

Hierarchical Bayesian Model Updating for Structural Identification

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ABSTRACT

A new probabilistic finite element (FE) model updating technique based on Hierarchical Bayesian modeling is proposed for identification of civil structural systems under changing ambient/environmental conditions. The performance of the proposed technique is investigated for (1) uncertainty quantification of model updating parameters, and (2) probabilistic damage identification of the structural systems. Accurate estimation of the uncertainty in modeling parameters such as mass or stiffness is a challenging task. Several Bayesian model updating frameworks have been proposed in the literature that can successfully provide the “parameter estimation uncertainty” of model parameters with the assumption that there is no underlying inherent variability in the updating parameters. However, this assumption may not be valid for civil structures where structural mass and stiffness have inherent variability due to different sources of uncertainty such as changing ambient temperature, temperature gradient, wind speed, and traffic loads. Hierarchical Bayesian model updating is capable of predicting the overall uncertainty/variability of updating parameters by assuming time-variability of the underlying linear system. A general solution based on Gibbs Sampler is proposed to estimate the joint probability distributions of the updating parameters. The performance of the proposed Hierarchical approach is evaluated numerically for uncertainty quantification and damage identification of a 3-story shear building model. Effects of modeling errors and incomplete modal data are considered in the numerical study.

Keywords: Hierarchical Bayesian Model Updating, Damage Identification, Uncertainty Quantification, Continuous Structural Health Monitoring, Prediction Error Correlation, Environmental Condition Effects

1. INTRODUCTION

Model updating techniques based on vibration data (e.g., modal parameters) have provided promising results for damage identification of civil structures. Modal parameters such as natural frequencies and mode shapes can be accurately identified from ambient vibration or forced vibration tests. However, the former is more attractive for operational full-scale civil structures than the latter because the forced vibration tests often require suspending the structure’s operation. In addition, forced vibration tests are not practical for continuous structural health monitoring applications. The finite element (FE) model updating techniques using the identified modal parameters can potentially predict the existence, location, and severity of damage which is commonly defined as a change in the structures’ physical properties [1]. Reviews on vibration-based model updating and damage identification of structural systems have been provided in [2-

5]. The FE model updating methods can be divided into two broad categories of deterministic and probabilistic approaches. The deterministic FE model updating methods are well established in the literature [6-9], with several successful applications to civil structures [10-16].

The quality of structural identification results obtained from the deterministic FE model updating methods depends on (1) the accuracy and informativeness of measured vibration data (e.g., identified modal parameters), and (2) the accuracy of the initial FE model. In practice, the identified modal parameters of operational structures show significant variations from test to test, especially if the structure is being monitored over a long period of time. These variations can be due to measurement noise, estimation errors, and most importantly changing environmental/ambient conditions [17-24]. Modeling errors also add to the estimation uncertainties of identification results especially for complex civil structures that are usually modeled with many idealizations and simplifications [5, 25-27]. These sources of variability motivated researchers to incorporate the underlying structural uncertainties through the probabilistic FE model updating approaches, which can identify the updating model parameters and their estimation uncertainties. For reliable and robust structural health monitoring, it is important to provide a measure of confidence (uncertainty) on the damage identification results. Different probabilistic damage identification methods based on FE model updating have been used in the literature including Bayesian methods [28-31] and perturbation based methods [32-35]. Filtering methods (e.g., Kalman filters) have also been applied for online parameter identification of structural systems based on measured input-output time histories [36, 37]. The available Bayesian model updating frameworks can successfully predict the estimation uncertainties of the updating parameters (e.g., structural stiffness or mass), but do not consider the inherent variability of these parameters due to different sources of uncertainties such as changing ambient temperature, temperature gradient, wind speed, and traffic load.

This paper implements the concept of Hierarchical Bayesian modeling [38-40] to develop a new probabilistic FE model updating procedure that can predict the total uncertainty of the updating model parameters, including the parameter estimation uncertainty and more importantly the inherent variability of updating structural parameters. This proposed framework is extended for probabilistic damage identification of civil structures. Section 2 reviews the most commonly used Bayesian model updating framework in the literature, referred to as *classical Bayesian model updating* in this paper. Section 3 introduces the proposed Hierarchical Bayesian model updating procedure. The performance of the proposed method for structural uncertainty quantification and damage identification is evaluated through a numerical application in Section 4. The effects of modeling errors, incompleteness of modal data, error function correlations, and the number of data sets used in the updating process on the identification results are investigated. Finally, Section 5 provides the conclusions of this work.

2. CLASSICAL BAYESIAN FE MODEL UPDATING FRAMEWORK

2.1. Review of the Framework

Detailed and in-depth reviews on the framework can be found in [41-43]. The papers by Beck [28], Beck and Katafygiotis [29], and Sohn and Law [30] are the pioneering efforts in the probabilistic FE model updating using the Bayesian inference scheme. In the past decade, this method has been implemented for identification of several structural systems [27, 31, 44-50].

There are also a few studies that applied this Bayesian model updating procedure on full-scale civil structures [51-53].

Based on the Bayes theorem the posterior (updated) probability distribution function (PDF) of the updating structural parameters $\boldsymbol{\theta}$, and the model error parameters $\boldsymbol{\sigma}^2$, given a single data set \mathbf{D} can be expressed as:

$$p(\boldsymbol{\theta}, \boldsymbol{\sigma}^2 | \mathbf{D}) \propto p(\mathbf{D} | \boldsymbol{\theta}, \boldsymbol{\sigma}^2) p(\boldsymbol{\theta}, \boldsymbol{\sigma}^2) \quad (1)$$

where $p(\mathbf{D} | \boldsymbol{\theta}, \boldsymbol{\sigma}^2)$ is the so-called likelihood function and $p(\boldsymbol{\theta}, \boldsymbol{\sigma}^2)$ is the prior probability. A common type of measured data \mathbf{D} in structural identification applications includes the identified system eigenvalues (squares of circular natural frequencies) and mode shapes. To formulate the likelihood function, the error functions for a mode m are defined in Equations (2) and (3), and they are assumed to have zero-mean Gaussian distributions:

$$\tilde{\lambda}_m - \lambda_m(\boldsymbol{\theta}) = e_{\lambda_m} \square N(0, \sigma_{\lambda_m}^2) \quad (2)$$

$$\tilde{\Phi}_m - a_m \Phi_m(\boldsymbol{\theta}) = \mathbf{e}_{\Phi_m} \square N(\mathbf{0}, \boldsymbol{\Sigma}_{\Phi_m}) \quad (3)$$

The identified eigenvalues and mode shapes are shown as $\tilde{\boldsymbol{\lambda}}$ and $\tilde{\boldsymbol{\Phi}}$, respectively. Model calculated eigenvalues and mode shapes are shown by $\boldsymbol{\lambda}(\boldsymbol{\theta})$ and $\boldsymbol{\Phi}(\boldsymbol{\theta})$. Please note that the identified mode shapes and the model-calculated mode shapes at the measured DOFs are normalized to their unit length (i.e., unit L2 norm). In all equations of this paper, the model calculated mode shapes $\boldsymbol{\Phi}(\boldsymbol{\theta})$ contain only the components at the measured DOFs. a_m is the scaling factor of mode m and is set to be the dot product of the two unit-normalized mode shapes $\tilde{\boldsymbol{\Phi}}_m^T \boldsymbol{\Phi}_m(\boldsymbol{\theta})$. The likelihood function can be written as Equation (4) by assuming that the identified modal parameters are statistically independent, i.e., knowing the value of any observed modal parameter does not provide any information regarding the probability of observing other modal parameters.

$$p(\tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\Phi}} | \boldsymbol{\theta}, \boldsymbol{\sigma}) = \prod_{m=1}^{N_m} p(\tilde{\lambda}_m | \boldsymbol{\theta}, \sigma_{\lambda_m}^2) p(\tilde{\Phi}_m | \boldsymbol{\theta}, \boldsymbol{\Sigma}_{\Phi_m}) = \prod_{m=1}^{N_m} N(\tilde{\lambda}_m | \lambda_m(\boldsymbol{\theta}), \sigma_{\lambda_m}^2) N(\tilde{\Phi}_m | \Phi_m(\boldsymbol{\theta}), \boldsymbol{\Sigma}_{\Phi_m}) \quad (4)$$

In this equation, N_m is the total number of identified modes, $N(\tilde{\lambda} | \boldsymbol{\lambda}(\boldsymbol{\theta}), \sigma_{\lambda}^2)$ is the value of a Gaussian PDF with the mean $\boldsymbol{\lambda}(\boldsymbol{\theta})$ and the standard deviation σ_{λ} at $\tilde{\lambda}$, and similarly $N(\tilde{\boldsymbol{\Phi}} | \boldsymbol{\Phi}(\boldsymbol{\theta}), \boldsymbol{\Sigma}_{\Phi})$ is the vector of a multidimensional Gaussian PDF with the mean $\boldsymbol{\Phi}(\boldsymbol{\theta})$ and the covariance matrix $\boldsymbol{\Sigma}_{\Phi}$ at $\tilde{\boldsymbol{\Phi}}$. In most of the Bayesian FE model updating applications, no correlation is considered for the error functions of Equations (2) and (3). In [41, 54], this statistical independence assumption is regarded as the independency of information and not as an inherent property of the system. However, as mentioned in [55], accounting for the correlations between the error functions can affect the identification results.

In the case of having N_t independent number of measured data sets, the likelihood function of Equation (4) can be extended as:

$$p(\tilde{\lambda}_1 \dots \tilde{\lambda}_{N_t}, \tilde{\Phi}_1 \dots \tilde{\Phi}_{N_t} | \boldsymbol{\theta}, \boldsymbol{\sigma}) = \prod_{t=1}^{N_t} \prod_{m=1}^{N_m} N(\tilde{\lambda}_{tm} | \lambda_m(\boldsymbol{\theta}), \sigma_{\lambda_m}^2) N(\tilde{\Phi}_{tm} | \Phi_m(\boldsymbol{\theta}), \Sigma_{\Phi_m}) \quad (5)$$

where sub-index tm indicates the identified modal parameters of mode m from test t . The posterior probability distributions of the updating parameters can be obtained numerically by generating Markov Chain Monte Carlo (MCMC) samples from the posterior PDF [44, 45, 48, 50, 53, 56] or analytically through asymptotic approximations [27, 31, 46, 47, 49, 51]. If MCMC sampling techniques are used, the distribution of the samples from the posterior PDF can provide a measure of parameter estimation uncertainty. If an asymptotic approximation is used, the covariance matrix of updating model parameters can be estimated as the inverse Hessian of $-\text{Log}(p(\boldsymbol{\theta}, \boldsymbol{\sigma}^2 | \mathbf{D}))$ at the optimum of the updating parameters.

2.2. Interpretation of the Estimated Covariance Matrix

Although the classical Bayesian model updating procedure has been successfully implemented for identification of model parameters and optimal model class selection, the estimated covariance matrix does not provide the total variability of updating model parameters. This covariance matrix only represents the “parameter estimation uncertainty” which will decrease with increasing number of data sets [30, 53, 57], referred to as “noise mitigation” in [31]. The estimated uncertainties of the model parameters from a frequentist approach, however, can provide the overall variability of updating parameters and will converge with increasing number of data sets. A frequentist framework can consider the inherent variability of the updating model parameters and thus the properties of the corresponding distribution can be estimated. Statistics of the updating parameters in a frequentist approach are estimated from the deterministically identified parameters corresponding to different test data [22, 58, 59]. In [58], it is shown that in the absence of modeling errors the estimated mean and covariance matrix from the Bayesian and the frequentist frameworks can be related. In applications to civil structures, the physical modeling parameters such as mass, damping, stiffness, or boundary conditions show variations due to the changing environmental and ambient conditions [17-23]. For example, temperature variations affect the structural boundary conditions and the material properties, higher wind speed often results in reduced effective stiffness, or variations of traffic load (generally live loads) will affect the mass of the structure. These variations will result in different modal parameters from test to test. In other words, the N_t data sets that are used in Equation (5) are collected from a structure with time-varying properties.

The following single-degree-of-freedom (SDOF) example is designed to illustrate that the estimated standard deviation in the classical Bayesian approach underestimates the total uncertainty of updating parameters. The considered SDOF system consists of a known deterministic mass and an unknown variable stiffness $\theta \sim N(\mu_\theta, \sigma_\theta^2)$. The measured data include N_t identified natural frequencies of the system at different tests which is similar to structural identification applications.

Based on the error function of Equation (2), the posterior joint probability distribution of the stiffness and the variance of the error function σ_λ^2 , can be expressed as:

$$p(\theta, \sigma_\lambda^2 | \tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_{N_t}) \propto \frac{1}{\sigma_\lambda^{N_t}} \exp\left(-\frac{1}{2\sigma_\lambda^2} \sum_{t=1}^{N_t} (\tilde{\lambda}_t - \lambda(\theta))^2\right) \quad (6)$$

The sub-index t refers to a single test and the condition of the system during that test. Note that in this formulation, no inherent variability is considered for parameter θ . The most probable θ corresponds to the average of the identified system eigenvalues, i.e., $\hat{\theta} = \bar{\tilde{\lambda}}m$. The most probable variance of the error function can be calculated as:

$$\hat{\sigma}_\lambda^2 = \frac{1}{N_t} \sum_{t=1}^{N_t} (\tilde{\lambda}_t - \lambda(\hat{\theta}))^2 = \frac{1}{N_t} \sum_{t=1}^{N_t} (\tilde{\lambda}_t - \bar{\tilde{\lambda}})^2 \quad (7)$$

In this simple example, the standard deviation $\hat{\sigma}_\lambda$ can be related to the true standard deviation of the system stiffness σ_θ , as shown in Equation (8).

$$\hat{\sigma}_\lambda^2 = \frac{1}{N_t} \sum_{t=1}^{N_t} \left(\frac{\tilde{\theta}_t}{m} - \frac{\hat{\theta}}{m}\right)^2 = \frac{1}{m^2 N_t} \sum_{t=1}^{N_t} (\tilde{\theta}_t - \hat{\theta})^2 \equiv \frac{\sigma_\theta^2}{m^2} \quad (8)$$

The variance of the stiffness parameter $\hat{\sigma}_\theta^2$ can be written as:

$$\hat{\sigma}_\theta^2 = \hat{\sigma}_\lambda^2 \frac{m^2}{N_t} \equiv \frac{\sigma_\theta^2}{N_t} \quad (9)$$

which is different from the true standard deviation of θ or the estimated standard deviation through the frequentist approach. In the presence of variability in the structural parameters, a different approach should be implemented that can predict both the inherent variability and the parameter estimation uncertainties of the updating parameters. The Hierarchical Bayesian FE model updating is proposed in this paper to achieve this goal.

3. HIERARCHICAL BAYESIAN FE MODEL UPDATING

Researchers in the social and behavioral sciences have investigated methods to treat hierarchical data with different levels of variables in the same statistical model. For example, the hierarchical data for sociological survey analysis include measurements from individuals with different historical, geographic, or economic variables. To this end, the hierarchical modeling was proposed to account for the different grouping or times at which data are measured [40]. Similar analogy can be made for the collected measurements from a structure under different ambient and environmental conditions. This framework has been recently implemented for an uncertainty quantification application in structural dynamics [60].

In the proposed Hierarchical Bayesian model updating, the structural parameters are assumed to be distributed according to a probability model; a truncated Gaussian distribution (no negative stiffness) is assumed in this study, i.e., $\boldsymbol{\theta} \propto N(\boldsymbol{\mu}_\theta, \boldsymbol{\Sigma}_\theta)$. The error function vector, defined as the difference between the identified and model-calculated modal parameters, can be represented as a multivariate Gaussian distribution:

$$\mathbf{e}_t = \begin{bmatrix} \mathbf{e}_{\lambda_t} \\ \mathbf{e}_{\Phi_t} \end{bmatrix} \square N(\boldsymbol{\mu}_e, \boldsymbol{\Sigma}_e) \quad (10)$$

where \mathbf{e}_{λ_t} is the eigenvalue error vector of size N_m , and \mathbf{e}_{Φ_t} is the mode shape error vector of size $N_m \times N_s$. N_s is the number of model shape components. These errors are defined as the following for mode m :

$$e_{\lambda_m} = 1 - \frac{\lambda_m(\boldsymbol{\theta}_t)}{\tilde{\lambda}_{tm}} \quad (11)$$

$$\mathbf{e}_{\Phi_m} = \tilde{\Phi}_{tm} - a_m \Phi_m(\boldsymbol{\theta}_t) \quad (12)$$

Given the error function of Equation (10), the posterior probability distribution of the updating parameters can be expressed by Equations (13) for a single data set from test t .

$$p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0, \boldsymbol{\theta}_t, \boldsymbol{\mu}_e, \boldsymbol{\Sigma}_e \mid \tilde{\lambda}_t, \tilde{\Phi}_t) \propto p(\tilde{\lambda}_t, \tilde{\Phi}_t \mid \boldsymbol{\theta}_t, \boldsymbol{\mu}_e, \boldsymbol{\Sigma}_e) p(\boldsymbol{\theta}_t \mid \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0, \boldsymbol{\mu}_e, \boldsymbol{\Sigma}_e) \quad (13)$$

Note the differences between the error functions given by Equation (10) and Equations (2-3). The first difference is the sub-index t for the structural parameters and the identified modal parameters, which specifies the values of updating structural parameter during the collection of data set t . Also, the error functions are not assumed uncorrelated and with zero means. In the presence of modeling errors, prior assumptions for the prediction error correlations [55] and their variances [47, 54] affect the identification results. Therefore, updating the prediction error parameters provides a more robust identification. The prior probabilities are specified in two hierarchical stages of $p(\boldsymbol{\theta}_t \mid \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$ and hyper-prior probability distribution $p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0, \boldsymbol{\mu}_e, \boldsymbol{\Sigma}_e)$. In the case of having N_t independent data sets, the joint posterior PDF can be stated as

$$p(\boldsymbol{\Theta}, \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0, \boldsymbol{\mu}_e, \boldsymbol{\Sigma}_e \mid \tilde{\lambda}, \tilde{\Phi}) \propto \prod_{t=1}^{N_t} p(\tilde{\lambda}_t, \tilde{\Phi}_t \mid \boldsymbol{\theta}_t, \boldsymbol{\mu}_e, \boldsymbol{\Sigma}_e) p(\boldsymbol{\theta}_t \mid \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0, \boldsymbol{\mu}_e, \boldsymbol{\Sigma}_e) \quad (14)$$

where $\boldsymbol{\Theta} = \{\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_t, \dots, \boldsymbol{\theta}_{N_t}\}$. The graphical representation of the proposed Hierarchical Bayesian modeling is shown in Figure 1.

Non-informative priors are assumed for $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_e$, i.e., $p(\boldsymbol{\mu}_0, \boldsymbol{\mu}_e) \propto 1$. Depending on the selection of updating structural parameters, it is often reasonable to assume no correlation between these parameters and therefore the covariance matrix $\boldsymbol{\Sigma}_0$ can be presented as a diagonal matrix:

$$\boldsymbol{\Sigma}_0 = \text{Diag}(\sigma_{\theta_1}^2, \sigma_{\theta_2}^2, \dots, \sigma_{\theta_p}^2, \dots, \sigma_{\theta_{N_p}}^2) \quad (15)$$

with N_p = number of updating structural parameters in $\boldsymbol{\theta}$. Note that the formulations can be extended for correlated structural parameters by updating all components of the full covariance matrix $\boldsymbol{\Sigma}_0$. An Inverse Gamma probability distribution is assumed for the prior probability of the

$\sigma_{\theta_p}^2$:

$$p\left(\sigma_{\theta_p}^2\right) \sim \text{InverseGamma}\left(\alpha, 1/\beta\right) \quad (16)$$

where α and β can be taken identically for all the updating structural parameters. The prior probability distribution of Σ_e is assumed as Inverse Wishart distribution [61]:

$$p\left(\Sigma_e\right) \sim \text{InverseWishart}\left(\alpha_e \mathbf{I}\left(N_e\right), N_e\right) \quad (17)$$

where $N_e = N_m \times (N_s + 1)$ is the size of the error function vector and α_e is a constant.

Based on the considered priors, the joint posterior probability distribution of all the updating parameters can be stated as in Equation (18).

$$p\left(\Theta, \mu_{\theta}, \Sigma_{\theta}, \mu_e, \Sigma_e \mid \tilde{\lambda}, \tilde{\Phi}\right) \propto \frac{1}{|\Sigma_e|^{\frac{N_t + 2N_e + 1}{2}}} \frac{1}{\prod_{p=1}^{N_p} \left(\sigma_{\theta_p}^2\right)^{\frac{N_t + \alpha - 1}{2}}} \exp\left(-\frac{\sum_{t=1}^{N_t} \left\{J\left(\theta_t, \tilde{\lambda}_t, \tilde{\Phi}_t, \mu_e, \Sigma_e\right)\right\}}{2} - \sum_{p=1}^{N_p} \left\{\frac{\sum_{t=1}^{N_t} \left(\theta_{tp} - \mu_{\theta_p}\right)^2 + \frac{2}{\beta}}{2\sigma_{\theta_p}^2}\right\}\right) \quad (18)$$

where

$$J\left(\theta_t, \tilde{\lambda}_t, \tilde{\Phi}_t, \mu_e, \Sigma_e\right) = \left(\mathbf{e}_t - \mu_e\right)^T \Sigma_e^{-1} \left(\mathbf{e}_t - \mu_e\right) \quad (19)$$

The most common technique to solve Equation (18) is the Gibbs Sampler [62, 63].

3.1. Estimation of Posterior Probabilities Using Gibbs Sampler

In the application of Gibbs sampling techniques, samples are generated from the full conditional probability distribution of each parameter until convergence is reached. Convergence is achieved when the changes in statistics of generated samples becomes smaller than a prescribed threshold. The full conditional posterior probability distributions of all the updating parameters are presented in Equations (20) to (24).

$$p\left(\theta_t \mid \mu_{\theta}, \Sigma_{\theta}, \mu_e, \Sigma_e, \tilde{\lambda}, \tilde{\Phi}\right) \propto \exp\left(-J\left(\theta_t, \tilde{\lambda}_t, \tilde{\Phi}_t, \mu_e, \Sigma_e\right) - \sum_{p=1}^{N_p} \left\{\frac{\left(\theta_{tp} - \mu_{\theta_p}\right)^2}{\sigma_{\theta_p}^2}\right\}\right), \quad t = 1, \dots, N_t \quad (20)$$

$$p\left(\mu_{\theta} \mid \Theta, \Sigma_{\theta}, \mu_e, \Sigma_e, \tilde{\lambda}, \tilde{\Phi}\right) \propto \exp\left(-\sum_{p=1}^{N_p} \sum_{t=1}^{N_t} \left\{\frac{\left(\theta_{pt} - \mu_{\theta_p}\right)^2}{\sigma_{\theta_p}^2}\right\}\right) \quad (21)$$

$$p\left(\Sigma_{\theta} \mid \Theta, \mu_{\theta}, \mu_e, \Sigma_e, \tilde{\lambda}, \tilde{\Phi}\right) \propto \frac{1}{\prod_{p=1}^{N_p} \left(\sigma_{\theta_p}^2\right)^{\frac{N_t + \alpha - 1}{2}}} \exp\left(-\frac{1}{2} \sum_{p=1}^{N_p} \sum_{t=1}^{N_t} \left\{\frac{\left(\theta_{pt} - \mu_{\theta_p}\right)^2}{\sigma_{\theta_p}^2}\right\} - \sum_{p=1}^{N_p} \frac{1}{\beta \sigma_{\theta_p}^2}\right) \quad (22)$$

$$p(\boldsymbol{\mu}_e | \boldsymbol{\Theta}, \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0, \boldsymbol{\Sigma}_e, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\Phi}}) \propto \exp\left(-\frac{1}{2} \sum_{t=1}^{N_t} (\mathbf{e}_t - \boldsymbol{\mu}_e)^T \boldsymbol{\Sigma}_e^{-1} (\mathbf{e}_t - \boldsymbol{\mu}_e)\right) \quad (23)$$

$$p(\boldsymbol{\Sigma}_e | \boldsymbol{\Theta}, \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0, \boldsymbol{\mu}_e, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\Phi}}) \propto \frac{1}{|\boldsymbol{\Sigma}_e|^{\frac{N_t+2N_e+1}{2}}} \exp\left(-\frac{1}{2} \sum_{t=1}^{N_t} (\mathbf{e}_t - \boldsymbol{\mu}_e)^T \boldsymbol{\Sigma}_e^{-1} (\mathbf{e}_t - \boldsymbol{\mu}_e) - \frac{1}{2} \text{tr}(\alpha_e \boldsymbol{\Sigma}_e^{-1})\right) \quad (24)$$

where tr denotes the trace of a matrix.

Equations (21) to (24) can be alternatively written as

$$p(\boldsymbol{\mu}_0 | \cdot) \propto N\left(\frac{1}{N_t} \sum_{t=1}^{N_t} \boldsymbol{\theta}_t, \frac{1}{N_t} \boldsymbol{\Sigma}_0\right) \quad (25)$$

$$p(\sigma_{\theta_p}^2 | \cdot) \propto \text{InverseGamma}\left(\left(\frac{N_t}{2} + \alpha\right), \frac{1}{\beta} + \frac{1}{2} \sum_{t=1}^{N_t} (\theta_{pt} - \mu_{\theta_p})^2\right) \quad (26)$$

$$p(\boldsymbol{\mu}_e | \cdot) \propto N\left(\frac{1}{N_t} \sum_{t=1}^{N_t} \mathbf{e}_t, \frac{1}{N_t} \boldsymbol{\Sigma}_e\right) \quad (27)$$

$$p(\boldsymbol{\Sigma}_e | \cdot) \propto \text{InverseWishart}\left(\alpha_e \mathbf{I}(N_e) + \sum_{t=1}^{N_t} (\mathbf{e}_t - \boldsymbol{\mu}_e)(\mathbf{e}_t - \boldsymbol{\mu}_e)^T, N_t + N_e\right) \quad (28)$$

The Inverse Wishard distribution in Equation (28) can be justified by re-writing the first exponential term of Equation (24) as:

$$\sum_{t=1}^{N_t} (\mathbf{e}_t - \boldsymbol{\mu}_e)^T \boldsymbol{\Sigma}_e^{-1} (\mathbf{e}_t - \boldsymbol{\mu}_e) = \sum_{t=1}^{N_t} \text{tr}\left((\mathbf{e}_t - \boldsymbol{\mu}_e)^T \boldsymbol{\Sigma}_e^{-1} (\mathbf{e}_t - \boldsymbol{\mu}_e)\right) = \sum_{t=1}^{N_t} \text{tr}\left((\mathbf{e}_t - \boldsymbol{\mu}_e)(\mathbf{e}_t - \boldsymbol{\mu}_e)^T\right) \boldsymbol{\Sigma}_e^{-1} \quad (29)$$

It can be observed that the full conditional probability distributions of all the updating parameters except $\boldsymbol{\theta}_t$ are standard distributions. The posterior joint probability distribution of updating parameter can be accurately estimated if an adequate number of samples has been generated. Generating samples from Equations (25) to (28) is trivial due to their known distribution functions; however, generating samples from the conditional probability distributions of $\boldsymbol{\theta}_t$, Equation (20), requires using advanced sampling techniques such as Metropolis-Hasting [64, 65], adaptive Metropolis-Hastings [44, 66], or Translational Markov Chain Monte Carlo algorithm [48, 56]. It is worth noting that the $\boldsymbol{\theta}_t$ samples can be generated independently for each test data which is ideal for parallel computing to reduce the computational time [67]. Alternatively, the conditional probability distribution of $\boldsymbol{\theta}_t$ in Equation (20) can be approximated as a Gaussian distribution using Laplace asymptotic approximation to simplify the sampling process. The standard deviations of the generated samples reflect the ‘‘parameter estimation uncertainties’’ which will be reduced by increasing the number of data sets used in the updating process.

The required number of samples for converging to the joint probability distribution of Equation (18) can become very large in complex civil structures with a large number of updating

structural parameters. Therefore, a simplified and more computationally efficient procedure will be introduced in the next section for the MAP estimations of the updating parameters.

3.2. Proposed Simplified Approach for MAP Estimations

In this subsection, a simplified procedure is proposed which is based on the MAP values of the full conditional PDFs of each updating parameter in Equations (20), (25) to (28). This approach is similar to the Empirical Bayes method, which can approximately solve the Hierarchical Bayes models. The Empirical Bayes methods were mostly used before the advent of Markov Chain Monte Carlo simulation techniques [61]. In this procedure, there is no need to draw random samples from the full conditional probability functions. This simplified procedure neglects the parameter estimation uncertainties and provides the most probable posterior probability distribution of the updating structural parameters (based on the MAP of $\boldsymbol{\mu}_\theta$ and $\boldsymbol{\Sigma}_\theta$). This procedure is based on the assumptions that (1) $\boldsymbol{\theta}_t$ is globally identifiable for each data set, and (2) a large number of data sets is available. The proposed algorithm has the following steps:

- (a) Start with initial estimates for ${}^0\hat{\boldsymbol{\mu}}_\theta$, ${}^0\hat{\boldsymbol{\mu}}_e$, ${}^0\hat{\boldsymbol{\Sigma}}_e$ and ${}^0\hat{\boldsymbol{\Sigma}}_\theta$
- (b) At iteration j
 - i. Find the MAP of $\boldsymbol{\theta}_t$ in Equation (20) given the ${}^{j-1}\hat{\boldsymbol{\mu}}_\theta$, ${}^{j-1}\hat{\boldsymbol{\Sigma}}_e$, and ${}^{j-1}\hat{\boldsymbol{\Sigma}}_\theta$:

$${}^j\hat{\boldsymbol{\theta}}_t = \arg \min_{\boldsymbol{\theta}_t} \left((\mathbf{e}_t - {}^{j-1}\boldsymbol{\mu}_e)^T {}^{j-1}\boldsymbol{\Sigma}_e^{-1} (\mathbf{e}_t - {}^{j-1}\boldsymbol{\mu}_e) + \sum_{p=1}^{N_p} \left\{ \frac{(\theta_{tp} - {}^{j-1}\hat{\mu}_{\theta_p})^2}{{}^{j-1}\sigma_{\theta_p}^2} \right\} \right), \quad t = 1, \dots, N_t \quad (30)$$

- ii. Find the MAP of $\boldsymbol{\mu}_\theta$, $\boldsymbol{\mu}_e$ and mean of $\boldsymbol{\Sigma}_e$ in Equations (25), (27), and (28) given ${}^j\hat{\boldsymbol{\theta}}_t$:

$${}^j\hat{\boldsymbol{\mu}}_\theta = \frac{1}{N_t} \sum_{t=1}^{N_t} {}^j\hat{\boldsymbol{\theta}}_t \quad (31)$$

$${}^j\hat{\boldsymbol{\mu}}_e = \frac{1}{N_t} \sum_{t=1}^{N_t} {}^j\hat{\mathbf{e}}_t \quad (32)$$

$${}^j\hat{\boldsymbol{\Sigma}}_e \square \frac{\alpha_e \mathbf{I}(N_e) + \sum_{t=1}^{N_t} ({}^j\hat{\mathbf{e}}_t - {}^j\hat{\boldsymbol{\mu}}_e)({}^j\hat{\mathbf{e}}_t - {}^j\hat{\boldsymbol{\mu}}_e)^T}{N_t - 1} \quad (33)$$

where ${}^j\hat{\mathbf{e}}_t$ are the prediction errors at the optimum ${}^j\hat{\boldsymbol{\theta}}_t$ values.

- iii. Find the MAP of $\boldsymbol{\Sigma}_\theta$ in Equation (26) given ${}^j\hat{\boldsymbol{\theta}}_t$ and ${}^j\hat{\boldsymbol{\mu}}_\theta$:

$${}^j\hat{\sigma}_{\theta_p}^2 = \frac{\frac{1}{\beta} + \frac{\sum_{t=1}^{N_t} ({}^j\hat{\theta}_{pt} - {}^j\hat{\mu}_{\theta_p})^2}{2}}{\alpha + \frac{N_t}{2} - 1} \quad (34)$$

- (c) Check for the convergence of the updating parameters. Convergence is achieved when the relative changes in $\boldsymbol{\mu}_\theta$ and $\sigma_{\theta_p}^2$ becomes less than a prescribed threshold.

The MAP estimates from Equations (30) to (32), and (34) will maximize the posterior conditional PDFs of Equations (20), and (25) to (27). MAP estimate of the conditional distribution of Equation (28) cannot be easily computed; therefore, the mean estimate is used in Equation (33) which should be close to the MAP if sufficient number of data sets is available. After convergence, the MAP estimates of all the updating parameters provide the global maxima of the posterior joint PDF of Equation (18).

4. APPLICATION TO A NUMERICAL CASE STUDY

A three-story shear building model is used as a numerical case study to evaluate the performance of the proposed Hierarchical Bayesian FE model updating procedure. This 3-DOF structure is shown in Figure 2, where the mass of each story is set to 1.2 metric ton, and the stiffness of stories one to three are assumed to be random variables with truncated Gaussian distributions of $N(2000\text{kN/m}, 100^2\text{kN}^2/\text{m}^2)$, $N(1000\text{kN/m}, 50^2\text{kN}^2/\text{m}^2)$, and $N(1000\text{kN/m}, 20^2\text{kN}^2/\text{m}^2)$, respectively. The natural frequencies of the building at the mean values of story stiffnesses are 2.378 Hz, 6.498 Hz, and 8.876 Hz. In Section 4.1, the performance of the proposed Hierarchical Bayesian FE model updating framework is evaluated and the results are compared with those of the classical Bayesian model updating framework. In Section 4.2, the proposed framework is extended for probabilistic damage identification where the estimated variability/uncertainty of updating structural parameters is propagated in the damage identification results.

4.1. Performance Evaluation of the Hierarchical Bayesian FE Model Updating

The simulated measured data include 400 sets of natural frequencies and mode shapes which are generated by sampling the story stiffness values from their considered probability distributions. The histograms of the generated natural frequencies are shown in Figure 3. These generated modal parameters will be used as the measured data in the model updating processes. In Section 4.1.1, the performance of the proposed framework is evaluated with no modeling errors considered while in Section 4.1.2 the Hierarchical approach is implemented in the presence of modeling errors. The effects of modal data incompleteness are studied in Section 4.1.3. The proposed simplified approach of Section 3.2 is implemented for MAP estimations. The initial points for the mean and standard deviation of the updating structural parameters for all three stories are taken as 800 and 1000, respectively. The parameters α and β in Equation (16) are taken as 1 and 2, respectively.

4.1.1. Complete Modal Data and No Modeling Errors

In the absence of modeling errors, the prediction error parameters will all be zeros, since there exists a $\boldsymbol{\theta}_t$ that makes the error functions of Equation (11) and (12) zero. Therefore at optimal $\boldsymbol{\theta}_t$ values, we can assume that $\boldsymbol{\mu}_e$ is known and equal to zero, and $\boldsymbol{\Sigma}_e$ is a diagonal matrix (no correlation) with σ_e^2 variances for eigenvalue errors and $w\sigma_e^2$ variances for mode shape errors. w is the considered ratio between the mode shape error and eigenvalue error variances. Given these assumptions, the likelihood function becomes similar to the likelihood functions that have been commonly used in the literature [30, 31, 47, 52-54, 68, 69]. In this case, the objective function of Equation (19) will be simplified to Equation (35) and the conditional probability distribution of σ_e^2 can be written as Equation (36) if its prior is assumed to be uniform [70]:

$$J(\boldsymbol{\theta}_t, \tilde{\boldsymbol{\lambda}}_t, \tilde{\boldsymbol{\Phi}}_t) = \mathbf{e}_t^T \mathbf{e}_t = \sum_{m=1}^{N_m} \left\{ \left(1 - \frac{\lambda_m(\boldsymbol{\theta}_t)}{\tilde{\lambda}_m} \right)^2 + \frac{1}{w} (\tilde{\boldsymbol{\Phi}}_{tm} - a_{tm} \boldsymbol{\Phi}_m(\boldsymbol{\theta}_t))^T (\tilde{\boldsymbol{\Phi}}_{tm} - a_{tm} \boldsymbol{\Phi}_m(\boldsymbol{\theta}_t)) \right\} \quad (35)$$

$$p(\sigma_e^2 | \cdot) \propto \text{InverseGamma} \left(\left(\frac{N_t N_m (N_s + 1)}{2} + 1 \right), \frac{1}{2} \sum_{t=1}^{N_t} J(\boldsymbol{\theta}_t, \tilde{\boldsymbol{\lambda}}_t, \tilde{\boldsymbol{\Phi}}_t) \right) \quad (36)$$

The initial error standard deviation is considered to be $\text{Log}(\sigma_e) = -40$, and w is considered as $1/N_s$.

Seven different subsets of data with $n_t = \{2, 5, 20, 50, 100, 200, \text{ and } 400\}$ data set numbers are used for model updating. Table 1 reports the model updating results for the considered data subsets. The results include the MAP estimates of the mean and standard deviation of the three updating structural parameters and the error standard deviation σ_e . The means and the standard deviations of the three structural parameters are estimated accurately except for the first two cases with $n_t = 2$ or $n_t = 5$, where insufficient number of data sets is used in the updating process. The means and standard deviations of updating structural parameters are also estimated through a frequentist approach (i.e., minimizing the objective function of Equation (36) for each set of data) and match the values reported in Table 1. In the absence of modeling errors and dominant prior assumptions, the Hierarchical and frequentist approaches provide identical estimates for the means and standard deviations of structural parameters. The last column of Table 1 shows the standard deviation of the error functions, which is very close to zero implying good matches between the updated model and the data. This is due to the fact that no modeling error is considered in this section (σ_e is not exactly zero because of numerical round-off error [71, 72]). As desired, the statistical properties of the updating structural parameters converge to their true values by adding more data sets. The most probable posterior PDFs of the three updating structural parameters are shown in Figure 4 for the cases of using 5, 20, 50, and 400 data sets. It should be noted that these distributions correspond to the MAP estimates of means and standard deviations, i.e., $N(\hat{\boldsymbol{\mu}}_{\theta_p}, \hat{\boldsymbol{\sigma}}_{\theta_p}^2)$. However, the estimated means and standard deviations are also associated with uncertainties that are not included in the results reported in this section.

The number of data sets required for accurate estimation of updating parameters can be checked from the convergence of the statistics of the measured data (i.e., identified modal

parameter). Figure 5 plots the means and standard deviations of the identified natural frequencies for increasing number of data sets. From this plot, it can be concluded that in the presence of the considered stiffness variations, approximately 100 data sets or more are needed for unbiased estimations. Such large number of data sets can be collected in monitored structures in a matter of few days [10, 16, 23, 53] although a longer period of data collection is recommended to observe the full range of environmental and ambient variations. Note that this step for predicting the required amount of measured data can be performed independently from the model updating.

For comparison purposes, the corresponding model updating results from the classical Bayesian model updating framework are also provided in Table 2. The Adaptive Metropolis-Hastings algorithm of [66] is used to sample the posterior probability distributions of updating parameters. In this algorithm, the adaption is performed on the proposal probability distribution functions of the standard Metropolis-Hastings algorithm. Table 2 presents the estimated MAP and parameter estimation uncertainty of updating structural parameters for different number of data sets. Figure 6 also shows the posterior PDF of structural parameters in three cases of $n_t = \{5, 50, 400\}$. From Table 2 and Figure 6, it can be seen that (1) the estimated MAP values are in good agreement with the mean estimates of the Hierarchical framework and the exact values, and (2) the parameter estimation uncertainties do not represent the total uncertainty of structural parameters and always decrease with addition of data. In addition, the error standard deviation contains both the modeling errors and the variability of structural parameters. Therefore, it should not be used as a measure of goodness of fit. In the Hierarchical Bayesian FE model updating, parameter σ_e shows the level of mismatch between the model and the data due to only the effects of modeling errors, i.e., the error standard deviation gets close to zero in the absence of modeling error even if the identified modal parameters have large variations. This comparison highlights the benefits of the proposed Hierarchical Bayesian model updating framework.

4.1.2. Complete Modal Data and Considering Modeling Errors

The accuracy of FE model updating results can be significantly affected by modeling errors [25-27, 53, 57]. To represent the effects of modeling errors, the 3-story shear building is assumed to be on a flexible base with rotational and horizontal springs as shown in Figure 7. The identified modal parameters are generated from the structure with the flexible base, but the FE model used in the updating process is assumed to be on a rigid base as in Figure 2. Stiffnesses of the horizontal and rotational springs are constant and are assumed to be 20,000 [kN/m] and 200,000 [kN-m/rad]. The natural frequencies of the system at the mean story stiffness values are 2.285Hz, 6.387Hz, 8.742Hz, and 68.095Hz. In this section, two cases of identifications are performed, (1) based on the simplified assumption that the error terms are uncorrelated with zero-means and therefore, Σ_e is a diagonal matrix (see Subsection 4.1.1), and (2) based on the general updating process of Section 3 that does not consider any simplifying assumption for Σ_e and μ_e . Note that the parameter α_e in Equation (17) is taken as 10^{-10} .

The model updating results for the first case of identification (uncorrelated error terms) are provided in Table 3 for different number of data sets. From this table, it is observed that the mean and standard deviation of the first story's stiffness are underestimated while the statistics of higher story stiffness values are accurately identified. The bias in the predicted mean of the first story stiffness can be due to the compensation effects of modeling errors. The simplified structural model of Figure 2, which is used in the FE model updating process, cannot simulate

the flexibility of the structural base and therefore, the stiffness of the first story will be underestimated to compensate for the base flexibility. The estimated σ_e values are larger than those obtained in Table 1, implying higher modeling errors. The underestimated standard deviation of the first story stiffness can be due to the fact that the uncertainties of unmeasured data (modal parameters of mode 4 and mode shape components at the base for modes 1-3) are not accounted for in the updating process. This error is inevitable in real-world applications when the measured data is incomplete and the structural models are discretized and simplified. Therefore in real-world applications, the predicted uncertainties are expected to be underestimated. Table 4 reports the identification results when the full covariance matrix of the error functions is considered as an updating parameter (identification case 2). The last two columns show the L2 norm of the $\hat{\boldsymbol{\mu}}_e$ divided by N_e (12 in this section) that represent the average error biases at the optimal parameters, and $\frac{1}{2} \text{Log} \left(\text{tr}(\hat{\boldsymbol{\Sigma}}_e) / N_e \right)$ which is an equivalent measure to the $\text{Log}(\hat{\sigma}_e)$ in Table 3. It can be seen that the estimated MAP values are very close to the results of Tables 3-4 and the average error variances are much smaller than their equivalent values in Table 3. In other words, the model uncertainties can be reduced by considering both $\boldsymbol{\mu}_e$ and $\boldsymbol{\Sigma}_e$ as updating parameters.

The updating structural parameters are also estimated through a frequentist approach. The MAP values are reported in Table 5, which are in good agreement with the MAP values from the proposed Hierarchical framework (Table 3) although the bias in the mean estimates is smaller in the Hierarchical updating process. Based on Tables 3-5, it can be observed that MAP estimates from the Hierarchical Bayesian and frequentist approaches are not identical but close in the presence of modeling errors.

4.1.3. Incomplete Modal Data and Considering Modeling Errors

In this section, in addition to the considered modeling errors, model updating is performed when using only the modal parameters of the first two vibration modes. Similar to Section 4.1.2, two cases of identifications are performed using (1) uncorrelated zero-mean error terms, and (2) correlated error terms with unknown $\boldsymbol{\mu}_e$ and $\boldsymbol{\Sigma}_e$. Table 6 shows the results for the first identification case, which are very similar to the results obtained in previous subsection, except for the underestimated mean and standard deviation of θ_2 . Please note that the bias and underestimation in the means and standard deviations are not observed when using incomplete modal data in the absence of modeling errors, provided that the problem is globally identifiable. Table 7 reports the second case of identification results. Unlike the previous section, these updating results have smaller bias compared to those listed in Table 6, especially for θ_2 .

4.2. Hierarchical Bayesian FE Model Updating for Damage Identification

The proposed Hierarchical Bayesian FE model updating framework is extended to be used for probabilistic damage identification of civil structures. In this section, the posterior probability distributions of updating parameters have been estimated using the Gibbs Sampler technique that was reviewed in Section 3.1. Therefore, the parameter estimation uncertainties are also available

from the Gibbs samples. Damage factor (DF) of the structural component i is defined as the loss of stiffness from the reference state to the current state.

$$DF_i = \frac{\theta_i^r - \theta_i^c}{\theta_i^r} \quad (37)$$

where superscript r refers to the reference state and superscript c refers to the current state of the structure. Equation (38) provides the probability of damage exceeding a given damage factor df given the measured data in both the reference and current states of the structure.

$$P[\theta_i^r - \theta_i^c \geq df \times \theta_i^r \mid \mathbf{D}^r, \mathbf{D}^c] = 1 - CDF\left(df \times \theta_i^r \mid \mu_{\theta_i}^r - \mu_{\theta_i}^c, \sqrt{(\sigma_{\theta_i}^r)^2 + (\sigma_{\theta_i}^c)^2}\right) = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{df \times \theta_i^r - (\mu_{\theta_i}^r - \mu_{\theta_i}^c)}{\sqrt{2((\sigma_{\theta_i}^r)^2 + (\sigma_{\theta_i}^c)^2)}}\right) \quad (38)$$

where CDF is the cumulative Gaussian distribution function, and erf is Gauss Error Function.

400 sets of data are generated from the structure in the healthy/reference state and another 400 sets of data are generated from the structure at the damaged state. Damage is considered to be 5% loss of stiffness in the first story, i.e., the mean of the first story stiffness is reduced to 1900 [kN/m] while its standard deviation is kept the same as in the undamaged state. The natural frequencies of the damaged structure at the mean values of story stiffnesses are 2.357 Hz, 6.439 Hz, and 8.809 Hz. The natural frequency of the first mode is affected most by the damage and it is reduced only by 0.92% from the reference state. This reduction is within the variation range of the natural frequencies of the structure in the healthy state as shown in Figure 3, which makes the damage identification challenging. This scenario is often observed in real-world civil structures where the variations of modal parameters due to changing environmental and ambient condition are relatively large and sensitivity of modal parameters to structural damage is small [17-23].

As it can be observed from Equations (25, 26), the mean and variances of the updating structural parameters are associated with parameter estimation uncertainties obtained from the Gibbs samples. Figure 8 shows the posterior probability distributions of the mean and the standard deviation of the first story stiffness based on different number of data sets at the undamaged state. It can be seen that the estimation uncertainties will be significantly reduced by using more data sets in the updating process. By increasing the number of data sets, the posterior probability distribution of each parameter becomes closer to a Dirac Delta function at its true value in the absence of measurement noise and modeling errors. In the presence of parameter estimation uncertainties, the updating structural parameters should be defined as PDFs with uncertainties in their estimated means and standard deviations. Therefore, in the damage identification process, the estimation uncertainties of these parameters should be propagated to the damage probabilities in Equation (38).

The damage identification results using the data of the damaged state (in the presence of 5% damage in the first story) are shown in Figure 9. This figure shows the damage exceedance probabilities for damage factor range of [-0.05 0.20] in the first story using $n_t = 5, 10, 50, 400$ data sets. It is observed that the most probable damage estimates at 50% confidence level are close to the exact damage value (5%) for different number of data sets while the variability of damage probability estimates decrease drastically with increasing number of data sets. The probability distributions of the most probable damage factors (corresponding to 50% confidence level) considering estimation uncertainties are shown in Figure 10. The obtained results underline

the fact that damage can be accurately predicted even when the changes in the measured data due to damage are smaller than those due to environmental conditions provided a sufficient number of measured data sets is available. The proposed framework is well suited for damage assessment of operational civil structures where changing environmental conditions can significantly affect the identified modal parameters. It should be noted that damage can still be estimated based on only few measured data sets. However, the corresponding large estimation uncertainties for the predicted damages should then be considered in any decision making or detection analysis [73, 74]. Finally, it can be seen from Figure 9 that even with negligible parameter estimation uncertainty (estimation uncertainty that can be reduced by adding more data) damage is still estimated probabilistically due to inherent variability of structural parameters [75, 76].

5. SUMMARY AND CONCLUSION

In this paper, a Hierarchical Bayesian FE model updating process is proposed for uncertainty quantification and damage identification of structural systems. The proposed framework can predict the overall uncertainties of the updating parameters accurately in the absence of modeling error and reasonably well in the presence of modeling errors. The inherent variability of the structural mass or stiffness properties is often due to changing ambient temperature, temperature gradient, wind speed, and traffic load that can affect the structural mass or stiffness. This framework can also estimate the posterior distribution of considered error functions, which represent the misfit between the data and the model predicted responses. The analytical formulation of the proposed framework for model updating is presented first and then two techniques are described for estimating the joint posterior probability distribution of updating parameters. The first technique is based on a standard Gibbs sampler while the second approach is more simplified and provides only the MAP estimates of the updating parameters.

The performance of the proposed Hierarchical Bayesian framework for model updating and uncertainty quantification is evaluated by means of numerical application to a three-story shear building model. It is shown that the identified statistical properties of updating parameters are the same as those obtained using a frequentist approach in the absence of modeling errors. However, the two approaches provide similar but slightly different estimates in the presence of modeling errors. The effects of modeling errors and modal data incompleteness are also investigated and it is observed that these factors will introduce bias in the estimated mean and underestimation in the standard deviations of the updating structural parameters. Therefore, the predicted uncertainties are expected to be underestimated in real-world applications.

In the presence of modeling errors, two sets of identifications are performed using (1) zero-mean uncorrelated error functions, and (2) correlated error functions with unknown means and correlations (used as updating parameters). Updating the mean vector and covariance matrix of the error functions has improved the identification results in the presence of modeling errors. More discussion on the topic requires further investigation on more complicated structures using both numerically simulated and experimental data. In this case, efficient sampling techniques are required to be able to handle non-standard conditional probability distributions.

Finally, the proposed Hierarchical framework is extended for probabilistic damage identification. The identified damages are associated with inherent uncertainties due to the variability of structural stiffness and parameter estimation uncertainty. It is observed that the latter can be reduced by using more data in the updating process. The obtained results underline

the fact that in the absence of modeling errors, damage can be accurately predicted even when the changes in measured data due to damage are smaller than those due to environmental/ambient conditions provided a sufficient number of measured data sets is available. Compared to the classical Bayesian model updating methods, the proposed framework is better suited for damage assessment of operational civil structures where changing environmental conditions can significantly affect the identified modal parameters.

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REFERENCES

- [1] M.I. Friswell, J.E. Mottershead, Finite element model updating in structural dynamics, Kluwer Academic Publishers, Boston; Dordrecht, 1995.
- [2] J.E. Mottershead, M.I. Friswell, Model Updating In Structural Dynamics - A Survey, *J. Sound Vibr.*, 167 (1993) 347-375.
- [3] H. Sohn, C.R. Farrar, F.M. Hemez, D.D. Shunk, D.W. Stinemates, B.R. Nadler, J.J. Czarnecki, A review of structural health monitoring literature: 1996-2001, Los Alamos National Laboratory Los Alamos, NM, 2004.
- [4] E.P. Carden, P. Fanning, Vibration based condition monitoring: A review, *Struct. Health Monit.*, 3 (2004) 355-377.
- [5] M.I. Friswell, Damage identification using inverse methods, *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 365 (2007) 393-410.
- [6] K. Alvin, Finite element model update via Bayesian estimation and minimization of dynamic residuals, *Aiaa J.*, 35 (1997) 879-886.
- [7] J.E. Mottershead, M. Link, M.I. Friswell, The sensitivity method in finite element model updating: A tutorial, *Mech. Syst. Signal Proc.*, 25 (2011) 2275-2296.
- [8] M. Baruch, Optimal correction of mass and stiffness matrices using measured modes, *Aiaa J.*, 20 (1982) 1623-1626.
- [9] C. Farhat, F.M. Hemez, Updating finite element dynamic models using an element-by-element sensitivity methodology, *Aiaa J.*, 31 (1993) 1702-1711.
- [10] A. Teughels, G. De Roeck, Structural damage identification of the highway bridge Z24 by FE model updating, *J. Sound Vibr.*, 278 (2004) 589-610.

- [11] O. Huth, G. Feltrin, J. Maeck, N. Kilic, M. Motavalli, Damage identification using modal data: Experiences on a prestressed concrete bridge, *J. Struct. Eng.*, 131 (2005) 1898-1910.
- [12] E. Reynders, G.D. Roeck, P. Gunders Bakir, C. Sauvage, Damage identification on the Tilff bridge by vibration monitoring using optical fiber strain sensors, *Journal of Engineering Mechanics*, 133 (2007) 185-193.
- [13] E. Asgariéh, B. Moaveni, A. Stavridis, Nonlinear finite element model updating of an infilled frame based on identified time-varying modal parameters during an earthquake, *J. Sound Vibr.*, 333 (2014) 6057-6073.
- [14] S.N. Pakzad, S.G. Shahidi, Generalized Response Surface Model Updating Using Time Domain Data, *J. Struct. Eng.*, 140 (2013) 130717215900003.
- [15] B. Moaveni, X.F. He, J.P. Conte, J.I. Restrepo, Damage identification study of a seven-story full-scale building slice tested on the UCSD-NEES shake table, *Struct. Saf.*, 32 (2010) 347-356.
- [16] B. Moaveni, I. Behmanesh, Effects of changing ambient temperature on finite element model updating of the Dowling Hall Footbridge, *Eng. Struct.*, 43 (2012) 58-68.
- [17] P. Cornwell, C.R. Farrar, S.W. Doebling, H. Sohn, Environmental variability of modal properties, *Experimental Techniques*, 23 (1999) 45-48.
- [18] H. Sohn, M. Dzwonczyk, E.G. Straser, A.S. Kiremidjian, K.H. Law, T. Meng, An experimental study of temperature effect on modal parameters of the Alamosa Canyon Bridge, *Earthquake Engineering & Structural Dynamics*, 28 (1999) 879-897.
- [19] S. Alampalli, Effects of testing, analysis, damage, and environment on modal parameters, *Mech. Syst. Signal Proc.*, 14 (2000) 63-74.
- [20] B. Peeters, G. De Roeck, One-year monitoring of the Z 24-Bridge: environmental effects versus damage events, *Earthquake Engineering & Structural Dynamics*, 30 (2001) 149-171.
- [21] J.F. Clinton, S.C. Bradford, T.H. Heaton, J. Favela, The observed wander of the natural frequencies in a structure, *Bulletin of the Seismological Society of America*, 96 (2006) 237-257.
- [22] B. Moaveni, J.P. Conte, F.M. Hemez, Uncertainty and Sensitivity Analysis of Damage Identification Results Obtained Using Finite Element Model Updating, *Comput.-Aided Civil Infrastruct. Eng.*, 24 (2009) 320-334.
- [23] P. Moser, B. Moaveni, Environmental effects on the identified natural frequencies of the Dowling Hall Footbridge, *Mech. Syst. Signal Proc.*, 25 (2011) 2336-2357.
- [24] D.F. Giraldo, S.J. Dyke, J.M. Caicedo, Damage detection accommodating varying environmental conditions, *Structural Health Monitoring-an International Journal*, 5 (2006) 155-172.
- [25] B. Goller, G. Schueller, Investigation of model uncertainties in Bayesian structural model updating, *J. Sound Vibr.*, 330 (2011) 6122-6136.
- [26] I. Behmanesh, B. Moaveni, Bayesian FE Model Updating in the Presence of Modeling Errors, Model Validation and Uncertainty Quantification, Volume 3, Springer, 2014, pp. 119-133.

- [27] J. Ching, J.L. Beck, Bayesian analysis of the Phase II IASC-ASCE structural health monitoring experimental benchmark data, *J. Eng. Mech.-ASCE*, 130 (2004) 1233-1244.
- [28] J.L. Beck, Statistical system identification of structures, *Proceedings of International Conference on Structural Safety and Reliability*, ASCE, 1989, pp. 1395-1402.
- [29] J.L. Beck, L.S. Katafygiotis, Updating models and their uncertainties. I: Bayesian statistical framework, *J. Eng. Mech.-ASCE*, 124 (1998) 455-461.
- [30] H. Sohn, K.H. Law, A Bayesian probabilistic approach for structure damage detection, *Earthquake Engineering & Structural Dynamics*, 26 (1997) 1259-1281.
- [31] M.W. Vanik, J.L. Beck, S.K. Au, Bayesian probabilistic approach to structural health monitoring, *J. Eng. Mech.-ASCE*, 126 (2000) 738-745.
- [32] J.D. Collins, G.C. Hart, T. Haselman, B. Kennedy, Statistical identification of structures, *Aiaa J.*, 12 (1974) 185-190.
- [33] H.H. Khodaparast, J.E. Mottershead, M.I. Friswell, Perturbation methods for the estimation of parameter variability in stochastic model updating, *Mech. Syst. Signal Proc.*, 22 (2008) 1751-1773.
- [34] E. Jacquelin, S. Adhikari, M.I. Friswell, A second-moment approach for direct probabilistic model updating in structural dynamics, *Mech. Syst. Signal Proc.*, 29 (2012) 262-283.
- [35] J.R. Fonseca, M.I. Friswell, J.E. Mottershead, A.W. Lees, Uncertainty identification by the maximum likelihood method, *J. Sound Vibr.*, 288 (2005) 587-599.
- [36] E.N. Chatzi, A.W. Smyth, The unscented Kalman filter and particle filter methods for nonlinear structural system identification with non-collocated heterogeneous sensing, *Structural control and health monitoring*, 16 (2009) 99-123.
- [37] E.N. Chatzi, A.W. Smyth, S.F. Masri, Experimental application of on-line parametric identification for nonlinear hysteretic systems with model uncertainty, *Struct. Saf.*, 32 (2010) 326-337.
- [38] W.R. Gilks, S. Richardson, D.J. Spiegelhalter, *Markov chain Monte Carlo in practice*, Chapman & Hall, Boca Raton, FL, 1998.
- [39] D. Gamerman, H.F. Lopes, *Markov chain Monte Carlo: stochastic simulation for Bayesian inference*, Taylor & Francis, Boca Raton, 2006.
- [40] J. Gill, *Bayesian methods: a social and behavioral sciences approach*, Chapman & Hall/CRC, Boca Raton, FL, 2002.
- [41] J.L. Beck, Bayesian system identification based on probability logic, *Struct. Control. Health Monit.*, 17 (2010) 825-847.
- [42] K.V. Yuen, S.C. Kuok, *Bayesian Methods for Updating Dynamic Models*, *Appl. Mech. Rev.*, 64 (2011).
- [43] K.-V. Yuen, *Bayesian methods for structural dynamics and civil engineering*, John Wiley & Sons, 2010.

- [44] J.L. Beck, S.K. Au, Bayesian updating of structural models and reliability using Markov chain Monte Carlo simulation, *J. Eng. Mech.-ASCE*, 128 (2002) 380-391.
- [45] K.V. Yuen, J.L. Beck, S.K. Au, Structural damage detection and assessment by adaptive Markov chain Monte Carlo simulation, *Structural Control and Health Monitoring*, 11 (2004) 327-347.
- [46] J. Ching, J.L. Beck, New Bayesian model updating algorithm applied to a structural health monitoring benchmark, *Struct. Health Monit.*, 3 (2004) 313-332.
- [47] K. Christodoulou, C. Papadimitriou, Structural identification based on optimally weighted modal residuals, *Mech. Syst. Signal Proc.*, 21 (2007) 4-23.
- [48] J.Y. Ching, Y.C. Chen, Transitional markov chain monte carlo method for Bayesian model updating, model class selection, and model averaging, *J. Eng. Mech.-ASCE*, 133 (2007) 816-832.
- [49] K. Christodoulou, E. Ntotsios, C. Papadimitriou, P. Panetsos, Structural model updating and prediction variability using Pareto optimal models, *Comput. Meth. Appl. Mech. Eng.*, 198 (2008) 138-149.
- [50] M. Muto, J.L. Beck, Bayesian updating and model class selection for hysteretic structural models using stochastic simulation, *J. Vib. Control*, 14 (2008) 7-34.
- [51] E. Ntotsios, C. Papadimitriou, P. Panetsos, G. Karaiskos, K. Perros, P.C. Perdikaris, Bridge health monitoring system based on vibration measurements, *Bull. Earthq. Eng.*, 7 (2009) 469-483.
- [52] E. Simoen, J.P. Conte, B. Moaveni, G. Lombaert, Uncertainty Quantification in the Assessment of Progressive Damage in a 7-Story Full-Scale Building Slice, *Journal of Engineering Mechanics*, 139 (2013) 1818-1830.
- [53] I. Behmanesh, B. Moaveni, Probabilistic identification of simulated damage on the Dowling Hall footbridge through Bayesian finite element model updating, *Structural Control and Health Monitoring*, 22 (2015) 463-483.
- [54] B. Goller, J.L. Beck, G.I. Schueller, Evidence-Based Identification of Weighting Factors in Bayesian Model Updating Using Modal Data, *J. Eng. Mech.-ASCE*, 138 (2012) 430-440.
- [55] E. Simoen, C. Papadimitriou, G. Lombaert, On prediction error correlation in Bayesian model updating, *J. Sound Vibr.*, 332 (2013) 4136-4152.
- [56] P. Angelikopoulos, C. Papadimitriou, P. Koumoutsakos, Bayesian uncertainty quantification and propagation in molecular dynamics simulations: a high performance computing framework, *The Journal of chemical physics*, 137 (2012) 144103.
- [57] J.A. Goulet, I.F.C. Smith, Structural identification with systematic errors and unknown uncertainty dependencies, *Comput. Struct.*, 128 (2013) 251-258.
- [58] S.K. Au, Connecting Bayesian and frequentist quantification of parameter uncertainty in system identification, *Mech. Syst. Signal Proc.*, 29 (2012) 328-342.

- [59] B. Moaveni, A.R. Barbosa, J.P. Conte, F.M. Hemez, Uncertainty analysis of system identification results obtained for a seven-story building slice tested on the UCSD-NEES shake table, *Struct. Control. Health Monit.*, 21 (2014) 466-483.
- [60] G. Ballesteros, P. Angelikopoulos, C. Papadimitriou, P. Koumoutsakos, Bayesian Hierarchical Models for Uncertainty Quantification in Structural Dynamics, *Vulnerability, Uncertainty, and Risk: Quantification, Mitigation, and Management*, ASCE, 2014, pp. 1615-1624.
- [61] M.S. Hamada, A. Wilson, C.S. Reese, H. Martz, *Bayesian reliability*, Springer Science & Business Media, 2008.
- [62] S. Geman, D. Geman, Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images, *Pattern Analysis and Machine Intelligence*, IEEE Transactions on, (1984) 721-741.
- [63] A.E. Gelfand, Sampling-Based Approaches to Calculating Marginal Densities, *Journal of the American Statistical Association*, 85 (1990) 398-409.
- [64] N. Metropolis, A.W. Rosenbluth, M.N. Rosenbluth, A.H. Teller, E. Teller, Equation of state calculations by fast computing machines, *The journal of Chemical Physics*, 21 (1953) 1087-1092.
- [65] W.K. Hastings, Monte Carlo sampling methods using Markov chains and their applications, *Biometrika*, 57 (1970) 97-109.
- [66] C. Andrieu, J. Thoms, A tutorial on adaptive MCMC, *Statistics and Computing*, 18 (2008) 343-373.
- [67] P.E. Hadjidoukas, P. Angelikopoulos, C. Papadimitriou, P. Koumoutsakos, Π4U: A high performance computing framework for Bayesian uncertainty quantification of complex models, *Journal of Computational Physics*, 284 (2015) 1-21.
- [68] L. Mthembu, T. Marwala, M.I. Friswell, S. Adhikari, Model selection in finite element model updating using the Bayesian evidence statistic, *Mech. Syst. Signal Proc.*, 25 (2011) 2399-2412.
- [69] H.F. Lam, H.Y. Peng, S.K. Au, Development of a practical algorithm for Bayesian model updating of a coupled slab system utilizing field test data, *Eng. Struct.*, 79 (2014) 182-194.
- [70] A. Gelman, Prior distributions for variance parameters in hierarchical models (comment on article by Browne and Draper), *Bayesian analysis*, 1 (2006) 515-534.
- [71] MathWorks, *MATLAB User's Guide*, MathWorks Inc., Natick, MA, 2014.
- [72] MathWorks, *Recognize and Avoid Round-Off Errors*, MathWorks Inc., Natick, MA, 2014.
- [73] D.W. North, A Tutorial Introduction to Decision Theory, *Systems Science and Cybernetics*, IEEE Transactions on, 4 (1968) 200-210.
- [74] T. Stepinski, T. Uhl, W.J. Staszewski, *Advanced structural damage detection: from theory to engineering applications*, Wiley, Chichester, West Sussex, United Kingdom, 2013.
- [75] A.D. Kiureghian, O. Ditlevsen, Aleatory or epistemic? Does it matter?, *Struct. Saf.*, 31 (2009) 105-112.

[76] S. Sankararaman, S. Mahadevan, Separating the contributions of variability and parameter uncertainty in probability distributions, *Reliability Engineering & System Safety*, 112 (2013) 187-199.

Table 1. MAP estimates from the Hierarchical framework with no modeling errors

	θ_1		θ_2		θ_3		-
	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	$Log(\hat{\sigma}_e)$
<i>Exact</i>	2000	100	1000	50	1000	20	-
$n_t = 2$	2005.3	25.5	1048.5	50.1	999.2	4.4	-19.8
$n_t = 5$	2008.6	86.3	1021.0	44.9	992.0	8.0	-19.8
$n_t = 20$	1988.6	106.2	1009.2	47.7	999.9	19.7	-19.8
$n_t = 50$	2000.0	99.0	1000.0	49.5	1000.0	19.8	-19.9
$n_t = 100$	2000.9	101.5	1002.8	49.4	998.7	18.7	-19.9
$n_t = 200$	1997.7	103.0	1002.8	51.4	1000.1	20.3	-19.9
$n_t = 400$	1999.7	100.1	1004.1	49.3	999.2	19.5	-19.9

Table 2. Updating results from the classical Bayesian framework

	θ_1		θ_2		θ_3		-
	MAP	STD ($\hat{\sigma}_\theta$)	MAP	STD ($\hat{\sigma}_\theta$)	MAP	STD ($\hat{\sigma}_\theta$)	$Log(\hat{\sigma}_e)$
<i>Exact</i>	2000	100	1000	50	1000	20	-
$n_t = 2$	2004.8	31.1	1048.2	16.1	999.3	10.7	-4.42
$n_t = 5$	2006.9	19.5	1021.0	9.8	991.8	6.8	-4.34
$n_t = 20$	1984.9	12.8	1009.9	6.5	1000.1	4.7	-4.00
$n_t = 50$	1997.5	7.1	1000.0	3.5	999.8	2.5	-4.13
$n_t = 100$	1998.0	5.1	1003.0	2.6	998.5	1.8	-4.12
$n_t = 200$	1995.1	3.6	1002.8	2.0	999.9	1.3	-4.12
$n_t = 400$	1997.4	2.6	1004.0	1.3	998.9	0.9	-4.16

Table 3. MAP estimates from the Hierarchical framework with modeling errors; Case 1

	θ_1		θ_2		θ_3		-
	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	$Log(\hat{\sigma}_e)$
<i>Exact</i>	2000	100	1000	50	1000	20	-
$n_t = 2$	1819.5	19.8	1044.6	52.3	998.3	0.8	-6.1
$n_t = 5$	1821.6	69.4	1016.8	45.4	991.3	7.1	-6.3
$n_t = 20$	1805.3	86.1	1004.8	48.0	999.1	19.3	-6.3
$n_t = 50$	1814.7	80.2	995.4	50.1	999.2	19.4	-6.3
$n_t = 100$	1815.3	82.2	998.2	50.0	997.9	18.2	-6.3
$n_t = 200$	1812.8	83.6	998.2	52.1	999.3	19.8	-6.3
$n_t = 400$	1814.4	81.2	999.5	50.0	998.4	19.0	-6.3

Table 4. MAP estimates from the Hierarchical framework with modeling errors; Case 2

	θ_1		θ_2		θ_3		-	-
	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	$\frac{\ \hat{\mu}_e\ }{12}$	$\frac{1}{2} Log \left(\frac{tr(\hat{\Sigma}_e)}{12} \right)$
<i>Exact</i>	2000	100	1000	50	1000	20	-	-
$n_t = 20$	1803.3	86.3	1006.4	48.0	998.0	19.6	6.4×10^{-4}	-8.6
$n_t = 50$	1812.6	80.5	997.1	50.0	998.0	19.6	6.5×10^{-4}	-8.3
$n_t = 100$	1813.3	82.5	999.9	49.9	996.7	18.5	6.5×10^{-4}	-8.4
$n_t = 200$	1810.7	83.8	999.9	52.0	998.2	20.1	6.5×10^{-4}	-8.3
$n_t = 400$	1812.3	81.4	1001.2	49.9	997.2	19.3	6.5×10^{-4}	-8.3

Table 5. MAP estimates from the frequentist framework with modeling errors

	θ_1		θ_2		θ_3	
	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)
<i>Exact</i>	2000	100	1000	50	1000	20
$n_t = 2$	1804.1	14.4	1005.8	50.5	998.1	4.4
$n_t = 5$	1810.1	73.5	978.0	45.9	991.0	7.9
$n_t = 20$	1795.2	87.5	966.1	46.7	998.7	19.6
$n_t = 50$	1805.9	84.4	956.7	49.9	998.9	19.7
$n_t = 100$	1806.2	86.1	959.5	49.6	997.6	18.6
$n_t = 200$	1803.4	88.3	959.7	52.2	999.1	20.1
$n_t = 400$	1804.9	86.1	960.9	50.1	998.1	19.3

Table 6. MAP estimates based on using the first two modes in the updating process; Case 1

	θ_1		θ_2		θ_3		-
	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	$Log(\hat{\sigma}_\epsilon)$
<i>Exact</i>	2000	100	1000	50	1000	20	-
$n_t = 2$	1822.9	29.8	990.0	0.7	998.1	0.7	-5.6
$n_t = 5$	1824.8	68.6	965.1	0.4	991.1	0.5	-5.5
$n_t = 20$	1809.1	84.8	955.6	26.7	998.7	19.0	-5.8
$n_t = 50$	1818.7	77.6	947.3	32.0	998.7	18.8	-5.8
$n_t = 100$	1819.2	79.5	949.7	32.0	997.5	17.7	-5.8
$n_t = 200$	1816.9	80.8	949.8	34.5	998.8	19.2	-5.8
$n_t = 400$	1818.5	78.5	950.8	31.2	997.9	18.4	-5.8

Table 7. MAP estimates based on using the first two modes; Case 2

	θ_1		θ_2		θ_3		-	-
	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	Mean ($\hat{\mu}_\theta$)	STD ($\hat{\sigma}_\theta$)	$\frac{\ \hat{\boldsymbol{\mu}}_e\ }{8}$	$\frac{1}{2} \text{Log} \left(\frac{\text{tr}(\hat{\boldsymbol{\Sigma}}_e)}{8} \right)$
<i>Exact</i>	2000	100	1000	50	1000	20	-	-
$n_t = 20$	1807.2	85.7	986.6	43.8	998.8	19.7	1.3×10^{-3}	-8.8
$n_t = 50$	1816.5	80.0	978.0	45.8	998.9	19.6	1.3×10^{-3}	-8.8
$n_t = 100$	1817.1	81.9	980.5	45.6	997.6	18.5	1.3×10^{-3}	-8.8
$n_t = 200$	1814.6	83.3	980.6	47.6	999.1	20.0	1.3×10^{-3}	-8.8
$n_t = 400$	1816.2	81.0	981.7	45.7	998.1	19.2	1.3×10^{-3}	-8.8

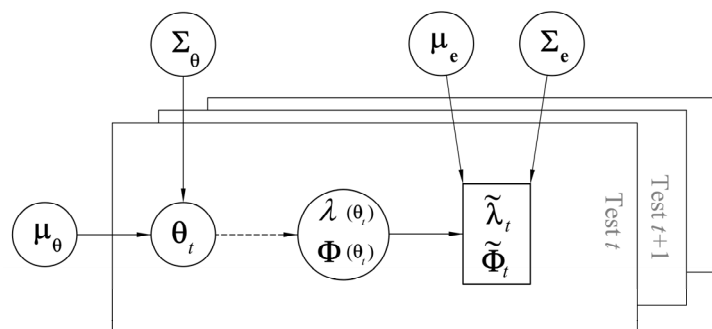


Figure 1. Graphical representation for the proposed Hierarchical Bayesian modeling

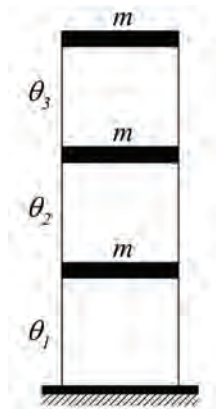


Figure 2. Three-story shear building model

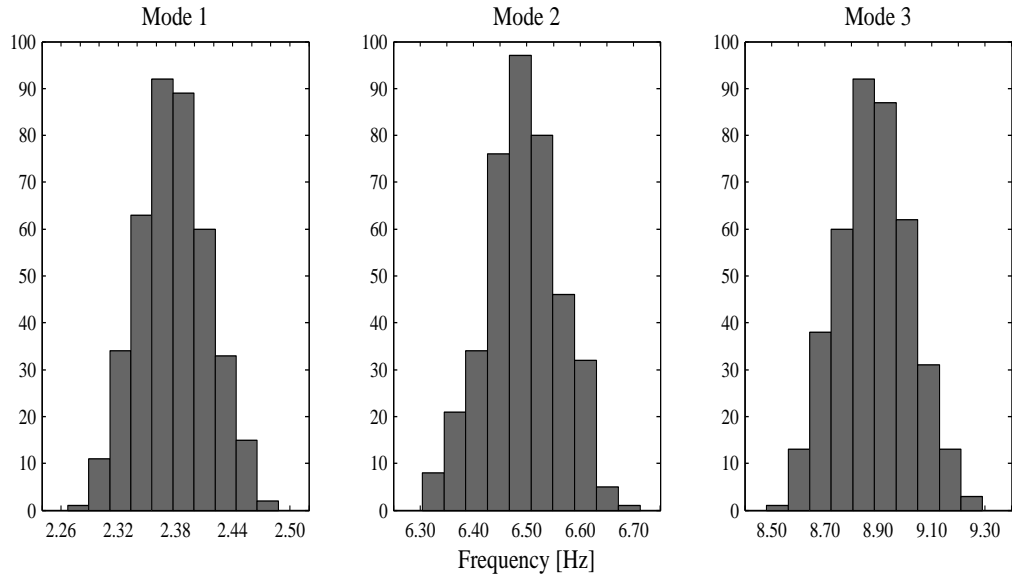


Figure 3. Histogram of identified natural frequencies

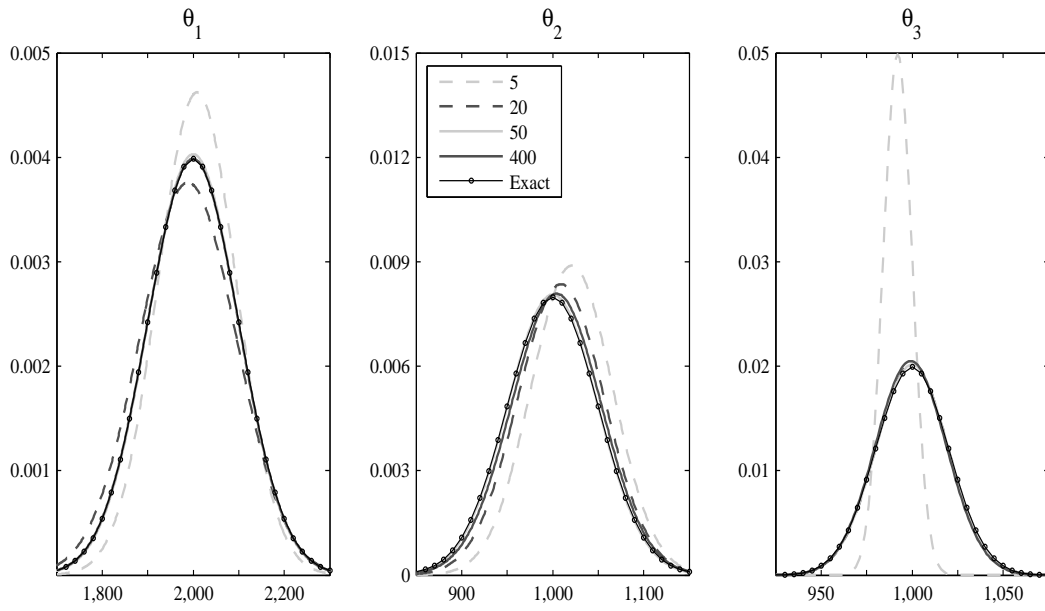


Figure 4. Most probable posterior PDFs using Hierarchical Bayesian framework

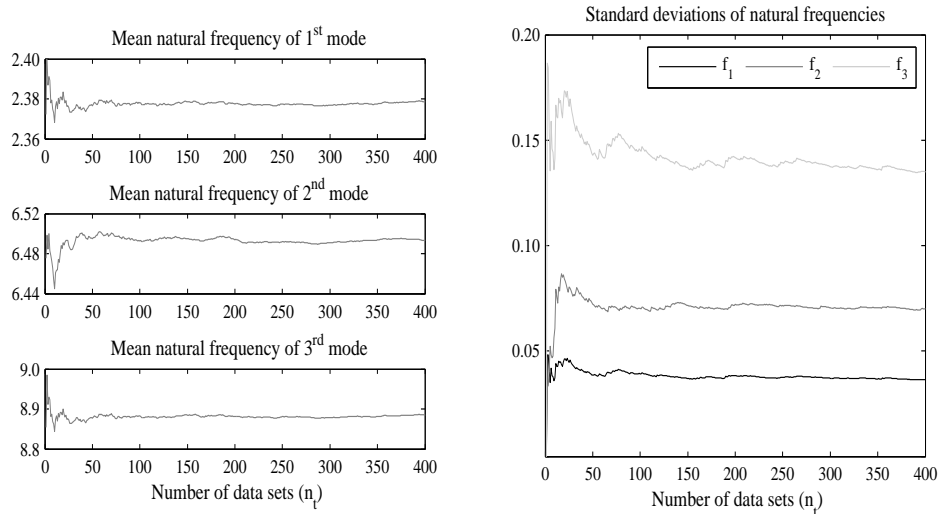


Figure 5. Convergence of the identified natural frequency statistics

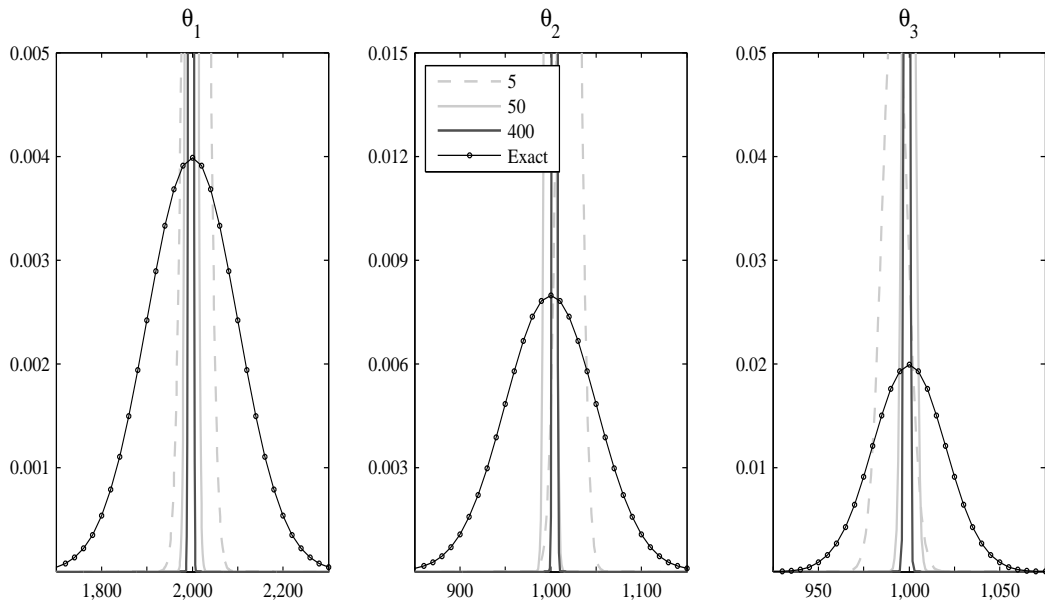


Figure 6. Posterior PDFs from the classical Bayesian framework

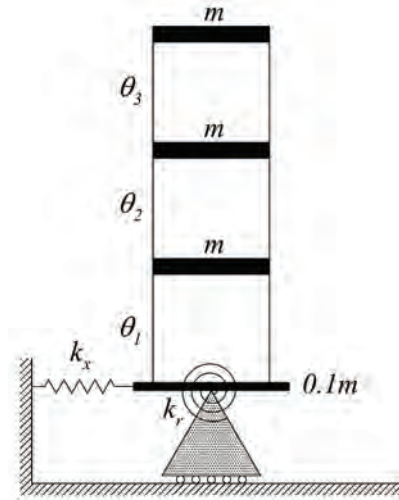


Figure 7. Flexible-base three-story shear model building

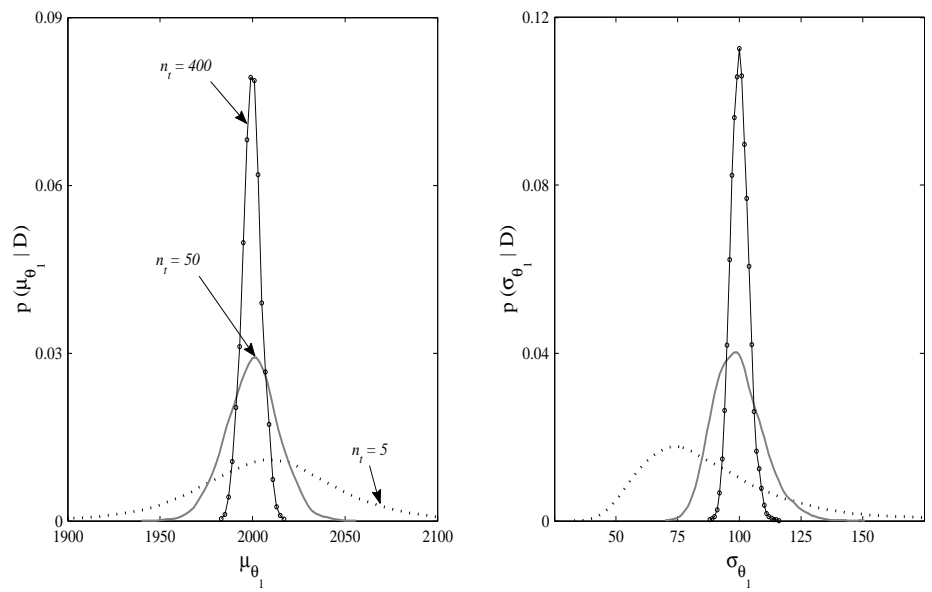


Figure 8. Parameter estimation uncertainties for the mean and standard deviation of θ_1 in the undamaged state

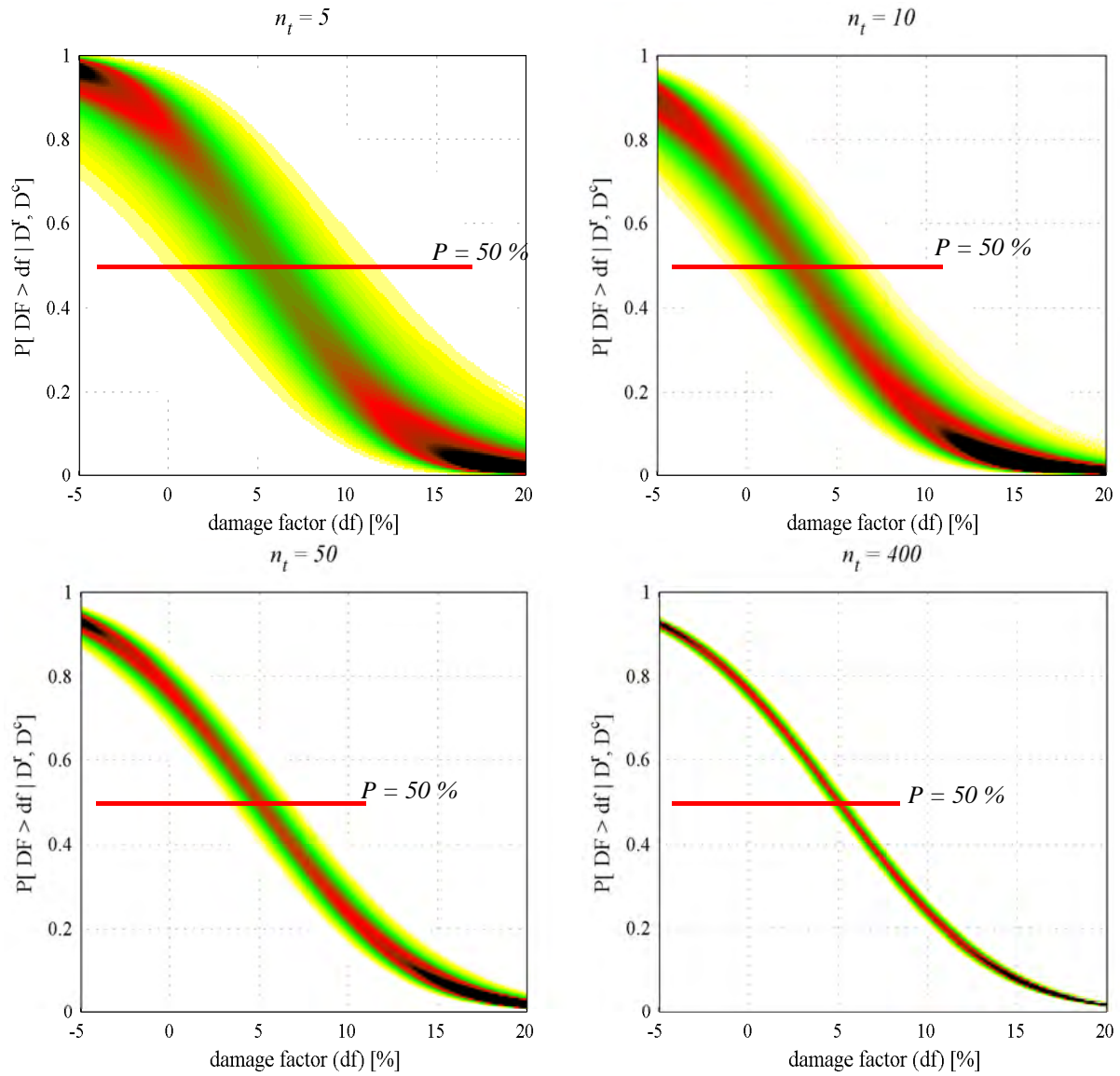


Figure 9. Probability of damage given the baseline data and the data in the damaged state

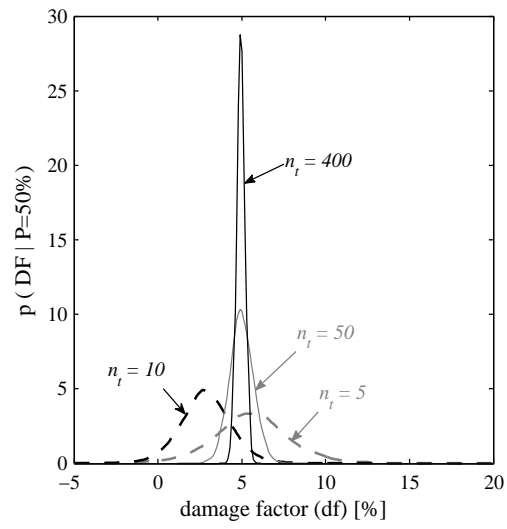


Figure 10. Distribution of most probable damage based of different number of data sets