on SG with prescribed boundary conditions.

Example 4.3.2 (Constant Functions) Fix $t > 0$, $x \in SG$, and $c > 0$. The STFT $V_t c(x, \lambda)$ is a multiple of the Fourier transform of the heat kernel $p_b(t, x, y)$ seen in Lemma 4.2.5:

$$\begin{aligned}
 V_t c(x, \lambda) &= \int_{SG} c p_b(t, x, y) \overline{\varphi_\lambda(y)} d\mu(y) \\
 &= c \int_{SG} \left(\sum_{\lambda' \in \Lambda} e^{-\lambda' t} \varphi_{\lambda'}(x) \overline{\varphi_{\lambda'}(y)} \right) \overline{\varphi_\lambda(y)} d\mu(y) \\
 &= c \sum_{\lambda' \in \Lambda} e^{-\lambda' t} \varphi_{\lambda'}(x) \int_{SG} \varphi_{\lambda'}(y) \overline{\varphi_\lambda(y)} d\mu(y) \\
 &= c e^{-\lambda t} \varphi_\lambda(x). \quad \diamond
 \end{aligned}$$

Example 4.3.3 (Estimates on $V_t \varphi_{\lambda'}(x, \lambda)$) Fix $\lambda' \in \Lambda$ and $x \in SG$. Then,

$$\begin{aligned}
 V_t \varphi_{\lambda'}(x, \lambda) &= \int_{SG} \varphi_{\lambda'}(y) p_b(t, x, y) \overline{\varphi_\lambda(y)} d\mu(y) \\
 &= \int_{SG} \varphi_{\lambda'}(y) \left(\sum_{\lambda'' \in \Lambda} e^{-\lambda'' t} \varphi_{\lambda''}(x) \overline{\varphi_{\lambda''}(y)} \right) \overline{\varphi_\lambda(y)} d\mu(y) \\
 &= \sum_{\lambda'' \in \Lambda} e^{-\lambda'' t} \varphi_{\lambda''}(x) \int_{SG} \varphi_{\lambda'}(y) \overline{\varphi_{\lambda''}(y) \varphi_\lambda(y)} d\mu(y).
 \end{aligned}$$

Apply Cauchy-Schwartz and known bounds on $\|\varphi_\lambda\|_\infty$ seen in Theorem 2.5.6:

$$\begin{aligned}
 \sum_{\lambda'' \in \Lambda} e^{-\lambda'' t} \varphi_{\lambda''}(x) \int_{SG} \varphi_{\lambda'}(y) \overline{\varphi_{\lambda''}(y) \varphi_\lambda(y)} d\mu(y) &\leq \sum_{\lambda'' \in \Lambda} e^{-\lambda'' t} \varphi_{\lambda''}(x) (\|\varphi_\lambda\|_\mu^2 \|\varphi_{\lambda'} \overline{\varphi_{\lambda''}}\|_\mu^2) \\
 &\leq \sum_{\lambda'' \in \Lambda} e^{-\lambda'' t} (C(\lambda')^{d_S \bar{\delta}/2} (\lambda'')^{d_S \bar{\delta}/2}) \\
 &= C(\lambda')^{d_S \bar{\delta}/2} \sum_{\lambda'' \in \Lambda} e^{-\lambda'' t} (\lambda'')^{d_S \bar{\delta}/2}. \quad \diamond
 \end{aligned}$$

Example 4.3.4 (A Solution to the Heat Equation) Let $u(x, t)$ be a function on $SG \times (0, \infty)$. Then $u(x, t)$ is a *solution to the heat equation* (see [62][Definition 5.1.8]) on $(0, \infty)$ if and only if:

1. it is continuous on $(0, \infty) \times SG$,
2. $u(\cdot, x) \in C^1((0, \infty))$ for any $x \in SG$,
3. $u_t \in \mathcal{D}_\mu$ for any $t \in (0, \infty)$,
4. it satisfies the PDE

$$\partial_t u(x, t) = \Delta u(x, t)$$

for any $(x, t) \in (SG, (0, \infty))$.

Fix $\lambda \in \Lambda$. Because the heat kernel $p_b(t, x, y)$ is a solution of the heat equation, for $x \in SG \setminus V_0$ and $t > 0$ we have:

$$\begin{aligned} \Delta V_t f(x, \lambda) &= \int_{SG} f(y) \overline{\varphi_\lambda(y)} \Delta p_b(t, x, y) d\mu(y) \\ &= \int_{SG} f(y) \overline{\varphi_\lambda(y)} \partial_t p_b(t, x, y) d\mu(y) \\ &= \partial_t V_t f(x, \lambda). \end{aligned}$$

Hence $V_t f(x, \lambda)$ is also solution of the heat equation on SG that satisfies the initial condition

$$\lim_{t \rightarrow 0} \|V_t f(x, \lambda) - f \overline{\varphi_\lambda}\|_\mu = 0, \quad \forall x \in SG \setminus V_0$$

with boundary conditions

$$\begin{cases} V_t(x, \lambda) = 0, & x \in V_0 \text{ and } b = D \\ \partial_t V_t(x, \lambda) = 0, & x \in V_0 \text{ and } b = N. \end{cases} \quad \diamond$$

4.4 Extension to Post-Critically Finite Fractals

The construction of the STFT V_t on SG can be extended to the class of post-critically finite fractals. We recall and follow the definition and notation given in Strichartz's *Differential Equations on Fractals* [86]. Let $w_j \in \{1, \dots, N\}$ and write $f_w = f_{w_1} \circ f_{w_2} \circ \dots \circ f_{w_m}$ as a particular composition of the contraction maps $\{f_i\}_{i=1}^N$ that define the fractal K . A fractal K is called *post-critically finite (pcf)* if it is connected and there exists a finite set $V_0 \subset K$ called the *boundary*, such that

$$f_w K \cap f_{w'} K \subset f_w V_0 \cap f_{w'} V_0 \quad \text{for } w \neq w' \text{ with } |w| = |w'|,$$

and the intersection is disjoint from V_0 . Further, it is required that each boundary point is the fixed point of one of the maps $\{f_i\}_{i=1}^N$. Pcf fractals are self-similar in the sense that they divide into smaller and smaller cells that intersect at isolated points; pcf fractals are connected but only slightly so in that removing these isolated points it becomes disconnected.

Example 4.4.1 (Hata's Tree-like Set) Let $X = \mathbb{C}$. Set $f_1(z) = c\bar{z}$ and $f_2(z) = (1 - |c|^2)\bar{z} + |c|^2$, where $|c|, |1 - c| \in (0, 1)$. The fractal

$$K_c = f_1(K_c) \cup f_2(K_c)$$

defined by these contractions maps is called Hata's tree-like set, see Figure 4.1. It is an example of a post-critically finite (pcf) fractal where the cells $f_w K$ subdivide into cells of smaller and smaller size and intersect only at isolated points. $\diamond$

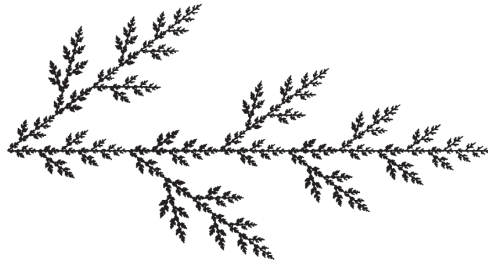


Figure 4.1: Hata's tree-like fractal K_c , $c = \frac{2}{5} + \frac{3}{10}i$.

Example 4.4.2 The Sierpinski carpet (SC): Let $X = \mathbb{C}$ and set $p_0, p_1, \dots, p_7$ be the eight vertices of the unit square and their midpoints on the boundary. Set $f_i(z) = (z - p_i)/3 + p_i$ for $i = 0, 1, \dots, 7$. The fractal,

$$K = \bigcup_{i=0}^7 f_i(K)$$

is the Sierpinski carpet (SC) seen in Figure 4.2. The SC is *not* a pcf fractal. $\diamond$

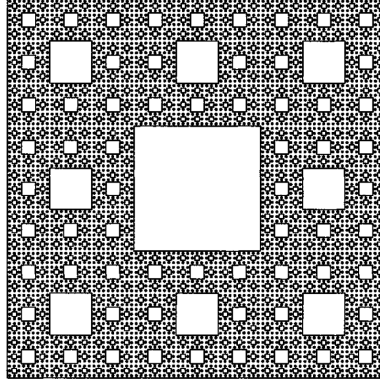


Figure 4.2: Unlike Hata's tree and SG , the Sierpinski carpet SC is not a pcf fractal. Intuitively, the boundaries of the cells $f_w K$ do not intersect at a finite number of points.

The STFT V_t on SG extends to pcf fractals due to an analogous theory of Laplacians via a sequence of graphs and Dirichlet forms. Crucial properties of the Laplacian and heat kernels that are required for the STFT theory on SG carryover to pcf fractals. Indeed, the ingredients in Proposition 4.2.1 and Theorem 4.2.2 were:

1. the existence of a countable orthonormal basis of eigenfunctions $\{\phi_\lambda\}_{\lambda \in \Lambda}$ of a Laplacian Δ for $L^2(K, \mu)$,
2. uniform bounds on heat kernels $p_b(t, x, y) = \sum_{\lambda \in \Lambda} e^{-\lambda t} \phi_\lambda(x) \phi_\lambda(y)$ as seen in Equation (2.27) and bounds on their diagonals as seen in Equation (2.28),
3. Weyl-type estimates on the asymptotic distribution of the eigenvalues via an eigenvalue counting function $\rho_b(x, \mu)$ as in Proposition 4.2.6.

We recall the construction of the Laplacian Δ for a pcf fractal. Given a vertex set

V_0 , construct a sequence of graphs $\Gamma_0, \Gamma_1, \dots$ as follows. Let Γ_0 be the complete graph on V_0 and having constructed Γ_{m-1} from vertex set V_{m-1} and let

$$V_m = \bigcup_i f_i V_{m-1}.$$

The edges in the graph Γ_m are defined by the relation $x \sim_m y$ if $x = f_i x'$ and $y = f_i y'$ where $x' \sim_{m-1} y'$. As with SG , for each graph in the sequence $\Gamma_0, \Gamma_1, \dots$, one considers the weighted graph Laplacian Δ_m given by

$$\Delta_m u(x) = \sum_{y \sim_m x} c_m(x, y)(u(y) - u(x))$$

where the weights (conductances) are chose to satisfy a non-decreasing and self-similarity constraint for the associated graph energy $\mathcal{E}_m(u) = \sum_{x \sim_m y} c(x, y)(u(x) - u(y))^2$. The Laplacian Δ for a pcf fractal is defined pointwise as the weighted limit of the graph Laplacians,

$$\Delta u(x) = \lim_{m \rightarrow \infty} \left(\int_K \psi_x^m(y) d\mu(y) \right)^{-1} \Delta_m u(x),$$

where $\psi_x^{(m)}(y)$ is a *piecewise harmonic spline* satisfying $\psi_x^{(m)}(y) = \delta_{xy}$, see [86, Chapters 2, 7]. As with SG , the Laplacian Δ can also be constructed weakly via a theory of Dirichlet forms, see [62, Chapter 3]. The eigenfunctions $\{\phi_\lambda\}_{\lambda \in \Lambda}$ form an orthonormal basis for $L^2(K, \mu)$ (Theorem 2.5.4), the heat kernels $p_b(t, x, y)$ are constructed explicitly in terms of the these eigenfunctions and corresponding eigenvalues seen in Equation (2.25) and Definition 5.1.1 in [62], and bounds on $p_b(t, x, y)$ in Equation (2.27) and Equation (2.28) are valid (Theorem 2.5.8). Furthermore, Kigami and Lapidus [63] showed that for the *lattice case* of a pcf fractal one has that the eigenvalue counting function $\rho_b(\mu, x)$ behaves like that on SG ,

$$\rho_b(\mu, x) = \left(G(\log(\frac{x}{2})) + o(1) \right) x^{d_S/2}$$

as $x \rightarrow \infty$ where G is a bounded positive periodic function. For *non-lattice* pcf

fractals, the limit

$$\lim_{x \rightarrow \infty} \frac{\rho_b(\mu, x)}{x^{d_S/2}}$$

exists and is independent of the boundary conditions. See Section 2.5 and Kigami [62, Chapter 4] for the formal statements and proofs. Therefore, the analysis in Sections 4.2.1 and 4.2.2 proceeds for pcf fractals.

Theorem 4.4.3 *The STFT V_t as defined in Equation (4.1) is valid for a pcf fractal K and satisfies the orthogonality relations seen in Theorem 4.2.4. If the spectral dimension $d_S < 2$, then the formula for the left inverse in Theorem 4.2.2 holds.*

Chapter 5

Finite Elements on Fractals

5.1 Introduction

This chapter continues the comparison between analysis on fractals and analysis on Euclidean $\mathbb{R}^d$. Here, the numerical algorithm of *finite elements* (FE) on fractal domains is analyzed and compared to the classical theory in Euclidean space $\mathbb{R}^d$. This procedure of numerical approximations to solutions of partial differential equations consists of the same overarching technique: orthogonal projection onto a spline space. We study this technique and its application to a prototype *Dirichlet problem* $\Delta u + qu = f$ with homogeneous Dirichlet boundary conditions $u|_{V_0} = 0$. This is based on an analysis in [37]. The differences between the fractal and classical theory of FE lie in the subtleties of constructing and understanding the analogues of key ingredients in classical partial differential equations such as the Laplacian operator, Sobolev inner products, and spline spaces. An application of FE on fractals is to approximate the eigenvalues and eigenfunctions of the Laplacian Δ on fractal geometries that are too complex to perform a spectral analysis analytically. We will implement the FE algorithm to approximate the eigenvalues and eigenfunctions of the Laplacian Δ on SG (by approximating solutions to the Helmholtz equation) that have been computed analytically.

As described in Chapter 2, analysis on fractals reformulates traditional analysis onto the "rough" self-similar structure of a fractal. Classic mathematical models assume the underlying space is "smooth" such as a manifold in $\mathbb{R}^d$. These models often consist of differential equations and describe physical phenomenon like heat flow or wave propagation. Is it possible to develop analogous analytic theories and models on an irregular domain like a fractal? If so, how do the theories, techniques, and models compare? The answer to the first question is in the affirmative [70]. In

particular, there is a rich theory of differential equations on fractals using analogous differential operators like the Laplacian. The answer to the second question is more subtle. This section focuses on the similarities and differences between fractal and classical analysis and differential equations, specifically numerical approximations to partial differential equations using FE.

Analysis and differential equations on fractals often begins on the well-understood fractal prototype SG and the Laplacian Δ . In Chapter 4 fractals were defined as the unique fixed point of a family of contraction maps. This captures the fractal as the *whole as a union of similar parts*. Recall the post-critically finite (pcf) fractals are those that are approximated or constructed in an appropriate sense by a sequence of graphs. Operators such as energy, Dirichlet forms, and Laplacians are constructed on the sequence of graphs and converge to well-defined operators on the fractal. Here, we focus on the Laplacian Δ and study it in the context of numerical differential equations. The first step is to note the similarity between the pointwise definition of the Laplacian Δ and the pointwise definition of the Laplacian (the second derivative operator) on $\mathbb{R}$. On the sequence of graphs $\{G_m\}_{m=1}^\infty$ approximating SG and expanding $\Delta = D - A$ pointwise,

$$\Delta f(x) = \sum_{y \sim x} f(x) - f(y)$$

where $x \sim y$ means there is an edge connecting vertices $x, y \in V_m$. On $\mathbb{R}$, expanding $\Delta = \frac{d^2}{dx^2}$,

$$\Delta f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) + f(x-h) - f(x)}{h^2}.$$

Both definitions are a sum over function value differences between x and its neighbors. This agrees with the intuition behind the Laplacian as the divergence of a gradient: we are measuring the flow (sourceness or sinkness) of a gradient approximated by a finite difference.

Equipped with the Laplacian, we can study differential equations on SG such as *Poisson's equation* $\Delta f(x, t) = g(x)$, the *heat equation* $\Delta f(x, t) = f_t(x, t)$, the *wave equation* $\Delta f(x, t) = f_{tt}(x, t)$, or the *Helmholtz equation* $\Delta f(x, t) = \lambda f(x, t)$.

The Helmholtz equation is the eigenvalue problem for the Laplacian Δ and is important to construct solutions to these three quintessential elliptic, parabolic, and hyperbolic differential equations above. In select cases, we can construct explicit solutions to these partial differential equations (PDEs) [22, 86]. The heat equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} \Delta f(x, t) = f_t(x, t), & (x, t) \in (SG \setminus V_0, \mathbb{R}_+) \\ f(x, t) = 0, & x \in V_0 \\ f(x, 0) = g(x), & x \in SG \end{cases}$$

has solution given by the exponential

$$f(x, t) = e^{-t\Delta}g(x) = \sum_{\ell=1}^{\infty} c_{\ell} e^{-t\lambda_{\ell}} \varphi_{\ell}$$

where the coefficients

$$c_{\ell} = \int_{SG} g(x) \varphi_{\lambda_{\ell}}(x) d\mu(x).$$

Details on the Laplacian Δ , the semigroup $e^{-t\Delta}$, the self-similar measure μ , and background on fractal analysis is in Chapter 2. We immediately see the importance of the eigenvalues and eigenfunctions in solving these PDEs. As in $\mathbb{R}^d$, we turn to a numerical analysis of the PDE when it is sufficiently complex or the geometry is sufficiently complex. Previous work in numerical analysis on fractals includes a theory of finite differences and FE [22, 37]. We follow the construction and theory in [37] where FE using splines spaces was developed. The examples studied here are the Dirichlet problems

$$\begin{cases} \Delta f(x, t) + q(x, t)f(x, t) = g(x, t) \\ f(x, t) = 0, & x \in V_0 \end{cases} \quad (5.1)$$

where $q(x, t)$ and $g(x, t)$ are continuous functions in $t > 0$ and $x \in SG$. All of the prototypical PDEs above come from choices for $q(x, t)$ and $g(x, t)$. For example, if

$q(x, t) = 0$ and $g(x, t) = f_{tt}(x, t)$ we get the *wave equation*. If $q(x, t) = 0$ and $g(x, t) = \lambda f(x)$ we get the *Helmholtz equation*. We will apply and implement in Python a FE method to the Helmholtz equation to obtain approximations and visualizations of the eigenvalues and eigenfunctions of the Laplacian Δ on SG . Like in $\mathbb{R}$, the FE approximations will be projections of the solution onto the spline spaces with a certain inner product. Furthermore, this method is based on reformulating the PDE as a *weak* PDE. In Section 5.2, the theory of FE is reviewed to draw clear comparisons with the corresponding theory on fractals. The outline of the following sections:

1. A background on classical FE. We outline the main components of FE using the example of Poisson's equation on the interval $[0, 1]$ (another fractal!) in order to make explicit connections and comparisons to FE on fractals. This involves weak differential equations, linear spline spaces, the Ritz-Galerkin approximation, and Cea's lemma.
2. The construction of FE on SG . Here, comparisons are made to classical FE. We see the major steps are the same:
 - (a) Reformulate the differential equation as a weak differential equation.
 - (b) Choose a finite dimensional subspace of the solution space.
 - (c) Project onto this subspace.

This leads to a system of equations whose solution is the vector of basis coefficients to construct the approximation.

3. An application of FE to the Helmholtz equation $\Delta f = \lambda f$. The algorithm is applied to the Helmholtz equation and implemented in Python to obtain successive approximations to the eigenvalues and eigenfunctions of the Laplacian Δ on SG .

5.2 Finite Elements on $[0, 1]$

Here, we look at FE applied to Poisson's equation on the interval $[0, 1]$. We consider the mixed *boundary-value problem (BVP)*

$$\begin{cases} -f''(x) = g(x), & x \in (0, 1) \\ f(0) = f'(1) = 0. \end{cases}$$

This is the *strong form* of the differential equation. Multiply both sides by a sufficiently smooth function $v(x)$ such that $v(0) = 0$ and integrate-by-parts,

$$-\int_0^1 f''(x)v(x)dx = \int_0^1 g(x)v(x)dx \implies \int_0^1 f'(x)v'(x)dx = \int_0^1 g(x)v(x)dx.$$

Note, the right-hand side is an $L^2(0, 1)$ inner product. Define the bilinear form

$$a(f, v) := \int_0^1 f'(x)v'(x)dx.$$

This bilinear form is indeed another inner product called the *energy inner product* $\langle \cdot, \cdot \rangle_e$ on the space of functions

$$V := \{v \in L^2(0, 1) | a(v, v) < \infty, v(0) = 0\}.$$

Consequently, we also have the *energy norm* $\|v\|_a := \sqrt{a(v, v)}$. The BVP for Poisson's equation can now be re-stated in the *weak form* to find a function $f \in V$ such that

$$a(f, v) = \langle g, v \rangle_{L^2(0, 1)}, \quad \forall v \in V.$$

Thus the weak form of the BVP for Poisson's equation is found by multiplying by a function v in the appropriate function space V and integrating-by-parts. Note that a solution to the strong form of this boundary-value problem for Poisson's equation will be a solution to the weak form. However, it is not clear that solutions to the weak form will necessarily be solutions to the strong form. Luckily, with a smoothness

assumption on f this is true.

Theorem 5.2.1 (Weak Solutions, Theorem 4.2 [2]) *If $g \in C^0(0,1)$ and $f \in C^2(0,1) \cap V$ satisfy $a(f, v) = \langle g, v \rangle_{L^2(0,1)}$, for all $v \in V$, then f is a solution to the strong form of the corresponding BVP of Poisson's equation.*

This weak formulation is imperative to numerically approximate solutions to the differential equation. This is because the problem of restricting the weak form to a finite-dimensional subspace of V is well-posed. Once we are situated in a finite-dimensional subspace, we can choose a basis, formulate existence and uniqueness as a linear algebra problem, and explicitly construct a solution in the subspace that is the best approximation to the true solution. This process is formalized by first introducing the *Ritz-Galerkin approximation* and *Cea's lemma*. Let $a : V \times V \rightarrow \mathbb{R}$ be the bilinear form from the BVP for Poisson's equation, and let S be a finite-dimensional subspace of V . The Ritz-Galerkin approximation f_S is the solution to the weak form restricted to S ,

$$a(f_S, v) = \langle g, v \rangle_{L^2(0,1)}, \quad \forall v \in S.$$

Theorem 5.2.2 (Ritz-Galerkin Approximation, Theorem 4.5 [2]) *If $g \in L^2(0,1)$, then the Ritz-Galerkin approximation f_S exists and is unique.*

Lemma 5.2.3 (Cea's Lemma, Lemma 4.6 and 4.7 [2]) *If f is a solution to the weak form of the BVP for Poisson's equation and f_S is the Ritz-Galerkin approximation, then $a(f - f_S, v) = 0$ for all $v \in S$ and $\|f - f_S\| \leq \|f - v\|_e$ for all $v \in S$.*

We summarize the above as stating the Ritz-Galerkin f_S approximation is the solution to the weak-form but restricted to a finite-dimensional subspace of V and is an orthogonal projection onto this subspace with respect to the inner product $a(\cdot, \cdot)$. The proof of this theorem consists of choosing a basis $\{\varphi_i\}_{i=1}^n$ for S , expressing f_S in terms of this basis $f_S = \sum_{i=1}^n c_i \varphi_i$ and plugging this expression into the weak form

to show the resulting linear system

$$\sum_{i=1}^n c_i a(\varphi_i, \varphi_j) = \langle g, \varphi_j \rangle$$

has a unique solution. Solving this linear system provides an explicit method to construct the Ritz-Galerkin approximation f_S . This begs the question: what basis should be chosen for S ? A common choice in numerical PDE is the space of *continuous piecewise linear splines* V_1^h on a mesh that divides $[0, 1]$ into n elements $[x_{i-1}, x_i]$ with spacing $h = x_i - x_{i-1}$. A feature of V_1^h is that we can determine the value of a function $f_h \in V_1^h$ by its values at nodes x_i in the mesh. We choose the basis for this space of piecewise linear splines V_1^h to be the set $\{\varphi_i\}_{i=1}^n$ such that $\varphi_i(x_j) = \delta_{ij}$ across all points in the mesh x_j for $j = 1, \dots, n$.

5.3 Finite Elements on Fractals

This section outlines the construction of FE on fractals. The FE algorithm on fractals is first described in [37] to approximate the solutions to the Dirichlet problem in Equation 5.1. The main steps mimic those for classical FE as shown in the previous section for Poisson's equation: reformulate the strong form of the differential equation into an appropriate weak form, orthogonally project the solution onto a finite-dimensional subspace, and solve a matrix equation for the coefficients of the best approximation to the true solution in this finite-dimensional space. To start this process, we introduce two definitions. Recall the *Dirichlet form* for continuous functions f and v on SG :

$$\mathcal{E}(f, v) := \lim_{m \rightarrow \infty} \left(\frac{5}{3}\right)^m \mathcal{E}_m(f, v).$$

where $\mathcal{E}_m(f, v)$ is the *graph Dirichlet form*,

$$\mathcal{E}_m(f, v) := \sum_{x \sim_m y} (f(x) - f(y))(v(x) - v(y)).$$

and $x \sim y$ means vertices $x, y \in V_m$ are connected by an edge at graph level m . This Dirichlet form is an inner product on SG if special care is given to the constant functions; if $f = c$ for some $c \in \mathbb{C}$, then $\mathcal{E}(f, v) = 0$ and so fails to be definite. Also recall the *Dirichlet energy* of a continuous function f on SG is

$$\mathcal{E}(f) := \lim_{m \rightarrow \infty} \left(\frac{5}{3}\right)^m \mathcal{E}_m(f),$$

where $\mathcal{E}_m(f) = \mathcal{E}_m(f, f)$ is the graph Dirichlet energy. These energies are the analogues $\int_{\mathbb{R}^d} |\nabla f(x)|^2 dx$ often referred to as Dirichlet energies in $\mathbb{R}^d$. The Dirichlet energy is a monotonically increasing sequence and if we include the limit $+\infty$ it is a well-defined limit for all continuous functions on SG [86, Chapter 2]. Recall the

space $\text{dom}_0(\mathcal{E})$ are functions with finite energy,

$$\text{dom}_0(\mathcal{E}) := \{f \in C^0(SG) \mid \mathcal{E}(f) < \infty, f(V_0) = 0\}.$$

This is the natural function space to look for solutions in the development of the weak form of a PDE and is the analogue of the space V from Section 5.2. We now return to the constant functions and the Dirichlet form.

Theorem 5.3.1 (Energy Inner Product, Theorem 1.4.2 [86]) *If $f, v \in \text{dom}_0(\mathcal{E})$, then $\mathcal{E}(f, v)$ exists and defines an inner product on the space of equivalence classes $\text{dom}_0(\mathcal{E}) \setminus \text{constants}$. Furthermore, $\text{dom}_0(\mathcal{E}) \setminus \text{constants}$ forms a Hilbert space on SG with this inner product.*

By modding out the constant functions and considering the space of equivalence classes this bilinear form is a true inner product. In practice, this distinction doesn't seem to be given attention in the same way that $L^2(0,1)$ is technically a Hilbert space on equivalence classes of functions. In FE on fractals, the Dirichlet form $\mathcal{E}(\cdot, \cdot)$ and Dirichlet energy $\mathcal{E}(\cdot)$ are analogous to the energy inner product $a(\cdot, \cdot)$ and energy norm $\|\cdot\|_a$ in classical FE.

We continue our goal of re-casting the Dirichlet problem into a weak form by defining the fractal Laplacian weakly. Recall that the Laplacian Δ was defined pointwise as a limit of re-normalized graph Laplacians. It can also be defined for a function $f \in \text{dom}_0(\mathcal{E})$ and a $g \in C^0(SG)$. We say

$$\Delta f = g \text{ weakly if } \mathcal{E}(f, v) = - \int_{SG} g(x)v(x)d\mu(x), \quad \forall v \in \text{dom}_0(\mathcal{E}).$$

The weak definition of the Laplacian resembles the first step in deriving the weak form of BVP for Poisson's equation when integrating-by-parts. Both equate a bilinear form on a space of functions with finite energy that are zero on the boundary with an L^2 inner product. The last ingredient is to define an analogue of the classical

Sobolev inner product on a bounded domain $\Omega \subset \mathbb{R}$,

$$\langle f, v \rangle_{H^1(\Omega)} := \int_{\Omega} f(x)v(x)dx + \int_{\Omega} \nabla f(x) \cdot \nabla v(x)dx.$$

This Sobolev inner product on SG is defined for $f, v \in \text{dom}_0(\mathcal{E})$ and has the form

$$\langle f, v \rangle_q := \int_{SG} q(x)f(x)v(x)d\mu(x) + \mathcal{E}(f, v).$$

Like the Sobolev inner product $\langle f, v \rangle_{H^1(\Omega)}$ the $\langle f, v \rangle_q$ is the sum of an L^2 inner product and an energy inner product. The $\langle f, v \rangle_q$ plays the role of the bilinear form $a(\cdot, \cdot)$ in the statement of the weak form. This is made clear by applying the following result from [37].

Theorem 5.3.2 (Weak Form, Theorem 2.1.1 [37]) *A continuous $f \in C^0(SG)$ satisfies $\Delta f + qf = g$ if and only if f vanishes on V_0 and $\langle f, v \rangle_q = \int_{SG} gvd\mu$ for all $v \in \text{dom}_0\mathcal{E}$.*

Thus, the weak form of the Dirichlet problem on SG is

$$\langle f, v \rangle_q = \langle g, v \rangle_{L^2(SG, \mu)}, \quad \forall v \in \text{dom}_0\mathcal{E}.$$

Like in the classical setting, we have equivalence of the weak form and the strong form. As in Section 5.2, we proceed to the next stage which projects a solution of the Dirichlet problem onto a finite-dimensional subspace, which is the analogue of a spline space.

Theorem 5.3.3 (Ritz-Galerkin on SG , Theorem 2.1.2 [37]) *Let $W \subset \text{dom}_0(\mathcal{E})$ be a finite-dimensional subspace and define $P_W f$ as the solution to the weak formulation of the Dirichlet problem restricted to W . Then $\langle f - P_W f, v \rangle = 0$ for all $v \in W$. Furthermore, $\|f - P_W f\|_q \leq \|f - v\|$ for all $v \in W$.*

$P_W f$ is the analogue of the Ritz-Galerkin approximation and is the best approximation in W to the solution f of the Dirichlet problem as an orthogonal projection

onto W with respect to $\langle \cdot, \cdot \rangle_q$. In Section 5.2, a crucial step to build the FE approximations was to choose a finite-dimensional subspace to project onto. In the fractal case, these are the *polyharmonic spline spaces*. Solutions to $\Delta f = 0$ are called *harmonic functions*. Solutions to $\Delta^2 f = 0$ are called *biharmonic functions*. The space of harmonic functions is denoted $\mathcal{H}_0$ and the space of biharmonic functions is denoted $\mathcal{H}_1$.

Recall V_m is the set of vertices in the level m graph approximation G_m to SG . To build the harmonic spline spaces we localize the above conditions on the Laplacian to each cell $f_w(SG)$ of G_m . These cells contain the edges and vertices that act as the elements and nodes in the FE method. The *harmonic spline space* $S(\mathcal{H}_0, V_m)$ is defined as

$$S(\mathcal{H}_0, V_m) := \{f \in C^0(SG) \mid \Delta(f \circ f_w) = 0\}.$$

In [37], a *biharmonic spline space* $S(\mathcal{H}_1, V_m)$ is defined similarly by localizing Δ^2 but requires matching conditions for normal derivatives where cells intersect. These harmonic spline spaces should be thought of as analogues of the piecewise linear and piecewise cubic splines: a linear function and a cubic polynomial are in the kernel of the Laplacian $\frac{d^2}{dx^2}$ and squared-Laplacian $(\frac{d^2}{dx^2})^2$ on an interval. Furthermore, like on the interval the harmonic functions in these spaces are minimizers of a localized energy constructed from $\mathcal{E}(\cdot)$ [37]. To simplify matters, we work with $S(\mathcal{H}_0, V_m)$ and explicitly construct a basis for this space. For every vertex $y \in V_m \setminus V_0$, let φ_y denote the piecewise harmonic function such that $\varphi_y(z) = \delta_{yz}$. There are exactly $\frac{3^m-3}{2}$ of these basis elements, $m > 1$.

The analogue of the Ritz-Galerkin approximation $P_W f$ is an element of $S(\mathcal{H}_0, V_m)$ and expressed in terms of the basis φ_y as

$$P_W f(z) = \sum_{y \in V_m \setminus V_0} c_y \varphi_y(z).$$

The goal is to determine the coefficients which are the values of $P_W f(x)$ in G_m . Plugging this representation into the weak form of the Dirichlet problem, multiplying

by another basis function φ_x , and integrating we obtain

$$\sum_{y \in V_m} c_y \int_{SG} \Delta \varphi_y(z) \varphi_x(z) d\mu(z) + \sum_{y \in V_m} c_y \int_{SG} q(z) \varphi_y(z) \varphi_x(z) d\mu(z) = \int_{SG} f(z) \varphi_x(z) d\mu(z).$$

Applying the weak definition of the Laplacian $\int_{SG} \Delta \varphi_y \varphi_x d\mu = \mathcal{E}(\varphi_y, \varphi_x)$ and substituting we obtain the matrix equation

$$(E + Q)c = F$$

that can be solved for c containing the coefficients c_y . The entries of these matrices and vectors are:

$$\begin{cases} E_{xy} = \mathcal{E}(\varphi_x, \varphi_y) \\ Q_{xy} = \int_{SG} q(z) \varphi_x(z) \varphi_y(z) d\mu(z) \\ F_x = \int_{SG} f(z) \varphi_x(z) d\mu(z). \end{cases}$$

5.4 Implementation for the Helmholtz Equation

The FE method outlined above is applied to the eigenfunction equation $\Delta f + qf = \lambda f$ where $f|_{V_0} = 0$. When $q = 0$, the eigenvalues and eigenfunctions can be derived analytically via *spectral decimation* [86, Chapter 3]. The finite difference method developed in [22] have provided accurate approximations of the eigenfunctions. As far as the author knows, when $q \neq 0$ no analytic or closed forms exist for the eigenvalues and eigenfunctions and numerical methods must be used to estimate these quantities. We take $q = 0$ and construct the first three approximations to solutions of the Helmholtz equation with Dirichlet boundary conditions using the piecewise harmonic splines. Mirroring the process in Section 5.3, plug $P_W f = \sum_{y \in V_m \setminus V_0} c_y \varphi_y(z)$ into a weak formulation of the Dirichlet problem for the Helmholtz equation, multiply by φ_x , and integrate to obtain the matrix equation

$$(E + Q)c = \lambda Gc$$

where

$$G_{xy} = \int_{SG} \varphi_x(z) \varphi_y(z) d\mu(z)$$

is the Gram matrix of basis functions. Therefore, the coefficients needed to construct $P_W f$ are the eigenvectors of $G^{-1}(E + Q)$ and the eigenvalues of this matrix approximate the true eigenvalues $\lambda \in \Lambda$ of the Laplacian Δ [37]. Since the basis $\{\varphi_y\}_{y \in V_m}$ are Kronecker delta's (again, like the piecewise linear tent functions in classical FE) this vector c is the Ritz-Galerkin approximation and the final output of the FE algorithm. This algorithm is implemented in Python and the output of the varying levels of approximation are plotted below. The varying levels $m = 1, 2, 3$ are thought of as refinements on meshes used in classical FE.

True λ	$m = 1$	$m = 2$	$m = 3$
16.81	20.45	17.46	16.94
55.88	112.51	63.53	57.73

Table 5.1: Convergence of FE approximations of λ_1 and λ_2 of Δ .

The implementation is divided into three steps. The first step creates the adjacency matrices and visualizes the first $m = 1, 2, 3$ graph approximations to SG . In the second step, the entries of the energy matrix E_{xy} and the Gram matrix G_{xy} are computed. These can be approximated numerically using the standard probability measure μ on SG . These integrals were computed theoretically in [87] and used in the implementation. The last step computes the eigenvalues and eigenvectors of $G^{-1}(E+Q)$ and uses the NetworkX graph library in Python to visualize the values of the eigenvectors on the graph approximations. In [37] are theoretical error estimates in terms of the energy norm, the L^2 -norm, and the level m of the approximation. This analysis yields results like in classical FE that bound the rate at which an error approaches zero in terms of mesh-spacing h^α . Numerical evidence for convergence to the true spectrum is provided in Table 5.1.

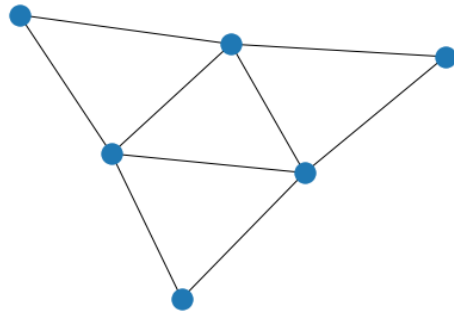


Figure 5.1: G_1 graph approximation to SG

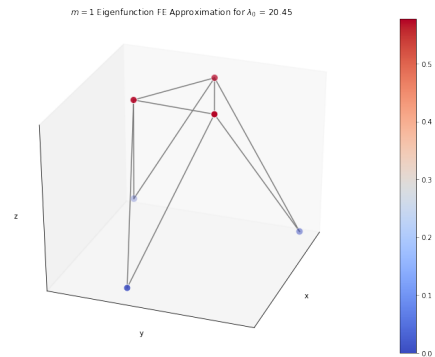


Figure 5.4: $m = 1$ approximation to $\varphi\lambda_1$

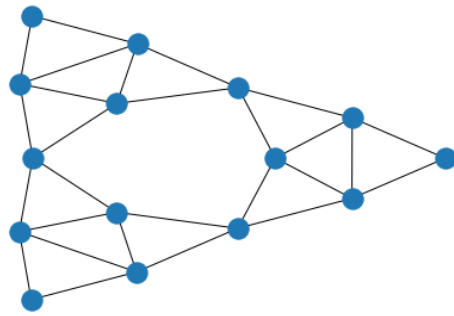


Figure 5.2: G_2 graph approximation to SG

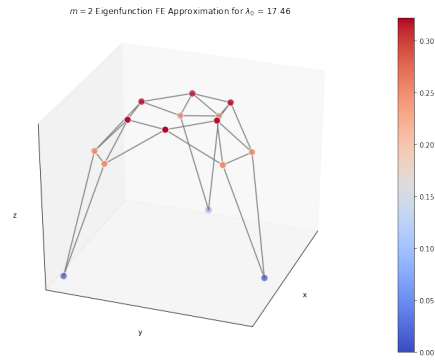


Figure 5.5: $m = 2$ approximation to $\varphi\lambda_1$

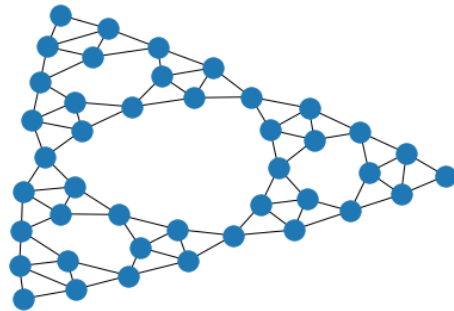


Figure 5.3: G_3 graph approximation to SG

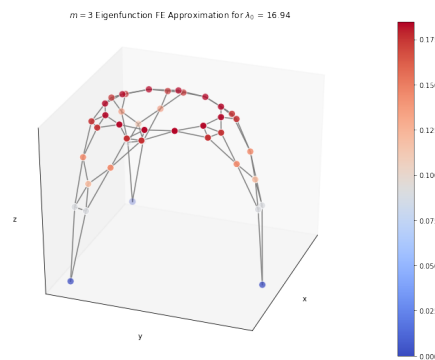


Figure 5.6: $m = 3$ approximation to $\varphi\lambda_1$

Figure 5.7: $\{G_0, G_1, G_2\}$ graph approximations to SG and corresponding FE eigenfunction approximations for λ_1 of Δ

$m = 1$ Eigenfunction FE Approximation for $\lambda_1 = 112.5$

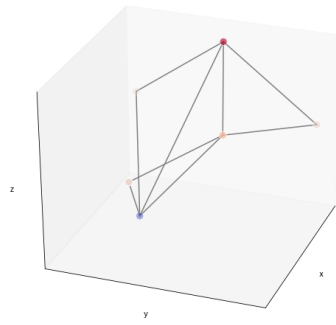


Figure 5.8: $m = 1$ approximation to φ_{λ_2}

$m = 1$ Eigenfunction FE Approximation for $\lambda_2 = 112.5$

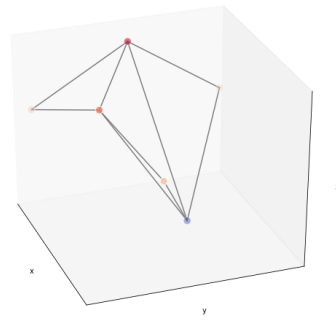


Figure 5.11: $m = 1$ approximation to φ_{λ_2}

$m = 2$ Eigenfunction FE Approximation for $\lambda_1 = 63.53$

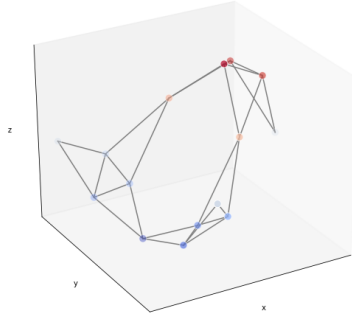


Figure 5.9: $m = 2$ approximation to φ_{λ_2}

$m = 2$ Eigenfunction FE Approximation for $\lambda_2 = 63.53$

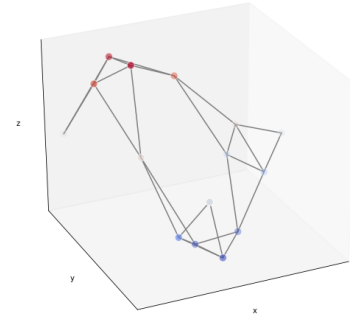


Figure 5.12: $m = 2$ approximation to φ_{λ_2}

$m = 3$ Eigenfunction FE Approximation for $\lambda_1 = 57.3$

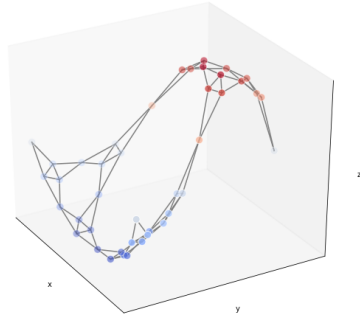


Figure 5.10: $m = 3$ approximation to φ_{λ_2}

$m = 3$ Eigenfunction FE Approximation for $\lambda_2 = 57.3$

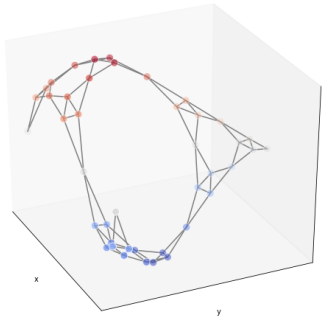


Figure 5.13: $m = 3$ approximation to φ_{λ_2}

Figure 5.14: FE eigenfunction approximations for λ_2 of Δ

Chapter 6

Spectral Properties of the Gram Matrix for Gabor Systems Generated by B-splines

6.1 Introduction

Formally introduced by Duffin and Schaeffer in their seminal paper *A Class of Non-harmonic Fourier Series* [28], frame theory centers around the study of overcomplete representations of functions via fundamental building blocks called frame vectors or frame elements. Whereas a basis provides a unique representation of a function, frames offer a flexible alternative in non-unique representations of a function. This allows for sparsity, redundancy, and stability of the representation. It draws on rich mathematical theory at the intersection harmonic analysis, numerical analysis, and sampling theory that underpin applications in signal processing, quantum information theory, and machine learning [15, 42, 65, 66, 76].

We consider a specific family of frames, *Gabor frames*, generated by a window $g \in L^2(\mathbb{R})$ and time-frequency parameters $a, b > 0$. The problem of characterizing the pairs (a, b) such that the system of time-frequency shifts

$$\mathcal{G}(g, a, b) := \{M_{\ell b}T_{ka}g = e^{2\pi i \ell b \cdot} g(\cdot - ka) : (\ell, k) \in \mathbb{Z}^2\} \quad (6.1)$$

is a frame for $L^2(\mathbb{R})$ is referred to as the *frame set problem* and is in general an unsolved and difficult problem in Gabor analysis [48, 54]. As discussed in Chapter 2, characterizing the frame set amounts to finding resolutions of a lattice from which we sample the STFT to reconstruct a function. Celebrated results in the 90's characterize the frame set when the window is a Gaussian or in related function

classes [23, 49, 80]. Since then progress has only been made in a handful of classes including the first-order B-spline (a characteristic function) where the frame set has been completely characterized [17, 21, 48]. Here, we zero-in on the case where the window g is a higher order B-spline and take its time-frequency shifts along the time-frequency lattice

$$\Lambda = a\mathbb{Z} \times b\mathbb{Z}.$$

This class of B-splines has many advantages as a choice of window including compact support, partitions of unity, linear independence, and regularity. Furthermore, finite subsets of Gabor systems generated by functions with compact support are a basis for their linear span, a solved special case of the HRT conjecture [56]. In applications, B-spline wavelet frame systems are central to finite element methods and successfully applied in molecular dynamics simulations, motion smoothing, and recurrent neural networks [10, 50, 58, 92].

Our approach investigates the structure and spectrum of the infinite Gram matrix $G(a, b) : \ell^2(\mathbb{Z}^2) \rightarrow \ell^2(\mathbb{Z}^2)$ with entries given by the inner products $\langle g_\lambda, g_{\lambda'} \rangle_{L^2(\mathbb{R})}$, $\lambda, \lambda' \in \Lambda$, and the finite matrices corresponding to a $n^2 \times n^2$ set of lattice points centered at the origin. Each of these finite matrices is viewed as a finite Gram matrix $G(a, b, n) \in \mathbb{C}^{n^2 \times n^2}$. This investigation was motivated by the frame set problem. The idea: numerically and analytically compute the spectrum of the $G(a, b, n)$ as $n \rightarrow \infty$ to gain insight into the spectrum of $G(a, b)$. These truncations $G(a, b, n)$ are *finite sections* of $G(a, b)$. Since the Gram operator $G(a, b)$ and frame operator $S(a, b)$ have a spectrum that coincide except possibly at zero, we can learn the influence of (a, b) on the spectrum of $S(a, b)$ and hence the invertibility of $S(a, b)$. This an incredibly delicate problem discussed in Section 6.4. Ultimately, we expect that the spectral analysis of these Gram matrices will lead to new insights for the frame set of higher order B-spline. This is due to

1. the frame operator and Gram operator share a spectrum except possibly at zero: $\{0\} \subset \text{Spec}(G(a, b)) \subset \{0\} \cup [A, B]$ [53] and,
2. the lower frame bound is the (reciprocal) operator norm of the pseudo-inverse

$A = \|G^\dagger(a, b)\|^{-1}$ and the upper frame bound is the operator norm $B = \|G(a, b)\|$ [1].

We summarize our contributions as follows:

1. In Section 6.2, under a specific sampling scheme of the time-frequency lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, the Gram matrix $G(a, b, n)$ has remarkable structure amenable to spectral analysis such as having Toeplitz blocks.
2. In Section 6.3, an asymptotic analysis and description of the spectrum of the blocks within $G(a, b, n)$ and $G(a, b)$ corresponding to samplings of lines on the lattice Λ via the Toeplitz structure and Szego-type limit theorems.

6.2 Lattice Sampling and Gram Matrix Structure

Recall a system $\{g_\lambda\}_{\lambda \in \Lambda} \subset L^2(\mathbb{R})$ forms a *frame* for $L^2(\mathbb{R})$ if there exist constants $A, B > 0$ such that for each $f \in L^2(\mathbb{R})$,

$$A\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, g_\lambda \rangle|^2 \leq B\|f\|^2.$$

The constants A, B are called *frame bounds*, and estimating the optimal frame bounds is generally an unresolved problem whose solution can impact the numerical accuracy and speed of reconstructing and representing a function. The frame inequality above implies the existence of *dual frames* $\{h_\lambda\}_{\lambda \in \Lambda} \subset L^2(\mathbb{R})$ such that each $f \in L^2(\mathbb{R})$ has the following representation or reconstruction:

$$f = \sum_{\lambda \in \Lambda} \langle f, h_\lambda \rangle g_\lambda = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle h_\lambda.$$

The parameters (a, b) in Equation (6.1) control the resolution of the time-frequency lattice that we sample with the hopes of reconstructing a function from its time-frequency shifts of the window g . The set

$$\mathcal{F}(g) = \{(a, b) \in \mathbb{R}_+^2 : \mathcal{G}(g, a, b) \text{ is a frame for } L^2(\mathbb{R})\}$$

is referred to as the *frame set* of g and the problem of characterizing the frame set is an open problem for many classes of window functions g . In the cases where g is a member of

1. $\{e^{-\pi x^2}, \frac{1}{\cosh x}, \chi_{[0, \infty)} e^{-x}, e^{-|x|}\}$
2. the totally positive functions of finite type
3. the totally positive functions of exponential type

then the frame set has a simple characterization as the region under the hyperbola $ab < 1$, $\mathcal{F}(g) = \{(a, b) \in \mathbb{R}_+^2, ab < 1\}$. Furthermore, if the window is the indicator function $g = \chi_{[0, c]}$, the frame set is also known (referred to as Janssen's tie [60])

and completely described by Dai and Sun [21]. The indicator function has Fourier transform

$$\widehat{\chi}_{[-\frac{1}{2}, \frac{1}{2}]}(\omega) = \text{sinc}(\omega),$$

which has slow decay, is not integrable, and consequently has poor frequency resolution. We study the case where the window is the second-order B-spline,

$$g(x) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x) * \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x) = \begin{cases} 1 - |x|, & x \in [-1, 1] \\ 0, & x \notin [-1, 1]. \end{cases} \quad (6.2)$$

Complete knowledge of the regions under the hyperbola $ab < 1$ that belong to the frame set $\mathcal{G}(g, a, b)$ is a fundamental open problem in Gabor analysis. It is known that $\mathcal{G}(g, a, b)$ is an open set in $\mathbb{R}^2$ [29]. Recent work by Atindehou et al. [3] discovered new regions by explicitly finding a dual window for the second order B-spline with these new time-frequency parameters (a, b) . There are few previous works that investigate connection between the spectrum of the Gram matrix of Gabor systems and the corresponding frame set problem. In [1], Adcock and Huybrechs provide a numerical analysis of regularized SVD decomposition of the finite sections of the Gram operator to estimate frame coefficients. In [53], Harrison characterizes the null space of the Gram operator for exponential frames and provides a theoretical and a numerical analysis of its spectrum. While we don't study the region $ab < 1$ directly, as far as the authors know this is the first approach in studying the spectrum of the Gram matrix directly and relate it to the frame bounds.

Fix $a, b > 0$ and consider the time-frequency shifts of the B-spline,

$$\{g_{\lambda_{ij}}\}_{i,j=1}^n = \{g(x - y_j)e^{2\pi i \omega_i x}\}_{i,j=1}^n$$

in which the time-frequency shifts belong to a set $\Lambda_{n^2} = \{\lambda_{ij} = (\omega_i, y_j)\}_{i,j=1}^n$ of enumerated elements centered at the origin, where

$$\omega_i = b(\lfloor n/2 \rfloor + i - n), \quad i = 1, 2, \dots, n$$

change-of-variables in the inner product seen in Equation (6.3).

Proposition 6.2.1 (Centered Sub-lattice) *Suppose we have a square subset of the lattice $\Lambda_M \subset \Lambda_{n^2} \subset a\mathbb{Z} \times b\mathbb{Z}$ where $|\Lambda_M| = M$. Let*

$$\tilde{\lambda}_{ij} = (\omega_i + tb, y_j + sa) = \lambda_{ij} + (tb, sa)$$

$$\tilde{\lambda}_{i'j'} = (\omega_{i'} + tb, y_{j'} + sa) = \lambda_{i'j'} + (tb, sa)$$

for $j, j' \in \{1, 2, \dots, n\}$ and $i, i' \in \{1, 2, \dots, n\}$ and $s, t \in \mathbb{Z}$. This shift $\Lambda_M + (tb, sa)$ introduces entry-wise phase factors into $G(a, b, n)$. The shifting factor s and the parameter b can be chosen so that $G(a, b, n)$ is unchanged. Furthermore, the Gershgorin discs remain unchanged independent of (a, b) and (s, t) .

Proof: Compute as,

$$\begin{aligned} & \langle g_{\tilde{\lambda}_{ij}}, g_{\tilde{\lambda}_{i'j'}} \rangle_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} g(x - (y_j + s)) e^{2\pi i(\omega_i + t)x} g(x - (y_{j'} + s)) e^{-2\pi i(\omega_{i'} + t)x} dx \\ &= \int_{\mathbb{R}} g(x - s - (aj)) e^{2\pi i(bi + t)x} g(x - s - (aj')) e^{-2\pi i(bi' + t)x} dx \quad \square \\ &= \int_{\mathbb{R}} g(u - (aj)) e^{2\pi i(bi)(u+s)} g(u - (aj')) e^{-2\pi i(bi')(u+s)} du \\ &= (e^{2\pi ibs(i-i')}) \langle g_{\lambda_{ij}}, g_{\lambda_{i'j'}} \rangle_{L^2(\mathbb{R})}. \end{aligned}$$

Recall that a square *Toeplitz matrix* $A \in \mathbb{C}^{n \times n}$ is a matrix whose entries obey the relation

$$A_{ij} = A_{(i+1)(j+1)} = a_{i-j}, \text{ for some } a_{i-j} \in \mathbb{C},$$

and $i, j = 0, 1, \dots, n-1$. In Section 6.3 we study the asymptotics of the eigenvalues of $G(a, b, n)$ and the spectrum of $G(a, b)$ that builds on Toeplitz structure seen below in Theorem 6.2.2. If a matrix is decomposed into blocks so that the entries themselves matrices, then the block matrix is called *block-Toeplitz* if

$$A_{ij} = A_{(i+1)(j+1)} = a_{i-j}, \text{ for some } a_{i-j} \in \mathbb{C}^{m \times m}.$$

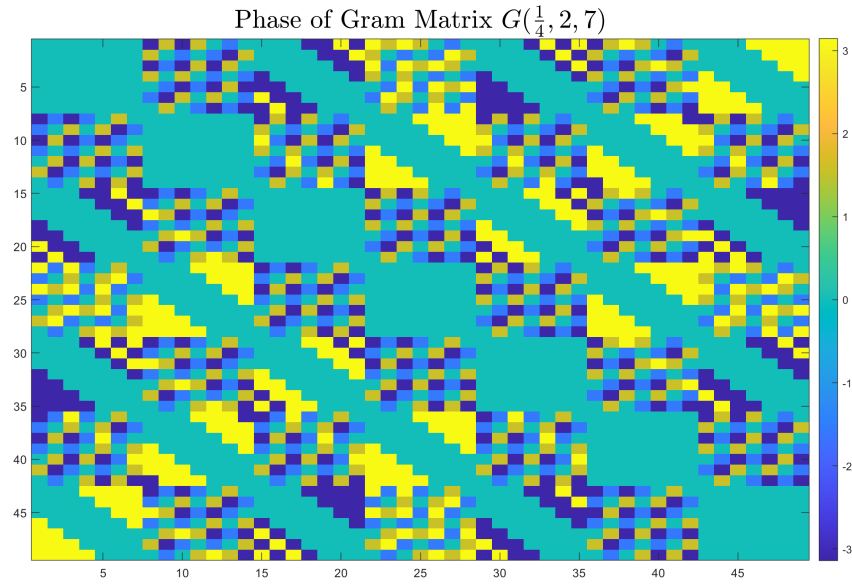
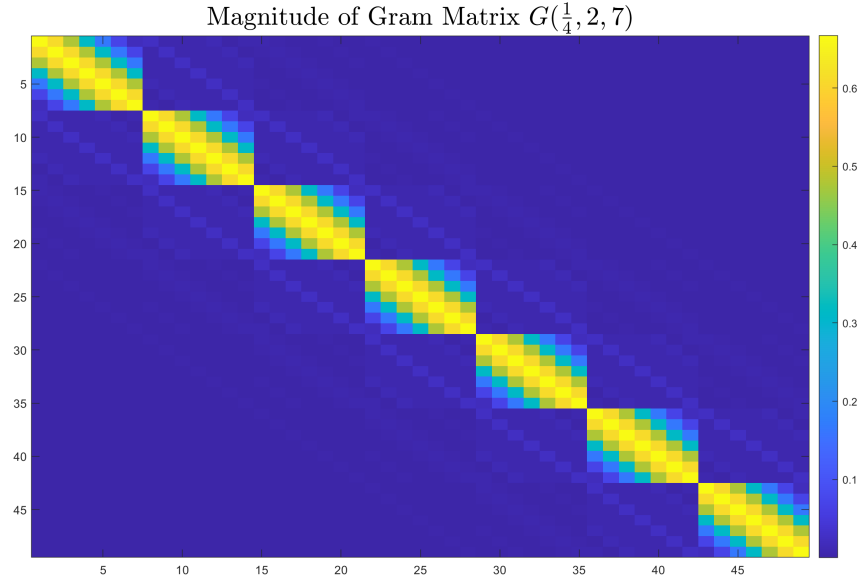


Figure 6.1: The magnitude and phase of the finite Gram matrix $G(\frac{1}{4}, 2, 7) \in \mathbb{C}^{49 \times 49}$ with blocks $G^{ii'}(\frac{1}{4}, 2, 7) \in \mathbb{C}^{7 \times 7}$ for $i, i' \in \{1, 2, \dots, 7\}$.

If the matrices in the block-Toeplitz matrix are themselves Toeplitz, then the matrix is *Toeplitz-block-Toeplitz*.

Theorem 6.2.2 (Toeplitz Blocks) *Each of the blocks $G^{ii}(a, b, n) \in \mathbb{C}^{n \times n}$ for $i \in \{1, 2, \dots, n\}$ within the finite Gram matrix $G(a, b, n) \in \mathbb{C}^{n^2 \times n^2}$ are identical, real-valued, banded, and Toeplitz. Further, $G(a, b, n)$ itself is block-Toeplitz and its magnitude-squared $|G(a, b, n)|^2$ is Toeplitz-block-Toeplitz.*

Proof: The inner products that form the diagonal blocks contain phase factors that cancel and the integral only depends on the shifting parameter. Fix $i \in \{1, \dots, n\}$ and pick $j, j' \in \{1, \dots, n\}$. Then, we have that the entries of the block $G^{ii}(a, b, n)$ are:

$$\begin{aligned} G_{jj'}^{ii} &= \langle g_{\lambda_{ij}}, g_{\lambda_{i'j'}} \rangle_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} g(x - y_j) e^{2\pi i \omega_i x} g(x - y_{j'}) e^{-2\pi i \omega_i x} dx \\ &= \int_{\mathbb{R}} g(x - y_j) g(x - y_{j'}) dx. \end{aligned}$$

Since the inner product is independent of index i , the blocks on the diagonal of $G(a, b, n)$ are identical. Since the phase factors cancel, they are also real-valued. Furthermore,

$$\int_{\mathbb{R}} g(x - y_j) g(x - y_{j'}) dx = \begin{cases} \int_{y_{j'}-1}^{1+y_j} (x - y_{j'} + 1)(-x + y_j + 1) dx & |y_{j'} - y_j| \leq 2 \\ 0 & |y_{j'} - y_j| \geq 2. \end{cases}$$

Since the integral is zero when $|y_{j'} - y_j| \geq 2$, $G(a, b, n)$ is banded. This also applies to the off-diagonal blocks when $i \neq i'$. The integral for the case $|y_{j'} - y_j| \leq 2$ is seen to be:

$$\int_{y_{j'}-1}^{1+y_j} (x - y_{j'} + 1)(-x + y_j + 1) dx = \frac{1}{6}(y_j - y_{j'} + 2)^3.$$

Therefore, the entry $G_{jj'}^{ii}$ is a function of the quantity $y_j - y_{j'}$, hence Toeplitz. Let $\ell = i - i'$ and $k = j - j'$ for $i, j \in \{1, \dots, n\}$. Then, after a change of variables, the entries of the blocks $G^{ii'}(a, b, n)$ for $i' \neq i \in \{1, \dots, n\}$ on the off-diagonals are given

by:

$$\begin{aligned} G_{jj'}^{ii'} &= \int_{\mathbb{R}} g(x - y_j) e^{2\pi i \omega_i x} g(x - y_{j'}) e^{-2\pi i \omega_{i'} x} dx \\ &= e^{2\pi i b \ell y_j} \int_{\mathbb{R}} g(x) g(x + a k) e^{-2\pi i b \ell x} dx. \end{aligned}$$

Hence, for fixed j and j' , the $G_{jj'}^{ii'} = G_{jj'}^{i-i'} = G_{jj'}^{\ell}$ are identical; therefore G is block-Toeplitz. Taking the magnitude or magnitude-squared eliminates the phase factor $e^{2\pi i b \ell y_j}$ and the resulting entries in the blocks $G_{jj'}^{ii'} = G_{j-j'}^{i-i'} = G_k^{\ell}$ depend only on ℓ and k ; therefore $|G(a, b, n)|^2$ is Toeplitz-block-Toeplitz. $\square$

Clearly the Toeplitz structure extends to the infinite blocks $G^{ii}(a, b)$ in the infinite Gram matrix $G(a, b)$. This allows us to associated a finite Fourier series constructed from the bands of $G^{ii}(a, b)$ which provides complete information about the spectrum seen in Section 6.3. In addition to the Toeplitz structure the $G(a, b, n)$ and $G(a, b)$ are Hermitian along their anti-diagonal - called *per-Hermitian* - even when considered as block matrices.

Theorem 6.2.3 (Per-Hermitian Blocks) *Each of the $G^{ii'}(a, b, n) \in \mathbb{C}^{n \times n}$ for $i, i' \in \{1, 2, \dots, n\}$ are per-Hermitian, $\overline{G_{jj'}^{ii'}} = G_{n-j'+1, n-j+1}^{ii'}$. Furthermore, the $G^{ii'}(a, b, n)$ are identical when reflected across the anti-diagonal, $G^{ii'} = G^{(n-i'+1)(n-i+1)}$. That is when considering $G(a, b, n)$ as a block matrix it is per-symmetric therefore per-Hermitian.*

Proof: Because a Gram matrix is Hermitian, we only need to show that $G_{jj'}^{ii'} = G_{n-j+1, n-j'+1}^{ii'}$. Because $y_j \in a\{-\lfloor n/2 \rfloor, \dots, \lfloor n/2 \rfloor\}$ is symmetric about the origin so that $|y_j| = |y_{n-j+1}|$, we have

$$\begin{aligned} G_{n-j+1, n-j'+1}^{ii'} &= \int_{\mathbb{R}} g(x - y_{n-j+1}) e^{2\pi i \omega_i x} g(x - y_{n-j'+1}) e^{-2\pi i \omega_{i'} x} dx \\ &= \int_{\mathbb{R}} g(x + y_j) g(x + y_{j'}) e^{-2\pi i (\omega_{i'} - \omega_i) x} dx \\ &= \mathcal{F}\{g(x + y_j) g(x + y_{j'})\}(\omega_{i'} - \omega_i), \end{aligned}$$

where we observed that the integral is just the Fourier transform of the product

$g(x + y_j)g(x + y_{j'})$ evaluated at $\omega_{i'} - \omega_i$. Without loss of generality, $|y_j| < |y_{j'}|$, and we have a parabola centered at $x = \frac{y_j + y_{j'}}{2}$:

$$\begin{aligned} g(x + y_j)g(x + y_{j'}) &= (x - y_j + 1)(-x + y_{j'} + 1) \\ &= -x^2 + (y_j + y_{j'})x + (y_j y_{j'} + y_{j'} - y_j + 1). \end{aligned}$$

When $y_j = -y_{j'}$, the parabola is centered at $x = 0$ and the integrand is $g(x + y_j)g(x - y_j)$.

Using the time-frequency shift property of the Fourier transform, we then have

$$\mathcal{F}\{g(x + y_j)g(x + y_{j'})\}(\omega_{i'} - \omega_i) = e^{2\pi i(\frac{y_j + y_{j'}}{2})} \mathcal{F}\{g(x + y_j)g(x - y_j)\}(\omega_{i'} - \omega_i).$$

We also have that shifting the integrand in the other direction, $g(x - y_j)g(x - y_{j'})$, conjugates the above:

$$\mathcal{F}\{g(x - y_j)g(x - y_{j'})\}(\omega_{i'} - \omega_i) = e^{-2\pi i(\frac{y_j + y_{j'}}{2})} \mathcal{F}\{g(x + y_j)g(x - y_{j'})\}(\omega_{i'} - \omega_i).$$

Therefore,

$$\begin{aligned} G_{n-j+1, n-j'+1}^{ii'} &= \mathcal{F}\{g(x + y_j)g(x + y_{j'})\}(\omega_{i'} - \omega_i) \\ &= \overline{\mathcal{F}\{g(x - y_j)g(x - y_{j'})\}(\omega_{i'} - \omega_i)} \\ &= \overline{G_{jj'}^{ii'}}. \end{aligned}$$

By construction, the only difference between two block matrices are the phase factors that appear in the inner product,

$$\begin{aligned} G_{jj'}^{ii'} &= \langle g_{\lambda_{ij}}, g_{\lambda_{i'j'}} \rangle_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} g(x - y_j)g(x - y_{j'})e^{2\pi i(\omega_i - \omega_{i'})} dx, \end{aligned}$$

and,

$$G_{jj'}^{(n-i'+1)(n-i+1)} = \langle g_{\lambda_{(n-i'+1)j}}, g_{\lambda_{(n-i+1)j'}} \rangle_{L^2(\mathbb{R})}$$

$$= \int_{\mathbb{R}} g(x - y_j) g(x - y_{j'}) e^{2\pi i(\omega_{(n-i'+1)} - \omega_{(n-i+1)})} dx.$$

Again, because $\omega_i \in b\{-\lfloor n/2 \rfloor, \dots, \lfloor n/2 \rfloor\}$ is symmetric about the origin so that $|\omega_i| = |\omega_{n-i+1}|$ and $|\omega_{i'}| = |\omega_{n-i'+1}|$. Thus,

$$\begin{aligned} G_{jj'}^{(n-i'+1)(n-i+1)} &= \int_{\mathbb{R}} g(x - y_j) g(x - y_{j'}) e^{2\pi i(\omega_{(n-i'+1)} - \omega_{(n-i+1)})} dx \\ &= \int_{\mathbb{R}} g(x - y_j) g(x - y_{j'}) e^{2\pi i(-\omega_{i'} - (-\omega_i))} \\ &= \int_{\mathbb{R}} g(x - y_j) g(x - y_{j'}) e^{2\pi i(\omega_i - \omega_{i'})} \\ &= G_{jj'}^{ii'}. \end{aligned} \quad \square$$

6.3 Spectral Asymptotics

Due to the Toeplitz structure in the finite Gram matrices $G(a, b, n)$ seen in Section 6.2, we can understand the asymptotic behavior of its eigenvalues and spectrum. This is due to a Fourier series one can construct from the Toeplitz bands and a class of *asymptotically equivalent* circulant matrices whose eigenvalues we explicitly provide. In this section we focus on the spectrum of the finite and infinite blocks $G^{ii}(a, b, n)$ and $G^{ii}(a, b)$ for $i \in \{1, 2, \dots\}$ to make conclusions about the spectra of the finite and infinite Gram matrices $G(a, b, n)$ and $G(a, b)$, respectively. For example, since the $G^{ii}(a, b, n)$ are principle sub-matrices of $G(a, b, n)$, the eigenvalues interlace and we can obtain upper and lower estimates on the minimum and maximum eigenvalues $\gamma_1(a, b, n)$ and $\gamma_n(a, b, n)$ of $G(a, b, n)$. Consequently, these provide upper and lower estimates on the frame bounds associated with the finite Gabor frame $\mathcal{G}(g, a, b, n)$,

$$\mathcal{G}(g, a, b, n) := \{M_{\ell b} T_{ka} g = e^{2\pi i \ell b \cdot} g(\cdot - ka) : (\ell b, ka) \in \Lambda_{n^2}\}.$$

Let $\mathbf{T} := \{t \in \mathbb{C} : |t| = 1\}$ be the complex unit circle. The *Wiener algebra* W is the set of functions $f : \mathbf{T} \rightarrow \mathbb{C}$ with absolutely convergent Fourier series. Thus, each function $f \in W$ has the representation and norm:

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}_n t^n, \quad \|f\|_W = \sum_{n=-\infty}^{\infty} |\hat{f}_n| < \infty.$$

The Fourier coefficients $\{\hat{f}_n\}_{n=-\infty}^{\infty}$ are of course computed as:

$$\hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta,$$

where the identification $t \sim e^{i\theta}$, $\theta \in [0, 2\pi)$ is made. We can associate $f \in W$ with an

infinite Toeplitz matrix built from the coefficients $\{\hat{f}_n\}_{n=-\infty}^{\infty}$,

$$T(f) = \begin{bmatrix} \hat{f}_0 & \hat{f}_{-1} & \hat{f}_{-2} & \dots \\ \hat{f}_1 & \hat{f}_0 & \hat{f}_{-1} & \dots \\ \hat{f}_2 & \hat{f}_1 & \hat{f}_0 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}.$$

It is known that $T(f)$ is a bounded linear operator, $T(f) \in \mathcal{B}(\ell^p(\mathbb{Z}_+))$, with estimate $\|T\|_p \leq \|f\|_W$ [12]. Furthermore, a classic result [11] in analysis of Toeplitz operators is that the spectrum of $T(f)$ is completely and beautifully characterized by the range of the function $f \in W$,

$$\text{sp } T(f) = f(\mathbf{T}) \cup \{\lambda \in \mathbb{C} - f(\mathbf{T}) : \text{wind}(f - \lambda) \neq 0\}.$$

As seen in Theorem 6.2.2, the infinite blocks $G^{ii}(a, b)$ for $i \in \mathbb{Z}$ within $G(a, b)$ are not just Toeplitz but real-valued and banded with only finitely many non-zero off-diagonal entries. Therefore, the corresponding Fourier series has only finitely many non-zero coefficients. These banded Toeplitz matrices correspond to *Laurent polynomials* $g \in W$, i.e. the functions represented by a finite Fourier series,

$$g(t) = \sum_{j=-r}^s g_j t^j, \quad t \in \mathbf{T}.$$

When $T(g)$ is banded and real-valued, we get a further refinement on its spectrum: it has no eigenvalues and the spectrum is the line segment $[\min g, \max g]$ [11]. This leads to Theorem 6.3.1. Note that these blocks on the diagonal $G^{ii}(a, b)$ correspond to sampling along vertical lines of the lattice Λ .

Theorem 6.3.1 (Spectrum of $G^{ii}(a, b)$ I) *The infinite blocks $G^{ii}(a, b) \in \mathcal{B}(\ell^2(\mathbb{Z}))$ for $i \in \mathbb{Z}$ on the diagonal of $G(a, b)$ have no eigenvalues and their spectrum is the interval*

$$\text{Spec}(G^{ii}(a, b)) = [\inf_x g^{ii}(x), \sup_x g^{ii}(x)]$$

where $g^{ii} \in W$ is

$$g^{ii}(x) = 2/3 + \sum_{k=1}^{\lfloor 1/a \rfloor} g_k \cos(2\pi kx) + \frac{1}{3} \sum_{k=\lfloor 1/a \rfloor + 1}^{\lfloor 2/a \rfloor} g_k \cos(2\pi kx)$$

and the coefficients are the inner products $\langle g_{\lambda_{ij}}, g_{\lambda_{i'j'}} \rangle_{L^2(\mathbb{R})}$ on the Toeplitz bands of $G^{ii}(a, b)$ indexed by $k = |j - j'|$. These coefficients have the form

$$g_k = \begin{cases} (ak)^3 - 2(ak)^2 + \frac{4}{3}, & k \leq \lfloor \frac{1}{a} \rfloor \\ (2 - ak)^3, & \lfloor \frac{1}{a} \rfloor \leq k \leq \lfloor \frac{2}{a} \rfloor \\ 0, & k \geq \lfloor \frac{2}{a} \rfloor. \end{cases}$$

Proof: By Theorem 6.2.2, the $G^{ii}(a, b)$ are Toeplitz, symmetric, real-valued, and banded. They therefore have no eigenvalues and the spectrum is given by a Laurent polynomial constructed from the bands of the $G^{ii}(a, b)$. The entries $G_{jj'}^{ii}$ for $j, j' \in \mathbb{Z}$ are given by integrating one shifted B-spline against another as seen in Equation (6.3) and has piecewise form. If $y_j \leq y_{j'}$, then:

$$G_{jj'}^{ii} = \begin{cases} \int_{y_{j'}-1}^{y_j} (x+1-y_j)(x+1-y_{j'})dx \\ \quad + \int_{y_j}^{y_{j'}} (x-y_{j'}+1)(-x+y_j+1)dx \\ \quad + \int_{y_{j'}}^{y_j+1} (-x+1+y_j)(-x+1+y_{j'})dx & 0 \leq y_j - y_{j'} < 1 \\ \int_{y_{j'}-1}^{y_j+1} (x-y_{j'}+1)(-x+y_j+1) & 1 \leq y_j - y_{j'} < 2 \\ 0 & 2 \leq y_j - y_{j'}. \end{cases}$$

If $y_{j'} \leq y_j$, then due to symmetry the entries $G_{jj'}^{ii}$ have the same form but with j and j' swapped. Since $y_j = -a(\lfloor n/2 \rfloor + (j-1))$, we have $y_j - y_{j'} = a(j - j')$. Index

the Toeplitz coefficients as $k = j - j'$. Then, solving these integrals yields:

$$G_{jj'}^{ii} = G_{j-j'}^{ii} = G_k^{ii} = \begin{cases} (ak+2)^3/6 & -2/a \leq k \leq -1/a \\ (ak+1)^2(-ak+2)/3 - (ak)^3/6 - ak(ak+1) & -1/a \leq k \leq 0 \\ (-ak+1)^2(ak+2)/3 + (ak)^3/6 + ak(-ak+1) & 0 \leq k \leq 1/a \\ (-ak+2)^3/6 & 1/a \leq k \leq 2/a \\ 0 & |k| \geq 2/a. \end{cases}$$

Because the coefficients for positive and negative index k are equal, the Laurent polynomial $g^{ii}(x) \in W$ associated to $G^{ii}(a, b)$ is real-valued. The $g^{ii}(x)$ has the form:

$$\begin{aligned} g^{ii}(x) &= \sum_{k=-\lfloor 2/a \rfloor}^{\lfloor 2/a \rfloor} G_k^{ii} e^{2\pi i k x} \\ &= 2/3 + 2 \sum_{k=1}^{\lfloor 2/a \rfloor} G_k^{ii} \cos(2\pi k x) \\ &= 2/3 + \sum_{k=1}^{\lfloor 1/a \rfloor} \left((ak)^3 - 2(ak)^2 + \frac{4}{3} \right) \cos(2\pi k x) + \frac{1}{3} \sum_{k=\lfloor 1/a \rfloor + 1}^{\lfloor 2/a \rfloor} (2 - ak)^3 \cos(2\pi k x). \end{aligned}$$

$G^{ii}(a, b) \in \mathcal{B}(\ell^p(\mathbb{Z}))$ as the Fourier series is finite with finite coefficients. $\square$

For example, if we take $a = 1/4$ and $b = 2$ (known to *not* be in the frame set for the 2nd-order B-spline), then $g^{ii}(x)$ (Figure 6.2a) has the form:

$$g^{ii}(x) = 2/3 + \sum_{k=1}^4 \left(\frac{k^3}{64} - \frac{k^2}{8} + \frac{4}{3} \right) \cos(2\pi k x) + \frac{1}{3} \sum_{k=5}^8 \left(2 - \frac{k}{4} \right)^3 \cos(2\pi k x).$$

As all the coefficients are positive, the $g^{ii}(x)$ attains its maximum when $\cos(2\pi k x) = 1$. Using a standard counting argument we can explicitly compute the maximum and minimum of $g^{ii}(x)$ in certain cases. Recall the formulas for the sums of natural numbers, their squares, and cubes

$$\sum_{k=1}^n k = \frac{n(n+1)}{2},$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2,$$

and the alternating sums of natural numbers, their squares, and cubes

$$\sum_{k=1}^n (-1)^k k = \begin{cases} -\left(\frac{n+1}{2}\right)^2 + \left(\frac{n-1}{2}\right)\left(\frac{n-1}{2} + 1\right), & n \text{ even} \\ -\left(\frac{n}{2}\right)^2 - \left(\frac{n}{2}\right)\left(\frac{n}{2} + 1\right), & n \text{ odd} \end{cases}$$

$$\sum_{k=1}^n (-1)^k k^2 = \begin{cases} \frac{n(n+1)}{2}, & n \text{ even} \\ -\frac{n(n+1)}{2}, & n \text{ odd} \end{cases}$$

$$\sum_{k=1}^n (-1)^k k^3 = \begin{cases} \frac{2n^3+3n^2-1}{4}, & n \text{ even} \\ -\frac{n^2(2n+3)}{4}, & n \text{ odd.} \end{cases}$$

Corollary 6.3.2 (Spectrum of $G^{ii}(a, b)$ II) *When $a = \frac{1}{p}$ for $p \in \mathbb{Z}_+$, the Laurent polynomial $g^{ii}(x)$ associated to the Toeplitz $G^{ii}(a, b)$ has a maximum at $x = m$ for $m \in \mathbb{Z}$,*

$$\|G^{ii}(a, b)\| = \max_x g^{ii}(x) = g^{ii}(m) = p.$$

The $g^{ii}(x)$ has a minimum at $x = m/2$ for $m \in \mathbb{Z}$ odd,

$$\min_x g^{ii}(x) = g^{ii}\left(\frac{m}{2}\right) = 0.$$

The spectrum of the $G^{ii}(a, b)$ is therefore the interval

$$\text{Spec}(G^{ii}(a, b)) = [g\left(\frac{m}{2}\right), g(m)] = [0, p].$$

Proof: Without loss of generality, take $m = 0$. Then,

$$\begin{aligned} g^{ii}(0) &= \frac{2}{3} + \sum_{k=1}^p (ak)^3 - 2 \sum_{k=1}^p (ak)^2 + \frac{4p}{3} + \frac{1}{3} \left(\sum_{k=1}^{2p} (2-ak)^3 - \sum_{k=1}^p (2-ak)^3 \right) \\ &= \frac{2}{3} + a^3 \sum_{k=1}^p k^3 - 2a^2 \sum_{k=1}^p k^2 + \frac{4p}{3} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{3} \left(-a^3 \sum_{k=1}^{2p} k^3 + 6a^2 \sum_{k=1}^{2p} k^2 - 12a \sum_{k=1}^{2p} k + 8(2p) \right) \\
& - \frac{1}{3} \left((-a^3 \sum_{k=1}^p k^3 + 6a^2 \sum_{k=1}^p k^2 - 12a \sum_{k=1}^p k + 8p) \right).
\end{aligned}$$

Apply the formulas for the sum of natural numbers, their squares, and cubes. The above becomes:

$$\begin{aligned}
g^{ii}(0) &= \frac{2}{3} + a^3 \left(\frac{p^2(p+1)^2}{4} \right) - 2a^2 \left(\frac{p(p+1)(2p+1)}{6} \right) + \frac{4p}{3} \\
&+ \frac{1}{3} \left(-a^3 \left(\frac{(2p)^2(2p+1)^2}{4} \right) + 6a^2 \left(\frac{2p(2p+1)(2(2p)+1)}{6} \right) - 12a \left(\frac{2p(2p+1)}{2} \right) + 8(2p) \right) \\
&+ a^3 \left(\frac{p^2(p+1)^2}{4} \right) - 6a^2 \left(\frac{p(p+1)(2p+1)}{6} \right) + 12a \left(\frac{p(p+1)}{2} \right) - 8p.
\end{aligned}$$

Simplifying:

$$\begin{aligned}
g^{ii}(0) &= \frac{2}{3} + \frac{a^3 p^4}{4} + \frac{a^3 p^3}{2} + \frac{a^3 p^2}{4} - \frac{2a^2 p^3}{3} - a^2 p^2 - \frac{a^2 p}{3} + \frac{4p}{3} \\
&- \frac{1}{12} a^3 (2p)^4 - \frac{a^3 (2p)^3}{6} - \frac{a^3 (2p)^2}{12} + \frac{2a^2 (2p)^3}{3} + a^2 (2p)^2 + \frac{a^2 (2p)}{3} - 2a(2p)^2 - 2a(2p) + \frac{8(2p)}{3} \\
&+ \frac{1}{12} a^3 p^4 + \frac{a^3 p^3}{6} + \frac{a^3 p^2}{12} - \frac{2a^2 p^3}{3} - a^2 p^2 - \frac{a^2 p}{3} + 2ap^2 + 2ap - \frac{8p}{3}.
\end{aligned}$$

Combining like terms and simplifying, we have

$$g^{ii}(0) = \frac{1}{a}.$$

To compute the minimum without loss of generality, take $x = 1/2$. Then,

$$\begin{aligned}
g^{ii}(1/2) &= \frac{2}{3} + \sum_{k=1}^p (-1)^k (ak)^3 - 2 \sum_{k=1}^p (-1)^k (ak)^2 + \sum_{k=1}^p (-1)^k \frac{4}{3} \\
&+ \frac{1}{3} \left(\sum_{k=1}^{2p} (-1)^k (2-ak)^3 - \sum_{k=1}^p (-1)^k (2-ak)^3 \right) \\
&= \frac{2}{3} + a^3 \sum_{k=1}^p (-1)^k k^3 - 2a^2 \sum_{k=1}^p (-1)^k k^2 + \sum_{k=1}^p (-1)^k \frac{4}{3} \\
&+ \frac{1}{3} \left(-a^3 \sum_{k=1}^{2p} (-1)^k k^3 + 6a^2 \sum_{k=1}^{2p} (-1)^k k^2 - 12a \sum_{k=1}^{2p} (-1)^k k + \sum_{k=1}^2 p(-1)^k 8 \right. \\
&\left. - (-a^3 \sum_{k=1}^p (-1)^k k^3 + 6a^2 \sum_{k=1}^p (-1)^k k^2 - 12a \sum_{k=1}^p (-1)^k k + \sum_{k=1}^p (-1)^k 8) \right).
\end{aligned}$$

Apply the formulas for alternating sums of natural numbers, their squares, and cubes. These formulas depend even and odd cases. When p is odd, the above simplifies to:

$$\begin{aligned} g^{ii}(1/2) &= 2a\left(\frac{2}{a} + 1\right) + \frac{\left(a\left(\frac{2}{a} + 3\right)\right)}{3} - \frac{\left(a\left(\frac{4}{a} + 3\right)\right)}{3} - 4a\left(\frac{\frac{1}{a} + 1}{a} - \frac{1}{a^2}\right) \\ &\quad + 4a\left(\frac{\left(\frac{1}{2a} + 1\right)}{2a} - \frac{1}{4a^2}\right) - 2a\left(\frac{1}{a} + 1\right) + \frac{2}{3} \\ &= 0. \end{aligned}$$

The computation for the even case proceeds similarly. $\square$

Corollary 6.3.3 (Upper Frame Bound) *When $a = \frac{1}{p}$ for $p \in \mathbb{Z}_+$, the upper frame bound of the Gabor system $\mathcal{G}(g, a, b)$ is $B = p$.*

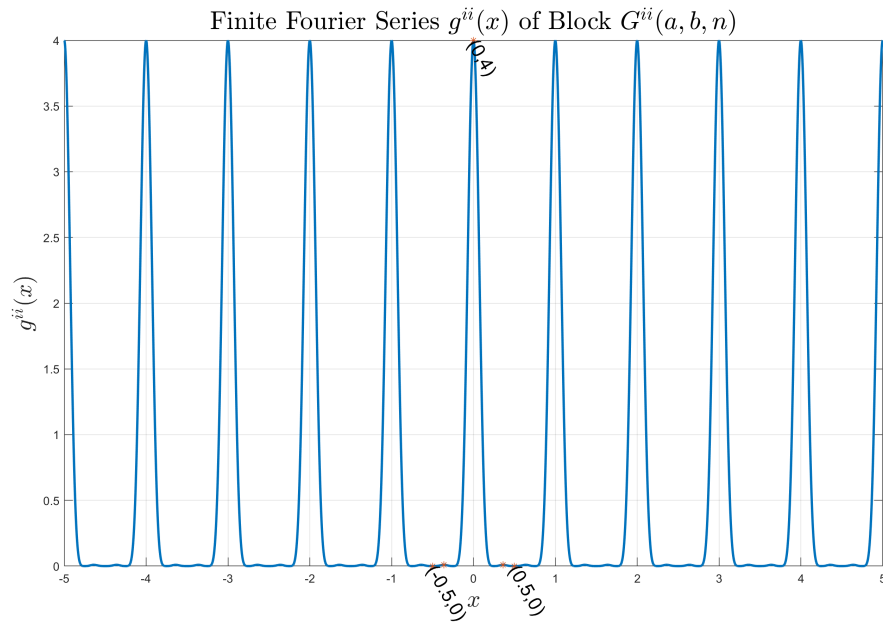
That $B = p$ follows from the fact that the upper frame bound is known to be the operator norm $B = \|G(a, b)\|$. Estimates on the operator norm of the pseudo-inverse $\|G^\dagger(a, b)\|$ are valuable as its reciprocal is the lower frame bound $A = \|G^\dagger(a, b)\|^{-1}$, see [1]. Furthermore, in the case where $(a, b) \in \mathcal{F}(g)$ and $a = \frac{1}{p}$ for $p \in \mathbb{Z}_+$, Corollary 6.3.3 provides the lower frame bound for the dual frame $\{h_\lambda\}_{\lambda \in \Lambda}$.

Analysis of the Laurent polynomial $g^{ii}(x)$ including its maximum and minimum provides information on the spectrum of the infinite Gram matrix $G(a, b)$ and its finite counterpart $G(a, b, n)$ hence information on their frame bounds. This is summarized in Corollaries 6.3.3 and 6.3.4 and is due to:

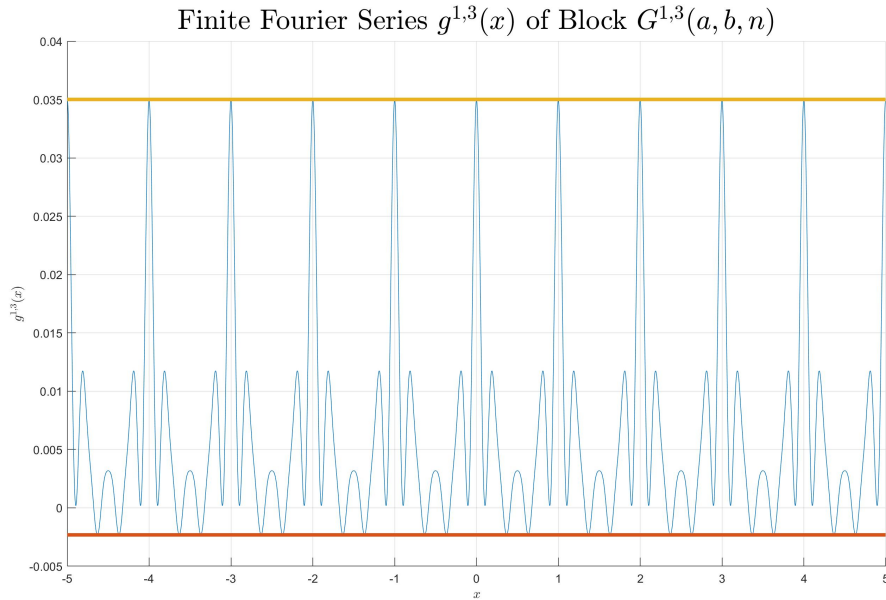
1. Classic Szego-type limit theorems [43] that state the minimum and maximum eigenvalues, $\gamma_1^{ii}(a, b, n)$ and $\gamma_n^{ii}(a, b, n)$, of the Toeplitz $G^{ii}(a, b, n)$ belong to the interval $[\inf_x g^{ii}(x), \sup_x g^{ii}(x)]$ with

$$\begin{aligned} \lim_{n \rightarrow \infty} \gamma_1^{ii}(a, b, n) &= \inf_x g^{ii}(x) \\ \lim_{n \rightarrow \infty} \gamma_n^{ii}(a, b, n) &= \sup_x g^{ii}(x). \end{aligned}$$

2. The blocks $G^{ii}(a, b, n)$ are principle sub-matrices and hence the eigenvalues



(a) The Laurent polynomial $g^{ii}(x) \in W$ associated to the Toeplitz block $G^{ii}(a, b) \in \mathcal{B}(\ell^p(\mathbb{Z}))$. The spectrum of $G^{ii}(a, b)$ is the interval $[\min g^{ii}(x), \max g^{ii}(x)]$, analytically and numerically computed to be $[0, 4]$ in the case where $a = 1/4, b = 2$.



(b) The Laurent polynomial $g^{13}(x) \in W$ associated to the Toeplitz block $|G^{13}(a, b)|^2 \in \mathcal{B}(\ell^p(\mathbb{Z}))$. The spectrum of $|G^{13}(a, b)|^2$ is the interval $[\min g^{13}(x), \max g^{13}(x)]$, numerically computed to be $\approx [-.002, .035]$ in the case where $a = 1/4, b = 2$.

interlace with $G(a, b, n)$,

$$\gamma_1(a, b, n) \leq \gamma_1^{ii}(a, b, n) \leq \dots \leq \gamma_n^{ii}(a, b, n) \leq \gamma_n(a, b, n).$$

Corollary 6.3.4 (Finite Frame Bound Estimates) *Let $A(a, b, n) = \gamma_1(a, b, n) \in \mathbb{R}$ and $B(a, b, n) = \gamma_n(a, b, n) \in \mathbb{R}$ be the lower and upper frame bounds associated to the finite Gabor frame $\mathcal{G}(g, a, b, n)$. There exists an $N \in \mathbb{N}$ such that $\forall n \geq N$, we have the upper and lower estimates on $A(a, b, n)$ and $B(a, b, n)$:*

$$\begin{aligned} A(a, b, n) &\leq \inf_x g^{ii}(x) \leq \gamma_1^{ii}(a, b, n), \\ \gamma_n^{ii}(a, b, n) &\leq \sup_x g^{ii}(x) \leq B(a, b, n). \end{aligned}$$

This analysis can be further refined with the introduction of a family of circulant matrices that are asymptotically equivalent to the Toeplitz block $G^{ii}(a, b)$ and allow estimates on the eigenvalues $\gamma_1(a, b, n)$ and $\gamma_n(a, b, n)$ and hence on the frame bounds $A(a, b, n)$ and $B(a, b, n)$. This is especially useful in the cases where we cannot explicitly calculate $\inf_x g^{ii}(x)$ and $\sup_x g^{ii}(x)$ and therefore cannot explicitly calculate $\text{Spec}(G^{ii}(a, b))$. Two sequences of matrices $\{G_n\}_{n=1}^\infty, \{H_n\}_{n=1}^\infty$ are said to be *asymptotically equivalent*, $G_n \sim H_n$, if:

1. G_n and H_n are uniformly bounded in the $\|M\|_2 := \sqrt{\lambda_{\max}(M^*M)}$ norm,

$$\|G_n\|_2, \|H_n\|_2 \leq P < \infty,$$

2. $|G_n - H_n| \rightarrow 0$, where $|M|$ is the scaled Hilbert-Schmidt norm $|M| := (\frac{1}{n} \sum_{i,j=1}^n |M_{ij}|^2)^{1/2}$.

Asymptotically equivalent sequences of matrices have eigenvalues that behave similarly for large n . Indeed, for a continuous function $f(x)$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} |f(\gamma_m) - f(\psi_m)| = 0$$

and the eigenvalues are said to be *asymptotically absolutely equally distributed*; the

average difference in the eigenvalues of asymptotically equivalent matrices tends to zero [43]. In our case where $\{G_n\}_{n=1}^\infty$ is a sequence of banded Toeplitz matrices and $\{H_n\}_{n=1}^\infty$ is a sequence of circulant matrices constructed from the Toeplitz $\{G_n\}_{n=1}^\infty$, we have that the eigenvalues individually converge [93]. These circulant matrices are constructed by shifting and padding the rows of the banded Toeplitz matrix appropriately [43]: they are circular convolutions of the B-spline. The advantage of working with circulant matrices to study the spectrum of a banded Toeplitz matrix is that their eigenvalues $\{\psi_m\}_{m=1}^n$ can be computed explicitly as the DFT of a row of the H_n ,

$$\psi_m = \sum_{k=0}^{n-1} h_k e^{-2\pi i m k / n}, \quad m \in \{1, 2, \dots, n\},$$

where h_k with $k = 0, \dots, n-1$ are entries in a row of the H_n . Let $\text{Comb}(x) = \sum_{\ell=-\infty}^{\infty} \delta(x - \ell)$ be the *Dirac comb*. The Dirac distribution and Dirac comb are best understood in the distributional sense, see [44, Chapter 1] and [74, Section 4] for a good introduction. Here, $\text{Comb}(x)$ is only used to periodize and sample a function. We use the following relation between the DFT (see Equation (2.13)) of a sampled function $\mathbf{f} = (f\chi_{[-\frac{1}{2}, \frac{1}{2}]}\text{Comb})(m)$ and samples of the Fourier transform $\mathcal{F}\{f\}(m)$ for $m \in \{0, 1, \dots, n-1\}$ [74, Sections 5, 6]:

$$\begin{aligned} \text{DFT}\{\mathbf{f}\}(m) &= (\mathcal{F}\{f\} * \text{sinc} * \text{Comb})\left(\frac{m}{n}\right) \\ &= \sum_{\ell=-\infty}^{\infty} (\mathcal{F}\{f\} * \text{sinc})\left(\frac{m}{n} - \ell\right). \end{aligned}$$

Theorem 6.3.5 (Circulant Matrix Eigenvalues) *Let $H^{ii}(a, b, n) \in \mathbb{C}^{n \times n}$ be the circulant matrix constructed from the Toeplitz block $G^{ii}(a, b, n)$. Then, the eigenvalues $\{\psi_m^{ii}(a, b, n)\}_{m=1}^n$ of $H^{ii}(a, b, n)$ have the form*

$$\psi_m^{ii}(a, b, n) = \frac{1}{a} \left(\sum_{\ell=-\infty}^{\infty} \text{sinc}^2\left(\frac{1}{a}\left(\frac{m}{n} - \ell\right)\right) \right)^2$$

and are the square of the discrete Fourier transform of an a -dilated B-spline and correspond to samples of the Laurent polynomial $g^{ii}(x)$ from Theorem 6.3.1.

Proof: Let

$$\mathbf{g}(m) := (g\chi_{[-1,1]}\text{Comb})(m)$$

for $m = 0, \dots, n-1$ be the discrete set of uniform samples of the B-spline g on the interval $[-1, 1]$,

$$\mathbf{g} = \left(g(-1), g(-1 + \frac{2}{n}), \dots, g(-1 + 2\frac{n-2}{n}), g(-1 + 2\frac{n-1}{n})\right).$$

A row of the circulant $H^{ii}(a, b, n)$ are samples of a circular convolution of g with itself [43]. We compute:

$$\begin{aligned} \psi_m^{ii}(a, b, n) &= \sum_{k=0}^{n-1} (g * g)(ak) e^{-2\pi i k m / n} \\ &= \sum_{k=0}^{n-1} (\sqrt{a}g(ax) * \sqrt{a}g(ax))(k) e^{-2\pi i k m / n} \\ &= \left(\sum_{k=0}^{n-1} \sqrt{a}g(ak) e^{-2\pi i k m / n} \right)^2 \\ &= \text{DFT}\{\sqrt{a}\mathbf{g}(a \cdot)\}^2(m). \end{aligned}$$

Thus, since $\mathcal{F}\{\sqrt{a}g(ax)\}(\cdot) = \frac{1}{\sqrt{a}}\text{sinc}^2(\frac{\cdot}{a})$, we have that

$$\begin{aligned} \mathcal{F}\{\sqrt{a}g(a \cdot)\}(\cdot) * n\text{sinc}(n \cdot) &= \frac{1}{\sqrt{a}}\text{sinc}^2(\frac{\cdot}{a}) * n\text{sinc}(n \cdot) \\ &= \frac{1}{\sqrt{a}}\text{sinc}^2(\frac{\cdot}{a}) \end{aligned}$$

when $n \geq \frac{1}{a}$, because

$$\begin{aligned} \mathcal{F}\left\{\frac{1}{\sqrt{a}}\text{sinc}^2(\frac{\cdot}{a}) * n\text{sinc}(n \cdot)\right\}(\cdot) &= \sqrt{a}g(a \cdot)\Pi(\frac{\cdot}{n}) \\ &= \sqrt{a}g(a \cdot). \end{aligned}$$

Therefore,

$$\text{DFT}\{\sqrt{a}\mathbf{g}(a \cdot)\}(m) = \frac{1}{\sqrt{a}}\left(\text{sinc}^2(\frac{\cdot}{a}) * \text{Comb}(\cdot)\right)\left(\frac{m}{n}\right)$$

$$= \frac{1}{\sqrt{a}} \sum_{\ell=-\infty}^{\infty} \text{sinc}^2\left(\frac{1}{a}\left(\frac{m}{n} - \ell\right)\right).$$

The convolution with the $\text{Comb}(\cdot)$ function above is just a periodization. The eigenvalues $\{\psi_m^{ii}(a, b, n)\}_{m=1}^n$ corresponding to samples of the Laurent polynomial $g^{ii}(x)$ is a standard calculation [43]. It relies on the way the rows of the circulant $H^{ii}(a, b, n)$ are constructed from rows of the Toeplitz $G^{ii}(a, b, n)$:

$$\begin{aligned} \psi_m^{ii}(a, b, n) &= \sum_{k=0}^{n-1} h_k e^{-2\pi i k m / n} \\ &= \sum_{k=0}^j g_{-k} e^{-2\pi i k m / n} + \sum_{k=n-j}^{n-1} g_{n-k} e^{-2\pi i k m / n} \\ &= \sum_{k=-j}^j g_k e^{-2\pi i k m / n} = g^{ii}\left(\frac{m}{n}\right). \end{aligned} \quad \square$$

Note, if $a \leq 1/2$, then the scaled and shifted $\text{sinc}^2(\frac{1}{a}(\cdot - \ell))$ share zeros with $\text{sinc}^2(\cdot)$ at the scaled integers $\frac{1}{a}\mathbb{Z}$. Hence, if $an|m$ for some $m \in \{0, 1, \dots, n-1\}$, then $\psi_m^{ii}(a, b, n) = 0$. Because the eigenvalues of the finite blocks $G^{ii}(a, b, n)$ interlace those of $G(a, b, n)$, it must be the case that if the minimum eigenvalue of $G^{ii}(a, b, n)$ tends to zero, so does that of $G(a, b, n)$. The minimum eigenvalue of $G^{ii}(a, b, n)$ tends to zero when $H^{ii}(a, b, n)$ has an eigenvalue of zero or tends to zero. This corresponds to when $g^{ii}(x)$ has a zero, see Figure 6.3. If $a = \frac{1}{4}$ and $n = 1024$, then

$$\psi_m^{ii}(a, b, n) = \frac{1}{4} \sum_{\ell=-\infty}^{\infty} \text{sinc}^2\left(\frac{m}{256} - 4\ell\right) \quad (6.4)$$

has zeros at $m = 256, 512, 768$. Hence, when $a = \frac{1}{4}$ it must be that the eigenvalue and frame bound $A(a, b, n) = \gamma_1(a, b, n) \rightarrow 0$. It is known that for finite Gabor frames, the lower frame bound must tend to zero [1]. Thus, Corollary 6.3.6 can be viewed as generalizing this result to Gabor systems in general and tying them to the eigenvalues of their Gram matrices.

Corollary 6.3.6 (Lower Frame Bound and Eigenvalue Decay) *If the minimum eigenvalue $\gamma_1^{ii}(a, b, n)$ of the of the finite block $G^{ii}(a, b, n) \in \mathbb{C}^{n \times n}$ tends to zero as $n \rightarrow \infty$, then (1) $a \in \mathbb{Q}$ and (2) the minimum eigenvalue $\gamma_1(a, b, n)$ of the finite Gram*

matrices $G(a, b, n)$ tend to zero as $n \rightarrow \infty$.

The $G^{ii'}(a, b, n)$ and $G^{ii'}(a, b)$ are in general not Toeplitz due to the phase factor seen in the proof of Theorem 6.2.2. However, taking the magnitude or magnitude-squared of the entries makes them Toeplitz and one can also compute exact expressions for the finite Fourier series associated to $|G^{ii'}(a, b, n)|^2$, for $i \neq i' \in \{1, 2, \dots, n\}$. Repeating the analysis of Theorem 6.3.5 by relating the DFT of the vectorized or sampled function to its continuum Fourier transform leads to Theorem 6.3.7. We also make use of the *Wiener-Khinchin theorem* [73, Chapter 4, 10]: the Fourier transform of the energy-density is the auto-correlation, $\mathcal{F}\{|\mathcal{F}\{f\}|^2\} = f * f = *f$.

Theorem 6.3.7 (Eigenvalue Estimates of $|G^{ii'}(a, b, n)|^2$) *The eigenvalues $\{\psi_m^{ii'}(a, b, n)\}_{m=1}^n$ of the circulant matrices $H^{ii'}(a, b, n)$ constructed from the Toeplitz blocks $|G^{ii'}(a, b, n)|^2$ are*

$$\psi_m^{ii'}(a, b, n) = \sum_{k'=-\infty}^{\infty} \text{Auto-corr}\{s\}\left(\frac{m}{n} - k'\right),$$

where

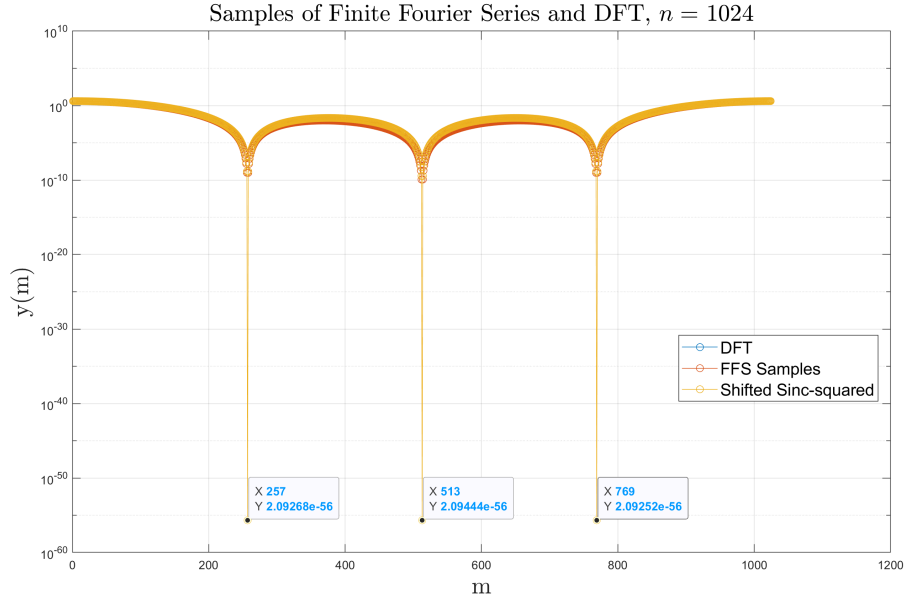
$$s(\cdot) := \frac{1}{a} \text{sinc}^2\left(\frac{\cdot}{a}\right) \text{sinc}^2\left(\frac{\cdot}{a} - b\ell\right)$$

and $\text{Auto-corr}\{s\}$ is the convolution of $s(\cdot)$ with itself,

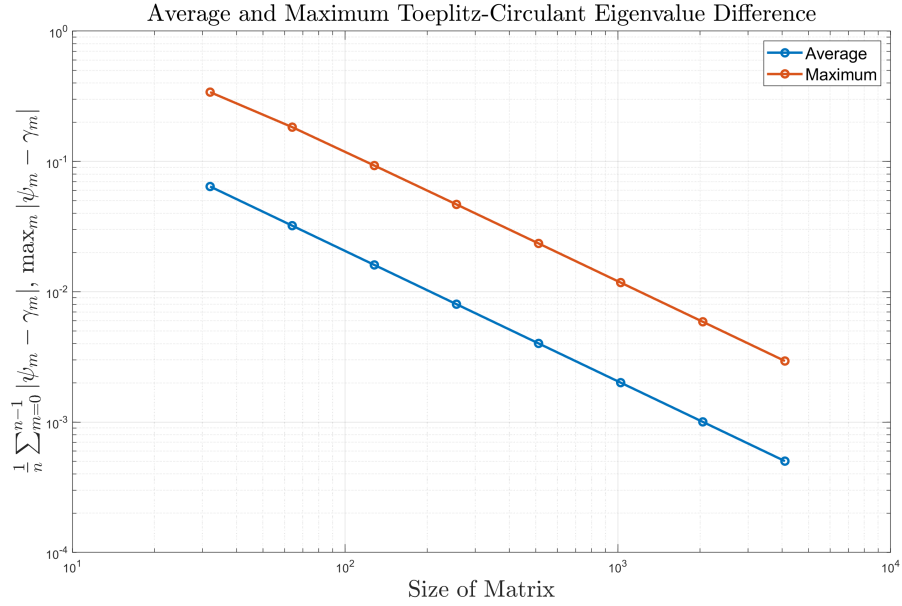
$$\text{Auto-corr}\{s\} = * \left(\frac{1}{a} \text{sinc}^2\left(\frac{\cdot}{a}\right) \text{sinc}^2\left(\frac{\cdot}{a} - b\ell\right) \right).$$

Proof: Fix an off-diagonal block and create a Toeplitz block by considering the magnitude-squared of the entries, $|G^{ii'}(a, b, n)|^2 = |G^{i-i'}(a, b, n)|^2 = |G^\ell(a, b, n)|^2$ for some $\ell \in \{1, 2, \dots\}$. These entries have the form,

$$\begin{aligned} |G_k^\ell(a, b, n)|^2 &= \left| \int_{\mathbb{R}} g(u) g(u + ak) e^{-2\pi i b \ell x} dx \right|^2 \\ &= |(\text{sinc}^2(\cdot) * (e^{2\pi i a k(\cdot)} \text{sinc}^2(\cdot)))(b\ell)|^2 \\ &= \left| \int_{\mathbb{R}} \text{sinc}^2(z) \text{sinc}^2(z - b\ell) e^{2\pi i a k(z - b\ell)} dz \right|^2 \\ &= \left| \int_{\mathbb{R}} \text{sinc}^2(z) \text{sinc}^2(z - b\ell) e^{2\pi i a k z} dz \right|^2 \\ &= \left| \frac{1}{a} \int_{\mathbb{R}} \text{sinc}^2\left(\frac{z'}{a}\right) \text{sinc}^2\left(\frac{z'}{a} - b\ell\right) e^{2\pi i k z} dz' \right|^2 \end{aligned}$$



(a) Samples of the Laurent polynomial $g^{ii}(x)$ (red), samples $\frac{1}{a} \left(\sum_{\ell=-\infty}^{\infty} \text{sinc}^2 \left(\frac{1}{a} \left(\frac{m}{n} - \ell \right) \right) \right)^2$ (orange), and the DFT of an a -dilated B-spline $\sqrt{a}g(ax)$ (blue) for $a = 1/4$, $b = 2$. Theorem 6.3.5 states that these three quantities are equal.



(b) The difference in eigenvalues $|\psi_m^{ii}(a, b, n) - \gamma^{ii}(a, b, n)|$ between the Toeplitz $G^{ii}(a, b, n)$ and circulant $H^{ii}(a, b, n)$ tends to zero as the size of the matrices grow. Since $\psi_m^{ii}(a, b, n)$ can be explicitly computed (Equation (6.4)) we can study the frame bounds of the Gabor system $\mathcal{G}(a, b, n)$.

Figure 6.3: Asymptotic behavior of the eigenvalues $\psi_m^{ii}(a, b, n)$ of Toeplitz blocks $G^{ii}(a, b, n)$ in the Gram matrix $G(a, b, n)$ of time-frequency shifts of the 2nd-order B-spline.

$$= |\mathcal{F}\{\frac{1}{a}\text{sinc}^2(\frac{\cdot}{a})\text{sinc}^2(\frac{\cdot}{a} - b\ell)\}(k)|^2.$$

Call $s(\cdot) := \frac{1}{a}\text{sinc}^2(\frac{\cdot}{a})\text{sinc}^2(\frac{\cdot}{a} - b\ell)$. In the continuum where ak is replaced by a $\omega \in \mathbb{R}$, we get the convolution of $\frac{1}{a}\text{sinc}^2(\frac{\cdot}{a})\text{sinc}^2(\frac{\cdot}{a} - b\ell)$ with itself,

$$\begin{aligned} \mathcal{F}\{|\mathcal{F}\{\frac{1}{a}\text{sinc}^2(\frac{\cdot}{a})\text{sinc}^2(\frac{\cdot}{a} - b\ell)\}|^2\} &= \text{Auto-corr}\{\frac{1}{a}\text{sinc}^2(\frac{\cdot}{a})\text{sinc}^2(\frac{\cdot}{a} - b\ell)\} \\ &= * \left(\frac{1}{a}\text{sinc}^2(\frac{\cdot}{a})\text{sinc}^2(\frac{\cdot}{a} - b\ell) \right). \end{aligned}$$

Thus, the eigenvalues $\psi_m^{ii'}(a, b, n)$ of the circulant matrix $H^{ii'}(a, b, n)$ associated with $|G^{ii'}(a, b, n)|^2$ are:

$$\begin{aligned} \psi_m^{ii'}(a, b, n) &= \text{DFT}\{|\mathcal{F}\{s\}|^2\}(m) \\ &= \mathcal{F}\{|\mathcal{F}\{s\}|^2\} * n\text{sinc}(n\cdot) * \text{Comb}(\frac{m}{n}) \\ &= \mathcal{F}\{|\mathcal{F}\{s\}|^2\} * \text{Comb}(\frac{m}{n}) \\ &= \sum_{k'=-\infty}^{\infty} \text{Auto-corr}\{s\}(\frac{m}{n} - k'). \end{aligned} \quad \square$$

Note that unlike the blocks $G^{ii}(a, b, n)$ on the diagonal which correspond to sampling vertical lines of Λ , the spectrum of the $|G^{ii'}(a, b, n)|^2$ are influenced by the parameter b as the phase factors in the inner product do not cancel one another.

6.4 Properties of the Gram Operator

This section contains results on the operators associated with the Gabor system $\{g_\lambda\}_{\lambda \in \Lambda}$. We are interested in what sense the $G(a, b, n)$ approximate $G(a, b)$. This is in order to obtain information about the spectrum of $G(a, b)$ and determine whether zero is in the spectrum and where it is in the spectrum. This is to determine invertibility of $G(a, b)$, hence invertibility of the frame operator $S(a, b)$, hence if $(a, b) \in \mathcal{F}(g)$. Unfortunately, the reasoning breaks down in a number of ways. This is an incredibly delicate problem and raises a number of questions in approximation theory and spectral theory:

1. By Theorem 2.4.11 and Theorem 2.4.14, $G(a, b)$ is never invertible. Is it possible that classifying zero in the spectrum influences whether $S(a, b)$ is invertible?
2. In what sense do the finite sections $G(a, b, n)$ approximate $G(a, b)$? Is there a particular truncation, norm, or method by which these finite matrices approximate their infinite matrices?
3. In what sense does the spectrum of $G(a, b, n)$ approximate $G(a, b)$? The spectrum of $G(a, b)$ can consist of more than just eigenvalues. Since the spectrum of $G(a, b, n)$ consists only of eigenvalues, is there anything that we can learn about the rest of the spectrum of $G(a, b)$ either via $G(a, b, n)$ or another approximation method?

If $G(a, b)$ is compact, it can be approximated by finite-dimensional Gram matrices in operator norm. We can say immediately if $\{g_\lambda\}_{\lambda \in \Lambda}$ forms a frame for $L^2(\mathbb{R})$, then the analysis and synthesis operators $T(a, b) : L^2(\mathbb{R}) \rightarrow \ell^2(\mathbb{Z}^2)$ and $T^*(a, b) : \ell^2(\mathbb{Z}^2) \rightarrow L^2(\mathbb{R})$ are not compact. This is because the frame operator $S(a, b) = T(a, b)T^*(a, b)$ is invertible. If either $T(a, b)$ or $T^*(a, b)$ is compact the composition is compact and therefore not invertible. This does not immediately imply $G(a, b) = T^*(a, b)T(a, b)$ is not compact. We can also make the following statement concerning the existence of an eigenvalue of zero for $G(a, b)$ and the operators $T(a, b)$ and $T^*(a, b)$: if $G(a, b)$ has an eigenvalue of zero, then $T(a, b)$ is not

injective and $T^*(a, b)$ is not surjective: if $T(a, b)$ is not injective then $T(a, b)x = 0$ for a $x \neq 0$ and $G(a, b)x = 0$. Therefore $G(a, b)$ is not injective. And if $G(a, b)$ is not injective then $G(a, b)x = 0$ for a $x \neq 0$, and

$$\begin{aligned} 0 &= \langle G(a, b)x, x \rangle_{\ell^2(\mathbb{Z}^2)} = \langle T^*(a, b)T(a, b)x, x \rangle_{\ell^2(\mathbb{Z}^2)} \\ &= \langle T(a, b)x, T(a, b)x \rangle_{\ell^2(\mathbb{Z}^2)} \\ &= \|T(a, b)x\|_{\ell^2(\mathbb{Z}^2)}^2. \end{aligned}$$

Hence $T(a, b)x = 0$ and $T(a, b)$ is not injective. $T(a, b)$ is injective if and only if $T^*(a, b)$ is surjective.

Recall that a Hilbert-Schmidt operator $A : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded operator that has finite Hilbert-Schmidt norm,

$$\|A\|_{HS} = \sum_{i \in I} \|Ae_i\|_{\mathcal{H}}^2.$$

Hilbert-Schmidt operators are a particularly nice class of compact operators. This is due in part to their connection to integral operators from PDEs and Fourier analysis with kernels $k(x, y) \in L^2(\mathbb{R} \times \mathbb{R})$ [67, Theorem 4.16]. Our first result is that $G(a, b)$ is not Hilbert-Schmidt.

Proposition 6.4.1 ($G(a, b)$ is not Hilbert-Schmidt) *The Gram operator $G(a, b)$ is not a Hilbert-Schmidt operator on $\ell^2(\mathbb{Z}^2)$.*

Proof: The diagonal entries of $G(a, b)$ are the $L^2(\mathbb{R})$ -norm of the B-spline g

$$G_{jj}^{ii} = \|g\|_{L^2(\mathbb{R})} = 2/3, \quad i, j = 1, 2, \dots$$

Consequently,

$$\sum_{k, \ell \in \mathbb{Z}} \|G(a, b)e_{k\ell}\|_{\ell^2(\mathbb{Z}^2)}^2 = \sum_{k, \ell \in \mathbb{Z}} \|g\|_{L^2(\mathbb{R})}^2 = \infty. \quad \square$$

The truncations or approximations $G(a, b, n)$ in the previous section are called *finite sections* of the operator $G(a, b)$. This problem of designing finite-dimensional approximations of the Gram operator $G(a, b)$ leads to delicate questions of stability

and convergence [12, 18]. Let $\{\mathcal{H}_n\}_{n=1}^\infty$ be a family of finite-dimensional subspaces of $\mathcal{H}$ for which

$$\mathcal{H}_1 \subset \mathcal{H}_2 \subset \dots \subset \mathcal{H}_n \uparrow \mathcal{H}. \quad (6.5)$$

Let P'_n denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_n$. Let $V : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded operator and assume that for each $n \in \mathbb{N}$ we have a bounded operator $V_n : \mathcal{H} \rightarrow \mathcal{H}$.

1. The sequence $\{V_n\}_{n=1}^\infty$ is an *approximation method* for the operator V if

$$V_n P'_n f \rightarrow V f, \quad \forall f \in \mathcal{H}.$$

2. An approximation method is *applicable* if there exists $n_0 \in \mathbb{N}$ such that for $f \in \mathcal{H}$ the equation.

$$V_n x = P'_n f$$

has a unique solution x_n for all $n \geq n_0$ and x_n converges to a solution of $Vx = f$.

3. The sequence $\{V_n\}_{n=1}^\infty$ is *stable* if there exists $n_0 \in \mathbb{N}$ such that the operators V_n are invertible on $\mathcal{H}_n$ for $n \geq n_0$ and

$$\sup_{n \geq n_0} \|V_n^{-1} P'_n\| < \infty.$$

Lemma 6.4.2 (Applicable Approximation Methods, Lemma 1.2 [18]) *An approximation method $\{V_n\}_{n=1}^\infty$ associated to an operator V is applicable if and only if V is invertible and $\{V_n\}_{n=1}^\infty$ is stable. In particular, this implies that if $\{V_n\}_{n=1}^\infty$ is applicable, then V is invertible and $V_n^{-1} P'_n f \rightarrow V^{-1} f, \quad \forall f \in \mathcal{H}.$*

An example of an applicable approximation method for V is the family of operators $\{P'_n V P'_n\}_{n=1}^\infty$, where P'_n are the orthogonal projections onto subspaces $\mathcal{H}_n$

satisfying Equation 6.5, called the *finite section method* [18]. The matrix formulation of $Vx = f$ with respect to the standard ONB is:

$$\begin{bmatrix} \langle Ve_1, e_1 \rangle & \langle Ve_2, e_1 \rangle & \dots \\ \langle Ve_1, e_2 \rangle & \langle Ve_2, e_2 \rangle & \dots \\ \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} \langle x, e_1 \rangle \\ \langle x, e_2 \rangle \\ \dots \end{bmatrix} = \begin{bmatrix} \langle f, e_1 \rangle \\ \langle f, e_2 \rangle \\ \dots \end{bmatrix}$$

The finite section method $P'_n V P'_n x_n = P'_n f$ has matrix form

$$\begin{bmatrix} \langle Ve_1, e_1 \rangle & \langle Ve_2, e_1 \rangle & \dots & \langle Ve_n, e_1 \rangle \\ \langle Ve_1, e_2 \rangle & \langle Ve_2, e_2 \rangle & \dots & \langle Ve_n, e_2 \rangle \\ \dots & \dots & \dots & \dots \\ \langle Ve_1, e_n \rangle & \dots & \dots & \langle Ve_n, e_n \rangle \end{bmatrix} \begin{bmatrix} \langle x, e_1 \rangle \\ \langle x, e_2 \rangle \\ \dots \\ \langle x, e_n \rangle \end{bmatrix} = \begin{bmatrix} \langle f, e_1 \rangle \\ \langle f, e_2 \rangle \\ \dots \\ \langle f, e_n \rangle \end{bmatrix}$$

If the finite section method applies, this finite matrix equation has a unique solution for n sufficiently large given by inverting the matrix. Then, $x_n = \sum_{k=1}^n \langle x_n, e_k \rangle e_k$ and $x_n \rightarrow V^{-1}f$. Note, the finite section method appeals to *pointwise convergence* of operators which is weaker than *operator (uniform) convergence* used in convergence of compact operators. Unfortunately, the finite section method does not apply because $G(a, b)$ is never invertible. The next proposition characterizes the way the $G(a, b, n)$ do not approximate $G(a, b)$. We do not have convergence in operator norm of the projections $P_n G(a, b)$ (therefore $G(a, b)$ is not compact) and we do not have convergence of the finite sections $P'_n G(a, b) P'_n$ in operator norm.

Proposition 6.4.3 (Finite Sections and Non-Compactness of $G(a, b)$) *The Gram operator $G(a, b)$ and the blocks $G^{ii}(a, b)$ for $i \in \mathbb{Z}$ are not compact operators on $\ell^2(\mathbb{Z}^2)$ and $\ell^2(\mathbb{Z})$. Furthermore the finite section approximations $P'_n G(a, b) P'_n$ do not approximate $G(a, b)$ in operator norm.*

Proof: The $P_n G(a, b)$ are projections onto the span of the first n columns of $G(a, b)$. As such, a matrix representation of $P_n G(a, b)$ is simply the first n columns of $G(a, b)$

followed by columns of zeros. We can choose a standard basis vector $e_{k,\ell}$ such that

$$\|(G(a, b) - P_n G(a, b))e_{k,\ell}\|_{\ell^2(\mathbb{Z}^2)}^2 \geq 2/3,$$

since the entries on the diagonal are $\|g\|_{L^2(\mathbb{R})} = 2/3$. Thus, the operator norm is bounded below:

$$\sup_{\|x\|_{\ell^2(\mathbb{Z}^2)}=1} \|(G(a, b) - P_n G(a, b))x\|_{\ell^2(\mathbb{Z}^2)} \geq 2/3, \quad \forall n \in \mathbb{Z}.$$

This applies to the blocks $G^{ii}(a, b)$ as the diagonal entries are also $G_{jj}^{ii}(a, b) = \|g\|_{L^2(\mathbb{R})} = 2/3$ for all $j \in \mathbb{Z}$. By the same reasoning,

$$\sup_{\|x\|_{\ell^2(\mathbb{Z}^2)}=1} \|(G(a, b) - P'_n G(a, b)P'_n)x\|_{\ell^2(\mathbb{Z}^2)} \geq 2/3, \quad \forall n \in \mathbb{Z}. \quad \square$$

Since $G(a, b)$ is not invertible, we cannot investigate the influence of (a, b) on the invertibility of $S(a, b)$ via the invertibility of $G(a, b)$. We also cannot apply the finite section method to approximate solutions to the system $G(a, b)x = \lambda x$ for a given $x \in \ell^2(\mathbb{Z}^2)$ and $\lambda \geq 0$. Since $0 \in \text{Spec}(G(a, b))$, we can ask where in the spectrum it lies and in particular if it is an eigenvalue. If (a, b) influences whether $G(a, b)$ is injective we can make statements about the operators $T(a, b)$ and $T^*(a, b)$ and therefore $S(a, b)$. Unfortunately, it is always the case that zero is an eigenvalue and $G(a, b)$ is not injective.

Proposition 6.4.4 (Eigenvalue Zero I) *If $\{g_\lambda\}_{\lambda \in \Lambda}$ forms a frame for $L^2(\mathbb{R})$, then $G(a, b)$ has an eigenvalue of zero.*

Proof: We know the Gram operator $G(a, b)$ is never invertible in the region $ab < 1$. We show that $G(a, b)$ is singular if and only if it has an eigenvalue of zero. Certainly if $G(a, b)$ has an eigenvalue of zero, it is not injective, and hence it is singular. Suppose that $G(a, b)$ is injective and hence does not have an eigenvalue of zero and

hence the kernel is zero. Then, since $G(a, b)$ is self-adjoint,

$$\text{Ker}(G(a, b))^\perp = \overline{\text{Range}(G(a, b)^*)} = \overline{\text{Range}(G(a, b))} = \text{Range}(G(a, b)),$$

as the range of $G(a, b)$ is closed if $\{g_\lambda\}_{\lambda \in \Lambda}$ is a frame [1]. We have

$$\begin{aligned} \ell^2(\mathbb{Z}^2) &= \text{Ker}(G(a, b))^\perp \oplus \text{Ker}(G(a, b)) \\ &= \text{Range}((G(a, b)) \oplus \text{Ker}(G(a, b))) \\ &= \text{Range}(G(a, b)). \end{aligned}$$

Hence, $G(a, b)$ is injective and surjective. $\square$

Note, Proposition 6.4.4 is not specific to the Gabor system $\{g_\lambda\}_{\lambda \in \Lambda}$. In order to make further statements about the spectrum of $G(a, b)$ and the eigenvalue of zero we need the result that $\{g_\lambda\}_{\lambda \in \Lambda}$ is complete. This is important in its own right as completeness is a necessary condition for the system to be a frame. The following notation is all equivalent: $\{g_\lambda\}_{\lambda \in \Lambda} = \{g_{k\ell}\}_{k, \ell \in \mathbb{Z}} = \{e^{2\pi i \ell b x} g(x - ka)\}_{k, \ell \in \mathbb{Z}} = \{M_{\ell b} T_{ka} g\}_{k, \ell \in \mathbb{Z}}$.

Proposition 6.4.5 ($\{g_\lambda\}_{\lambda \in \Lambda}$ is Complete) *The time frequency shifts of the B-spline $\{g_\lambda\}_{\lambda \in \Lambda}$ are complete in $L^2(\mathbb{R})$.*

Proof: Let $f := e^{-\pi x^2}$. We show that $\overline{\text{span}}\{M_{\ell b} T_{ka} f\}_{k, \ell \in \mathbb{Z}} \subset \overline{\text{span}}\{M_{\ell b} T_{ka} g\}_{k, \ell \in \mathbb{Z}}$.

Since $\overline{\text{span}}\{M_{\ell b} T_{ka} f\}_{k, \ell \in \mathbb{Z}} = L^2(\mathbb{R})$, we conclude $\overline{\text{span}}\{M_{\ell b} T_{ka} g\}_{k, \ell \in \mathbb{Z}} = L^2(\mathbb{R})$.

First, $\langle f, g_{k\ell} \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} e^{-\pi x^2} g(x - ka) e^{-2\pi i \ell b x} dx$. This is the Fourier transform of the product of the Gaussian and the shifted B-spline g . It is therefore equal to the convolution of the Fourier transforms. For each $k, \ell \in \mathbb{Z}$,

$$\begin{aligned} \langle f, g_{k\ell} \rangle &= \mathcal{F}\{fg(\cdot - ka)\}(\ell b) \\ &= (\mathcal{F}\{f\} * \mathcal{F}\{g(\cdot - ka)\})(\ell b) \\ &= (f * e^{2\pi i ka \cdot} \text{sinc}^2(\cdot))(\ell b) \\ &\neq 0, \end{aligned}$$

since this is a convolution of a positive function and a function that is non-zero almost everywhere. If $f \notin \overline{\text{span}}\{M_{\ell b}T_{ka}g\}_{k,\ell \in \mathbb{Z}}$, $f \in \overline{\text{span}}\{M_{\ell b}T_{ka}g\}_{k,\ell \in \mathbb{Z}}^\perp$, and so $\langle f, \sum_{k,\ell \in I} c_{k\ell} g_{k\ell} \rangle_{L^2(\mathbb{R})} = \sum_{k,\ell \in I} \bar{c}_{k\ell} \langle f, g_{k\ell} \rangle_{L^2(\mathbb{R})} = 0$ for all finite index sets $|I| < \infty$ and complex scalars $c_{k\ell} \in \mathbb{C}$. But $\langle f, g_{k\ell} \rangle \neq 0$, for all $k, \ell \in \mathbb{Z}$. So $f \in \overline{\text{span}}\{M_{\ell b}T_{ka}g\}_{k,\ell \in \mathbb{Z}}$. Then, $f_{k'\ell'} \in \overline{\text{span}}\{M_{\ell b}T_{ka}g\}_{k,\ell \in \mathbb{Z}}$, as $\langle f_{k'\ell'}, g_{k\ell} \rangle = \langle f, g_{k'-k, \ell'-\ell} \rangle$. Therefore we have, $\overline{\text{span}}\{M_{\ell b}T_{ka}f\}_{k,\ell \in \mathbb{Z}} \subset \overline{\text{span}}\{M_{\ell b}T_{ka}g\}_{k,\ell \in \mathbb{Z}}$. $\square$

Propositions 6.4.6 and 6.4.7 relate the existence of an eigenvalue of zero to (a, b) being in the frame set $\mathcal{F}(g)$.

Proposition 6.4.6 (Eigenvalue Zero II) *If $G(a, b)$ has an eigenvalue of zero and $\{g_\lambda\}_{\lambda \in \Lambda}$ is ω -independent, then $(a, b) \notin \mathcal{F}(g)$.*

Proof: Let $\{c_{k\ell}\}_{k,\ell \in \mathbb{Z}} \in \text{Ker}(G(a, b))$ be non-trivial and thus in the eigenspace corresponding to eigenvalue zero. Let $f \in L^2(\mathbb{R})$ be defined as

$$f = \sum_{k,\ell \in \mathbb{Z}} c_{k\ell} e^{2\pi i \ell b x} g(x - ka).$$

The f is square-integrable as $\{c_{k\ell}\}_{k,\ell \in \mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2)$ and $\|e^{2\pi i \ell b x} g(x - ka)\|_{L^2(\mathbb{R})} = 1$. If $\{g_\lambda\}_{\lambda \in \Lambda}$ is ω -independent, then certainly $f \neq 0$, see Section 2.4.3. Then, the (k', ℓ') -entry of $G(a, b)\{c_{k\ell}\}_{k,\ell \in \mathbb{Z}}$ is

$$\begin{aligned} 0 &= (G(a, b)\{c_{k\ell}\}_{k,\ell \in \mathbb{Z}})_{k'\ell'} \\ &= \sum_{k,\ell \in \mathbb{Z}} c_{k\ell} \langle g_{k\ell}, g_{k'\ell'} \rangle_{L^2(\mathbb{R})} \\ &= \langle f, g_{k'\ell'} \rangle_{L^2(\mathbb{R})}. \end{aligned}$$

Hence,

$$\sum_{k',\ell' \in \mathbb{Z}} |\langle f, g_{k'\ell'} \rangle_{L^2(\mathbb{R})}|^2 = 0,$$

and there is no positive lower frame bound. $\square$

Proposition 6.4.7 (Eigenvalue Zero III) *If $\{g_{k\ell}\}_{k,\ell \in \mathbb{Z}}$ does not form a frame for $L^2(\mathbb{R})$, then $G(a, b)$ has an eigenvalue of zero.*

Proof: Let $f = \sum_{k,\ell \in \mathbb{Z}} c_{k\ell} e^{2\pi i \ell b x} g(x - ka) \in L^2(\mathbb{R})$ be as above. If $\langle Sf, f \rangle_{L^2(\mathbb{R})} = 0$, then for each $k', \ell' \in \mathbb{Z}$:

$$\begin{aligned} 0 &= \langle f, g_{k'\ell'} \rangle_{L^2(\mathbb{R})} \\ &= \sum_{k,\ell \in \mathbb{Z}} c_{k\ell} \langle g_{k\ell}, g_{k'\ell'} \rangle_{L^2(\mathbb{R})} \\ &= (G(a, b) \{c_{k\ell}\}_{k,\ell \in \mathbb{Z}})_{k'\ell'}. \end{aligned}$$

Hence, $G(a, b)$ has an eigenvalue of zero. Now suppose a positive lower frame bound does not exist because there exists a non-zero sequence $\{f_n\}_{n=1}^\infty \subset L^2(\mathbb{R})$ such that

$$\lim_{n \rightarrow \infty} \langle Sf_n, f_n \rangle_{L^2(\mathbb{R})} = \lim_{n \rightarrow \infty} \sum_{k', \ell' \in \mathbb{Z}} |\langle f_n, g_{k'\ell'} \rangle_{L^2(\mathbb{R})}|^2 = 0.$$

Let $\{h_{k'\ell'}\}_{k', \ell' \in \mathbb{Z}} := \lim_{n \rightarrow \infty} \{|\langle f_n, g_{k'\ell'} \rangle_{L^2(\mathbb{R})}|^2\}_{k', \ell' \in \mathbb{Z}}$. Then, $\{h_{k'\ell'}\}_{k', \ell' \in \mathbb{Z}}$ must be the zero sequence. Therefore, for some $n \in \mathbb{N}$, $f_n \in \overline{\text{span}\{g_{k\ell}\}_{k,\ell \in \mathbb{Z}}}^\perp$ and $\{g_{k\ell}\}_{k,\ell \in \mathbb{Z}}$ is not complete. This is a contradiction. $\square$

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