A Fourth Grade Student's Exploration of Variable Notation Through a Function Representation Task

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Abstract

In this paper, I examine a fourth grade student's discourse and work with variable notation in the context of a function representation task. In spite of being unfamiliar with variable notation previous to this task, the student seemed to gain some awareness of the symbolizing significance of variable notation. Further, she was able to coordinate equations in slope-intercept form with other function representations that shared the same correspondence relationships. I argue that the student's statements and conjectures about the equations were acts of sense-making mediated by other function representations, particularly data tables, and by discourse during the interview. The student's achievements with this task documented in this case study provide evidence that upper elementary students can draw on personal resources to make sense of algebraic equations with variable notation.

Introduction

In a synthesis of research and policy regarding implementation of algebra throughout students' K – 12 education, the RAND mathematical study *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education* (2003) highlighted three expectations related to algebraic proficiency, one of which was "a robust understanding of the notion of function, including representing functions (for example, tabular, analytic, and graphical forms)" (p. 44). Teaching the concept of function through multiple representations has been advocated for 25 years or more (e.g., Brenner et al., 1992; Dubinsky & Harel, 1992; Schoenfeld, Smith, & Arcavi, 1993). Whereas many students seem to have an impoverished understanding of functions based in procedures of symbol manipulation (Oehrtman, Carlson, & Thompson, 2008), a goal in teaching functions through multiple representations is to encourage student understanding of the "notationally rich web of representations and applications of functions" (Kaput, 1991, p. 61). The case study presented in this paper evolved out of interviews with 4th grade students who had limited to no exposure to algebra around a task linking multiple representations of functions. Here, I examine how one student's interpretation of variable notation[1] was mediated by discourse during the interview, including utterances by the interviewer, utterances by the student, and interactions with the different function representations presented in the task.

In a landmark paper, Küchemann (1981) documented the ways in which students first engage with variable notation. Küchemann noted that student interpretations of letters fell into six categories: ignoring the letter, assigning it a value, to using it as shorthand notation for an

[1] In this paper, "variable notation" is the convention of using a letter to represent variables, which in turn refer to both varying and fixed unknown quantities (Blanton, 2008; Blanton, Levi, Crites, & Dougherty, 2011; Brizuela, Blanton, Sawrey, Newman-Owens, & Gardiner, 2015).

object, or using it as a specific unknown, a generalized number, or a variable. Through results of a written test on a variety of algebraic tasks, his summary conclusion was that over half of middle school students were not able to successfully work with the use of letters as specific unknown numbers, let alone generalized number or variable. MacGregor and Stacey (1997), in their work with 11 to 15 year-olds, delved into why students might interpret variable notation in these ways, concluding that everyday experiences and pragmatic reasoning, in addition to poorly-designed teaching materials, were main sources of student interpretations. In a more recent survey of middle school students, Knuth and colleagues (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005), through their own study of student interpretations, reported that students are more capable than these reports. Almost half of sixth grade students and close to 80 percent of 8th grade students seemed to understand letters in an algebraic expression as at least generalized numbers, potentially as a variable (Knuth et al., 2005). They credited this finding to students' classroom experiences learning about the use of variables and suggested that meaningful exposure to variable notation at earlier grades would likely benefit students.

Recent work in student understandings of variable notation comes from a perspective that recognizes symbolization and generalization as the heart of algebraic reasoning (Kaput, Blanton, & Moreno, 2008). From this point of view, studies have found that young students can and do use variable notation to represent their generalizations of functions. Elementary students' learning and adopting variable notation has been found in work which emphasized: young students' abilities to generalize functional relationships (Brizuela et al., 2015; Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015a), encouraging students to think about mathematical processes rather than products (Warren, Cooper, & Lamb, 2006), the effectiveness of teaching algebra in the elementary curriculum (Blanton, Stephens, Knuth, Gardiner, Isler, & Kim, 2015b), and unifying student discourses about functions (Nachieli & Tabach, 2012). In each of these studies, students were introduced to equations with variable notation in classroom lessons, and student-produced equations are motivated by an intent to generalize from values in a data table or on a graph. This method has proved powerful at encouraging children across younger grades (kindergarten through 7th grade are represented in these studies) to adopt equations with variable notation as a way of expressing generalizations.

In the work reported here, rather than arriving at variable notation after exploring other representations, students with limited to no formal exposure to variable notation were given a task, called the Function Puzzle, which required them to interpret algebraic equations without express directions on how to do so. This approach is similar to that of Knuth et al. (2005), Küchemann (1981), and MacGregor and Stacey (1997). Here, students were given four types of function representations (natural language, tabular, Cartesian graphs, and equations that included variable notation) with the assignment of finding connections across the different types. The Function Puzzle did not necessarily give students insight into different affordances of the representations, nor did it motivate them to generalize relationships from data, but as a potential opening activity for exploring functions, it primed them to consider connections across representations of functions (Brenner et al., 1992) and encouraged them to draw on personal resources (Moschkovich, 2007; Pratt & Noss, 2009) to construct their own understandings. Similar to some students in both Küchemann's (1981) and MacGregor's and Stacey's (1997) studies, several students in this study seemed to ignore the independent variable when they worked with the equations. The phenomenon was recurrent enough to warrant investigation through a case study of Kara (a pseudonym), one of the task participants.

This case study presents a “thick description” (Geertz, 1973) of Kara’s discourse during the interview about and experiences with representations of function in the Function Puzzle, where the analysis looks for meaningfulness not only in what Kara said and did, but in how the experience of the interview and the artifact of the Function Puzzle was a part of and influenced what Kara said and did. In keeping with the phenomenon of interest, student’s awareness and impressions of variable notation, I pay particular attention to how Kara talked about equation representations that included variable notation. In those utterances, she both said and did not say the independent variable. As will be developed, my argument is that Kara’s statements and conjectures about these equations were acts of sense-making through which Kara began to build understandings of the symbolizing significance of variable notation. My analysis will address the research question: how did discourse during the interview, including utterances by the interviewer, utterances by Kara, and interaction with function representations, mediate Kara’s awareness of variable notation?

Theoretical Perspective

Fundamentally, I take the perspective that individuals are the architects of their own understandings, and construct knowledge in personally meaningful ways (Piaget, 1970; von Glaserfeld, 1991) through experiences situated in cultural, historical, and institutional contexts (Cole, 1996; Forman, 2003; Vygotsky, 1978, 1987; Wertsch, 1991). This situatedness implies that both local communities, such as a classroom, and broader arenas, such as conventional symbolizing systems or national education standards, may impact learners. Cognitive development, therefore, includes processes of self-organization (constructivism) and enculturation (socioculturalism) (Cobb, 1994).

Similarly, both individual and sociocultural understandings form the basis for students' mathematical activity. For example, learners gain mathematical content knowledge (Hill, Rowan, & Ball, 2005; Ma, 2010) and understandings of classroom practices and norms (Forman & Ansell, 2001; Lave & Wenger, 1991; Yackel & Cobb, 1996), but also experience mathematics through extensive symbol systems (Cobb, 2002; Nemirovsky, 1994; Radford, 2014) and discourse practices (Michaels, O'Connor, & Resnick, 2007; Sfard, 2001, 2012; Zack & Graves, 2001). Learning is not only a process of increasingly proficient participation in mathematical practices, it is mediated by that participation (Halliday, 1993; Vygotsky, 1987; Wells, 1999). Participation in classroom discussion, for example, is not only something students learn to do, they come to be more proficient at it through their participation in discussions. More broadly, a student's understanding of mathematics and what is means to do mathematics is mediated by how they experience mathematics, whether through discussion, worksheets, projects, or other means.

The mediational triangle as described by Cole (1996) and shown here in Figure 1 is one way to represent mediating influences. In this simplified representation, there are three actors: the subject, the object, and the mediating artifact. A mediating artifact can be a physical object, like an axe in the system of person, tree, and axe, but is more broadly conceptualized as a "product of human history" (Cole, 1996, p. 118), a description which encompasses culturally-established patterns of behavior or systems of meaning in addition to material objects (such as a system of student, mathematics, and classroom discussion). The sides of the triangle represent relationships among the actors. The direct relationship between the subject and object can be thought of as a particular way that the subject might understand and interact with the object in the absence of influences (a purely hypothetical relationship, since a subject never interacts with an object without any mediation). In the mediated relationship, on the other hand, the subject's relationship to (and understanding of) the mediating artifact and the relationship of that artifact to the object influences how the subject comes to understand and interact with the object. Mediating artifacts, therefore, add new dimensions to the relationship between subject and object.

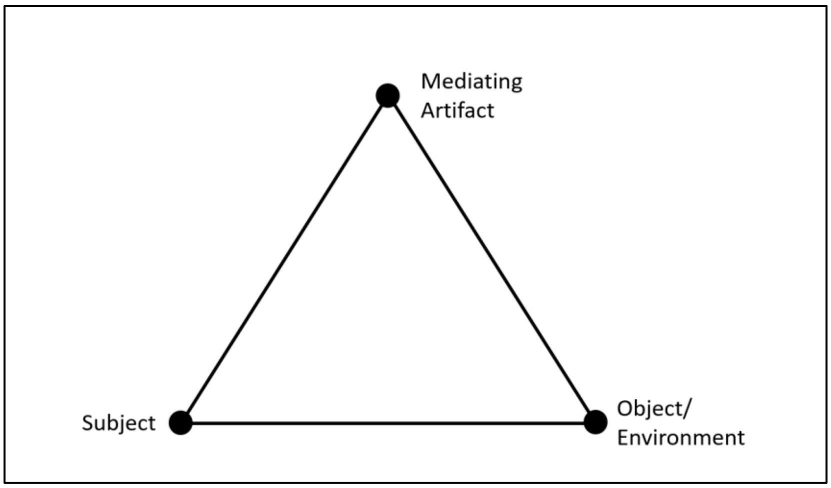

Figure 1: The basic mediational triangle per Cole, 1996, p. 119

In this work, I examine the ways discourse in the interview mediated Kara's sense-making of variable notation in the Function Puzzle task. The three particular aspects of discourse are utterances by the interviewer, utterances by Kara, and interactions with function representations as a symbol system. In a sociocultural perspective, symbolizing and symbol practices are subsumed as a specialized form of discourse (Lerman, 2001; Moschkovich, 1996; Sfard, 2012), as symbolizing is a form of communication. Here, however, I differentiate between Kara's discourse as talk and Kara's symbolizing as work with the function representations to examine each form's unique contributions to Kara's interpretations of variable notation.

Discourse as Talk or Conversation

Discourse as explicit communication between participants in a conversation centers around spoken words, talk, and conversation, as opposed to broader discourses that could include "those who are long gone whose ideas are instantiated in cultural ways of being, doing, and speaking" (Zack & Graves, 2001, p. 266). It is the immediate negotiation of meaning as it evolves in conversations with particular goals and focuses of attention (Moschkovich, 1996, 2007).

Two aspects from semiotic mediation are relevant here. The first is that an individual's interpretation of meaning, or meaning-making, is not adopted from social interaction alone, but is incorporated into that person's worldview as established by their past experiences (Wells, 2007). In this sense, understandings of discourse norms (Sfard, 2001; Yackel & Cobb, 1996) and forms of discursive activity (Moschkovich, 2007) inform how an individual might use discourse for their own sense-making. Secondly, there is the self-mediative role that speech can play in a person's understandings (Teasley, 1995; Wertsch, 1991). Vygotsky wrote, "Speech does not merely serve as the expression of developed thought. Thought is restructured as it is transformed into speech. It is not expressed but completed in the word" (1987, p. 251 as cited in Wells, 2007, p. 264). In this analysis, I look for evidence that discourse or conversation with the interviewer influenced how Kara worked with the representations in front of her, and also how her own utterances may have been a resource for new understandings.

Symbolizing

Mathematics is rife with established specialized notation and representations. Viewing these established systems as opaque or immutable attends to a limited perspective on their utility for meaning-making. Although a symbol system is indeed "a rule-governed set of elements" (Nemirovsky, 1994, p. 390), individuals understand, interact with, and perceive these symbols in ways that are personally meaningful. Attending simply to symbols as a system with fixed rules ignores how individuals use that system for their own purposes.

"Symbol-use," on the other hand, refers to the use of a given symbol system for a purpose in "a chain of meaningful events" (Nemirovsky, 1994, p. 390). Drawing meaningful use from variable notation could include generalizing a relationship between two quantities, such as the relationship between time and the height of water in a tank as it is emptied or filled, or more broadly, understanding that an algebraic expression does not just connect the domain to co-domain, but it is a mathematical object that can be manipulated, transformed, or analyzed for a specific intent. In other and recent literature, "symbolizing" is interchangeable with Nemirovsky's "symbol-use" (e.g., Brizuela, 2006; Cobb, 2002; Cobb, Yackel, & McClain, 2000; Sfard, 2000).

For the learner, engaging with a new symbol system may be learning the rules of the system and aligning one's practices to match those rules or it may be a process of symbolizing, where negotiating meaning between the symbol system and the situation context is one's central purpose (Kaput, 1998; Moschkovich, 1990; Nemirovsky, 1994). It may be a little bit of both. In her interview, Kara was both sense-making across various function representations, which may have included working to understand the rules of those symbol systems and interpreting those symbols in terms of a context of water in a bucket. My analysis examines whether the function representations had a mediational influence on her interpretation of variable notation in equation representations.

Methods

The Function Puzzle Task Design

I designed the Function Puzzle to elicit a "sense of functions" by which students would "incorporat[e] many ideas and skills …to tie together graphical and analytical representations" (Eisenberg, 1992, p. 154). The inspiration for this puzzle came from a theoretical belief in young student's capacity to engage in algebraic activity (Carraher & Schliemann, 2007), Tufts University's Early Algebra activities such as "Who Shares Your Function" (http://ase.tufts.edu/education/earlyalgebra/materials.asp), and Jennifer Sauriol's dissertation work (2013) to reorient ninth-grade algebra curriculum to encourage students to visualize functional relationships, shifting away from equation manipulation.

In this function representations task, students are given 16 cards (see Figure 2) with representations of four different functions. The representation types are natural language descriptions, data tables, Cartesian graphs, and equations that include variable notation.[2] The natural language cards described changes over time in the height of water in a tank. The represented functions are affine functions (functions of the form $y = Aw + B$, where w and y are variables, and A and B are constants): one constant ($y = 7$), one purely linear ($y = 3w$), and two with a non-zero translation ($y = 3w + 4$ and $y = 30 - 2w$). In keeping with terminology used in elementary schools, the relationships are respectively called constant, multiplicative, positive, and negative in this work. The design of the cards is minimalist, such that many cards can be quickly considered. For example, the cards do not mention whether water fills or drains at a constant rate, nor what units for time and height belong on the graph and the data table. Additionally, multiplication is represented in the equations as number directly followed by a letter (i.e., $3w$ for "three times *w*"), recognizing that this convention may be unfamiliar to elementary school students.

[2] The independent variable was represented by *w* instead of the more conventional use of *x* because preliminary screening showed that elementary students interpreted *x* as representing multiplication.

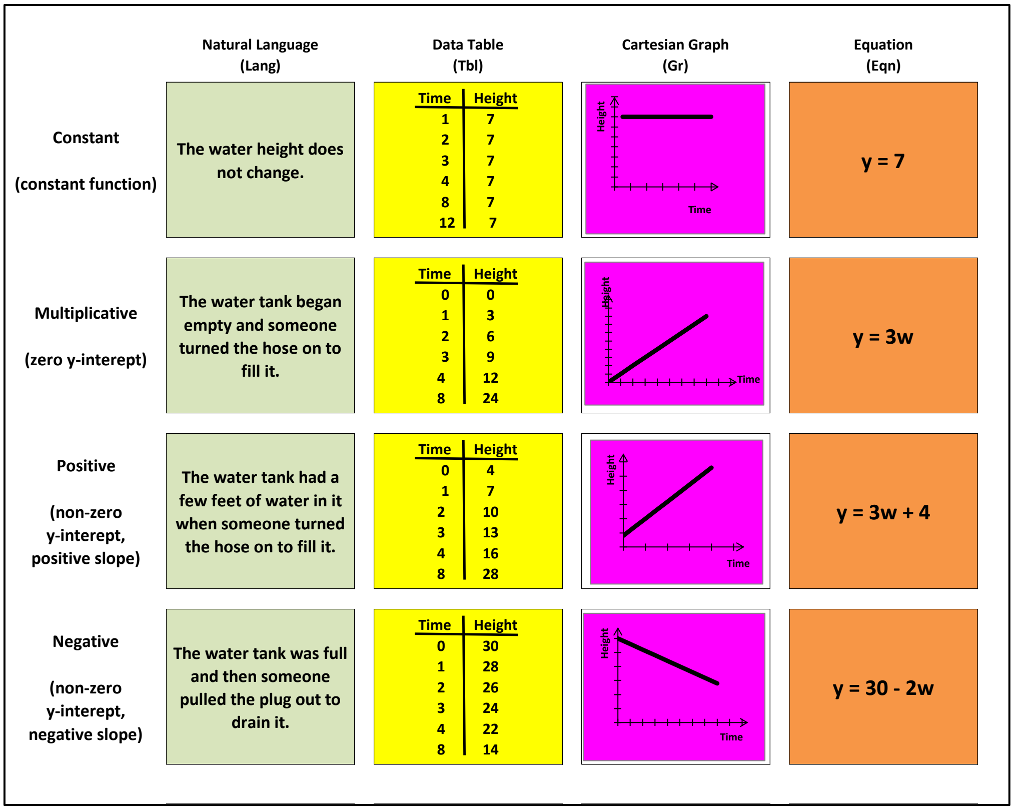

Figure 2: Function Puzzle Cards

The students are encouraged to make four sets, a set having one card of each color (or type of representation), and to share "a reason the cards belong together." The fact that there are four correspondence relationships (Smith, 2003) across each set of representations was not shared with the students. For this work, a student is considered to have "solved" the Function Puzzle if the shared attribute in each of their final sets of 4 cards is the same correspondence relationship. Figure 1 shows a possible configuration for "the solution" to the Function Puzzle.

The Function Puzzle design, where a final solution contains all 16 cards, constrains options that the students have available to create sets. This aspect entails that participants are not only choosing cards that belong in any given set, but they may be "not choosing" cards which they think "don't belong." In other words, students may be choosing the least wrong card rather than what they view to be a correct card. Further, reasoning through the appropriateness of their choices may happen *a posteriori*, on reflection of the sets they have already created, which in itself, is an opportunity for students to exercise sense-making.

Data Collection and Case Selection

For this study, the Function Puzzle was presented in a semi-clinical, one-on-one interview. In the interview protocol (see Appendix 1), the students completed the Function Puzzle and then were asked questions about their mathematical experiences in school and their thoughts about mathematics in general. Throughout each interview, the protocol was loosely followed, with the intention of following topics and ideas expressed by the student participant. Generally, students were invited to use any rationale to put sets of cards together, as long as there was one card of each color in a set. As they finished each set, students were asked to explain why they put the set of cards together, but requests for extended justification were limited to establish a "taken-as-shared" (Cobb, Stephan, McClain, & Gravemeijer, 2001, p. 119) atmosphere, minimizing the interviewer's influence in the student's execution of the task.[3] The protocol questions about mathematical experiences were included to potentially gauge whether and how students' epistemological and emotional outlook towards mathematics might influence their approach to novel mathematical experiences like the Function Puzzle. In this paper, those questions served as the source for Kara's description of herself as a learner of mathematics.

Students were selected for participation from one of three fourth-grade classrooms in a K-8 school through random draw from students who had returned consent forms. The school has over 1200 students and roughly 130 of them were in fourth grade. Almost half of the school (43%) is identified as "high needs," which includes characteristics such as free or reduced lunch, learning disability, limited English language proficiency, or combinations of such characteristics.

[3] On reflection, this choice has left some gaps in the narrative of Kara's work with the task.

The district has a relatively stable student population, where there is less than 5 percent attrition rate in the upper grades (3rd through 7th) (Massachusetts Department of Elementary and Secondary Education, 2014). The mathematics curriculum for fourth grade at this school was established by the cohort of fourth grade teachers, in alignment with mathematics standards from the Common Core State Standards Initiative (CCSSI; CCSSI, 2010) and by using online resources for class work and project ideas. The curriculum heavily emphasized development of students' computation skills and mathematics vocabulary. While the students' math curriculum included working with word problems and data tables, it did not include Cartesian graphs or equations with variable notation (C. Olszowy, personal communication, May, 2014).

In the time made available by the school, I conducted twelve interviews over three days towards the end of the school year at the participating school. Two video cameras were used to capture the interviews: one of the work space in front of the student, and one of the student, including face and work space. Most interviews were between 15 and 30 minutes where the time was generally split between completing the puzzle and answering questions about mathematical experiences and perspectives. In the beginning of the interview, before any directions were given, the students were asked if they were familiar with what was shown on the cards. All students expressed familiarity with the data tables and the natural language statements. None of the students were familiar with the Cartesian graphs, and students recognized the equations as equations, but claimed to be unfamiliar with equations with letters.

Out of the twelve students who participated in the interviews, six solved the puzzle, creating four sets of cards whose shared attribute was a correspondence relationship. Seven out of the twelve were noted to overlook the independent variable, the *w*, meaning they either did not say the *w* out loud when they read the equation cards, or they evaluated the expression as if the *w* were not there. The assumption here is not that the students did not see or consider the *w*, but that if someone were listening to what a student said as they read or evaluated an equation card, that listener would not know there was a *w* in the written equation. Four students both solved the puzzle and seemingly overlooked the independent variable. Kara's interview was chosen for this analysis because, of the four students who both solved the puzzle and were noted as not always attending to the independent variable, she most consistently shared what she was thinking as she worked through the Function Puzzle task. Further, she was not the quickest at the task, nor was she the slowest. It may be worth noting that Kara identified herself as a top performer in mathematics and had not had any trouble with topics in mathematics in her fourth-grade year. She is recognized for her mathematical abilities by others at the school, as her third-grade Massachusetts Comprehensive Assessment System (MCAS) exam scores were high enough to qualify her to participate in Competitive Math League (CML) and she earned a medal for her high scores on the CML tests. One implication of developing a case study on a strong student is that the results may be more indicative of what students can possibly achieve with this task, as opposed to what most students will typically do.

Analysis Methods

This case study is a descriptive, instrumental case study wherein "it is hoped that the detail provided by the description will generate new insights into, and a better understanding of, the nature of the phenomenon under investigation" (Willig, 2013, p. 103). Specifically, the object of study is Kara's work with and talk about the equation cards, and mediating influences of discourse during the interview and function representations on her understandings of variable notation.

The analytical techniques of constructivist grounded theory as described by Charmaz (2010) were used for Kara's work on the Function Puzzle. In contrast to the positivist grounded theory of Glazer (1978, 1992) or the prescriptive techniques of Strauss and Corbin (1990), Charmaz fully recognizes that "[d]ata are narrative constructions" (p. 187), impossible to fully untangle from broader contexts of participants' or researchers' points of view. Originally intended to explicate and map complex social processes (Willig, 2013), grounded theory's emergent analytical techniques are well-suited for examining young students' often idiosyncratic and nonconventional mathematical experiences and understandings.

Kara's interview was transcribed verbatim (see Appendix 2). Microanalysis (Nemirovsky, Kelton, & Rhodehamel, 2013) of non-verbal aspects of the interview, such as touching and moving of cards were also included in the transcript, as they were considered relevant to gain insight into Kara's thinking. From the transcript and viewing and reviewing of the interview video, a narrative account of Kara working on the Function Puzzle with close attention to her work and discourse about the equation cards was created. All of the interview artifacts—the videos, the transcript, and the narrative—were then used to build perspectives inductively (Charmaz, 2010) about Kara's understandings of the equation cards and variable notation. Transcript alone was not sufficient to complete this work, as both Kara and the interviewer used gesture or words like "this" and "that" to obliquely reference cards from the Function Puzzle task.

In order to address this study's research question ("How did discourse during the interview, including utterances by the interviewer, utterances by Kara, and interactions with function representations, mediate Kara's work with variable notation in linear equations?") I carried out a line-by-line coding of the interview transcripts that captured and categorized Kara's discourse about the equation cards using emergent codes. Specifically, I focused on how (a) utterances by the interviewer, (b) Kara's own utterances, and (c) representations of functions other than the equation cards (i.e., natural language, data tables, and Cartesian graphs), mediated her work with variable notation. The first step in this process was to identify every utterance she made related to the equation cards. The second step was to characterize in what ways (a), (b), and (c) mediated her work with variable notation.

Results

The narrative account used in my analysis and described above is the first part of the Results section. While this account does include some analysis and conjectures regarding Kara's thinking, the main task of this section is to portray the original enactment of the interview as that underscores "what makes her actions meaningful and how they are a part of a personal history with a past and a future" (Nemirovsky, 1994, p. 392).

Also note that I use a naming convention to more easily reference the Function Puzzle cards. The convention is based on the relationship type (constant, multiplicative, positive, negative) and the representation type (Lang for language, Eqn for equation, Table for data table, Graph for Cartesian graph). Coordinating relationship and representation types creates the name for a particular card. For example, the card that shows "$y = 3w + 4$" is called positive-Eqn, as it is the equation representation of the function with positive slope. Another convention used is to capture the progression of the interview in minutes in the header of each section.

Narrative of Kara's Function Puzzle Work

Kara's first set (1:27 – 2:35 in interview). Students were invited to use any rationale when combining cards into a set, as long as the sets were composed of one card of each color. Before I had finished introducing the Function Puzzle, Kara said, "I think I found a match," and she paired together the constant-Lang and constant-Tbl cards, added constant-Gr, then looked through the equation cards and found constant-Eqn (*y = 7*) (see Figure 3). It took Kara 30 seconds to build the set from her first exclamation to having all four cards in front of her. Each of these cards represent a constant function *y* = 7.

Kara used both language and gesture to justify her set, saying, “The water height does not change [touched constant-Lang], does not change [ran finger down the height column of constant-Tbl], does not change [ran finger along horizontal function on constant-Gr], and the height is seven [ran finger down the height column of constant-Tbl], so *y* equals seven [touched each symbol in the equation as she said it].” I interpret Kara’s statement “so *y* equals seven” as she touched constant-Eqn as a justification of her choosing constant-Eqn to belong with that set of cards. Those cards were put to the side, and Kara looked for other associations to make from the remaining 12 cards.

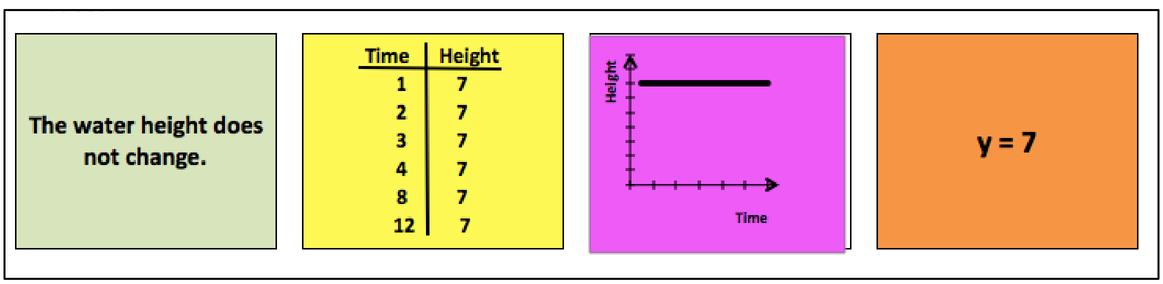

Figure 3: Kara’s first set. All four cards represent a constant function, y= 7.

Kara’s second set (2:35 – 6:04 in interview). In making a second set of cards, Kara first chose the natural language card that was closest to her, multiplicative-Lang, which describes filling an empty tank, and almost immediately paired it with multiplicative-Gr. The data table card negative-Tbl was closest to her, and she spent 35 seconds looking between this card and multiplicative-Gr, at times counting tick marks on the graph’s axes, or moving her finger from a tick mark on the *y*-axis across to the function line. Eventually, Kara included negative-Tbl in the set (the first three cards from left to right of Figure 4).

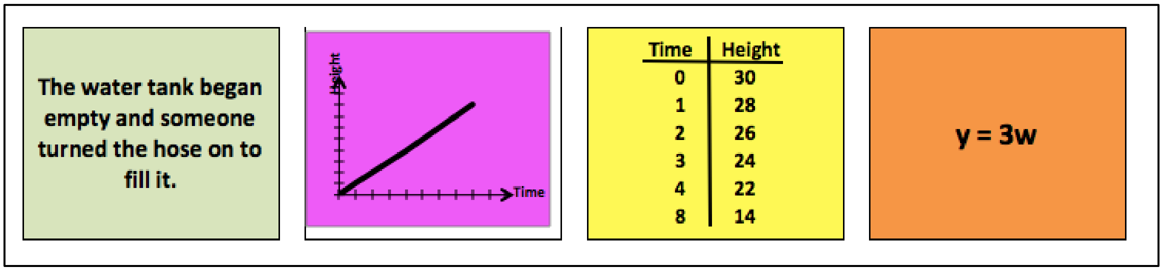

Figure 4: Kara's second set. The data table represents a function with negative slope and non-zero intercept, y = 30 – 2w; the other three cards represent a multiplicative function, y = 3w.

With the partial set of three cards lined up in front of her, Kara touched each remaining equation card, then asked, "The *y* equals the height, right?" After the interviewer responded, "Yup, it does," Kara contemplated the equation cards for about 30 seconds, before suggesting, "This one [positive-Eqn] could also go with that [the constant function cards in Figure 3] because *y* = 3 + 4 [ran thumb along the equation as she speaks]." Although the card showed *y* = 3*w* + 4, Kara did not say the *w* when she read the card, similar to the adolescents in Küchemann's (1981) study. Her suggestion that positive-Eqn belonged with the set she had created earlier implied that Kara thought positive-Eqn and constant-Eqn were interchangeable in this set. The opportunity to explore Kara's interpretation of these cards was missed by the interviewer, who thought not saying *w* was simply an oversight by Kara and pointed out the *w* on positive-Eqn saying, "Except that there's that *w* in there." The interviewer's remark was enough for Kara to retract her suggestion that positive-Eqn belonged with the constant function set.

Kara eventually settled on a set of multiplicative-Lang, multiplicative-Gr, negative-Tbl, and multiplicative-Eqn (see Figure 4 above). She made a justification for connections among the first three cards by inferring that the bottom left corner of the graph indicated zero and thus an empty tank, and that the function line ended at the eighth tick mark of both axes and there was a time value of eight in the table. When it came to how $\boxed{\text{multiplicative-Eqn}}$ belonged with the set, she said "And… I don't know how I got that," emphasizing her discomfort by covering the card with her right hand and laughing in a self-deprecating way. As a note, this situation is an example of *a postieri* reasoning, where Kara's idea of how the cards belong in the set changed as she talked through the set she created.

Before the interviewer made any substantive response, Kara switched out the equation cards, putting $\boxed{\text{negative-EQN}}$ in place of $\boxed{\text{multiplicative-EQN}}$. When the interviewer asked why she made the change, she suggested, "Because '30 – 2*w*,' I'm gonna guess, is either 28 [pointed at 28 in the height column of the table] or… like, it's probably one of these [waved her hand over the right –hand column of $\boxed{\text{negative-Tbl}}$]." Here she read the *w* aloud as part of the expression, and evaluated the expression as "twenty-eight," as if *w* was not in the expression or equal to 1, and 28 matched a value on $\boxed{\text{negative-Tbl}}$ at (1, 28). In this instance, even though Kara included the *w* when she read the expression, it is not evident whether she did or did not consider *w* in her calculation of the expression's value. Kara did make a mild allusion to potentially alternate values for 30 – 2*w* when she stated that it was "either 28 or one of these [other values]."

The interviewer accepted Kara's response, but then asked how $\boxed{\text{negative-Tbl}}$ fit Kara's previous description of an empty tank, since that had been her justification for $\boxed{\text{multiplicative-Gr}}$. This question led Kara to reassess $\boxed{\text{negative-Tbl}}$, and she changed that card out for $\boxed{\text{multiplicative-Tbl}}$ because it was the only table, according to Kara, that began empty; the first entry in $\boxed{\text{multiplicative-Tbl}}$ is (0, 0). As she was justifying her choice of data table, she remarked, "The height is multiplied by three every time," and she ran her fingers across each row, from left to right. In actuality, multiplying time (on the left-hand side) by three every time produced values for height (on the right-hand side). This type of faulty analysis, where students interchange the product and one of the factors, has been documented elsewhere (e.g., Clement, 1982). On the other hand, Kara was noticing the consistent multiplying-by-three pattern in the table. Without comment or prompting from the interviewer, she said, "and this [negative-Eqn] wouldn't make any sense. So I think it would be this [multiplicative-Eqn] because it's multiplied by three every time [touched the equation on the card while speaking]." Kara made the association between the table, where height values were three times larger than time values, and the equation $y = 3w$, which may indicate that she recognized the notation $3w$ as multiplication of three and w (see Figure 5).

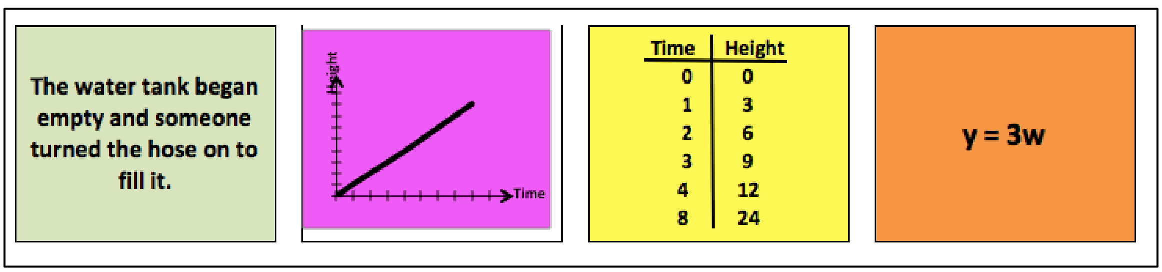

Figure 5: Kara's second set, revised. All cards represent a multiplicative function y = 3w.

Kara's third set (6:04 – 7:25 in interview). It took Kara less than 30 seconds to pull the next set of four cards from the remaining eight, pulling together positive-Lang, positive-Tbl, positive-Gr, and positive-Eqn (see Figure 6). When she was justifying how positive-Eqn was connected to the set, she said, "And the '3 + 4' 'cuz it's seven [touched the 7 in row (1, 7) in positive-Tbl]," and gave a little self-conscious giggle. When the interviewer asked for clarification of the connection, Kara reiterated her point by reading, "the '$3w + 4$' [touched positive-Eqn], I was thinking, because of the seven there [touched row (1, 7) in positive-Tbl]." In this instance, Kara evaluated y as equal to seven, or three plus four, which could result from ignoring the *w* or assigning it a value of 1. Assigning it a value of 1 would link to the row Kara touched in the table.

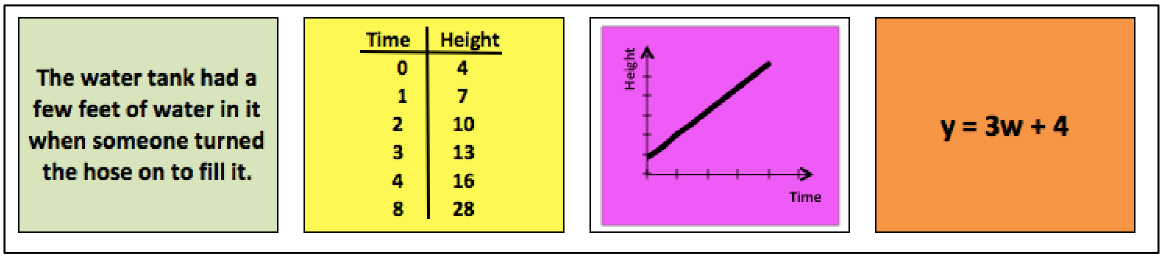

Figure 6: Kara's third set. All cards represent a function with positive slope and non-zero intercept, y = 3w + 4.

Almost without hesitation, however, Kara changed out positive-Eqn for negative-Eqn, suggesting, "or, I could do the '30 – 2' [touched negative-Eqn] because of the '28' there [pointed to '28' at (8, 28) of positive-Tbl]." In the table, the height value of 28 corresponds to a time value of 8. If she were considering time in calculating height (i.e., *w* = 1) she would not have removed positive-Eqn, where height is seven when time is one, matching the table value (1, 7) for negative-Eqn which would not match the table value (8, 28). This choice seems in line with what Küchemann (1981) described, and what MacGregor and Stacey (1997) noted with some Year 7 students, where students ignore the letter. MacGregor and Stacey noted that some older students seemed to assign letters a value of 1, but Kara does not seem to be doing that here, as noted above. Kara considered this set of cards complete (see Figure 7), and moved on to verifying that she would be satisfied with the last four cards as a set.

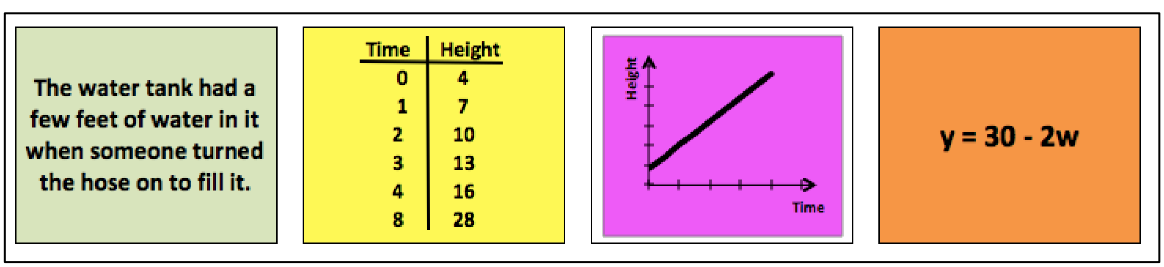

Figure 7: Kara's third set, revised. Three cards represent a function with positive slope and non-zero intercept, y = 3w + 4 while the equation represents a function with negative slope and non-zero intercept, y = 30 - 2w.

Kara's fourth set (7:25 – 9:12 in interview). She looked over the cards in Figure 8, murmuring to herself for about 45 seconds, and eventually told the interviewer, "I think these all make sense." In that private murmuring, she touched three of the cards, negative-Lang, negative-Tbl, and negative-Gr, but did not touch positive-Eqn. The interviewer then asked how the equation fit with the rest of the cards. Kara contemplated the set for over 20 seconds, then replied, "I don't know."

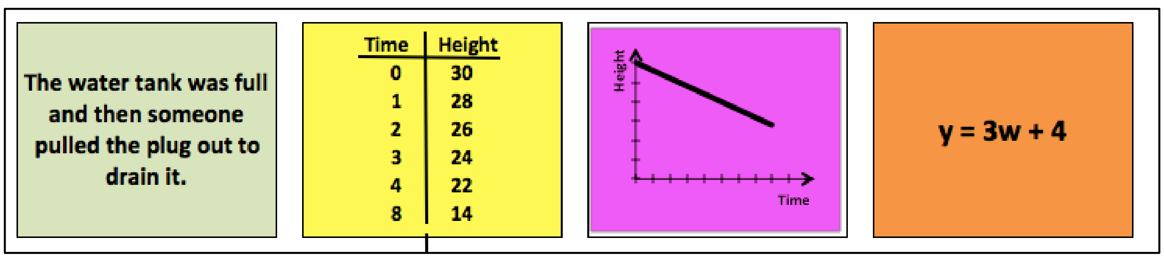

Figure 8: Kara's fourth set. Three cards represent a function with a negative slope and non-zero intercept, y = 30 – 2w; the equation card represents a function with positive slope and non-zero intercept, y = 3w + 4.

The interviewer pulled over Kara's third set (Figure 7) so Kara could look at both sets together. Kara did not take long (less than 10 seconds) to switch the two equation cards, saying, "This one (negative-Eqn) makes more sense down here (with first three cards of Figure 8). When asked by the interviewer, "and why is that?" She responded:

> Because ... 30 - 2*w* equals 28 and 2*w* could be like... it could be like two times two, so, four, and "24" and "14" [put her finger on these numbers in the left hand column of negative-Tbl]. Or it could just be plain two, and "22" [put her finger on "22" in the left

hand column of $\boxed{\text{negative-Tbl}}$]. Or it could be times three, "26" [put her finger on “26” from the left hand column of $\boxed{\text{negative-Tbl}}$].

This series of “it could be like” phrases are Kara’s thoughts about different possible values for 2*w* from $\boxed{\text{negative-Eqn}}$. While the specifics of her calculations are not completely clear, Kara explicitly connected *2w* to multiplication when she said, “it could be two times two … or it could be times three,” indicating she did seem to think that *w* could have more than one value and those values would be multiplied by two. The mention of “plain two” as a part of the above response is interesting because if she consistently or rigorously believed that *2w* represented two times a value for *w* then 22 would result from *w* = 1, and maybe she would have said, “times one” instead of “plain two.” Perhaps she was reading *2w* as two place values: two tens and *w* ones. This interpretation would match with her statement, “Or it could just be plain two, and ‘22’.” Perhaps she was both using the 2 as the tens place and to multiply by *w*. This interpretation matches with her saying “two times two, and 24” and “two times three, 26,” but does not explain the connection between “two times two” and “14.” In each of these cases, it seems that the correspondence relationship between *y* and *w* was not part of Kara’s interpretation of the connections between cards. Although she mentioned values of 2 and 3 for *w* explicitly, her calculation with 2 resulted in “24” or “14” while (2, 26) is the row in the table. Similarly, her calculation with 3 resulted in “26” while (3, 24) is the row in the data table. Finally, any of these ways of linking the algebraic expression with values from the data table may be a trial-and-error approach to sense-making around coordinating the cards together.

Lastly, the interviewer asked about the other half of the switch: how $\boxed{\text{positive-Eqn}}$ fits in with the set in Figure 7 and Kara responded, “Because um, the ‘7’ as I said earlier.” This response did not demonstrate consideration of the *w* but was a return to an earlier justification from her work with this card on the third set.

Results Summary

Kara's work on the Function Puzzle. Kara created four sets of cards in roughly eight minutes (see Table 1). In the final configuration, the four cards in each of the sets had a shared the same correspondence relationship. In building each of her sets, Kara started with a natural language card. In all cases except her second set, a data table card was chosen second, followed by a Cartesian graph card. In the second set, a Cartesian graph was chosen second and a data table was chosen third. In all cases, the equation card was selected last. Interestingly, the progression through natural language, data table, graph then equation is a parallel sequence to functional thinking studies mentioned earlier (Blanton et al., 2015a; Blanton et al., 2015b; Nachieli & Tabach, 2012; Warren et al., 2006). In those studies, the task starts with a presentation of a situation in natural language, then a data table of values is produced, which motivates the development of a symbolic generalization. What is of further interest here is that the data tables were the linchpin to link representations together: Kara would link a data table to a natural language description, then, in her first, third, and fourth sets, she used values in the data tables to choose graphical and equation representations to add to the particular set. One interpretation of this sequence is that the natural language description established a context for Kara to imagine the dynamics of the situation, then a data table confirmed those dynamics with particular instantiations of the context. When considering the graphs and equations, representations that Kara was least familiar with, Kara perhaps had two tasks in front of her: figuring out which representation fits in a set and the representations' symbol system rules.

Table 1: Time Kara Spent Creating Sets During Her Interview

	Type of Correspondence Relationship	Interview Time	Duration of work	Number of Card Revisions
First Set	Constant	1:27 – 2:03	0:36	0
Second Set	Multiplicative	2:03 – 5:42	3:39	3
Third Set	Positive Slope	5:42 – 7:07	1:25	1
Fourth Set	Negative Slope	7:07 – 9:12	2:05	1

Kara built the constant set first and quickly, making no revisions to her initial card choices. She was quick to point out “no change” on each of the cards in her constant set, indicating that constancy was a salient quality of each of those representations to her, and she picked out representations with this quality quickly from the entire set of 16 cards. She spent the most time and made the most revisions on her second set, the multiplicative set, perhaps because there were many cards to choose from as compared with the third and fourth sets. She spoke about all three available equation cards as she built her second set, and made three revisions to her card choices before finishing the set. The third set came together in about one and one half minutes, and the fourth set in two minutes. In each of these cases, she made one revision before deciding the set was finished and spoke about both positive-Eqn and negative-Eqn while working on each of these sets.

Kara’s talk about the equation cards. Over the course of the interview, there were eleven instances where Kara articulated conjectures or statements regarding an equation card (Table 2). Across the interview, she pronounced the independent variable three times when she read the card aloud (at 4:42, 6:49, and 8:50), whereas she did not say the independent variable at three other opportunities (at 3:32, 6:32, and 6:56). She also indicated that the dependent variable could or did have multiple values three times (at 4:42, 5:24, and 8:50). In the four instances Kara indicated awareness of variable notation, either through reading the independent variable out loud or indicating that the dependent variable could have multiple values (at 4:42, 5:24, 6:49, and 8:50), she was speaking in direct response to a request for clarification by the interviewer.

Table 2: Kara's utterances about the equation cards.

Equation	**Time**	**Set**	**Utterance**
y = 7	2:03	First	"so the *y* = seven [constant-Eqn]. [pointing finger at the equation as she reads]."
y = 3w+4	3:32	Second	"This one [positive-Eqn] could also go with that [constant set in Figure 2] because [reading] '*y* equals 3 plus 4.'"*
y = 3w	4:29	Second	"And, I don't know how I got that [covering multiplicative-Eqn],"
y = 30 - 2w	4:42	Second	"Because *30 – 2w*, I'm gonna guess, is either 28 or… like, it's probably one of these [gesturing vaguely to negative-Tbl]."
y = 3w	5:24	Second	"and this [negative-Eqn] wouldn't make any sense. So I think it would be this [multiplicative-Eqn] because it's multiplied by three every time."
y = 3w+4	6:32	Third	"And the *3 + 4* [on positive-Eqn] 'Cuz it's seven [touching 7 in positive-Tbl at (1,7)]"
y = 3w+4	6:49	Third	"the *3w + 4*, I was thinking, because of the seven there [in positive-Tbl at (1,7)]."
y = 30 - 2w	6:56	Third	"Or I could do the *30 – 2* because of the *28* there [in negative-Tbl at (8, 28)]."
y = 3w+4	8:22	Fourth	"I don't know." [In response to the interviewer's question of how positive-Eqn fits with the rest of the set.]
y = 30 - 2w	8:50	Fourth	"Because *30 – 2w* equals 28 and *2w* could be two times two, so four… or it could be just plain two…or it could be times three…" [edited]
y = 3w+4	9:06	Fourth	"Because um, the '7' as I said earlier."

Discussion

Kara's talk, gestures such as touching cards, and her card selections in making sets during the interview are the data available to gauge her sense-making around the Function Puzzle. Throughout Kara's interview, she intermittently talked in ways that indicated she was considering possible implications of variable notation, in spite of having had no formal education or experiences in algebraic symbolizing. Three ways in which she indicated awareness of variable notation in her talk were:

- saying the w when she read the equation aloud,
- indicating that y could have multiple values, and
- indicating that w has a role in evaluating y.

This section will summarize the ways in which these ways of talking about the equations were mediated by utterances by the interviewer, Kara's own utterances, and interactions with the function representations.

Mediating Influences of the Interviewer's Utterances

There were two ways that the interviewer's utterances mediated Kara's awareness of variable notation. I will talk about each influence in this section. In one sense, the interviewer's utterances implied that variable notation was a part of the Function Puzzle task space. For example, when Kara was building the second set and suggested that $y = 3w + 4$ could belong with the constant set, the interviewer suggested, "Yes, but there's that w in there," which indicated to Kara that "$3w + 4$" was different from "$3 + 4$." Similarly, when Kara was reviewing the fourth set of cards on her own, she touched all of the cards in the set except for the equation card. The interviewer, noticing that omission, brought Kara's attention to the equation card by asking, "Can you tell me how the equation card fits in?" In both of these cases, the interviewer's utterances implied that the equations and independent variable were important and needed to be considered. The interviewer's utterances therefore mediated the way that Kara focused her attention during the task, and this may have impacted how she considered the equations and variable notation.

In another, the interviewer's requests for clarification seemed to initiate a heightened attention to detail in Kara, which was reflected in how she read the equations cards aloud. As noted in the results section and in Table 2, Kara did not say the independent variable aloud half of the times she read the equations that contained a *w*. For example, she read $30 - 2w$ from negative-Eqn as "thirty minus two." When she did not say the *w*, she calculated values for *y* as if the independent variable did not exist. For example, she thought *y* would be 7 for equation $y = 3w + 4$ (for example, at 6:32 in Table 2) and 28 for equation $y = 30 - 2w$ (for example, at 6:56 in Table 2). Küchemann noticed a similar phenomenon and classified this interpretation as "Letter not used" (1981, p. 106). As mentioned in the results section, MacGregor and Stacey (1997) concluded that some of the Year 7 students ignored the letter in the calculation, whereas older students often assigned it a value of 1.

The three instances where Kara did explicitly read the *w* were cases where the interviewer had asked for clarification of her thinking. In one case at 4:42, the interviewer asked her to share the reason she switched out negative-Eqn for multiplicative-Eqn, and Kara responded by suggesting that $30 - 2w$ could be 28, which coordinated with the ordered pair (1, 28) on negative-Tbl. In another at 6:49, the interviewer had asked Kara to reiterate how positive-Eqn ($y = 3w + 4$) connected to positive-Tbl, and Kara pointed to the 7 in the height column on the table and said, "the $3w + 4$, I was thinking, because of the seven there." In the third instance at 8:50, Kara had just switched negative-Eqn from her third set to her fourth set "because it made more sense," and the interviewer asked why that was the case. Kara began her explanation with, "Because. 30 – 2*w* equals..."

Saying the independent variable aloud was something Kara did when the interviewer asked for clarification or justification by asking "why." In fact, Kara consistently included the *w* in how she read the equation cards after a request for clarification. Sfard calls this kind of response " a mechanism of interaction" (2001, p. 39), where the social positioning of participants and/or the context of the situation invite a particular response. In this case, an implicit understanding in Kara's mind could have been that one responds to a query for clarification by paying close attention to the task at hand, which meant she read each of the symbols aloud when reading the equations on the cards. In this way, the interviewer's request precipitated pronunciation of the independent variable, which mediated increased sensitivity to its role.

Mediating Influences of Kara's Own Utterances

Although Kara's additional precision in reading the cards cannot be said itself to constitute a change in her understandings of how to read them (she did not consistently start to say the *w* at a certain point in the interview, for example), this precision did seem to influence how she evaluated equations. Two of the times when she did read the *w* were the instances that she seemed to give *y* multiple values. For example, when she suggested that 30 – 2*w* might be 28, and said the *w*, she went on to say, "[it's] either 28 or I'm gonna guess, one of these [other values in the data table on negative-Tbl]," indicating that 30 – 2*w* might be one of the other values in the height column. There was therefore a chain of events: Kara would adopt a more conventional way of saying what was written on the equation cards, which then may have influenced how she interpreted the significance of the symbols in the equation, as if "[t]hought is restructured as it is transformed into speech" (Vygotsky, 1987, p. 251, as cited in Wells, 2007, p. 252).

Mediating Influences of Function Representations

In Kara's first set, the constancy of the water height was readily noticeable to Kara in the natural language, data table, and graph cards, as can be inferred from how quickly she pulled the cards together (30 seconds) and how she justified her choices my moving her finger along the representation to accentuate "no change." When she got to justifying her choice of $\boxed{\text{constant-Eqn}}$, she said, "so *y* equals seven." Her use of "so" seems to indicate that she is making a conclusion about the choice of equation, whereas the other representations were the data or facts that led to "$y = 7$." In considering $\boxed{\text{constant-Eqn}}$, Kara saw that *y* does not change; it is seven. The set of cards seemed to mediate her understanding that *y* represented height for this context, something she clarified with the interviewer in her work on the second set.

In constructing her second set, Kara had noticed and said aloud the consistent pattern of multiplying by three across each row in $\boxed{\text{multiplicative-Tbl}}$. In finalizing her equation card choice, Kara had the equations $y = 3w$, $y = 3w + 4$, and $y = 30 - 2w$ in front of her. Inferring her perspective, $\boxed{\text{positive-Eqn}}$ was the best choice of the three, as it was the only equation card (at this point in her work) that did not include addition or subtraction and, conveniently, it also had a three on it. In other words, it is possible that rather than choosing $y = 3w$, Kara was not choosing $y = 3w + 4$ or $y = 30 - 2w$. This choice or not-choice, paired with the three-fold relationship between columns in the data table, could then suggest to Kara that $3w$ is a way of notating three times a number.[4] In this sense, the numerical pattern in the data table may have mediated her understandings about how multiplication is notated for letters representing variables.

In the explanation that Kara provided to establish that negative-Eqn belonged with the other negative-slope cards, she both said the *w* and gave it a role in evaluating the equation card. In that explanation, Kara coordinated between the equation card negative-Eqn and several values on the data table card: 28, 24, 14, 22, and 26, saying, "2*w* could be like two times two, so, four, and '24' and '14' [put her finger on these numbers in negative-Tbl]. Or it could just be plain two, and '22' [put her finger on "22" in negative-Tbl]. Or it could be times three, '26' [put her finger on "26" in negative-Tbl]." As highlighted in the results section, there are several possible interpretations for Kara's thoughts at that moment. She made some association with multiplication when she suggested the *2w* could be "two times two", "plain two", or "times three." Alternatively, she may have interpreted the "2" of the 2*w* as two tens, evidenced by when she included 24, 22, and 26 as part of her justification for linking negative-Eqn with negative-Tbl. Or, it could have been some combination of these conjectures. In this moment, Kara was having to revise her third set to use negative-Eqn in the fourth set, and she has already justified the third set. Thus the challenge was not simply reasoning how negative-Eqn fits in the fourth set, but coming to have a more rigorous argument than what she used for the third. This additional rigor stands in contrast to the first time she paired negative-Eqn with negative-Tbl, in building the second set. At that point, she only made vague reference to possible multiple values, saying, "30 – 2 *w* is either 28 or one of those [other values in negative-Tbl]." Kara pushed into

[4] In fourth grade generally and in Kara's school specifically, multiplication is represented by the symbol "x." Some teachers may show alternate formats of representing multiplication, such as * or ·, but do not represent it without a symbol.

new territory with her justification by noticing the *w*, assigning it multiple values, and attributing the notation of "2*w*" to some kind of multiplying-by-two process. There is evidence of all three mediating influences in this final example: the interviewer asked her to clarify her justification, she read the *w* aloud in response to that request, and she linked several of the height values on the negative-Tbl.

Conclusion

Carraher, Schliemann, and Schwartz (2008) point out, "Teachers need to introduce unfamiliar terms, representations, and techniques, despite the irony that in the beginning students will not understand such things as they were intended" (p. 237). Warren, Cooper, and Lamb (2006) asserted that students need activities that require moving among different representations of functions, as student learning about functions is complex and non-linear. Knuth and colleagues (2005) felt that students would benefit from meaningful exposure to the use of letters as variables in their preparation for algebra. The Function Puzzle addresses both of these concerns, as four types of representations are presented together, and it is the students' task to meaningfully make connections among them. Further, the presentation of the four function representation types together is unique to functional thinking studies with young children (e.g., Blanton et al., 2015a; Blanton et al., 2015b; Nachieli & Tabach, 2012; Warren et al., 2006), where teaching variable notation usually follows other representations of functions. Even with this very different context, Kara became more aware of variable notation through her experience with the Function Puzzle.

Her work with the equation representations was quite surprising, considering she was unfamiliar with variable notation. In her eight minutes of completing and justifying the Function Puzzle, Kara was able to intuit that 3*w* was a likely representation for "three times something," 2*w* was a likely representation for “two times something, “and she indicated a willingness to consider the dependent variable, *y*, as having multiple values, making some effort to give the independent variable a role in calculating values for the dependent variable. Additionally, she was able to reason how given equations fit with other function representations. Here, I argued that this new awareness was mediated by aspects of her interview. Specifically, utterances by the interviewer maintained Kara’s attention on sense-making around the equations and triggered Kara to use precision in justifying her answers, which led her to saying the *w* out loud. Saying the *w* out loud then seemed to impact how her interpretation of *y* as having multiple values. Finally, Kara worked to resolve the information in the data tables with the formats of the equations.

Mediation through spoken discourse is perhaps not such a novel finding, but the role of conventional function representations in scaffolding someone’s understanding of those forms deserves some extra attention. Radford (2014) makes an interesting claim via Vygotsky (1994) that conventional forms “exert a real influence” (p. 274) on learner’s thinking. I claim it is not so much that conventional forms exert influence, but that these forms have matured through historical use to be easily recognizable, even by young learners. Kaput, Blanton, and Moreno (2008) put it this way: “And, of course, the main reason that they have become conventional is that they are very useful across a wide variety of situations. Each is powerful in its own way. Each is a highly efficient way of symbolizing, the end result of an historic process of refinement – contributing to algebra’s identity as a cultural artifact” (p. 22).

Kara’s success with this task was two-fold: she linked together sets of cards by their underlying mathematical relationships, and through that activity she broadened her awareness of variable notation. Kara reasoned and talked through which representations might belong together and successfully solved the Function Puzzle, potentially gaining some insights into the nature of letters in algebraic equations and the symbolizing significance of letters in equations. Although Kara's achievement should not be overly generalized, it does indicate that upper elementary students have the personal resources to make sense of function representations and the connections between them.

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Appendix 1:

Interview Protocol for Function Puzzle

Pilot Work, June 2014

INTERVIEW PROTOCOL: Elementary Student Ideas about Functions

PRE. (Before turning on video) ***I would like you to pick a pretend name to use in the video. What name would you like to choose?*** (turn on video)

CARD TASK: Materials: 16 function cards with 4 kinds of representations (table, graph, equation, natural lang) for 4 functions (3 linear, 1 quadratic); pencil & paper

1. Show cards with function representations. ***These are different ways of showing relationships*** Show the different types of cards without naming them. Encourage the student to supply the name for the different representations(i.e.: ***There is this kind*** *(gesture or hold up).* ***Have you worked with these before? What would you call something like this?***

2. ***The idea is for you to find a matched set of cards: one of each kind of card that goes together. You are free to ask me any questions that might help you put the set together. There are actually different ways of doing this task, so you have a lot of freedom in how you put things together. Do you have any general questions before you start?***

3. ***Some people like to work silently, while others like to talk as they pull their ideas together. It is helpful to me if you talk as you think about putting these cards together, but don't feel like you have to.***

4. Let the students put the cards together as sets. Encourage them to keep working for at least 5 minutes (10 minutes max?). If they appear to be stuck after 2 minutes (?), talk with them about what information the representations might be showing.

REFLECTION ON CARD TASK (& GRADUALLY MOVE TOWARDS DISCUSSION OF MATHEMATICS IN GENERAL)

5. **D*escribe your sets to me.***

6. ***Tell me how you did this task. How did you decide what to put together?***

7. ***Were there cards that were easy to put together? Which ones? Why?***

8. ***What do you think of this task? If you were explaining it to a friend, how would you describe it? What is an activity that is similar?*** Look for opportunity to explore attitude, affect & technical aspects;(enjoyable,required thinking, would be helpful do put work on paper, mathematical) ***Do you do tasks like this in math class?***

MATHEMATICS: In the classroom

10. ***Some XX- graders (*year younger*) are wondering what it will be like to do math in XX grade. What would you tell them about the kinds of things that you do in math?***

11. ***Do you work alone or as groups?***

12. ***Suppose you are working in a small group and there is more than one answer. How do you know which is the right answer? Is it possible that more than one answer could be right?***

13. ***Does it seems that some kids are better at math than others? How can you tell?***

14. ***What are some of the ways your teacher helps students understand math better? Do you think it's important to her that you guys understand?***

MATHEMATICS: personal relationship

15. ***Is the math you cover in class ever difficult for you? What are some of the ways you try to understand it?***

16. ***Do you get a lot of math homework? Is it something you do right away or do you dread doing it when you get home?***

17. ***Does working hard help you do well at math?***

18. ***Do you feel like you can explain your ideas and answers to your teacher or friends?***

19. ***What do you think about math in general? Is it enjoyable? Important? Useful?***

CLOSING .

20. ***Is there anything else you would like to tell me or show me?***

21. ***I need some more students to try this task. Which classmate should I ask? Why?***

Appendix 2:

Transcript of Kara’s Interview with the Function Puzzle

[01.00] NOTES: Condition A, Kara, June 16, 2014
[02.01] NOTES:
[03.02] NOTES: Introduction to 1:44 (Kara stops me to say she has found a match)
[04.03] NOTES: Completes 4 sets in about 10 minutes
[05.04] NOTES: Full interview is 17 minutes
[06.05] NOTES: All four cards (G, P, O, Y) in her sets represent the same function.
[07.06] NOTES:
[08.07] NOTES: Excerpts with her interpreting equation cards:
[09.08] NOTES: 3:09 - 4:50
[10.09] NOTES: 6:10 - 7:08
[11.10] NOTES: 8:00 - 9:12
[00:34.24] NOTES Intro: finished introducing the four types of representations
[00:45.26] NOTES Intro:four different math relationships
[00:53.03] NOTES Intro:find a set that belongs together.
[01:02.09] NOTES Intro:a logical reason that they belong together. Like, you can tell that the relationship that is described in this table goes with the equation or goes with the math statement... so, it's important to take your time...
[00:01:24.29] NOTES Intro:make a set, describe why you put it together...

[01:27.26] *NOTES: Found a match*
[01:56.08] *NOTES: adds equation;*
[01:56.08] *NOTES: *****FIRST SET; describes set. {G-A, Y-A, P-A, O-A} *******
[01:58.04] INT: OK, describe the set for me.
[02:03.12] Kara: Um, the water does height does not change [reading G], does not change [run finger down right-hand column (values of seven)], does not change [run finger along horizontal line on graph], and, uh, the height is seven [with emphasis, running finger down values of seven in the table], **so "y = seven" [touching each symbol of the equation as she reads it].**
[02:10.26] NOTES: Int pulls set A to the side; she speads out Y, spreads out P, drags P-B closest to her, pulls G-B out (which happens to be closest G to her); looks at Y-D (closest; could be reading values in table), counts tick marks
[02:54.25] NOTES: She spends a lot of time counting the tick marks on P-B, comparing to Y-D; Y-D is the closest table to the set she has pulled together. add Y-D to set, compares to P-B
[03:06.63] Kara: I think this so far...
[03:10.15] INT: OK, Alright.
[03:15.00] Kara: **The "y" [touches O-B (furthest from her)] equals the height [touches the word "height" on Y-D], right?**
[03:18.00] INT: Um, Yup. It does.
[03:23.04] Kara: **[Touches O-D (closest), O-C (middle) and considers O-C for a long time.]**
[03:32.00] Kara: **This one [O-C; picks up and holds up to set A, to the side] could also go with that, because "y = 3 + 4".**
[03:37.00] INT: 3 + 4... Yup. Except that there's that "w" in there.
[03:40.00] Kara: Yeah. ...**[touches O-C again, almost picks up, moves to O-B (furthest)]**
[03:51.00] Kara: **I think this is a match. [adds O-B]**
*[3:52] NOTES: ***** SECOND SET; set is {G-B, P-B, Y-D, O-B} *******

[03:55.00] INT: OK. So, describe the set to me.
[03:57.00] Kara: Um. [Reads G-B] "The water tank began empty and someone turned the hose on to fill it." So it began empty [finger on bottom left corner of graph P-B] and then, it got filled [runs finger up the line of the graph]. 'Cuz all of these either began fille- full or at least a little bit full. [looking at and touching P-C, P-D for comparison]
[04:11.00] INT: I see. Yup. That would make sense, OK.
[04:13.00] Kara: And, um, this [touching Y-B], it goes up to the eighth one [graph P-B has eight tick marks on axes 8th line].
[04:19.00] INT: Yup.
[04:21.00] Kara: So...
[04:22.00] INT: On both of 'em, OK.
[04:24.00] Kara: And that's- that's what I was thinking for that [Y-D, pointing at value time=8], too. [little huff of air (doubtling own proposal?)
[04:28.00] INT: I see, OK.
[04:29.00] Kara: And, **I don't know how I got that.** [Laugh] [**covers equation card O-B**; gesture reinforcing her statement.]
[04:32.00] INT: Alright. Um. OK.
[04:35.00] Kara: [Touching Y-D with left hand while right hand is on O-B (she is referencing Y-D to remind herself of the connection she made? PROXIMITY MATTERS] **Actually, [switching out O-B with O-D, two hands] I think it might be this [O-D]...**
[04:38.00] INT: OK. And why do you think- Why did you make the change?
[04:40.19] *NOTES: talk about equation; she says that she's guessing a value for the RHS of equation*
[04:42.00] Kara: **Because: 30 - 2w [touching symbols as she reads the equation]...**
[04:44.00] INT: Yup.
[04:46.00] Kara: **I'm gonna guess is either 28 or...like, it's probably one of these...** so...
[04:50.00] INT: OK. Alright. Well here's what's interesting. When you said here, the tank began empty, someone turned on the hose to fill it. And you said, you said, here it is down at zero but, um, is this beginning empty?
[05:09.25] Kara: Oh, no. oooh. It has to be this one [Y-B] because this is the only one that begins empty. Wait. This is the beginning? [statement/question, not a query, necessarily]
[05:22.00] INT: Tell me what you think. Like- See if you can reason through if that's the beginning.
[05:28.00] Kara: It's multiplied by three every time. [the rows of Y-B]
[05:34.00] Kara: The height is multiplied by three every time. **AND this [O-D] wouldn't make any sense. So I think it would be this [O-B] because it's multiplied by three every time. ["it's" refers to the "w" in "y=3w", but she does not say that]**
[05:42.00] INT: OK. Great. Alright, let's put this set to the side. and see if you can make the next two. Well, I know you CAN... (laughs)
[05:49.08] Kara: She works on set; Touches G-C, **O-D (proximity)**; draws P-C over to pair with G-C. Inserts Y-C between G-C and P-C; **touches O-D, O-C (further), chooses O-C**
[06:07.00] *NOTES: ***** THIRD SET; in order {G-C, Y-C, P-C, O-C} *****
[06:10.00] INT: OK. Tell me about this set.
[06:11.00] Kara: Um. [Reads.] "Water tank had a few feet of water in it." ..Few feet [pointing at row (0,4)]
[06:16.00] INT: 'kay

[06:18.00] Kara: And this one [Y-D] starts with 30, and that's not a few feet.
[06:20.00] INT: laughs. That's a lot
[06:22.00] Kara: yeah.
[06:23.00] INT: uh-huh.
[06:25.00] Kara: uh. And this [P-C] starts at the first line, so it's only a *few* feet.
[06:31.00] INT: Yup, ok.
[06:32.00] Kara: **And the "3 + 4" 'Cuz it's seven. ["it's" refers to row (1,7) in Y-C; has to skip OVER P-C to do this]**
[06:38.00] INT: OK.
[06:39.00] Kara: Yeah. [laughs, somewhat unsure]
[06:40.00] INT: alright. So, you're going, um... "3 + 4" so... can you explain it to me again? 'Cuz I don't want to put words in your mouth.
[06:49.00] Kara: **The "3w + 4" [touching O-C] I was thinking because of the seven there [Y-C].**
[06:53.00] INT: OK. Yup. Alright, that
[06:56.00] Kara: **Or, I could do the 30 - 2 [O-D] because of the 28 there [Y-C].**
[06:58.18] *NOTES: Does she make the change because I asked her to explain her thinking with "3+4"? THIRD SET REVISED*
[07:00.00] INT: Mmmm. So, which one do you think works- which one do you think makes more sense?
[07:04.00] Kara: **Probably ...this... one [O-D]**
[07:07.00] INT: OK. Alright. So let's pull those {G-C, Y-C, P-C, O-D} to the side. And then, now you're left with these four...
[07:12.20] *NOTES: ***** checking FINAL SET {G-D, Y-D, P-D, O-C} *****
[07:12.21] Kara: I want to see if those make sense.
[07:14.26] INT: Yeah, right. And if they don't, you can change out anything.
[07:19.03] Kara: Well, that makes sense [thumb on Y-D], and then [finger on G-D] "someone pulled the plug out to drain it." That can't make sense, because it ends in 14. They ALL end in, like, a higher number. None of them end in zero.
[07:41.07] Kara: [thumb on graph line, counting with fingers] so... it could be eight... It's twenty six plus eight...
[07:59.00] Kara: I think these all make sense
[08:02.08] *NOTES: Int bring attn to equation*
[08:01.00] INT: OK. How about can **you, um, tell me how the equation fits in with the description? like, the rest of the descriptions?**
[08:08.00] Kara: Oh. Um.
[08:22.00] Kara: I don't know. [Laugh.]
[08:24.00] INT: Mkay. Well you were debating between these two, so let's look at this- these two together [pulls set {G-C, Y-C, P-C, O-D} above "last set"]. and see.
[08:33.00] Kara: **[Before talking, switch out O-C, pull down O-D]. This one makes more sense down here.**
[08:36.00] INT: Ok. And why is that?
[08:37.00] **Kara: [Moves O-C into row with other C cards] Because ... um.. "30 - 2w" equals 28 and 2w could be like... it could be like two times two, so, four, and ... "24" and "14"... [she puts her finger on these numbers from the LH column of Y-D.]**
[08:51.00] INT: 'kay

[08:52.00] Kara: Or it could just be plain two, and "22" [she puts her finger on this number from the LH column of Y-D.]
[08:56.00] INT: Yup.
[08:57.00] Kara: Or it could be times three, "26"
[09:00.00] INT: I see. Ok. And then, does this one [pointing at O-D] seem to work with this [run finger back and forth above {G-C, Y-C, P-C}]?
[09:06.00] **Kara: Yes. [touching O-C with thumb] Because um, the "7" [touching Y-C] as I said earlier.**
[09:12.00] INT: Yup. ..Yup. Great. OK! So let's pull all these back together. You did a great job. I didn't tell the other girls, but these, um, have you everdone problems like this before? A little bit? But these problems are, arrree, sort of like, middle school problems? So, they're meant to be kind of tough. Um. Which set was the easiest to get together? [laughs as Kara reaches for the constant set].
[09:38.06] INT: This one definitely [referencing Set A with her hand].
[09:40.23] INT: And tell me why.
[09:41.25] Kara: Because it says, "the water height does not change." [thumb on G-A] It's all sevens [finger runs down Y-A]; it does not change, and it changes in all of these [runs finger up and down along RH column of other tables]; and it's a straight line so it doesn't change [finger on horizontal line of P-A, looks at interviewer for emphasis], and [Reads] "y= 7"; Seven. [points at LH column of Y-A].
[09:54.11] INT: Yup. Well, it's funny because before I even explained- finished explaining, you knew that those go together. So I figured that was the one you were going to pick. And then which were the most- were challenging?
[10:06.14] Kara: I think, probably these two [points at rows of cards C & D]? were the most challenging? This one [points at row B] was like, in the middle.
[10:13.04] INT: OK. That makes sense. Can you- is there anything in particular that made it challenging?
[10:17.26] Kara: **Um. Yes, because I couldn't figure out the equations and these [Y-C & Y-D] were kinda tough, too.**
[10:27.29] INT: Yeah. OK. Great. Um- Do you think there'd be other kids in your class that would enjoy trying the puzzle?
[10:36.17] Kara: Mhmm. [affirmative] A couple. yeah.
[10:38.18] INT: Good! We'll see. I hope everybody does. So far, everyone has liked it. Alright, I'm gonna shift gears a little bit here and just ask, um, about math in general? Um.. Let's say that someone from Mrs. O's class is coming up to Mrs. H's class next year and they were kind of wondering what math in fourth grade is like. How would you describe it?