e. Proposition 10: universality of speed dictated by height of fall

3. The theories of the cycloidal and small arc pendulum in Part II are idealized in more than one respect
   a. As always in the Galilean tradition, resistance effects are ignored, though these effects can be made small, and of course the clocks themselves can be adjusted to compensate for them (and to compensate for the resistance of the rest of the mechanism)
   b. The pendulum itself is then idealized by ignoring the bulk -- i.e. the quantity of matter -- of its string and by treating the bob as what Euler later called a "point mass"
   c. Friction between the evolute wall and the string also ignored
4. Theory of evolutes in Part III gives a theory of the curvature of a large variety of curves -- i.e. of how these curves are defined as a sequence of osculating circles
5. The section from Part IV develops a theory that drops the idealization of the pendulum itself, but still without taking into account resistance and friction effects
   a. Of practical as well as theoretical interest, because pertinent to way clocks tuned, via small masses added on the end, resulting in multiple bobs
   b. Of course, also yields results for pendular motion involving a rigid body oscillating about a point, with no string
   c. And provides more exact way of calculating the length of simple pendulum with heavy non-point mass on the end
6. As Huygens remarks at the outset, the theoretical issue concerns the center of oscillation, a problem that Mersenne had proposed to him and to others in the 1640's, with no successful respondents
7. I assigned this section, in spite of its greater difficulty, with two things in mind:
   a. To show how far Huygens was able to advance theoretical mechanics beyond the point where it stood in the mid-1640's, leading toward the general theory of the motion of rigid bodies developed by Euler and d'Alembert in the 18th century
   b. As an example of Huygens's willingness to begin dropping idealizations, and hence to work in a realm where small discrepancies from an idealization become evidence for a refined theory and success of refined theory then becomes evidence for principles underlying both theories!

B. The Compound Pendulum and its Significance
1. The problem: given a pendulum with several bobs on a string, what is the length of a simple pendulum with the same total weight and the same period
   a. Mersenne had given the problem to Huygens in his youth (and to Descartes, who failed)
   c. (Cannot be solved from Newton's laws of motion by themselves)
2. Mersenne knew from experiments that the center of gravity of the bodies is not the answer: the required simple pendulum is longer than this
   a. Mersenne, of course, did not know why: some of the motion (energy) is going not into translation, but into rotation of the masses about their common center of gravity

b. The equivalent simple pendulum, which has all of its motion (energy) going into translation, must therefore have more translational motion (energy) than the compound pendulum has

3. Huygens's Proposition V gives the correct solution for the center of oscillation of the compound pendulum: $x = \Sigma(B_i r_i^2)/\Sigma(B_i r_i) = \Sigma(B_i r_i^2)/(r_{cg} * \Sigma(B_i))$
   a. This theoretical result is thus in full accord with the known fact that the required simple pendulum is longer than the length to the center of gravity
   b. The obvious practical significance of the result is that it provides a principled basis for tuning pendulum clocks by adding small weights, which Huygens proceeded to describe in detail
4. Huygens's proof of this result is based on two hypotheses, one of which is generalized from the collision paper, and the other is from Galileo, both preceded by a string of definitions
   a. Huygens's Torricellian principle reformulated, here stated as follows: "if any number of weights [connected or otherwise] begin to move by the force of their gravity, the center of gravity composed of them cannot ascend higher than where it was located when the motion began."
   b. Galilean: "Abstracting from the air and from every other manifest impediment, the center of gravity of an oscillating pendulum crosses through equal arcs in descending and ascending"
5. A reductio ad absurdum proof: if the velocity of the compound pendulum is greater than (or less than) the velocity of the corresponding simple pendulum, Proposition III, derived from the two hypotheses, is contradicted

   III: "If any magnitudes all descend or ascend, albeit through unequal intervals, the heights of descent or ascent of each, multiplied by the magnitude itself, yield a sum of products equal to that which results from the multiplication of the height of descent or ascent of the center of gravity of all the magnitudes times all the magnitudes"

   a. In effect, a special instance of a conservation of mechanical energy result, for requiring that the velocity acquired be sufficient to raise all of the masses back to their original height
   b. Amounts to requiring the total of $B*v^2$ to be the same in the simple and compound pendulums
6. {The key enabling theorem became a point of controversy from 1691 until Lagrange -- not whether it is true, but whether it is sufficiently transparent to serve as basis for such an important result

   IV: "Assume that a pendulum is composed of many weights, and beginning from rest, has completed any part of its whole oscillation. Imagine next that the common bond between the weights has been broken and that each weight converts its acquired velocity upwards and rises as high as it can. Granting all this, the common center of gravity will return to the same height which it had before the oscillation began."

   a. Jacob Bernoulli in 1691 objected to this principle and proposed replacing it with the "principle of the lever" to obtain the same result from more secure foundations
   b. d'Alembert subsequently proposed the initial version of the principle named after him as still a more secure foundation from which to obtain the result
   c. Lagrange (1788) generalizes d'Alembert's principle as foundation for his equations of motion}

C. The Results on the Center of Oscillation

1. The compound pendulum, with a series of point masses on a massless string, is a special case of a more general problem: given any body fixed at a point and moving through an arc with respect to an axis of oscillation through this point, what is the equivalent simple pendulum
   a. I.e. the simple pendulum that will have the same period of oscillation
   b. Problem asks for the center of oscillation of an arbitrary body with respect to an axis, on a line from the axis through the center of gravity of the body
2. This center of oscillation matches the center of percussion -- e.g. the "sweet spot" on a baseball bat
   a. Period = $2\pi\sqrt{(I/mgh)}$, where h is the distance from the axis of rotation to the center of gravity and I is the moment of inertia of the body vis-a-vis an axis parallel to the axis of rotation, but through the center of gravity
   b. Thus $\ell = I/mh$ gives the distance in question and the length of the equivalent simple pendulum
3. The modern treatment of the problem, besides using the moment of inertia first introduced by Euler in 1750, is developed around the fact that energy is being absorbed both translationally and rotationally by the oscillating rigid body
   a. (The rotational energy in the physical body involves rotation about the center of gravity, and not the axis of oscillation)
   b. Accordingly, this problem is closely related to the contrast between rolling and falling on an inclined plane, for there too the difference arises because part of the energy is going into rotation
   c. {This was undoubtedly why Huygens became the first to appreciate (1693) the difference between rolling and falling in the years immediately after Bernoulli objected to his solution}
4. Rotational energy varies as the product of the angular velocity squared and the moment of inertia
   a. For the moment of inertia is the correlate of mass for rotational motion
   b. Moment of inertia = sum of mass times distance from c.g. squared
5. Thus Huygens is flirting with the notion of moment of inertia and a theory of rotational motion in the problem he is addressing here -- a notion and a theory elaborated in full by Euler around 1760
   a. Given the measurement of g via a cycloidal pendulum, Huygens might well have tried the experiment of measuring g by rolling a ball down a cycloidal trough
   b. If he had, the value of g he would have obtained would have been the equivalent of 701 cm/sec/sec, revealing the difference between rolling and falling
   c. Instead he exposed the distinction between rolling and falling analytically (see Notes, Class 6)

D. The Theory of the Physical Pendulum

1. The problem addressed by Huygens starting, with Proposition XIV, asks for the simple pendulum equivalent to a physical pendulum -- i.e. to a rigid body oscillating about an axis of rotation under the influence of gravity

   a. Poses the problem for a completely general shape
   b. Clearly thinking of it as a generalization of the compound pendulum problem, but now for an infinity of masses
2. He faces an obvious difficulty attempting such a generalization in the 1660's, before the calculus
   a. The problem is straightforward with the calculus

$$x = \int y^2 dm / \int y dm$$

   b. But it is not at all straightforward without it, requiring quadratures for figures
3. Huygens manages to get solutions for certain standard shapes by generalizing his result for the compound pendulum and arguing to limits for quadratures for these figures
   a. Viz., pyramid, cone, sphere, cylinder, paraboloid, hyperboloid, and semi-cone
   b. Not only sufficient cases to test the theory, but also to allow precise calculation of the length of simple pendulums with solids of various shapes for bobs
4. The results have both scientific and technological significance
   a. Provide basis for designing and tuning clocks using physical pendulums, and for treating the bob of a simple pendulum as other than a point mass
   b. Solves the problem posed by Mersenne for the range of cases of principal interest, leaving to the future the generalization that will yield the full solution
   c. In the history of mechanics, among the first efforts that treat rigid bodies, and not just point-masses or bodies that can be simply reduced to an equivalent point-mass
5. Huygens himself uses the results to argue for a new universal measure of length -- the hour-foot -- corresponding to 1/3 of the length of the pendulum for an exact 1 sec arc -- i.e 1.05518 Rhenish ft, 1.01968 Paris ft, 33.122 cm
   a. Huygens indicates that the center of oscillation of the bob must be taken into consideration to give a precise value of this universal measure
   b. (Shows that when the *Horologium Oscillatorium* was going to press, Huygens was not yet aware of Richer's findings that g not the same in Cayenne as in Paris)
   c. Ends with a discussion of ways of realizing Mersenne's measurement of free-fall g -- i.e. distance of fall in the first second -- more accurately, to confirm pendulum value: 15 Paris ft 1 in
   d. Value presented as including small correction for center of oscillation
6. A disappointment noted at the end of Part IV: no isochronism result for a real cycloidal pendulum
   a. Length to center of oscillation gives only very close to isochronism for cycloidal path, for length changes along the arc of a cycloid
   b. Searched for, but did not find result for real bobs, that is, the isochronous path
   c. Best proposal he found instead: have the axis of the physical bob always remain perfectly vertical by putting the bob on hinges and weighting the bottom of the bob