THE TIME-CHANGED Q-WIENER PROCESS AND ASSOCIATED STOCHASTIC DIFFERENTIAL EQUATIONS

A dissertation

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Abstract

The general narrative of this work involves constructing the time-changed Q-Wiener process and comparing it to the usual Q-Wiener process. Unlike its non-time-changed counterpart, the time-changed Q-Wiener process lacks the nice properties of having stationary and independent increments. Thus, we must turn to the martingale property of the time-changed Q-Wiener process in order to recover analogous results to the ones known for the Q-Wiener process, without a time change. These results include formulating stochastic differential equations (SDEs) driven by time-changed Q-Wiener processes, establishing existence of solutions to these SDEs, exploring the stability of those solutions, looking at associated Fokker-Planck-Kolmogorov equations, and making connections to stochastic partial differential equations.

To my family and friends.

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Chapter 1

Introduction

The work in this thesis stems from a general interest in the intersection of probability and partial differential equations (PDEs). One of the first well-known examples I encountered in this intersection was that of a Brownian motion, with mean zero and variance $2t$, having a probability density function $p(x,t)$ that is a solution to the diffusion equation: $\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2}$. (See Figure 1.1 for an image of some sample paths of Brownian motion.) This lead me to Meerschaert and Sikorskii's book "Stochastic Models for Fractional Calculus," where one looks at stable processes, rather than just Brownian motion, and the fractional Fokker-Planck-Kolmogorov (FPK) equations that their densities solve. [27] and see also [16–19].

Also of interest was the field of stochastic partial differential equations (SPDEs). There, the idea is to look at PDEs that have been perturbed by some white noise term, where this term is constructed using Brownian sheets. See [10] and [35] for further details. More accessible, however, is Gawarecki and Mandrekar's book on the related topic of "Stochastic Differential Equations in Infinite Dimensions." Now the equations are comprised of Hilbert-spaced valued objects and the white noise term is replaced by one involving a Q-Wiener process. [14]

This is really where much of this thesis work began. Q. Wu, P. Garmirian, and I combined this Q-Wiener process, an object constructed to be the Hilbert-spaced analog of Brownian motion, with a time change that is the inverse of a stable subordinator. Recall from earlier that stable processes were also a natural object of interest that lead to fractional FPK equations. Thus, we were able to combine different interests in the realm of probability and PDEs in this joint work. Before moving on, it should be noted that we were also very much inspired by the methods and results of Kei Kobayashi in his dissertation.

Thus the general motivation was to see how many of the existing results involving

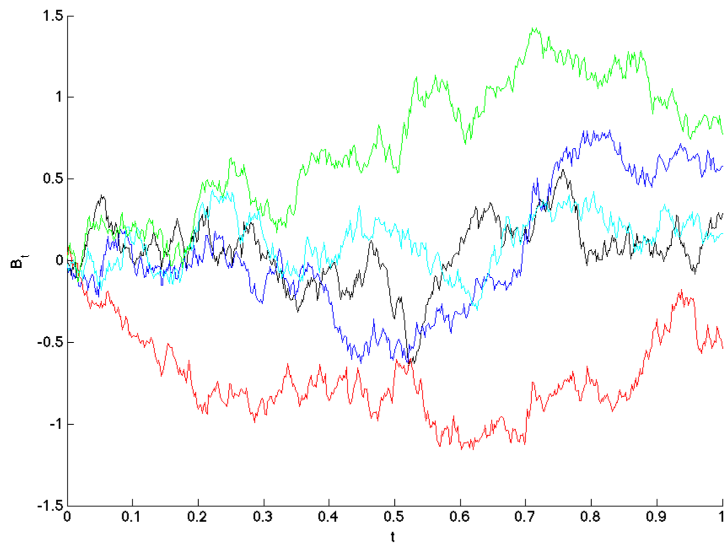

Figure 1.1: Sample Paths of Brownian Motion

the Q-Wiener process could be extended to the case of the time-changed Q-Wiener process, a process that no longer has nice stationary and independent increments. The key throughout is to take advantage of the martingale property of the time-changed Q-Wiener process.

In Chapter 2, the time-changed Q-Wiener process is introduced and its properties explored. In particular, it is shown that the time-changed Q-Wiener process is still a martingale with respect to the proper filtration. The associated quadratic variation and increasing processes are also discussed. In Chapter 3, we describe the class of integrands for which the Itô stochastic integral with respect to the time-changed Q-Wiener process exists. Moreover, we show that for these integrands, the stochastic integral is itself a martingale, and there is an Itô isometry that allows us to evaluate the second moment of the norm of the stochastic integral. This result is followed by two change of variable formulas, an Itô formula, and a duality result. Finally, we give conditions under which some SDEs driven by the time-changed Q-Wiener process have unique solutions.

One of the important results in Chapter 3 is determining the existence and uniqueness of a mild solution to a kind of SDE driven by the time-changed Q-Wiener process. Chapter 4 focuses on trying to get stability results for this solution and some of the problems that arise. Since finding a result in the Hilbert space case has remained elusive, the stability results are for the one-dimensional version of the SDE. A general Gronwall inequality is established for the mild solution composed with a different kind of time change. Finally, asymptotic and exponential stability of this time-changed mild solution is discussed.

Chapter 5 focuses on the strong solution to SDEs driven by the time-changed Q-Wiener process. The goal here is to look at the associated Fokker-Planck-Kolmogorov (FPK) equations, where the solutions are measures induced by the time-changed Q-Wiener processes. Because of the inclusion of the time change, the FPK equations include time-fractional pseudo-differential operators. Two different techniques are used in order to justify the form of these FPK equations.

Finally, Chapter 6 provides foundational work that should be useful in connecting SDEs driven by a time-changed Q-Wiener process with SPDEs. A direct connection has not yet been established, but some headway is made by giving conditions under which the integral with respect to a time-changed Q-Wiener process is equal to both the integral with respect to a time-changed cylindrical Wiener process and the integral with respect to a time-changed martingale measure.

Chapter 2

The Time-Changed Q-Wiener Process

The fundamental object of study in this thesis is the time-changed Q-Wiener process. The goal of this chapter is to introduce this object, give some motivation for its construction, and provide some results regarding its properties.

We begin by looking at the Q-Wiener process, without any time change. It is a Hilbert space-valued object that can be viewed as an infinite-dimensional analog of Brownian motion. Formally, it satisfies the following definition.

Definition 2.0.1 *[14] Let Q be a nonnegative definite, symmetric, trace-class operator on a separable Hilbert space K, let $\{f_j\}_{j=1}^{\infty}$ be an orthonormal basis in K diagonalizing Q, and denote the corresponding eigenvalues by $\{\lambda_j\}_{j=1}^{\infty}$. Let $\{w_j(t)\}_{t\geq 0}, j = 1, 2, \cdots$, be a sequence of independent Brownian motions defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Then the process*

$$W_t := \sum_{j=1}^{\infty} \lambda_j^{1/2} w_j(t) f_j$$

is called a Q-Wiener process in K.

Therefore, conceptually, the Q-Wiener process is constructed by placing independent Brownian motions in each of the orthogonal basis directions, along with a $\lambda_j^{1/2}$ coefficient that allows for the convergence of the sum. For more details on the properties of Q-Wiener processes, see [8, 14].

Before introducing the time-changed Q-Wiener process, we must explain what is meant by a time change in this context. The time change of interest for the bulk of this work is the first hitting time process of a β-stable subordinator defined as

$$E_t := E_\beta(t) = \inf\{\tau > 0 : U_\beta(\tau) > t\}, \tag{2.1}$$

where $U_\beta(t)$ is the β-stable subordinator which has index $\beta \in (0, 1)$ and Laplace transform

$$\mathbb{E}(e^{-uU_\beta(\tau)}) = e^{-\tau u^\beta}. \tag{2.2}$$

E_t is also called the inverse β-stable subordinator.

It is natural to wonder why we might be interested in the E_t process and not, for example, its inverse $U_\beta(t)$. Though they are both increasing processes that have self-similarity properties, the two behave very differently. $U_\beta(t)$ is a Lévy process, whose paths are strictly increasing and contain jumps. E_t, on the other hand, is no longer a Lévy process, as it does not have independent or stationary increments. (See [26] for argument.) However, it does have continuous paths with constant periods. Since the paths of E_t contain these "flat" regions that correspond to jumps in the $U_\beta(t)$ process, it follows that E_t, in some sense, grows at a slower rate than regular time t, and in particular, $\mathbb{E}(E_t) = \frac{t^\beta}{\Gamma(\beta+1)}$. Thus, it is natural to consider a process that has been time-changed by E_t if it is evolving according to a slower "clock." (See Figure 2.1 for a sample path of this time change.) For more information on stable subordinators and inverse stable subordinators, see [2] and [28], respectively.

Since we are interested in time-changing the Q-Wiener process, which is comprised of infinitely many Brownian motions, it is helpful to first consider a single time-changed Brownian motion. Let $B(t)$ denote a one-dimensional standard Brownian motion. Recall the following definition:

Definition 2.0.2 *A stochastic process X_t on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a square integrable martingale with respect to the filtration $\{\mathcal{F}_t\}$ if $\mathbb{E}(X_t^2) < \infty$ and $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$.*

It is a fact that $B(t)$ is a square integrable martingale with respect to its natural filtration. Now consider $Z_\beta(t) := B(E_t)$, a subordinated Brownian motion which has been time-changed by E_t. (Again, see Figure 2.1 for a sample path of this process. Notice that "flats" in E_t correspond to "flats" in $B(E_t)$.) One can recover the martingale property for this time-changed process. The following result of Magdziarz [25] shows that $Z_\beta(t)$ is a square integrable martingale with respect to the appropriate

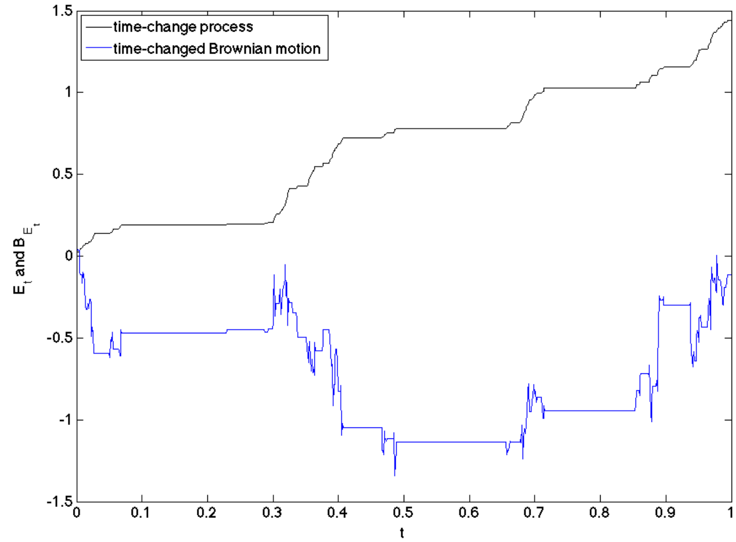

Figure 2.1: Sample Paths of the Time Change E_t and a Time-Changed Brownian Motion B_{E_t} with $\beta = 0.7$. (Image provided courtesy of Kei Kobayashi.)

right-continuous filtration,

$$\hat{\mathcal{F}}_t = \bigcap_{u>t}\{\sigma[B(s) : 0 \le s \le u] \vee \sigma[E_s : s \ge 0]\}, \tag{2.3}$$

where $\hat{\mathcal{F}}_0$ is assumed to be complete.

Theorem 2.0.3 *[25] The time-changed Brownian motion $Z_\beta(t)$ is a mean zero and square integrable martingale with respect to the filtration $\{\hat{\mathcal{F}}_{E_t}\}_{t\ge0}$. The quadratic variation process of $Z_\beta(t)$ is $\langle Z_\beta(t), Z_\beta(t)\rangle = E_t$.*

By incorporating the time-change E_t into the independent Brownian motions in Definition 2.0.1, we can now define a Hilbert space-valued time-changed Q-Wiener process as follows:

Definition 2.0.4 *Let Q be a nonnegative definite, symmetric, trace-class operator on a separable Hilbert space K, let $\{f_j\}_{j=1}^\infty$ be an orthonormal basis in K diagonalizing Q, and let the corresponding eigenvalues be $\{\lambda_j\}_{j=1}^\infty$. Let $\{w_j(t)\}_{t\ge0}, j = 1, 2, \cdots$, be a sequence of independent Brownian motions defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ which are* *independent of E_t. Then the process*

$$W_{E_t} := \sum_{j=1}^{\infty} \lambda_j^{1/2} w_j(E_t) f_j \tag{2.4}$$

is called a time-changed Q-Wiener process in K.

Whereas a one-dimensional Brownian motion $B(t)$ is a diffusion process, its corresponding time-changed version $Z_\beta(t)$ is a sub-diffusion process. Similarly, the time-changed Q-Wiener process is a sub-diffusion process in Hilbert space. Let μ_t be the Borel probability measure induced by the time-changed Q-Wiener process W_{E_t} on K, i.e., $\mathbb{E}(W_{E_t}) = \int_K x\mu_t(dx)$. Then the time-fractional Fokker-Planck-Kolmogorov (FPK) equation corresponding to the time-changed Q-Wiener process in the following theorem can be considered as a special case of Theorem 5.0.5 in Chapter 5.

Theorem 2.0.5 *Suppose μ_t is the probability measure induced by the time-changed Q-Wiener process W_{E_t} on K. Then μ_t satisfies the following time-fractional PDE*

$$D_t^\beta \mu_t = D_x^2 \mu_t,$$

where D_x^2 denotes the second-order Fréchet derivative in space and D_t^β is the Caputo time fractional derivative operator defined as

$$D_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\beta} \mathrm{d}\tau,$$

where β is the index associated with the β-stable subordinator $U_\beta(t)$ and $\Gamma(\beta)$ is the gamma function.

For more on the connection between sub-diffusions and fractional Fokker-Planck equations, see [30, 31].

However, before discussing FPK equations corresponding to the time-changed Q-Wiener process, it makes sense to first define integrals with respect to this process and then consider stochastic differential equations (SDEs) driven by it. Now, some of the "nice" properties of the non-time-changed Q-Wiener process break down with the inclusion of the time change E_t. For example, W_{E_t}, unlike W_t, no longer has independent or stationary increments. One property of W_t that can be recovered, though, even under an E_t time change, is the martingale property. Of course, it should be noted that one needs to be careful about which filtration is being used.

Before showing that the time-changed Q-Wiener process is a martingale with respect to an appropriate filtration, let us recall the definition of a martingale in a Hilbert space.

Definition 2.0.6 *[14] Let K be a separable Hilbert space endowed with its Borel σ-field $\mathcal{B}(K)$. Fix $T > 0$ and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, \mathbb{P})$ be a filtered probability space and $\{M_t\}_{t\leq T}$ be a K-valued process adapted to the filtration $\{\mathcal{F}_t\}_{t\leq T}$. Assume that M_t is integrable, i.e., $\mathbb{E}\|M_t\| < \infty$.*

a) If for any $0 \leq s \leq t$, $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$, $\mathbb{P}$-a.s., then M_t is called an $\mathcal{F}_t$-martingale.

b) If $\mathbb{E}\|M_T\|^2 < \infty$, the martingale M_t is called square integrable on $0 \leq t \leq T$.

Let $\mathcal{M}_T^2(K)$ denote the collection of all continuously square integrable martingales in the Hilbert space K. Note that the condition for $\{M_t\}_{t<T}$ to be a martingale in Hilbert space is equivalent to the following: for all $s < t$ and $h \in K$,

$$\mathbb{E}(\langle M_t, h\rangle|\mathcal{F}_s) = \langle M_s, h\rangle, \ \mathbb{P} - a.s.$$

Example 2.0.7 As mentioned before, the Q-Wiener process $W_t := \sum_{j=1}^{\infty} \lambda_j^{1/2} w_j(t) f_j$ introduced in Definition 2.0.1 is a square integrable martingale with respect to the filtration $\{\mathcal{F}_t\}$ generated by independent Brownian motions, i.e., $\sigma(w_j(s) : 0 \leq s \leq t, j = 1, 2, \cdots)$. ◊

The following theorem will establish that the time-changed Q-Wiener process is also a square-integrable martingale, i.e. $W_{E_t} \in \mathcal{M}_T^2(K)$ with respect to the appropriate filtration, which is a generalization of (2.3), defined as

$$\tilde{\mathcal{F}}_t = \bigcap_{u>t} \{\sigma[w_j(s) : 0 \leq s \leq u, j = 1, 2, ...] \vee \sigma[E_s : s \geq 0]\}, \tag{2.5}$$

where $\tilde{\mathcal{F}}_0$ is assumed to be complete and $\{w_j(t)\}_{t\geq 0}, j = 1, 2, \cdots$, is a sequence of independent Brownian motions which are independent of E_t.

Theorem 2.0.8 *(Chlebak, Garmirian, Wu) A time-changed Q-Wiener process defined by Definition 2.0.4 is a K-valued square integrable martingale with respect to the filtration $\mathcal{G}_t = \tilde{\mathcal{F}}_{E_t}$.*

Proof: From [25], all moments of the time change E_t are finite, i.e.

$$\mathbb{E}(E_t^n) = \frac{t^{n\beta} n!}{\Gamma(n\beta + 1)},$$

for $n = 1, 2, \cdots$. Let f_{E_t} be the density function for E_t. Then, compute the second moment for the time-changed Brownian motions $w_j(E_t)$,

$$\mathbb{E}(w_j^2(E_t)) = \int_0^\infty \mathbb{E}(w_j^2(\tau)) f_{E_t}(\tau) \mathrm{d}\tau = \int_0^\infty \tau f_{E_t}(\tau) \mathrm{d}\tau = \frac{t^\beta}{\Gamma(\beta + 1)}.$$

Thus,

$$\begin{aligned}\mathbb{E}\|W_{E_t}\|_K^2 &= \mathbb{E}\left\langle \sum_{j=1}^\infty \lambda_j^{1/2} w_j(E_t) f_j, \sum_{i=1}^\infty \lambda_i^{1/2} w_i(E_t) f_i \right\rangle_K = \mathbb{E} \sum_{j=1}^\infty \lambda_j w_j^2(E_t) \\ &= \sum_{j=1}^\infty \lambda_j \mathbb{E}(w_j^2(E_t)) = \frac{t^\beta}{\Gamma(\beta + 1)} \sum_{j=1}^\infty \lambda_j < \infty.\end{aligned}$$

The sum is finite since Q is a trace-class operator. Also, since the Q-Wiener process W_t is a square integrable martingale in the Hilbert space K, then for any $h \in K$, the process X_t defined by

$$X_t := \langle W_t, h \rangle_K$$

is a real-valued square integrable martingale with respect to the filtration $\tilde{\mathcal{F}}_t$. This means that in order to prove the time-changed Q-Wiener process W_{E_t} is a square integrable martingale, it suffices to verify that the time-changed real-valued process

X_{E_t} defined by

$$X_{E_t} := \langle W_{E_t}, h\rangle_K,$$

is a square integrable martingale with respect to the filtration $\mathcal{G}_t = \tilde{\mathcal{F}}_{E_t}$. Define the sequence of $\{\tilde{\mathcal{F}}_\tau\}$-stopping times

$$T_n = \inf\{\tau > 0 : |X(\tau)| \geq n\}.$$

It is known that the stopped process $X(T_n \wedge \tau)$ is a bounded martingale with respect to $\tilde{\mathcal{F}}_\tau$. Thus, by Doob's optional sampling theorem,

$$\mathbb{E}(X(T_n \wedge E_t) \mid \mathcal{G}_s) = X(T_n \wedge E_s) \tag{2.6}$$

for $s < t$. The right hand side of (2.6) converges to $X(E_s)$ as $n \to \infty$. For the left hand side,

$$|X(T_n \wedge E_t)| \leq \sup_{0\leq s\leq t} |X(E_s)|.$$

Then, applying Doob's Maximum inequality yields

$$\begin{aligned}
\mathbb{E}(\sup_{0\leq s\leq t}|X(E_s)|^2) &= \mathbb{E}(\sup_{0\leq s\leq E_t}|X(s)|^2) \\
&= \int_0^\infty \mathbb{E}(\sup_{0\leq s\leq \tau}|X(s)|^2 \mid E_t = \tau) f_{E_t}(\tau)\mathrm{d}\tau \\
&= \int_0^\infty \mathbb{E}(\sup_{0\leq s\leq \tau}|X(s)|^2) f_{E_t}(\tau)\mathrm{d}\tau \leq 4\int_0^\infty \mathbb{E}(|X(\tau)|^2) f_{E_t}(\tau)\mathrm{d}\tau \\
&= 4\int_0^\infty \mathbb{E}(|\langle W_\tau, h\rangle_K|^2) f_{E_t}(\tau)\mathrm{d}\tau \leq 4\int_0^\infty \mathbb{E}(\|W_\tau\|_K^2\|h\|_K^2) f_{E_t}(\tau)\mathrm{d}\tau \\
&= 4\|h\|_K^2 \int_0^\infty \mathbb{E}\|W_\tau\|^2 f_{E_t}(\tau)d\tau = 4\|h\|_K^2\mathbb{E}\|W_{E_t}\|_K^2 < \infty.
\end{aligned}$$

Also by Holder's inequality,

$$\mathbb{E}(\sup_{0\leq s\leq t}|X_{E_s}|) \leq (\mathbb{E}(\sup_{0\leq s\leq t}|X_{E_s}|)^2)^{1/2}.$$

Thus,

$$\mathbb{E}(\sup_{0\le s\le t} |X_{E_s}|) < \infty.$$

By the dominated convergence theorem,

$$\mathbb{E}(X(T_n \wedge E_t) \mid \mathcal{G}_s) \longrightarrow \mathbb{E}(X(E_t) \mid \mathcal{G}_s), \text{ as } n \to \infty.$$

Therefore, from (2.6),

$$\mathbb{E}(X(E_t) \mid \mathcal{G}_s) = X(E_s),$$

which implies that $X(E_t) = \langle W_{E_t}, h\rangle_K$ is a martingale with respect to the filtration $\tilde{\mathcal{F}}_{E_t}$. Therefore, W_{E_t} is a square integrable martingale in the Hilbert space K. □

Since W_{E_t} is a square integrable martingale, we can find its associated increasing and quadratic variation processes. These will become useful in later chapters. Recall the following definition and lemma concerning the increasing processes and the quadratic variation processes of square integrable martingales in a Hilbert space. These will be applied to the time-changed Q-Wiener process W_{E_t} in Proposition 2.0.11.

Definition 2.0.9 *[14] Let $M_t \in \mathcal{M}_T^2$. Denote by $\langle M\rangle_t$ the unique adapted continuous increasing process starting from 0 such that $\|M_t\|_K^2 - \langle M\rangle_t$ is a continuous martingale. The quadratic variation process $\langle\langle M\rangle\rangle_t$ of M_t is an adapted continuous process starting from 0, with values in the space of nonnegative definite trace-class operators on K, such that for all $h, g \in K$,*

$$\langle M_t, h\rangle_K \langle M_t, g\rangle_K - \big\langle \langle\langle M\rangle\rangle_t(h), g\big\rangle_K$$

is a martingale.

Lemma 2.0.10 *[14] The quadratic variation process of a martingale $M_t \in \mathcal{M}_T^2$*

exists and is unique. Moreover,

$$\langle M \rangle_t = tr(\langle\langle M \rangle\rangle_t).$$

Proposition 2.0.11 *(Chlebak, Garmirian, Wu) The increasing process and quadratic variation process of the time-changed Q-Wiener process in Definition 2.0.4 are respectively*

$$\langle W_E \rangle_t = tr(Q)E_t \text{ and } \langle\langle W_E \rangle\rangle_t = QE_t.$$

Proof: Let Q be a nonnegative definite, symmetric, trace-class operator on a separable Hilbert space K and let $\{f_j\}_{j=1}^{\infty}$ be an orthonormal basis in K diagonalizing Q with corresponding eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$. Then,

$$\begin{aligned} \|W_{E_t}\|_K^2 &= \left\langle \sum_{j=1}^{\infty} \lambda_j^{1/2} w_j(E_t) f_j, \sum_{i=1}^{\infty} \lambda_i^{1/2} w_i(E_t) f_i \right\rangle_K \\ &= \sum_{j=1}^{\infty} \lambda_j w_j^2(E_t). \end{aligned}$$

On the other hand,

$$tr(Q)E_t = E_t \sum_{j=1}^{\infty} \lambda_j.$$

Define the process N_{E_t} as

$$N_{E_t} := \|W_{E_t}\|_K^2 - tr(Q)E_t = \sum_{j=1}^{\infty} \lambda_j (w_j^2(E_t) - E_t),$$

which can be considered as a time-change of N_t where

$$N_t = \|W_t\|_K^2 - tr(Q)t = \sum_{j=1}^{\infty} \lambda_j (w_j^2(t) - t).$$

From Definition 2.0.9, N_t is a real-valued martingale since $tr(Q)t$ is the unique increasing process of the Q-Wiener process. Similarly to Theorem 2.0.8, the time-changed process N_{E_t} is a martingale. Furthermore, W_{E_t} is a martingale and there is an increasing process $\langle W_E \rangle_t$ such that $\|W_{E_t}\|_K^2 - \langle W_E \rangle_t$ is a martingale. Finally, since $\|W_{E_t}\|_K^2$ is a real-valued submartingale, from uniqueness of the Doob-Meyer decomposition [33],

$$\langle W_E \rangle_t = tr(Q)E_t. \tag{2.7}$$

Again, from Theorem 2.0.8 and Lemma 2.0.10, the quadratic process $\langle\langle W_E \rangle\rangle_t$ of the time-changed Q-Wiener process W_{E_t} exists, is unique, and satisfies

$$tr\big(\langle\langle W_E \rangle\rangle_t\big) = \langle W_E \rangle_t. \tag{2.8}$$

Therefore, from (2.7) and (2.8),

$$\langle\langle W_E \rangle\rangle_t = QE_t,$$

thus completing the proof. □

Chapter 3

SDEs Driven by the Time-Changed Q-Wiener Process

In this chapter, we begin by developing the Itô stochastic integral with respect to the time-changed Q-Wiener process in Hilbert space. Then, the following results are proven: the time-changed Itô isometry, two change of variable formulas, the time-changed Itô formula, a mild solution result, a duality result, and a strong solution result. The time-changed Itô isometry will be a useful tool in Chapter 4 where the stability of a mild solution to an SDE that contains a time change is discussed. The main purpose of the two change of variable formulas is to generate the duality result, which in turn is used to get the strong solution result. The time-changed Itô formula will be used in Chapter 5 to help justify the form of the FPK equations associated with given strong solutions to SDEs that include a time change. These results were joint work between P. Garmirian, Q. Wu, and myself.

Although not explicitly mentioned in this chapter, other consulted references concerning stochastic calculus with martingales include [20, 29].

3.1 Stochastic integral with respect to the time-changed Q-Wiener process

In order to construct an Itô stochastic integral with respect to the time-changed Q-Wiener process, we briefly recall Itô stochastic integrals with respect to a Q-Wiener process without a time change as in [8, 14].

Let K and H be two separable Hilbert spaces, and Q be a symmetric, nonnegative definite trace-class operator on K. Let $\{f_j\}_{j=1}^{\infty}$ be an orthonormal basis (ONB) in K such that $Qf_j = \lambda_j f_j$, where these eigenvalues $\lambda_j > 0$ for all $j = 1, 2, \cdots$. Then, the separable Hilbert space $K_Q = Q^{1/2}K$ with an ONB $\{\lambda_j^{1/2} f_j\}_{j=1}^{\infty}$ is endowed with the following scalar product

$$\langle u, v\rangle_{K_Q} = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \langle u, f_j\rangle_K \langle v, f_j\rangle_K.$$

Let $\mathcal{L}_2(K_Q, H)$ be the space of Hilbert-Schmidt operators from K_Q to H. The Hilbert-Schmidt norm of an operator $L \in \mathcal{L}_2(K_Q, H)$ is given by

$$\|L\|^2_{\mathcal{L}_2(K_Q,H)} = \|LQ^{1/2}\|^2_{\mathcal{L}_2(K,H)} = tr((LQ^{1/2})(LQ^{1/2})^*).$$

The scalar product between two operators $L, M \in \mathcal{L}_2(K_Q, H)$ is defined by

$$\langle L, M\rangle_{\mathcal{L}_2(K_Q,H)} = tr((LQ^{1/2})(MQ^{1/2})^*).$$

Define $\Lambda_2(K_Q, H)$ as the class of $\mathcal{L}_2(K_Q, H)$-valued processes which are measurable mappings from $([0,T] \times \Omega, \mathcal{B}([0,T]) \times \mathcal{F})$ to $(\mathcal{L}_2(K_Q, H), \mathcal{B}(\mathcal{L}_2(K_Q, H)))$ adapted to the filtration $\{\mathcal{F}_t\}_{t \le T}$, and satisfying the condition

$$\mathbb{E} \int_0^T \|\Phi(t)\|^2_{\mathcal{L}_2(K_Q,H)} dt < \infty.$$

Note that $\Lambda_2(K_Q, H)$ is a Hilbert space if it is equipped with the norm

$$\|\Phi\|_{\Lambda_2(K_Q,H)} := \left(\mathbb{E} \int_0^T \|\Phi(t)\|^2_{\mathcal{L}_2(K_Q,H)} dt\right)^{1/2}.$$

The following lemma from [14] can be considered as a definition of a stochastic integral with respect to the Q-Wiener process:

Lemma 3.1.1 *[14] Let W_t be a Q-Wiener process in a separable Hilbert space K, $\Phi \in \Lambda_2(K_Q, H)$, and $\{f_j\}_{j=1}^{\infty}$ be an ONB in K consisting of eigenvectors of Q. Then,*

$$\int_0^t \Phi(s) \mathrm{d}W_s = \sum_{j=1}^{\infty} \int_0^t (\Phi(s)\lambda_j^{1/2} f_j) \mathrm{d}\langle W_s, \lambda_j^{1/2} f_j\rangle_{K_Q}.$$

In order to incorporate the time-change E_t into the Itô stochastic integral, we need to consider a different class of integrands.

Definition 3.1.2 *Let $\tilde{\Lambda}_2(K_Q, H)$ denote the class of $\mathcal{L}_2(K_Q, H)$-valued processes which are measurable mappings from $([0,T] \times \Omega, \mathcal{B}([0,T]) \times \mathcal{G})$ to $(\mathcal{L}_2(K_Q, H), \mathcal{B}(\mathcal{L}_2(K_Q, H)))$ adapted to the filtration $\{\mathcal{G}_t := \tilde{\mathcal{F}}_{E_t}\}_{t \le T}$, and satisfying*

$$\mathbb{E} \int_0^T \|\Phi(t)\|^2_{\mathcal{L}_2(K_Q,H)} dE_t < \infty.$$

Then $\tilde{\Lambda}_2(K_Q, H)$ can be viewed as a separable Hilbert space when equipped with the norm

$$\|\Phi\|_{\tilde{\Lambda}_2(K_Q,H)} := \left(\mathbb{E} \int_0^T \|\Phi(t)\|^2_{\mathcal{L}_2(K_Q,H)} dE_t \right)^{1/2}.$$

Thus, Itô stochastic integrals with respect to the time-changed Q-Wiener process can be introduced.

Definition 3.1.3 *Let W_{E_t} be a time-changed Q-Wiener process in a separable Hilbert space K, $\Phi \in \tilde{\Lambda}_2(K_Q, H)$, and let $\{f_j\}_{j=1}^{\infty}$ be an ONB in K consisting of eigenvectors of Q. Then,*

$$\int_0^t \Phi(s) \mathrm{d}W_{E_s} = \sum_{j=1}^{\infty} \int_0^t (\Phi(s) \lambda_j^{1/2} f_j) \mathrm{d}\langle W_{E_s}, \lambda_j^{1/2} f_j \rangle_{K_Q}.$$

Now that the Itô integral with respect to the time-changed Q-Wiener process has been established, the next step is to derive the Itô isometry. The general idea is to first prove the result for elementary processes as described in the following definition.

Definition 3.1.4 *Let $\mathcal{E}(\mathcal{L}(K, H))$ denote the class of bounded elementary processes that are $\{\mathcal{G}_t\}$-adapted and of the form*

$$\Phi(t, \omega) = \phi(\omega) 1_{\{0\}}(t) + \sum_{j=0}^{n-1} \phi_j(\omega) 1_{(t_j, t_{j+1}]}(t), \tag{3.1}$$

where $0 \le t_0 \le t_1 \le \cdots \le t_n = T$ and ϕ, ϕ_j, $j = 0, 1, \cdots, n-1$ are respectively $\mathcal{G}_0$-measurable and $\mathcal{G}_{t_j}$-measurable $\mathcal{L}_2(K_Q, H)$-valued random variables such that $\phi(\omega)$ and $\phi_j(\omega)$ are linear, bounded operators from K to H.

Since $\mathcal{E}(\mathcal{L}(K, H))$ is a dense subset of $\tilde{\Lambda}_2(K_Q, H)$, which is a complete space, we can use elementary processes to approximate an arbitrary operator $\Phi \in \tilde{\Lambda}_2(K_Q, H)$.

Thus, the idea behind showing the time-changed Itô isometry is to first prove that it holds for elementary processes and then extend the result to general $\Phi \in \tilde{\Lambda}_2(K_Q, H)$. Before proceeding to the proof of the Itô isometry for an elementary process $\Phi(t,\omega)$, we need to show the following useful lemma.

Lemma 3.1.5 *Let* $\{f_j\}_{j=1}^{\infty}$ *be an ONB in* K *consisting of eigenvectors of* Q *and* $\mathcal{G}_s = \tilde{\mathcal{F}}_{E_s}$ *be the filtration used. Then, for* $\ell \neq \ell'$ *and* $t > s > 0$,

$$\mathbb{E}\left(\mathbb{E}\left(\left\langle W_{E_t} - W_{E_s}, f_\ell\right\rangle_K \left\langle W_{E_t} - W_{E_s}, f_{\ell'}\right\rangle_K \middle| \mathcal{G}_s\right)\right) = 0.$$

Proof: From Definition 2.0.4 in Chapter 2,

$$W_{E_t} = \sum_{j=1}^{\infty} \lambda_j^{1/2} w_j(E_t) f_j.$$

Multiplying the terms and using the definition of W_{E_t},

$$\begin{aligned}
&\mathbb{E}\left(\mathbb{E}\left(\left\langle W_{E_t} - W_{E_s}, f_\ell\right\rangle_K \left\langle W_{E_t} - W_{E_s}, f_{\ell'}\right\rangle_K \middle| \mathcal{G}_s\right)\right)\\
&= \mathbb{E}\Big\{\mathbb{E}\Big(\left\langle W_{E_t}, f_\ell\right\rangle_K \left\langle W_{E_t}, f_{\ell'}\right\rangle_K - \left\langle W_{E_s}, f_\ell\right\rangle_K \left\langle W_{E_t}, f_{\ell'}\right\rangle_K\\
&\quad - \left\langle W_{E_s}, f_{\ell'}\right\rangle_K \left\langle W_{E_t}, f_\ell\right\rangle_K + \left\langle W_{E_s}, f_\ell\right\rangle_K \left\langle W_{E_s}, f_{\ell'}\right\rangle_K \Big| \mathcal{G}_s\Big)\Big\}\\
&= \mathbb{E}\left(\mathbb{E}\left(\lambda_\ell^{1/2} w_\ell(E_t) \lambda_{\ell'}^{1/2} w_{\ell'}(E_t) | \mathcal{G}_s\right)\right) - \mathbb{E}\left(\mathbb{E}\left(\lambda_\ell^{1/2} w_\ell(E_s) \lambda_{\ell'}^{1/2} w_{\ell'}(E_t) | \mathcal{G}_s\right)\right)\\
&\quad - \mathbb{E}\left(\mathbb{E}\left(\lambda_{\ell'}^{1/2} w_{\ell'}(E_s) \lambda_\ell^{1/2} w_\ell(E_t) | \mathcal{G}_s\right)\right) + \mathbb{E}\left(\lambda_\ell^{1/2} w_\ell(E_s) \lambda_{\ell'}^{1/2} w_{\ell'}(E_s)\right)\\
&:= I_1 - I_2 - I_3 + I_4.
\end{aligned}$$

Since w_ℓ is independent of $w_{\ell'}$, conditioning on E_t to compute the first term yields

$$\begin{aligned}
I_1 &= \mathbb{E}\left(\mathbb{E}\left(\lambda_\ell^{1/2} w_\ell(E_t) \lambda_{\ell'}^{1/2} w_{\ell'}(E_t) | \mathcal{G}_s\right)\right)\\
&= \mathbb{E}(\lambda_\ell^{1/2} w_\ell(E_t) \lambda_{\ell'}^{1/2} w_{\ell'}(E_t))\\
&= \lambda_\ell^{1/2} \lambda_{\ell'}^{1/2} \int_0^\infty \mathbb{E}(w_\ell(\tau) w_{\ell'}(\tau)) f_{E_t}(\tau) \mathrm{d}\tau\\
&= \lambda_\ell^{1/2} \lambda_{\ell'}^{1/2} \int_0^\infty 0 \cdot f_{E_t}(\tau) \mathrm{d}\tau = 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned} I_2 &= \mathbb{E}\Bigg(\mathbb{E}\Bigg(\lambda_\ell^{1/2} w_\ell(E_s)\lambda_{\ell'}^{1/2} w_{\ell'}(E_t)\Bigg|\mathcal{G}_s\Bigg)\Bigg) \\ &= \lambda_\ell^{1/2}\lambda_{\ell'}^{1/2}\mathbb{E}\Bigg(\mathbb{E}\Bigg(w_\ell(E_s)w_{\ell'}(E_t)\Bigg|\mathcal{G}_s\Bigg)\Bigg) \\ &= \lambda_\ell^{1/2}\lambda_{\ell'}^{1/2}\mathbb{E}\Bigg(\mathbb{E}\Bigg(w_\ell(E_s)\Big(w_{\ell'}(E_t) - w_{\ell'}(E_s)\Big) + w_\ell(E_s)w_{\ell'}(E_s)\Bigg|\mathcal{G}_s\Bigg)\Bigg) \\ &= \lambda_\ell^{1/2}\lambda_{\ell'}^{1/2}\mathbb{E}\Bigg(w_\ell(E_s)\mathbb{E}\Bigg(w_{\ell'}(E_t) - w_{\ell'}(E_s)\Bigg|\mathcal{G}_s\Bigg) + w_{\ell'}(E_s)w_\ell(E_s)\Bigg) \\ &= 0 \end{aligned}$$

since $\mathbb{E}(w_{\ell'}(E_t) - w_{\ell'}(E_s)|\mathcal{G}_s) = 0$ by the martingale property of W_{E_t}, and by the same conditioning argument previously used in computing term I_1, $\mathbb{E}(w_{\ell'}(E_s)w_\ell(E_s)) = 0$. Similarly, the third term, I_3, and the fourth term, I_4, are also equal to 0. The proof is complete. □

Theorem 3.1.6 *(Chlebak, Garmirian, Wu) Let $\Phi \in \mathcal{E}(\mathcal{L}(K,H))$ be a bounded elementary process. Then, for $t \in [0,T]$,*

$$\mathbb{E}\left\| \int_0^t \Phi(s)dW_{E_s}\right\|_H^2 = \mathbb{E}\int_0^t \|\Phi(s)\|^2_{\mathcal{L}_2(K_Q,H)} dE_s < \infty.$$

Proof: First, without loss of generality, assume that $t = T$. Then, for the bounded elementary process, Φ, defined in (3.1),

$$\begin{aligned} \mathbb{E}\left\| \int_0^T \Phi(s)dW_{E_s}\right\|_H^2 &= \mathbb{E}\left\| \sum_{j=0}^{n-1} \phi_j(W_{E_{t_{j+1}}} - W_{E_{t_j}})\right\|_H^2 \\ &= \sum_{j=0}^{n-1} \mathbb{E}\left\| \phi_j(W_{E_{t_{j+1}}} - W_{E_{t_j}})\right\|_H^2 \\ &\quad + \sum_{i\neq j=0}^{n-1} \mathbb{E}\Big\langle \phi_j(W_{E_{t_{j+1}}} - W_{E_{t_j}}), \phi_i(W_{E_{t_{i+1}}} - W_{E_{t_i}})\Big\rangle_H \\ &:= I + II. \end{aligned}$$

Let $\{e_m\}_{m=1}^{\infty}$ in H and $\{f_\ell\}_{\ell=1}^{\infty}$ in K be ONBs. For fixed j, I_j is denoted by

$$\begin{aligned}
I_j &= \mathbb{E}\left\|\phi_j(W_{E_{t_{j+1}}} - W_{E_{t_j}})\right\|_H^2 = \mathbb{E}\sum_{m=1}^{\infty}\left\langle\phi_j(W_{E_{t_{j+1}}} - W_{E_{t_j}}), e_m\right\rangle_H^2 \\
&= \sum_{m=1}^{\infty}\mathbb{E}\left(\mathbb{E}\big(\langle\phi_j(W_{E_{t_{j+1}}} - W_{E_{t_j}}), e_m\rangle_H^2 | \mathcal{G}_{t_j}\big)\right) \\
&= \sum_{m=1}^{\infty}\mathbb{E}\left(\mathbb{E}\left(\left\langle W_{E_{t_{j+1}}} - W_{E_{t_j}}, \phi_j^* e_m\right\rangle_K^2 \middle| \mathcal{G}_{t_j}\right)\right) \\
&= \sum_{m=1}^{\infty}\mathbb{E}\left(\mathbb{E}\left(\left(\sum_{\ell=1}^{\infty}\left\langle W_{E_{t_{j+1}}} - W_{E_{t_j}}, f_\ell\right\rangle_K\left\langle\phi_j^* e_m, f_\ell\right\rangle_K\right)^2 \middle| \mathcal{G}_{t_j}\right)\right) \\
&= \sum_{m=1}^{\infty}\mathbb{E}\left(\mathbb{E}\left(\left(\sum_{\ell=1}^{\infty}\left\langle W_{E_{t_{j+1}}} - W_{E_{t_j}}, f_\ell\right\rangle_K^2\left\langle\phi_j^* e_m, f_\ell\right\rangle_K^2\right) \middle| \mathcal{G}_{t_j}\right)\right) \\
&\quad + \sum_{m=1}^{\infty}\mathbb{E}\Bigg(\mathbb{E}\Bigg(\Bigg(\sum_{\ell\neq\ell'=1}^{\infty}\langle W_{E_{t_{j+1}}} - W_{E_{t_j}}, f_\ell\rangle_K\langle\phi_j^* e_m, f_\ell\rangle_K \\
&\quad \times \langle W_{E_{t_{j+1}}} - W_{E_{t_j}}, f_{\ell'}\rangle_K\langle\phi_j^* e_m, f_{\ell'}\rangle_K\Bigg)\Bigg|\mathcal{G}_{t_j}\Bigg)\Bigg) \\
&:= J_1 + J_2.
\end{aligned}$$

Since ϕ_j^* is $\mathcal{G}_{t_j}$-measurable and $W_{E_{t_{j+1}}}$ is a discrete martingale with respect to $\mathcal{G}_{t_j}$, the first term, J_1, becomes

$$\begin{aligned}
J_1 &= \sum_{m=1}^{\infty}\mathbb{E}\left(\sum_{\ell=1}^{\infty}\left\langle\phi_j^* e_m, f_\ell\right\rangle_K^2\mathbb{E}\left(\left\langle W_{E_{t_{j+1}}} - W_{E_{t_j}}, f_\ell\right\rangle_K^2\middle|\mathcal{G}_{t_j}\right)\right) \\
&= \sum_{m=1}^{\infty}\mathbb{E}\left\{\sum_{\ell=1}^{\infty}\left\langle\phi_j^* e_m, f_\ell\right\rangle_K^2\left(\mathbb{E}\left(\left\langle W_{E_{t_{j+1}}}, f_\ell\right\rangle_K^2\middle|\mathcal{G}_{t_j}\right) - \left\langle W_{E_{t_j}}, f_\ell\right\rangle_K^2\right)\right\} \\
&= \sum_{m=1}^{\infty}\mathbb{E}\left\{\sum_{\ell=1}^{\infty}\left\langle\phi_j^* e_m, f_\ell\right\rangle_K^2\left(\mathbb{E}\left(\left\langle W_{E_{t_{j+1}}}, f_\ell\right\rangle_K^2\middle|\mathcal{G}_{t_j}\right)\right\} \\
&\quad - \sum_{m=1}^{\infty}\mathbb{E}\left\{\sum_{\ell=1}^{\infty}\left\langle\phi_j^* e_m, f_\ell\right\rangle_K^2\left\langle W_{E_{t_j}}, f_\ell\right\rangle_K^2\right)\right\} \\
&= \mathbb{E}\left\{\sum_{m=1}^{\infty}\left(E_{t_{j+1}} - E_{t_j}\right)\sum_{\ell=1}^{\infty}\lambda_\ell\left\langle\phi_j^* e_m, f_\ell\right\rangle_K^2\right\} \\
&= \mathbb{E}\left\{\left(E_{t_{j+1}} - E_{t_j}\right)\sum_{m,\ell=1}^{\infty}\left\langle\phi_j(\lambda_\ell^{1/2} f_\ell), e_m\right\rangle_H^2\right\} \\
&= \mathbb{E}\left\{\left(E_{t_{j+1}} - E_{t_j}\right)\|\phi_j\|_{\mathcal{L}_2(K_Q,H)}^2\right\}.
\end{aligned}$$

Also using the $\mathcal{G}_{t_j}$-measurability of ϕ_j^* and Lemma 3.1.5, the second term, J_2, becomes

$$J_2 = \sum_{m=1}^{\infty} \mathbb{E}\Bigg\{ \sum_{\ell \neq \ell' = 1}^{\infty} \Big\langle \phi_j^* e_m, f_\ell \Big\rangle_K \Big\langle \phi_j^* e_m, f_{\ell'} \Big\rangle_K \\ \times \mathbb{E}\Bigg(\Big\langle W_{E_{t_{j+1}}} - W_{E_{t_j}}, f_\ell \Big\rangle_K \Big\langle W_{E_{t_{j+1}}} - W_{E_{t_j}}, f_{\ell'} \Big\rangle_K \Bigg| \mathcal{G}_{t_j} \Bigg) \Bigg\} = 0.$$

Thus,

$$I = \sum_{j=0}^{n-1} I_j = \sum_{j=0}^{n-1} \mathbb{E}\Bigg\{ \Big(E_{t_{j+1}} - E_{t_j} \Big) \|\phi_j\|^2_{\mathcal{L}_2(K_Q,H)} \Bigg\} \\ = \mathbb{E} \int_0^T \|\Phi(s)\|^2_{\mathcal{L}_2(K_Q,H)} dE_s < \infty.$$

On the other hand, without loss of generality, assume that $i < j$. From Lemma 3.1.5,

$$II = \mathbb{E} \sum_{m=1}^{\infty} \mathbb{E}\Bigg(\sum_{\ell,\ell'=1}^{\infty} \Big\langle W_{E_{t_{j+1}}} - W_{E_{t_j}}, f_\ell \Big\rangle_K \Big\langle \phi_j^* e_m, f_\ell \Big\rangle_K \\ \times \Big\langle W_{E_{t_{i+1}}} - W_{E_{t_i}}, f_{\ell'} \Big\rangle_K \Big\langle \phi_i^* e_m, f_{\ell'} \Big\rangle_K \Bigg| \mathcal{G}_{t_j} \Bigg) = 0.$$

Therefore, the desired result is derived. □

Theorem 3.1.7 *(Time-changed Itô Isometry) For $t \in [0,T]$, the stochastic integral $\Phi \to \int_0^t \Phi(s) \mathrm{d}W_{E_s}$ with respect to a K-valued time-changed Q-Wiener process W_{E_t} is an isometry between $\tilde{\Lambda}_2(K_Q,H)$ and its image in the space of continuous square-integrable martingales $\mathcal{M}_T^2(H)$, i.e.,*

$$\mathbb{E}\Bigg\| \int_0^t \Phi(s) \mathrm{d}W_{E_s} \Bigg\|_H^2 = \mathbb{E} \int_0^t \|\Phi(s)\|^2_{\mathcal{L}_2(K_Q,H)} \mathrm{d}E_s < \infty. \tag{3.2}$$

Proof: For elementary processes $\Phi \in \mathcal{E}(\mathcal{L}(K,H)$, Theorem 3.1.6 establishes the desired equality (3.2) and thus the square-integrability of the integral $\int_0^t \Phi(s) \mathrm{d}W_{E_s}$.

Furthermore, since the time-changed Q-Wiener process, W_{E_t}, is a K-valued martingale, for any $h \in H$ and $s < t$,

$$\begin{aligned}
\mathbb{E}\left(\left\langle \int_0^t \Phi(r)dW_{E_r}, h\right\rangle_H \middle| \mathcal{G}_s\right) &= \mathbb{E}\left(\left\langle \sum_{j=0}^{n-1} \phi_j(W_{E_{t_{j+1}\wedge t}} - W_{E_{t_j\wedge t}}), h\right\rangle_H \middle| \mathcal{G}_s\right) \\
&= \sum_{j=0}^{n-1} \mathbb{E}\left(\left\langle W_{E_{t_{j+1}\wedge t}} - W_{E_{t_j\wedge t}}, \phi^*(h)\right\rangle_K \middle| \mathcal{G}_s\right) \\
&= \sum_{j=0}^{n-1} \left\langle W_{E_{t_{j+1}\wedge s}} - W_{E_{t_j\wedge s}}, \phi^*(h)\right\rangle_K \\
&= \left\langle \sum_{j=0}^{n-1} \phi(W_{E_{t_{j+1}\wedge s}} - W_{E_{t_j\wedge s}}), h\right\rangle_H = \left\langle \int_0^s \Phi(s)dW_{E_s}, h\right\rangle_H,
\end{aligned}$$

which implies that the stochastic integral $\int_0^t \Phi(s)\mathrm{d}W_{E_s}$ is a square-integrable martingale. Therefore, the desired result holds when $\Phi(s)$ is an elementary process.

Now, let $\{\Phi_n\}_{n=1}^{\infty}$ be a sequence of elementary processes approximating $\Phi \in \tilde{\Lambda}_2(K_Q, H)$. Assume that $\Phi_1 = 0$ and $\|\Phi_{n+1} - \Phi_n\|_{\tilde{\Lambda}_2(K_Q,H)} < \frac{1}{2^n}$. Then, by Doob's maximal inequality,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathbb{P}\left(\sup_{t\leq T}\left\| \int_0^t \Phi_{n+1}(s)dW_{E_s} - \int_0^t \Phi_n(s)dW_{E_s}\right\|_H > \frac{1}{n^2}\right) \\
&\leq \sum_{n=1}^{\infty} n^4 \mathbb{E}\left\| \int_0^T (\Phi_{n+1}(s) - \Phi_n(s))dW_{E_s}\right\|_H^2 \\
&= \sum_{n=1}^{\infty} n^4 \mathbb{E}\int_0^T \|\Phi_{n+1}(s) - \Phi_n(s)\|^2_{\mathcal{L}(K_Q,H)} dE_s \\
&\leq \frac{T^\beta}{\Gamma(\beta+1)} \sum_{n=1}^{\infty} \frac{n^4}{2^n} < \infty.
\end{aligned}$$

By Borel-Cantelli, it follows that for some $k(\omega) > 0$,

$$\sup_{t\leq T}\left\| \int_0^t \Phi_{n+1}(s)dW_{E_s} - \int_0^t \Phi_n(s)dW_{E_s}\right\|_H \leq \frac{1}{n^2}, \quad n > k(\omega),$$

holds $\mathbb{P}$-almost surely. Therefore, for every $t \leq T$,

$$\sum_{n=1}^{\infty}\left(\int_0^t \Phi_{n+1}(s)dW_{E_s} - \int_0^t \Phi_n(s)dW_{E_s}\right) \to \int_0^t \Phi(s)dW_{E_s} \text{ in } L^2(\Omega, H),$$

which also converges $\mathbb{P}$-almost surely to a continuous version of the integral.

So, the map $\Phi \to \int_0^t \Phi(s) dW_{E_s}$, viewed as an isometry from elementary processes to the space of continuous square-integrable martingales, has an extension to $\Phi \in \tilde{\Lambda}_2(K_Q, H)$ by the completeness property of H. □

The following two change of variable formulas concern the Itô stochastic integral related to the time-change E_t. They are needed later and can be considered as the Hilbert space extensions of formulas in [22].

Theorem 3.1.8 *(1st change of variable formula) Let W_t be a Q-Wiener process in a separable Hilbert space K, $\Phi \in \tilde{\Lambda}_2(K_Q, H)$, and E_t be the inverse of a β-stable subordinator. Then, with probability one, for all $t \geq 0$,*

$$\int_0^{E_t} \Phi(s) \mathrm{d}W_s = \int_0^t \Phi(E_s) \mathrm{d}W_{E_s}.$$

Proof: Let $\{f_j\}_{j=1}^{\infty}$ be an ONB in the separable Hilbert space K consisting of eigenvectors of Q. Then, it follows from Lemma 3.1.1 that

$$\int_0^{E_t} \Phi(s) \mathrm{d}W_s = \sum_{j=1}^{\infty} \int_0^{E_t} (\Phi(s) \lambda_j^{1/2} f_j) \mathrm{d}\langle W_s, \lambda_j^{1/2} f_j \rangle_{K_Q}.$$

For any $h \in H$, applying the first change of variable formula of the real-valued stochastic integral from [22] for the third equality below yields

$$\begin{aligned}
\left\langle \int_0^{E_t} \Phi(s) \mathrm{d}W_s, h \right\rangle_H &= \left\langle \sum_{j=1}^{\infty} \int_0^{E_t} (\Phi(s) \lambda_j^{1/2} f_j) \mathrm{d}\langle W_s, \lambda_j^{1/2} f_j \rangle_{K_Q}, h \right\rangle_H \\
&= \sum_{j=1}^{\infty} \int_0^{E_t} \langle (\Phi(s) \lambda_j^{1/2} f_j), h \rangle_H \mathrm{d}\langle W_s, \lambda_j^{1/2} f_j \rangle_{K_Q} \\
&= \sum_{j=1}^{\infty} \int_0^{t} \langle (\Phi(E_s) \lambda_j^{1/2} f_j), h \rangle_H \mathrm{d}\langle W_{E_s}, \lambda_j^{1/2} f_j \rangle_{K_Q} \\
&= \left\langle \sum_{j=1}^{\infty} \int_0^{t} (\Phi(E_s) \lambda_j^{1/2} f_j) \mathrm{d}\langle W_{E_s}, \lambda_j^{1/2} f_j \rangle_{K_Q}, h \right\rangle_H \\
&= \left\langle \int_0^{t} \Phi(E_s) \mathrm{d}W_{E_s}, h \right\rangle_H.
\end{aligned}$$

Theorem 3.1.9 *(2nd change of variable formula) Let W_t be a Q-Wiener process in a separable Hilbert space K and $\Phi \in \tilde{\Lambda}_2(K_Q, H)$. Let U_t be a β-stable subordinator* *with* $\beta \in (0,1)$ *and* E_t *be its inverse stable subordinator. Then, with probability one, for all* $t \geq 0$,

$$\int_0^t \Phi(s)\mathrm{d}W_{E_s} = \int_0^{E_t} \Phi(U_{s-})\mathrm{d}W_s.$$

Proof: Let $\{f_j\}_{j=1}^{\infty}$ be an ONB in the separable Hilbert space K consisting of eigenvectors of Q. Applying Definition 3.1.3 yields,

$$\int_0^t \Phi(s)\mathrm{d}W_{E_s} = \sum_{j=1}^{\infty} \int_0^t (\Phi(s)\lambda_j^{1/2} f_j)\mathrm{d}\langle W_{E_s}, \lambda_j^{1/2} f_j\rangle_{K_Q}.$$

For any $h \in H$, applying Definition 3.1.3 in the first equality and using the second change of variable formula for real-valued stochastic integrals from [22] in the third equality below yield

$$\begin{aligned}
\left\langle \int_0^t \Phi(s)\mathrm{d}W_{E_s}, h \right\rangle_H &= \left\langle \sum_{j=1}^{\infty} \int_0^t (\Phi(s)\lambda_j^{1/2} f_j)\mathrm{d}\langle W_{E_s}, \lambda_j^{1/2} f_j\rangle_{K_Q}, h \right\rangle_H \\
&= \sum_{j=1}^{\infty} \int_0^t \langle(\Phi(s)\lambda_j^{1/2} f_j), h\rangle_H \mathrm{d}\langle W_{E_s}, \lambda_j^{1/2} f_j\rangle_{K_Q} \\
&= \sum_{j=1}^{\infty} \int_0^{E_t} \langle(\Phi(U_{s-})\lambda_j^{1/2} f_j), h\rangle_H \mathrm{d}\langle W_s, \lambda_j^{1/2} f_j\rangle_{K_Q} \\
&= \left\langle \sum_{j=1}^{\infty} \int_0^{E_t} (\Phi(U_{s-})\lambda_j^{1/2} f_j)\mathrm{d}\langle W_s, \lambda_j^{1/2} f_j\rangle_{K_Q}, h \right\rangle_H \\
&= \left\langle \int_0^{E_t} \Phi(U_{s-})\mathrm{d}W_s, h \right\rangle_H.
\end{aligned}$$

This completes the proof. □

3.2 Time-changed Itô formula in Hilbert space

The technique used to develop the time-changed Itô formula in this section is inspired by the proof of the standard Itô formula of Theorem 2.9 in [14].

Theorem 3.2.1 *(Time-changed Itô formula) Let* Q *be a symmetric, nonnegative definite trace-class operator on a separable Hilbert space* K*, and let* $\{W_{E_t}\}_{0\leq t\leq T}$ *be a time-changed* Q*-Wiener process on a filtered probability space* $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{0\leq t\leq T}, \mathbb{P})$.

Assume that a stochastic process $X(t)$ *is given by*

$$X(t) = X(0) + \int_0^t \psi(s)\mathrm{d}s + \int_0^t \gamma(s)\mathrm{d}E_s + \int_0^t \phi(s)\mathrm{d}W_{E_s},$$

where $X(0)$ *is a* $\mathcal{G}_0$*-measurable* H*-valued random variable,* $\psi(s)$ *and* $\gamma(s)$ *are* H*-valued* $\mathcal{G}_s$*-measurable* $\mathbb{P}$*-a.s. integrable processes on* $[0,T]$ *such that*

$$\int_0^T \|\psi(s)\|_H \mathrm{d}s < \infty \qquad \textit{and} \qquad \int_0^T \|\gamma(s)\|_H \mathrm{d}E_s < \infty,$$

and $\phi \in \tilde{\Lambda}_2(K_Q, H)$*. Also assume that* $F : H \to \mathbb{R}$ *is continuous and its Fréchet derivatives* $F_x : H \to \mathcal{L}(H,\mathbb{R})$ *and* $F_{xx} : H \to \mathcal{L}(H,\mathcal{L}(H,\mathbb{R}))$ *are continuous and bounded on bounded subsets of* H*. Then,*

$$\begin{aligned} F(X(t)) - F(X(0)) = &\int_0^t \big\langle F_x(X(s)), \psi(s)\big\rangle_H \mathrm{d}s \\ &+ \int_0^{E_t} \big\langle F_x(X(U(s-))), \gamma(U(s-))\big\rangle_H \mathrm{d}s \\ &+ \int_0^{E_t} \big\langle F_x(X(U(s-))), \phi(U(s-))\mathrm{d}W_s\big\rangle_H \\ &+ \frac{1}{2}\int_0^{E_t} tr(F_{xx}(X(U(s-)))(\phi(U(s-))Q^{1/2})(\phi(U(s-))Q^{1/2})^*)\mathrm{d}s, \end{aligned}$$

$\mathbb{P}$*-a.s. for all* $t \in [0,T]$*.*

Proof: First, the desired Itô formula is reduced to the case where $\psi(s) = \psi$, $\gamma(s) = \gamma$, and $\phi(s) = \phi$ are constant processes for $s \in [0,T]$. Let $C > 0$ be a constant, and define the stopping time

$$\tau_C = \inf\Bigg\{t \in [0,T] : \max\Bigg(\|X(t)\|_H, \int_0^t \|\phi(s)\|_H \mathrm{d}s, \int_0^t \|\gamma(s)\|_H \mathrm{d}E_s, \\ \int_0^t \|\phi(s)\|^2_{\mathcal{L}_2(K_Q,H)} \mathrm{d}E_s\Bigg) \ge C\Bigg\}.$$

Then, define $X_C(t)$ as

$$X_C(t) = X_C(0) + \int_0^t \psi_C(s)\mathrm{d}s + \int_0^t \gamma_C(s)\mathrm{d}E_s + \int_0^t \phi_C(s)\mathrm{d}W_{E_s},\ t \in [0,T],$$

where $X_C(t) = X(t\wedge\tau_C), \psi_C(t) = \psi(t)1_{[0,\tau_C]}(t), \gamma_C(t) = \gamma(t)1_{[0,\tau_C]}(t)$, and $\phi_C(t) =$

$\phi(t)1_{[0,\tau_C]}(t)$. It is enough to prove the Itô formula for the processes stopped at τ_C. Since

$$\mathbb{P}\bigg(\int_0^T \|\psi_C(s)\|_H \mathrm{d}s < \infty\bigg) = 1, \quad \mathbb{P}\bigg(\int_0^T \|\gamma_C(s)\|_H \mathrm{d}E_s < \infty\bigg) = 1, \quad \text{and}$$
$$\mathbb{E}\int_0^T \|\phi_C(s)\|^2_{\mathcal{L}_2(K_Q,H)} \mathrm{d}E_s < \infty,$$

it follows that ψ_C, γ_C, and ϕ_C can be approximated respectively by sequences of bounded elementary processes $\psi_{C,n}$, $\gamma_{C,n}$, and $\phi_{C,n}$ such that as $n \to \infty$

$$\begin{aligned} &\int_0^t \|\psi_{C,n}(s) - \psi_C(s)\|_H \mathrm{d}s \to 0, \quad \int_0^t \|\gamma_{C,n}(s) - \gamma_C(s)\|_H \mathrm{d}E_s \to 0, \quad \text{and} \\ &\bigg\|\int_0^t \phi_{C,n}(s) dW_{E_s} - \int_0^t \phi_C(s) \mathrm{d}W_{E_s}\bigg\|_H \to 0, \quad \mathbb{P}-\text{a.s.} \end{aligned} \tag{3.3}$$

uniformly in $t \in [0,T]$. Let

$$X_{C,n}(t) = X(0) + \int_0^t \psi_{C,n}(s)\mathrm{d}s + \int_0^t \gamma_{C,n}(s)\mathrm{d}E_s + \int_0^t \phi_{C,n}(s)\mathrm{d}W_{E_s}.$$

Then, as $n \to \infty$

$$\sup_{t\le T} \|X_{C,n}(t) - X_C(t)\|_H \to 0, \ \mathbb{P}-\text{a.s.}. \tag{3.4}$$

Assume the Itô formula for $X_{C,n}(t)$ holds $\mathbb{P}$-a.s. for all $t \in [0,T]$, i.e.,

$$\begin{aligned} F(X_{C,n}(t)) - F(X(0)) &= \int_0^t \langle F_x(X_{C,n}(s)), \phi_{C,n}(s)\mathrm{d}W_{E_s}\rangle_H \\ &+ \int_0^t \langle F_x(X_{C,n}(s)), \psi_{C,n}(s)\rangle_H \mathrm{d}s + \int_0^t \langle F_x(X_{C,n}(s)), \gamma_{C,n}(s)\rangle_H \mathrm{d}E_s \\ &+ \int_0^t \frac{1}{2} tr[F_{xx}(X_{C,n}(s))(\phi_{C,n}(s)Q^{1/2})(\phi_{C,n}(s)Q^{1/2})^*]\mathrm{d}E_s \\ &:= I^1_{C,n} + I^2_{C,n} + I^3_{C,n} + I^4_{C,n}. \end{aligned} \tag{3.5}$$

By using the continuity of F and the continuity and local boundedness of F_x and

F_{xx}, we need to show that the following holds $\mathbb{P}$-a.s. for all $t \in [0, T]$, i.e.,

$$\begin{aligned} F(X_C(t)) - F(X(0)) &= \int_0^t \langle F_x(X_C(s)), \phi_C(s) \mathrm{d}W_{E_s} \rangle_H \\ &+ \int_0^t \langle F_x(X_C(s)), \psi_C(s) \rangle_H \mathrm{d}s + \int_0^t \langle F_x(X_C(s)), \gamma_C(s) \rangle_H \mathrm{d}E_s \\ &+ \frac{1}{2} \int_0^t tr[F_{xx}(X_C(s))(\phi_C(s) Q^{1/2})(\phi_C(s) Q^{1/2})^*] \mathrm{d}E_s \\ &:= I_C^1 + I_C^2 + I_C^3 + I_C^4. \end{aligned} \tag{3.6}$$

Consider term by term the difference between both sides of (3.5) and (3.6). Due to the continuity of F and almost sure convergence in (3.4), the left hand side of (3.5) converges to the left hand side of (3.6) $\mathbb{P}$-a.s for all $t \le T$, i.e.,

$$F(X_{C,n}(t)) \to F(X_C(t)), \quad \mathbb{P}-\text{a.s. as } n \to \infty. \tag{3.7}$$

Turn to the first terms in both right hand sides of (3.5) and (3.6),

$$\begin{aligned} \mathbb{E}|I_{C,n}^1 - I_C^1|^2 &= \mathbb{E}\left| \int_0^t \left(\phi_{C,n}^*(s) F_x(X_{C,n}(s)) - \phi_C^*(s) F_x(X_C(s)) \right) \mathrm{d}W_{E_s} \right|^2 \\ &\le 2\mathbb{E} \int_0^t \|(\phi_{C,n}^*(s) - \phi_C^*(s)) F_x(X_{C,n}(s))\|_{\mathcal{L}_2(K_Q,\mathbb{R})}^2 \mathrm{d}E_s \\ &\quad + 2\mathbb{E} \int_0^t \|\phi_C^*(s)(F_x(X_{C,n}(s)) - F_x(X_C(s)))\|_{\mathcal{L}_2(K_Q,\mathbb{R})}^2 \mathrm{d}E_s \\ &\le 2\mathbb{E} \int_0^t \left(\|\phi_C^*(s) - \phi_{C,n}^*\|_{\mathcal{L}_2(K_Q,H)}^2 \|F_x(X_{C,n}(s)\|_H^2 \right) \mathrm{d}E_s \\ &\quad + 2\mathbb{E} \int_0^t \left(\|\phi_C^*(s)\|_{\mathcal{L}_2(K_Q,H)}^2 \|F_x(X_{C,n}(s)) - F_x(X_C(s))\|_H^2 \right) \mathrm{d}E_s \\ &:= J_1 + J_2, \end{aligned}$$

where $\int_0^t \beta^*(s)\alpha(s)\mathrm{d}W_{E_s} := \int_0^t \langle \alpha(s), \beta(s)\mathrm{d}W_{E_s} \rangle_H$ and $\beta^*(s)$ is the adjoint operator of $\beta(s)$. Since F_x is bounded on bounded subsets of H, there exists an $M > 0$ such that the first term J_1 is bounded,

$$J_1 \le M\mathbb{E} \int_0^t \|\phi_C^*(s) - \phi_{C,n}^*\|_{\mathcal{L}^2(K_Q,H)}^2 \mathrm{d}E_s \to 0, \text{ as } n \to \infty.$$

For the second term, J_2, since $\phi_C(s)$ is square integrable in the space $\tilde{\Lambda}_2(K_Q, H)$

and F_x is bounded in H, $J_2 \to 0$ by applying the Lebesgue dominated convergence theorem. So, $I^1_{C,n}$ converges to I^1_C in mean square, i.e.,

$$\mathbb{E}|I^1_{C,n} - I^1_C|^2 \to 0, \tag{3.8}$$

and thus converges in probability. For the second terms, $I^2_{C,n}$ and I^2_C, of RHSs of (3.5) and (3.6), applying the conditions of (3.3) and (3.4) leads to

$$\begin{aligned} I^2_{C,n} - I^2_C = \int_0^t \Bigg(\Big\langle F_x(X_{C,n}(s)) - F_x(X_C(s)), \psi_{C,n}(s) \Big\rangle_H \\ + \Big\langle F_x(X_C(s)), \psi_{C,n}(s) - \psi_C(s) \Big\rangle_H \Bigg) \mathrm{d}s \to 0, \quad \mathbb{P}-\text{a.s.}. \end{aligned} \tag{3.9}$$

Similarly, for the third terms, $I^3_{C,n}$ and I^3_C, of RHSs of (3.5) and (3.6),

$$\begin{aligned} I^3_{C,n} - I^3_C = \int_0^t \Bigg(\Big\langle F_x(X_{C,n}(s)) - F_x(X_C(s)), \gamma_{C,n}(s) \Big\rangle_H \\ + \Big\langle F_x(X_C(s)), \gamma_{C,n}(s) - \gamma_C(s) \Big\rangle_H \Bigg) \mathrm{d}E_s \to 0, \quad \mathbb{P}-\text{a.s.}. \end{aligned} \tag{3.10}$$

Before proceeding to the fourth terms, $I^4_{C,n}$ and I^4_C, of RHSs of (3.5) and (3.6), note that

$$\|\phi_{C,n}(s) - \phi_C(s)\|_{\tilde{\Lambda}_2(K_Q,H)} \to 0,$$

which means there exists a subsequence n_k such that for all $s \le T$

$$\|\phi_{C,n_k}(s) - \phi_C(s)\|_{\mathcal{L}_2(K_Q,H)} \to 0, \quad \mathbb{P}-\text{a.s.}.$$

Thus, for the eigenvectors $\{f_j\}_{j=1}^{\infty}$ of Q and all $t \le T$,

$$\|\phi_{C,n_k}(s) f_j - \phi_C(s) f_j\|_H \to 0, \ \mathbb{P}-\text{a.s.} \tag{3.11}$$

On the other hand, for the ONB, $\{f_j\}_{j=1}^{\infty}$, in the Hilbert space K,

$$\begin{aligned}
&tr(F_{xx}(X_{C,n_k}(s))\phi_{C,n_k}(s)Q\phi^*_{C,n_k}(s))\\
&\qquad = tr(\phi^*_{C,n_k}(s)F_{xx}(X_{C,n_k}(s))\phi_{C,n_k}(s)Q)\\
&\qquad = \sum_{j=1}^{\infty}\lambda_j\langle F_{xx}(X_{C,n_k}(s))\phi_{C,n_k}(s)f_j,\phi_{C,n_k}(s)f_j\rangle_H,
\end{aligned}$$

where λ_j is the eigenvalue associated with eigenvector f_j of Q. Since $X_{C,n_k}(s)$ is bounded and F_{xx} is continuous, (3.11) implies that for $s \le T$

$$\begin{aligned}
&\langle F_{xx}(X_{C,n_k}(s))\phi_{C,n_k}(s)f_j,\phi_{C,n_k}(s)f_j\rangle_H\\
&\qquad\qquad \to \langle F_{xx}(X_C(s))\phi_C(s)f_j,\phi_C(s)f_j\rangle_H,\ \mathbb{P}-a.s..
\end{aligned}$$

By the Lebesgue dominated convergence theorem (DCT) (with respect to the counting measure), it holds a.e. on $[0,T]\times\Omega$ that

$$\begin{aligned}
&tr(F_{xx}(X_{C,n_k}(s))\phi_{C,n_k}(s)Q\phi^*_{C,n_k}(s))\\
&\qquad\qquad \to tr(F_{xx}(X_C(s))\phi_C(s)Q\phi^*_C(s)).
\end{aligned}\tag{3.12}$$

Moreover, the left hand side of (3.12) is bounded above by

$$\eta_n(s) := \|F_{xx}(X_{C,n_k}(s))\|_{\mathcal{L}(H)}\|\phi_{C,n_k}\|^2_{\tilde{\Lambda}_2(K_Q,H)},$$

and a.e. on $[0,T]\times\Omega$

$$\eta_n(s) \to \eta(s) = \|F_{xx}(X_C(s))\|_{\mathcal{L}(H)}\|\phi_C\|^2_{\tilde{\Lambda}_2(K_Q,H)}.$$

So, by the boundedness of F_{xx}, $\int_0^t \eta_n(s)dE_s \to \int_0^t \eta(s)dE_s$ and by the general Lebesgue DCT, it holds $\mathbb{P}$-a.s. that for $t \le T$

$$\begin{aligned}
I^4_{C,n_k} - I^4_C = &\int_0^t \frac{1}{2}tr[F_{xx}(X_{C,n_k}(s))(\phi_{C,n_k}(s)Q^{1/2})(\phi_{C,n_k}(s)Q^{1/2})^*]\mathrm{d}E_s\\
&-\int_0^t \frac{1}{2}tr[F_{xx}(X_C(s))(\phi_C(s)Q^{1/2})(\phi_C(s)Q^{1/2})^*]\mathrm{d}E_s \to 0.
\end{aligned}\tag{3.13}$$

Therefore, from (3.7), (3.8), (3.9), (3.10) and (3.13), the Itô formula for the process $X_{C,n}(t)$, (3.5), converges in probability to the Itô formula for the process $X_C(t)$, (3.6) and possibly for a subsequence, n_k, converges $\mathbb{P}$-a.s.

Second, the proof can be reduced to the case where

$$X(t) = X(0) + \psi t + \gamma E_t + \phi W_{E_t}$$

where ψ,γ, and ϕ are $\mathcal{G}_0$-measurable random variables independent of t. Define the function $u(t_1, t_2, x) : \mathbb{R}_+ \times \mathbb{R}_+ \times H \to \mathbb{R}$ as

$$u(t, E_t, W_{E_t}) = F(X(0) + \psi t + \gamma E_t + \phi W_{E_t}) = F(X(t)).$$

Now, we prove that the Itô formula holds for the function $u(t_1, t_2, x)$. First, let $0 = t_1 < t_2 < ... < t_n = t \le T$ be a partition of an interval $[0, t]$, then

$$\begin{aligned} u(t, E_t, W_{E_t}) - u(0,0,0) = &\sum_{j=1}^{n-1} [u(t_{j+1}, E_{t_{j+1}}, W_{E_{t_{j+1}}}) - u(t_j, E_{t_{j+1}}, W_{E_{t_{j+1}}})] \\ &+ \sum_{j=1}^{n-1} [u(t_j, E_{t_{j+1}}, W_{E_{t_{j+1}}}) - u(t_j, E_{t_j}, W_{E_{t_{j+1}}})] \\ &+ \sum_{j=1}^{n-1} [u(t_j, E_{t_j}, W_{E_{t_{j+1}}}) - u(t_j, E_{t_j}, W_{E_{t_j}})]. \end{aligned}$$

Also, denote $\Delta t_j = t_{j+1} - t_j$, $\Delta E_j = E_{t_{j+1}} - E_{t_j}$ and $\Delta W_j = W_{E_{t_{j+1}}} - W_{E_{t_j}}$. Let $\theta_j \in [0,1]$ be a random variable, and $\bar{t}_j = t_j + \theta_j(t_{j+1} - t_j)$, $\bar{E}_j = E_{t_j} + \theta_j(E_{t_{j+1}} - E_{t_j})$ and $\bar{W}_j = W_{E_{t_j}} + \theta_j(W_{E_{t_{j+1}}} - W_{E_{t_j}})$. Using Taylor's formula,

$$
\begin{aligned}
&u(t,E_t,W_{E_t}) - u(0,0,0)\\
&= \sum_{j=1}^{n-1} u_{t_1}(\bar{t}_j, E_{t_{j+1}}, W_{E_{t_{j+1}}})\Delta t_j + \sum_{j=1}^{n-1} u_{t_2}(t_j, \bar{E}_j, W_{E_{t_{j+1}}})\Delta E_j\\
&+ \sum_{j=1}^{n-1} [\langle u_x(t_j, E_{t_j}, W_{E_{t_j}}), \Delta W_j\rangle_K + \frac{1}{2}\langle u_{xx}(t_j, E_{t_j}, \bar{W}_j)(\Delta W_j), \Delta W_j\rangle_K]\\
&= \sum_{j=1}^{n-1} u_{t_1}(t_j, E_{t_{j+1}}, W_{E_{t_{j+1}}})\Delta t_j\\
&+ \sum_{j=1}^{n-1} u_{t_2}(t_j, E_{t_j}, W_{E_{t_{j+1}}})\Delta E_j + \sum_{j=1}^{n-1}\langle u_x(t_j, E_{t_j}, W_{E_{t_j}}), \Delta W_j\rangle_K\\
&+ \frac{1}{2}\sum_{j=1}^{n-1}\langle u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})(\Delta W_j), \Delta W_j\rangle_K\\
&+ \sum_{j=1}^{n-1}[u_{t_1}(\bar{t}_j, E_{t_{j+1}}, W_{E_{t_{j+1}}}) - u_{t_1}(t_j, E_{t_{j+1}}, W_{E_{t_{j+1}}})]\Delta t_j\\
&+ \sum_{j=1}^{n-1}[u_{t_2}(t_j, \bar{E}_{j+1}, W_{E_{t_{j+1}}}) - u_{t_2}(t_j, E_{t_j}, W_{E_{t_{j+1}}})]\Delta E_j\\
&+ \frac{1}{2}\sum_{j=1}^{n-1}\langle [u_{xx}(t_j, E_{t_j}, \bar{W}_j)(\Delta W_j) - u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})(\Delta W_j)], \Delta W_j\rangle_K\\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned}
\tag{3.14}
$$

By the uniform continuity of the mappings:

$$
[0,T]\times[0,T]\times[0,T] \ni (t,s,r) \to u_{t_1}(t, E_s, W_{E_r}) \in \mathbb{R}
$$
$$
[0,T]\times[0,T]\times[0,T] \ni (t,s,r) \to u_{t_2}(t, E_s, W_{E_r}) \in \mathbb{R}
$$

and the continuity of the map $[0,T] \ni t \to u_x(t, E_t, W_{E_t}) \in K^*$, the following holds $\mathbb{P}$-a.s.

$$
\begin{aligned}
I_1 &= \sum_{j=1}^{n-1} u_{t_1}(t_j, E_{j+1}, W_{E_{t_{j+1}}})\Delta t_j \to \int_0^t u_{t_1}(s, E_s, W_{E_s})\mathrm{d}s,\\
I_2 &= \sum_{j=1}^{n-1} u_{t_2}(t_j, E_j, W_{E_{t_{j+1}}})\Delta E_{t_j} \to \int_0^t u_{t_2}(s, E_s, W_{E_s})\mathrm{d}E_s,\\
I_3 &= \sum_{j=1}^{n-1}\langle u_x(s, E_s, W_{E_{t_j}}), \Delta W_j\rangle_K \to \int_0^t \langle u_x(s, E_s, W_{E_s}), \mathrm{d}W_{E_s}\rangle_K.
\end{aligned}
\tag{3.15}
$$

Also since the time change E_t has bounded variation,

$$\begin{aligned}
|I_5| &= \left|\sum_{j=1}^{n-1}[u_{t_1}(\bar{t}_j, E_{t_{j+1}}, W_{E_{t_{j+1}}}) - u_{t_1}(t_j, E_{t_{j+1}}, W_{E_{t_{j+1}}})]\Delta t_j\right| \\
&\le T \sup_{j\le n-1}\left|u_{t_1}(\bar{t}_j, E_{t_{j+1}}, W_{E_{t_{j+1}}}) - u_{t_1}(t_j, E_{t_{j+1}}, W_{E_{t_{j+1}}})\right| \\
&\to 0, \\
|I_6| &= \left|\sum_{j=1}^{n-1}[u_{t_2}(t_j, \bar{E}_{j+1}, W_{E_{t_{j+1}}}) - u_{t_2}(t_j, E_{t_j}, W_{E_{t_{j+1}}})]\Delta E_j\right| \\
&\le \sum_{j=1}^{n-1}|\Delta E_{t_j}| \sup_{j\le n-1}\left|u_{t_2}(t_j, \bar{E}_{j+1}, W_{E_{t_{j+1}}}) - u_{t_2}(t_j, E_{t_j}, W_{E_{t_{j+1}}})\right| \\
&\to 0.
\end{aligned} \tag{3.16}$$

Similarly, by the continuity of the map $K \ni x \to u_{xx}(t_1, t_2, x) \in \mathcal{L}(K, K)$,

$$\begin{aligned}
|I_7| &= \frac{1}{2}\left|\sum_{j=1}^{n-1}\langle[u_{xx}(t_j, E_{t_j}, \bar{W}_j)(\Delta W_j) \right. \\
&\qquad\qquad \left. - u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})(\Delta W_j)], \Delta W_j\rangle_K\right| \\
&\le \sup_{j\le n-1}\|u_{xx}(t_j, E_{t_j}, \bar{W}_j)(\Delta W_j) - u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})(\Delta W_j)\|_{\mathcal{L}(K,K)} \\
&\quad \times \sum_{j=1}^{n-1}\|\Delta W_j\|_K^2 \to 0
\end{aligned} \tag{3.17}$$

with probability one as $n \to \infty$ since the function u has the same smoothness as F and $\bar{W}_j \to W_{E_{t_j}}$ as the increments $t_{j+1} - t_j$ get smaller. It remains to deal with the fourth term, I_4. Let $1_j^N = 1_{\{\max\{\|W_{E_{t_i}}\|_K \le N,\ i\le j\}\}}$ which is $\mathcal{G}_{t_j}$-measurable. To handle

I_4, the following computations are helpful. First,

$$
\begin{aligned}
&\mathbb{E}\bigg(\langle 1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})(\Delta W_j), \Delta W_j\rangle_K \bigg| \mathcal{G}_{t_j}\bigg) \\
&= \mathbb{E}\Bigg(\Bigg\langle 1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}}) \sum_{k=1}^{\infty} \lambda_k^{1/2}(w_k(E_{t_{j+1}}) - w_k(E_{t_j}))f_k, \\
&\qquad\qquad \sum_{\ell=1}^{\infty} \lambda_\ell^{1/2}(w_\ell(E_{t_{j+1}}) - w_\ell(E_{t_j}))f_\ell\Bigg\rangle_K \Bigg| \mathcal{G}_{t_j}\Bigg) \qquad (3.18) \\
&= \sum_{k=1}^{\infty} \mathbb{E}\bigg(\lambda_k \langle 1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})f_k, f_k\rangle_K (w_k(E_{t_{j+1}}) - w_k(E_{t_j}))^2 \bigg| \mathcal{G}_{t_j}\bigg) \\
&= tr(1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})Q)\Delta E_j.
\end{aligned}
$$

Second, for the cross term arising in the computation below, without loss of generality, assume $i < j$, then

$$
\begin{aligned}
I^N := \mathbb{E}\Bigg\{&\bigg(\Big\langle 1_i^N u_{xx}(t_j, E_{t_j}, W_{E_{t_i}})(\Delta W_i), \Delta W_i\Big\rangle_K \\
&\qquad\qquad - tr\Big(1_i^N u_{xx}(t_j, E_{t_j}, W_{E_{t_i}})Q\Big)\Delta E_i\bigg) \\
&\times \bigg(\Big\langle 1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})(\Delta W_j), \Delta W_j\Big\rangle_K \\
&\qquad\qquad - tr\Big(1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})Q\Big)\Delta E_j\bigg)\Bigg\} \\
= \mathbb{E}\Bigg\{&\bigg(\Big\langle 1_i^N u_{xx}(t_j, E_{t_j}, W_{E_{t_i}})(\Delta W_i), \Delta W_i\Big\rangle_K \\
&\qquad\qquad - tr\Big(1_i^N u_{xx}(t_j, E_{t_j}, W_{E_{t_i}})Q\Big)\Delta E_i\bigg) \\
&\times \mathbb{E}\bigg(\Big\langle 1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})(\Delta W_j), \Delta W_j\Big\rangle_K \\
&\qquad\qquad - tr\Big(1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})Q\Big)\Delta E_j \bigg| \mathcal{G}_{t_{(i+1)}}\bigg)\Bigg\}.
\end{aligned}
$$

Apply the tower property of conditional expectation and (3.18) to yield

$$\begin{aligned} I^N &= \mathbb{E}\Bigg\{\bigg(\Big\langle 1_i^N u_{xx}(t_j, E_{t_j}, W_{E_{t_i}})(\Delta W_i), \Delta W_i\Big\rangle_K \\ &\qquad\qquad - tr\Big(1_i^N u_{xx}(t_j, E_{t_j}, W_{E_{t_i}})Q\Big)\Delta E_i\bigg) \\ &\quad \times \mathbb{E}\bigg(\mathbb{E}\Big(\Big\langle 1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})(\Delta W_j), \Delta W_j\Big\rangle_K | \mathcal{G}_{t_j}\Big) \\ &\qquad\qquad - tr\Big(1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})Q\Big)\Delta E_j \Big| \mathcal{G}_{t_{(i+1)}}\bigg)\Bigg\} \\ &= \mathbb{E}\Bigg\{\bigg(\Big\langle 1_i^N u_{xx}(t_j, E_{t_j}, W_{E_{t_i}})(\Delta W_i), \Delta W_i\Big\rangle_K \\ &\qquad\qquad - tr\Big(1_i^N u_{xx}(t_j, E_{t_j}, W_{E_{t_i}})Q\Big)\Delta E_i\bigg) \\ &\quad \times \mathbb{E}\bigg(\mathbb{E}\Big(tr\Big(1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})Q\Big)\Delta E_j \\ &\qquad\qquad - tr\Big(1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})Q\Big)\Delta E_j \Big| \mathcal{G}_{t_{(i+1)}}\bigg)\Bigg\} \\ &= 0. \end{aligned} \tag{3.19}$$

Third, let $f_{E_{t_{j+1}}, E_{t_j}}(\tau_1, \tau_2)$ be the joint density function of random variables, $E_{t_{j+1}}$ and E_{t_j}. Then, for $t_{j+1} > t_j$, let $\mathbb{D} = \{(\tau_1, \tau_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 \le \tau_2 \le \tau_1\}$,

$$\begin{aligned} &\mathbb{E}\bigg(\Big\langle 1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})(\Delta W_j), \Delta W_j\Big\rangle_K \\ &\qquad\qquad \times tr\Big(1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})Q\Big)\Delta E_j\bigg) \\ &= \iint_{\mathbb{D}} \mathbb{E}\bigg(\Big\langle 1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})(\Delta W_j), \Delta W_j\Big\rangle_K \\ &\qquad\qquad \times tr\Big(1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})Q\Big)\Delta E_j \\ &\qquad\qquad \Big| E_{t_{j+1}} = \tau_1, E_{t_j} = \tau_2\bigg) f_{E_{t_{j+1}}, E_{t_j}}(\tau_1, \tau_2)\mathrm{d}(\tau_1, \tau_2) \\ &= \iint_{\mathbb{D}} \mathbb{E}\bigg(tr\Big(1_j^N u_{xx}(t_j, \tau_2, W_{\tau_2})Q\Big)^2 (\tau_1 - \tau_2)^2\bigg) f_{E_{t_{j+1}}, E_{t_j}}(\tau_1, \tau_2)\mathrm{d}(\tau_1, \tau_2) \\ &= \mathbb{E}\bigg(tr\Big(1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})Q\Big)^2 (\Delta E_{t_j})^2\bigg), \end{aligned} \tag{3.20}$$

Finally, for $t_{j+1} > t_j$

$$\begin{aligned}
\mathbb{E}\|\Delta W_j\|_K^4 &= \mathbb{E}\|W_{E_{t_{j+1}}} - W_{E_{t_j}}\|_K^4 \\
&= \iint_{\mathbb{D}} \mathbb{E}\left(\|W_{\tau_1} - W_{\tau_2}\|_K^4 \,\middle|\, E_{t_{j+1}} = \tau_1, E_{t_j} = \tau_2 \right) \\
&\qquad \times f_{E_{t_{j+1}}, E_{t_j}}(\tau_1, \tau_2) \mathrm{d}(\tau_1, \tau_2) \\
&= \iint_{\mathbb{D}} 3(trQ)^2 (\tau_1 - \tau_2)^2 f_{E_{t_{j+1}}, E_{t_j}}(\tau_1, \tau_2) \mathrm{d}(\tau_1, \tau_2) \\
&= 3(trQ)^2 \mathbb{E}(E_{t_{j+1}} - E_{t_j})^2 = 3(trQ)^2 \mathbb{E}(\Delta E_j)^2.
\end{aligned} \tag{3.21}$$

Noting that u_{xx} is bounded on bounded subsets of H, apply (3.18), (3.19), (3.20) and (3.21) to yield

$$\begin{aligned}
&\mathbb{E}\left(\sum_{j=1}^{n-1} (\langle 1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})(\Delta W_j), \Delta W_j \rangle_K \right. \\
&\qquad\qquad \left. - tr\big(1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})Q\big)\Delta E_j) \right)^2 \\
&= \sum_{j=1}^{n-1} \mathbb{E}\left(\langle 1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})(\Delta W_j), \Delta W_j \rangle_K \right. \\
&\qquad\qquad \left. - tr\big(1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})Q\big)\Delta E_j \right)^2 \\
&\quad + \sum_{i \neq j = 1}^{n-1} \mathbb{E}\left\{ \left(\langle 1_i^N u_{xx}(t_i, E_{t_i}, W_{E_{t_i}})(\Delta W_i), \Delta W_i \rangle_K \right.\right. \\
&\qquad\qquad \left. - tr\big(1_i^N u_{xx}(t_i, E_{t_i}, W_{E_{t_i}})Q\big)\Delta E_i \right) \\
&\quad \times \left(\langle 1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})(\Delta W_j), \Delta W_j \rangle_K \right. \\
&\qquad\qquad \left.\left. - tr\big(1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})Q\big)\Delta E_j \right) \right\} \\
&= \sum_{j=1}^{n-1} \left\{ \mathbb{E}\langle 1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})(\Delta W_j), \Delta W_j \rangle_K^2 \right. \\
&\qquad\qquad \left. - \mathbb{E}\left(tr(1_j^N u_{xx}(t_j, E_{t_j}, W_{E_{t_j}})Q)^2 (\Delta E_j)^2 \right) \right\} \\
&\le \sup_{s \le t, \|h\|_H \le N} |u_{xx}(s, E_s, h)|_{\mathcal{L}(H)}^2 \sum_{j=1}^{n-1} \left(\mathbb{E}\|\Delta W_j\|_K^4 - (trQ)^2 \mathbb{E}(\Delta E_j)^2 \right) \\
&= 2 \sup_{s \le t, \|h\|_H \le N} |u_{xx}(s, E_s, h)|_{\mathcal{L}(H)}^2 (trQ)^2 \mathbb{E} \sum_{j=1}^{n-1} (\Delta E_j)^2 \to 0,
\end{aligned}$$

where the convergence is from the fact that the time-change E_t has finite bounded variation, and Fubini-Tonelli's theorem. Additionally,

$$\mathbb{P}\Bigg(\sum_{j=1}^{n-1}(1-1_j^N)\big\{\big\langle u_{xx}(t_j,E_{t_j},W_{E_{t_j}})(\Delta W_j),\Delta W_j\big\rangle_K - tr\big(1_j^N u_{xx}(t_j,E_{t_j},W_{E_{t_j}})Q\big)\Delta E_j\big\}\neq 0\Bigg)$$
$$\leq \mathbb{P}\Bigg(\sup_{s\leq t}\{\|W_{E_s}\|>N\}\Bigg)\to 0 \text{ as } N\to\infty.$$

Thus, it follows that

$$\begin{aligned} I_4 &= \frac{1}{2}\sum_{j=1}^{n-1}\langle u_{xx}(t_j,E_{t_j},W_{E_{t_j}})(\Delta W_j),\Delta W_j\rangle_K \\ &\to \frac{1}{2}\int_0^t tr[u_{xx}(s,E_s,W_{E_s})Q]\mathrm{d}E_s \end{aligned} \tag{3.22}$$

in probability. Combining (3.15), (3.17) and (3.22), and taking the limit on the right hand side of (3.14) yields the Itô formula for the function $u(W_{E_t})$:

$$\begin{aligned} u(t,E_t,W_{E_t}) &= u(0,0,0) + \int_0^t u_{t_1}(s,E_s,W_{E_s})\mathrm{d}s \\ &+ \int_0^t u_{t_2}(s,E_s,W_{E_s})\mathrm{d}E_s + \frac{1}{2}\int_0^t tr[u_{xx}(s,E_s,W_{E_s})Q]\mathrm{d}E_s \\ &+ \int_0^t \langle u_x(s,E_s,W_{E_s}),dW_{E_s}\rangle_K, \end{aligned} \tag{3.23}$$

in probability. Also note that

$$\begin{aligned} u_{t_1}(t,E_t,k) &= \langle F_x(X(0)+\psi t+\gamma E_t+\phi k),\psi\rangle_H \\ u_{t_2}(t,E_t,k) &= \langle F_x(X(0)+\psi t+\gamma E_t+\phi k),\gamma\rangle_H \\ u_x(t,E_t,k) &= \phi^* F_x(X(0)+\psi t+\gamma E_t+\phi k), \\ u_{xx}(t,E_t,k) &= \phi^* F_{xx}(X(0)+\psi t+\gamma E_t+\phi k)\phi, \end{aligned} \tag{3.24}$$

and

$$tr[F_{xx}(X(s))(\phi Q^{1/2})(\phi Q^{1/2})^*] = tr[\phi^* F_{xx}(X(s))\phi Q]. \tag{3.25}$$

Substitute (3.24) and (3.25) into (3.23) to yield

$$\begin{aligned} F(X(t)) = F(X(0)) &+ \int_0^t \langle F_x(X(s)), \psi(s)\rangle_H \mathrm{d}s \\ &+ \int_0^t \langle F_x(X(s)), \gamma(s)\rangle_H \mathrm{d}E_s + \int_0^t \langle F_x(X(s)), \phi(s)\mathrm{d}W_{E_s}\rangle_H \\ &+ \frac{1}{2}\int_0^t tr(F_{xx}(X(s))(\phi(s)Q^{1/2})(\phi(s)Q^{1/2})^*)\mathrm{d}E_s \end{aligned} \tag{3.26}$$

in probability. Consequently, there is a subsequence such that the equality (3.26) holds almost surely. Therefore applying the second change of variable formula yields the desired result. □

3.3 Stochastic differential equations (SDEs) driven by the time-changed Q-Wiener process

Let K and H be real separable Hilbert spaces, and let $M(t) := W_{E_t}$ be a time-changed K-valued Q-Wiener process on a complete filtered probability space $(\Omega, \mathcal{G}, \{\mathcal{G}\}_{t\le T}, \mathbb{P})$ with the filtration $\mathcal{G}_t = \tilde{\mathcal{F}}_{E_t}$ satisfying the usual conditions. Consider the following type of semilinear SDE driven by the time-changed Q-Wiener process on $[0,T]$ in H:

$$dX(t) = AX(t)\mathrm{d}t + B(t,X)\mathrm{d}M(t) \tag{3.27}$$

with initial condition $X(0) = x_0$, where A is the generator of a C_0-semigroup of operators $\{S(t), t \ge 0\}$.

Theorem 3.3.1 *Assume that the following hypotheses are satisfied:*

1. *$S(t)$ is a contraction C_0-semigroup generated by A.*

2. *$B(t,X) : \mathcal{D}(\mathbb{R}^+, H) \to \tilde{\Lambda}_2(K_Q, H)$ is non-anticipating where $\mathcal{D}(\mathbb{R}^+, H)$ denotes the H-valued cadlag adapted processes with $\mathbb{R}^+$ as the time interval.*

3. *(Local Lipschitz property) For every $r > 0$, there exists a constant $K_r > 0$ such*

that for every $x, y \in \mathcal{D}(\mathbb{R}^+, H)$ *and* $t \geq 0$

$$\|B(t,x) - B(t,y)\|^2_{\mathcal{L}_2(K_Q,H)} \leq K_r \sup_{s<t} \|x(s) - y(s)\|^2_H$$

on the set $\{\omega : \sup_{s<t} \max(\|x(s,\omega)\|_H, \|y(s,\omega)\|_H) \leq r\}$.

4. *(Growth condition) There exists a constant* $K_0 > 0$ *such that for every* $x \in \mathcal{D}(\mathbb{R}^+, H)$ *and* $t \geq 0$,

$$\|B(t,x)\|^2_{\mathcal{L}_2(K_Q,H)} \leq K_0(1 + \sup_{s<t} \|x(s)\|^2_H).$$

5. $\mathbb{E}(\|x_0\|^2_H) < \infty$.

Then, the SDE (3.27) *has a pathwise unique continuous solution of the form*

$$X(t) = S(t)X_0 + \int_0^t S(t-s)B(s,X)dW_{E_s}, \tag{3.28}$$

which is called the mild solution to the SDE.

Remark 3.3.2 Since W_{E_t} is a Hilbert space-valued square integrable martingale, Theorem 3.3.1 follows from [15] where a similar result is provided for a SDE driven by a general Hilbert space-valued martingale. ◊

Consider two types of SDEs in Hilbert space with the same initial condition x_0:

$$\mathrm{d}Y(t) = (AY(t) + F(t, Y(t)))\mathrm{d}t + B(t, Y(t))\mathrm{d}W_t, \tag{3.29}$$

and

$$\mathrm{d}X(t) = (AX(t) + F(E_t, X(t)))\mathrm{d}E_t + B(E_t, X(t))\mathrm{d}W_{E_t}. \tag{3.30}$$

The following theorem establishes a deep connection between the classic SDE (3.29) and the time-changed SDE (3.30).

Theorem 3.3.3 *(Duality) Let U_t be a β-stable subordinator and E_t be the inverse of U_t, which is a finite $\mathcal{G}_t = \tilde{\mathcal{F}}_{E_t}$ measurable time-change.*

- *If an H-valued process $Y(t)$ satisfies SDE (3.29), then the H-valued process $X(t) := Y(E_t)$ satisfies SDE (3.30).*
- *If an H-valued process $X(t)$ satisfies SDE (3.30), then the H-valued process $Y(t) := X(U_{t-})$ satisfies SDE (3.29).*

Proof: First, consider the integral form of SDE (3.29):

$$Y(t) = x_0 + \int_0^t \Big(AY(s) + F(s, Y(s))\Big)\mathrm{d}s + \int_0^t B(s, Y(s))\mathrm{d}W_s. \tag{3.31}$$

Suppose the H-valued process $Y(t)$ satisfies (3.31), and let $X(t) := Y(E_t)$. Applying the first change of variable formula gives

$$\begin{aligned} X(t) &= x_0 + \int_0^{E_t} \Big(AY(s) + F(s, Y(s))\Big)\mathrm{d}s + \int_0^{E_t} B(s, Y(s))\mathrm{d}W_s \\ &= x_0 + \int_0^t \Big(AY(E_s) + F(E_s, Y(E_s))\Big)\mathrm{d}E_s + \int_0^t B(E_s, Y(E_s))\mathrm{d}W_{E_s} \\ &= x_0 + \int_0^t \Big(AX(s) + F(E_s, X(s))\Big)\mathrm{d}E_s + \int_0^t B(E_s, X(s))\mathrm{d}W_{E_s}, \end{aligned}$$

which is the corresponding integral form of SDE (3.30).

Similarly, suppose the H-valued process $X(t)$ satisfies the SDE (3.30). Using the second change of variable formula yields

$$\begin{aligned} X(t) &= x_0 + \int_0^t \Big(AX(s) + F(E_s, X(s))\Big)\mathrm{d}E_s + \int_0^t B(E_s, X(s))\mathrm{d}W_{E_s} \\ &= x_0 + \int_0^{E_t} \Big(AX(U_{s-}) + F(E_{U_{s-}}, X(U_{s-}))\Big)\mathrm{d}s \\ &\qquad + \int_0^{E_t} B(E_{U_{s-}}, X(U_{s-}))\mathrm{d}W_s \\ &= x_0 + \int_0^{E_t} \Big(AX(U_{s-}) + F(s, X(U_{s-}))\Big)\mathrm{d}s + \int_0^{E_t} B(s, X(U_{s-}))\mathrm{d}W_s. \end{aligned}$$

Let $Y(t) := X(U_{t-})$, then

$$Y(t) = x_0 + \int_0^t \Big(AY(s) + F(s, Y(s)) \Big) \mathrm{d}s + \int_0^t B(s, Y(s)) \mathrm{d}W_s,$$

which is the integral form of SDE (3.29), thus completing the proof. □

It is known from [14] that under appropriate conditions, the strong solution, $Y(t)$, of the SDE (3.29) exists and is unique in the form

$$Y(t) = x_0 + \int_0^t (AY(s) + F(s, Y(s))) \mathrm{d}s + \int_0^t B(s, Y(s)) \mathrm{d}W_s$$

for all $t \le T$, $\mathbb{P}$-a.s. The existence and uniqueness of a strong solution to the time-changed SDE (3.30) is then established based on the duality theorem.

Theorem 3.3.4 *Assume that the following hypotheses are satisfied:*

1) *W_t is a K-valued Q-Wiener process on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \le T}, \mathbb{P})$ with the filtration $\mathcal{F}_t$ satisfying the usual conditions and E_t is the inverse β-stable subordinator which is independent of W_t.*

2) *A is a linear bounded operator.*

3) *The coefficients $F : \Omega \times [0,T] \times C([0,T], H) \to H$ and $B : \Omega \times [0,T] \times C([0,T], H) \to \mathcal{L}_2(K_Q, H)$, where $C([0,T], H)$ is the Banach space of H-valued continuous functions on $[0,T]$, satisfy the following conditions*

   (a) *F and B are jointly measurable, and for every $0 \le t \le T$, they are measurable with respect to the product σ-field $\mathcal{F}_t \times \mathcal{C}_t$ on $\Omega \times C([0,T], H)$, where $\mathcal{C}_t$ is a σ-field generated by cylinders with bases over $[0,t]$.*

   (b) *There exists a constant L such that for all $x \in C([0,T], H)$,*

$$\|F(\omega, t, x)\|_H + \|B(\omega, t, x)\|_{\mathcal{L}_2(K_Q, H)} \le L(1 + \sup_{0 \le s \le T} \|x(s)\|_H)$$

   for $\omega \in \Omega$ and $0 \le t \le T$.

(c) For all $x, y \in C([0,T], H)$*,* $\omega \in \Omega$*,* $0 \le t \le T$*, there exists* $K_0 > 0$ *such that*

$$\|F(\omega,t,x) - F(\omega,t,y)\|_H + \|B(\omega,t,x) - B(\omega,t,y)\|_{\mathcal{L}_2(K_Q,H)}$$
$$\le K_0 \sup_{0\le s\le T} \|x(s) - y(s)\|_H.$$

4) $\mathbb{E}\int_0^T \|B(t,Y(t))\|^2_{\mathcal{L}_2(K_Q,H)} dt < \infty$.

5) $Y(t)$ *is in the domain of* A $\mathrm{d}\mathbb{P} \times \mathrm{d}t$*-almost everywhere.*

6) x_0 *is a* $\mathcal{F}_0$*-measurable* H*-valued random variable.*

Then, the time-changed SDE (3.30) *has a unique strong solution,* $X(t)$*, satisfying*

$$X(t) = x_0 + \int_0^t (AX(s) + F(E_s, X(s)))\mathrm{d}E_s + \int_0^t B(E_s, X(s))\mathrm{d}W_{E_s}. \quad (3.32)$$

Proof: From [14], based on conditions in Theorem 3.3.4, there is a unique solution $Y(t)$ that satisfies the SDE (3.29). Moreover, it follows from Theorem 3.3.3 that $X(t) := Y(E_t)$ satisfies the SDE (3.30). Therefore, there exists a solution to the time-changed SDE (3.30).

Now suppose there exists another solution to the SDE (3.30). Call this solution $\hat{X}(t)$. Then, by Theorem 3.3.3, the process $\hat{Y}(t) := \hat{X}(U_{t-})$ is a solution to the SDE (3.29). Since the solution to the SDE (3.29) is unique from [14], it must be that $\hat{Y}(t) = Y(t)$. Thus $\hat{X}(U_{t-}) = Y(t)$ which implies that $\hat{X}(t) = Y(E_t) = X(t)$. Therefore, the solution $X(t)$ of the SDE (3.30) is unique and satisfies the desired integral equation (3.32). □

Chapter 4

Stability Results in the 1-dimensional Time-Changed Case

4.1 Introduction

In the previous chapter, we established the existence and uniqueness of a mild solution to an SDE including a time-changed Q-Wiener process. However, no other qualities of the solution were determined, as of yet. A natural question, then, is to ask how the solution behaves over time. Does it exhibit any kind of stability?

In [14], Gawarecki and Mandrekar explore exponential stability of mild solutions to stochastic differential equations that include a Q-Wiener process by using a Lyapunov function approach. More specifically, they are interested in mild solutions to SDEs of the form

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))dt + B(X(t))dW_t, \\ X(0) = x \in H, \end{cases} \tag{4.1}$$

where A is the generator of a C_0-semigroup $\{S(t), t \geq 0\}$ on H, W_t is a K-valued Q-Wiener process, and $F : H \to H$ and $B : H \to \mathcal{L}(K, H)$ are Bochner-measurable functions satisfying certain growth and boundedness conditions. They then establish conditions under which the mild solution to the SDE (4.1) is exponentially stable in the mean square sense.

Definition 4.1.1 *A mild solution $X^x(t)$ of (4.1) is exponentially stable in the mean square sense (m.s.s.) if for all $t \geq 0$ and $x \in H$,*

$$\mathbb{E} \|X^x(t)\|_H^2 \leq ce^{-\beta t} \|x\|_H^2, \quad c, \beta > 0.$$

One of their results states that the mild solution is exponentially stable in the m.s.s. if there exists a Lyaponuv function $\Lambda : H \to \mathbb{R}$ that satisfies some nice properties. [See Theorem 6.4 in [14] for more details.] Furthermore, for the simpler, linear case of (4.1) where $F := 0$ and $B(x) = B_0(x)$ with $B_0 \in \mathcal{L}(H, \mathcal{L}(K, H))$ and $\|B_0 x\|_H \le d\|x\|_H$, they show that the converse holds, provided that $\{S(t)\}$ is a pseudo-contraction semigroup.

Further results are established involving the nonlinear SDE (4.1), but for the purposes of this exposition, let us just consider the linear case. Consider the mild solution to the following SDE

$$\begin{cases} dX(t) = AX(t)dt + B_0 X(t) dW_{E_t}, \\ X(0) = x \in H, \end{cases} \tag{4.2}$$

with appropriate conditions. Can we say that this solution is exponentially stable in the m.s.s. if and only if a certain kind of Lyaponuv function exists?

It turns out that the above results cannot be easily extended to the time-changed case. One of the fundamental facts exploited in the non-time-changed case is the Markov property of the solution. However, in the time-changed case, the solution cannot be expected to be Markovian. The time change E_t introduces memory into the system and thus the solution. Therefore, getting any stability result requires taking into account the non-Markovian property of our solution. Some have dealt with stability of non-Markovian solutions and systems by using related Markovian structures. See [1, 34]. This will not be our approach here. It should also be noted that according to [36], "It is impossible to construct a suitable Lyapunov function and to find an appropriate fixed point theorem for stochastic partial differential equations with memory, even for constant delays, to deal with stability." However, this does not mean that one needs to completely relinquish the idea of stability as related to Lyapunov functions. For example, [24] explore stability in terms of a Lyapunov direct method, but instead of being concerned with exponential stability, they introduce the notion of Mittag-Leffler stability.

Here our approach is very different. We simply see under which conditions we can recover a stability result for our system. Unfortunately, many concessions are made. For example, we limit ourselves to the one-dimensional case, we force stricter conditions on our operator A, and we are unable to show results for $X(t)$ directly. Instead, the results are for a time-changed version of the mild solution, namely $X \circ \gamma(t)$. These results were generated with the help of Kei Kobayashi.

4.2 Time-Changed Gronwall Inequality

Consider the following SDE:

$$\begin{cases} dX(t) = AX(t)dt + B_0 X(t) dW_{E_t}, \\ X(0) = x_0, \end{cases} \tag{4.3}$$

where A is a bounded operator that is also the generator of an exponentially stable C_0-semigroup (i.e. there exists an eigenvalue of A, $-\lambda < 0$, such that $|S(t)| \le e^{-\lambda t}$, $\forall t \ge 0$; as we are in one dimension, here $A = -\lambda$), W is a one-dimensional Brownian motion, and B_0 is Lipschitz (i.e. $|B_0(x) - B_0(y)| \le L|x - y|$).

Then the corresponding mild solution to SDE (4.3) satisfies

$$X(t) = Z(t) + \int_0^t S(t-s) B_0 X(s) dM_s, \tag{4.4}$$

where $Z(t) := S(t)x_0$ and $M := W \circ E$. Recall from the previous section that M is a square-integrable martingale with respect to the filtration $\hat{\mathcal{F}}_{E_t}$ where

$$\hat{\mathcal{F}}_\tau = \bigcap_{u > \tau} \{\sigma[W(s) : 0 \le s \le u] \vee \sigma[E_s : s \ge 0]\}.$$

Let $R(t) = 2L^2[M]_t + g(t)$, where $g(t)$ is a continuous, nonnegative, increasing, deterministic function of t, and define $\gamma(u) = \inf\{t : R(t) > u\}$. Then $\gamma(u)$ is a new time change that can be used to get the following result.

Theorem 4.2.1 *Let $X(t)$ be the mild solution of (4.3) as given by (4.4). Assume*

$$\mathbb{E}\left[\int_0^\infty |X(s)|^2 d[M]_s\right] < \infty.$$

Then, the time-changed solution $X \circ \gamma(u)$ satisfies the following Gronwall-type inequality:

$$\mathbb{E}[|X \circ \gamma(u)|^2] \leq 2e^u \mathbb{E}[|Z \circ \gamma(u)|^2]. \tag{4.5}$$

Proof: First, it follows immediately that

$$|X(t)|^2 \leq 2|Z(t)|^2 + 2\left|\int_0^t S(t-s)B_0 X(s) dM_s\right|^2.$$

Thus, fixing $u > 0$, we can evaluate the above at time $\gamma(u)$, which is just a positive, finite random variable. Taking the expectation of both sides yields

$$\mathbb{E}|X \circ \gamma(u)|^2 \leq 2\mathbb{E}|Z \circ \gamma(u)|^2 + 2\mathbb{E}\left|\int_0^{\gamma(u)} S(\gamma(u)-s)B_0 X(s) dM_s\right|^2. \tag{4.6}$$

Now, notice that for a non-time-changed Brownian motion W_s, the following Itô isometry holds

$$\mathbb{E}\left[\left|\int_0^t \Phi(s) dW_s\right|^2\right] = \mathbb{E}\left[\int_0^t |\Phi(s)|^2 ds\right]. \tag{4.7}$$

for cadlag, adapted $\Phi : [0,t] \times \Omega \to \mathbb{R}$ such that the right-hand side is finite.

We now wish to bound the second term of the right-hand side of (4.6). By first conditioning on the entire path of the process U, we fix E and $\gamma(u)$ and get the following

$$\begin{aligned}
&\mathbb{E}\left|\int_0^{\gamma(u)} S(\gamma(u)-s)B_0 X(s) dM_s\right|^2 \\
&\quad = \mathbb{E}\left[\mathbb{E}\left[\left|\int_0^{\gamma(u)} S(\gamma(u)-s)B_0 X(s) dW_{E_s}\right|^2 \middle| U, E, \gamma(u)\right]\right].
\end{aligned}$$

Since E and $\gamma(u)$ are now nonrandom functions after conditioning, we can apply a usual change of variable formula to get

$$\mathbb{E}\left[\mathbb{E}\left[\left|\int_0^{\gamma(u)} S(\gamma(u)-s)B_0X(s)dW_{E_s}\right|^2\middle|U,E,\gamma(u)\right]\right]$$
$$=\mathbb{E}\left[\mathbb{E}\left[\left|\int_0^{E(\gamma(u))} S(\gamma(u)-U(s-))B_0X(U(s-))dW_s\right|^2\middle|U,E,\gamma(u)\right]\right].$$

Now, although E and $\gamma(u)$ are fixed, W is still random. Thus, we can use the Itô isometry (4.7) to conclude

$$\mathbb{E}\left[\mathbb{E}\left[\left|\int_0^{E(\gamma(u))} S(\gamma(u)-U(s-))B_0X(U(s-))dW_s\right|^2\middle|U,E,\gamma(u)\right]\right]$$
$$=\mathbb{E}\left[\mathbb{E}\left[\int_0^{E(\gamma(u))} |S(\gamma(u)-U(s-))B_0X(U(s-))|^2\,ds\middle|U,E,\gamma(u)\right]\right].$$

Again, we use the change of variable formula and get

$$\mathbb{E}\left[\mathbb{E}\left[\int_0^{E(\gamma(u))} |S(\gamma(u)-U(s-))B_0X(U(s-))|^2\,ds\middle|U,E,\gamma(u)\right]\right]$$
$$=\mathbb{E}\left[\mathbb{E}\left[\int_0^{\gamma(u)} |S(\gamma(u)-s)B_0X(s)|^2\,dE_s\middle|U,E,\gamma(u)\right]\right].$$

Using the bounds on $S(t)$ and B_0 yields

$$\mathbb{E}\left[\mathbb{E}\left[\int_0^{\gamma(u)} |S(\gamma(u)-s)B_0X(s)|^2\,dE_s\middle|U,E,\gamma(u)\right]\right]$$
$$\leq\mathbb{E}\left[\mathbb{E}\left[\int_0^{\gamma(u)} L^2e^{-2\lambda(\gamma(u)-s)}|X(s)|^2dE_s\middle|U,E,\gamma(u)\right]\right]$$
$$\leq\mathbb{E}\left[\mathbb{E}\left[\int_0^{\gamma(u)} L^2|X(s)|^2dE_s\middle|U,E,\gamma(u)\right]\right].$$

Undoing the conditioning on U, we are left with

$$\mathbb{E}\left[\mathbb{E}\left[\int_0^{\gamma(u)} L^2|X(s)|^2dE_s\middle|U,E,\gamma(u)\right]\right]=\mathbb{E}\int_0^{\gamma(u)} L^2|X(s)|^2d[M]_s.$$

Thus using the above and the definition of R,

$$\begin{aligned} \mathbb{E}[|X\circ\gamma(u)|^2] &\le 2\mathbb{E}[|Z\circ\gamma(u)|^2] + 2L^2\mathbb{E}\left[\int_0^{\gamma(u)} |X(s)|^2 d[M]_s\right] \\ &\le 2\mathbb{E}[|Z\circ\gamma(u)|^2] + \mathbb{E}\left[\int_0^{\gamma(u)} |X(s)|^2 dR(s)\right]. \end{aligned} \tag{4.8}$$

Now focusing on the last integral, using the change of variable formula (5.12) on p. 30 in [23] yields

$$\begin{aligned} \int_0^{\gamma(u)} |X(s)|^2 dR(s) &= \int_0^u |X\circ\gamma(s)|^2 dR\circ\gamma(s) \\ &= \int_0^u |X\circ\gamma(s)|^2 ds \quad \text{since } R\circ\gamma(s) = R(\inf\{r : R(r) > s\}) = s. \end{aligned} \tag{4.9}$$

Before going further, we must check that the hypotheses for (5.12) are valid. Specifically, (5.12) requires that R is an $\hat{\mathcal{F}}_{E_t}$-semimartingale, γ is an $\hat{\mathcal{F}}_{E_t}$-stopping time that is continuous and nondecreasing with $\gamma(0) = 0$, and $|X|^2$ is cadlag and $\hat{\mathcal{F}}_{E_t}$-adapted.

- Recall that $R(t) = 2L^2[M]_t + g(t)$. Since $M = W\circ E$, $[M]_t = E_t$. Thus, $R(t) = 2L^2E_t + g(t)$, which is an $\hat{\mathcal{F}}_{E_t}$-semimartingale since $R(t)$ is a nondecreasing process.

- $\gamma(u) = \inf\{t : R(t) > u\}$ and $R(0) = 0$ imply that $\gamma(0) = 0$. Moreover, $R(t)$ is continuous and strictly increasing. Thus, γ is also continuous and nondecreasing. Also, $\gamma(u)$ is an $\hat{\mathcal{F}}_{E_t}$-stopping time since the event $\{\gamma(u) \le t\} = \{R(t) \ge u\}$. Therefore for any fixed t, this set is measurable and thus in the $\hat{\mathcal{F}}_{E_t}$ σ-algebra since R is an $\hat{\mathcal{F}}_{E_t}$-semimartingale.

- Since a solution to (4.3) is continuous and $\hat{\mathcal{F}}_{E_t}$-adapted, so is $|X(s)|^2$. It should be noted that $|X\circ\gamma(u)|$ is also continuous.

Now, combining (4.8) and (4.9) yields

$$\begin{aligned} \mathbb{E}[|X\circ\gamma(u)|^2] &\le 2\mathbb{E}[|Z\circ\gamma(u)|^2] + \mathbb{E}\int_0^u [|X\circ\gamma(s)|^2] ds \\ &= 2\mathbb{E}[|Z\circ\gamma(u)|^2] + \int_0^u \mathbb{E}[|X\circ\gamma(s)|^2] ds, \end{aligned}$$

and (4.5) follows from Gronwall's inequality (7.4) on p. 42 in [23]. □

Remark 4.2.2 Notice that the Gronwall inequality holds when $R(t) = 2L^2 E_t + t$, as well as when $R(t) = 2L^2 E_t + t^\beta$. ◊

4.3 A Sufficient Condition for Stability of a γ-time-changed Solution in $L^2(\Omega, \mathbb{P})$

Although our original motivation was to get a sense of the longterm behavior of our solution $X(t)$ as given in (4.4), we were only able to obtain a Gronwall-type inequality by considering the time-changed solution $X \circ \gamma(u)$. At the very least, we want to use the bound in (4.5) to show that the time-changed solution $X \circ \gamma(u)$ decays to 0 as $u \to \infty$. Here, we let $\gamma(u)$ be the inverse of $R(t) = 2L^2 E_t + t$, and since u is simply a dummy variable, we will instead consider the limit as t goes to infinity of $X \circ \gamma(t)$.

Definition 4.3.1 *The solution $X \circ \gamma(t)$ is called asymptotically stable in $L^2(\Omega, \mathbb{P})$ if*

$$\lim_{t \to \infty} \mathbb{E}[|X \circ \gamma(t)|^2] = 0.$$

The following theorem gives a sufficient condition for the solution $X(\gamma(t))$ to be asymptotically stable in $L^2(\Omega, \mathbb{P})$.

Theorem 4.3.2 *Let $a = 2L^2$ and $b = 2\lambda$ where L is the bound associated with the Lipschitz property of B_0 and $-\lambda$ is an eigenvalue of A such that $|S(t)| \leq e^{-\lambda t}$ for all $t \geq 0$. Let $h(t) := e^t \mathbb{E}[e^{-b\gamma(t)}]$.*

(i) If $\lim_{t \to \infty} h(t) = 0$, then $X(\gamma(t))$ is asymptotically stable in $L^2(\Omega, \mathbb{P})$.

(ii) If condition (4.10) below is satisfied, then $\lim_{t \to \infty} h(t) = 0$:

$$b + s - 1 > |(s-1)a|^{1/\beta} \quad \textit{for sufficiently small } s \geq 0. \tag{4.10}$$

Notice in particular that the condition above implies $b > 1$ and thus $\lambda > 1/2$.

Proof (of (i)): From the Gronwall Inequality (4.5) in the previous section,

$$\mathbb{E}[|X \circ \gamma(t)|^2] \le 2e^t \mathbb{E}[|Z \circ \gamma(t)|^2]$$

where $R(t) := aE_t + t$ (with $a = 2L^2$), $\gamma(t) := \inf\{s : R(s) > t\}$ is the inverse of R, and $Z(t) := S(t)x_0$.

Now, since $|S(t)| \le e^{-\lambda t}$ for all $t \ge 0$,

$$\begin{aligned} 2e^t \mathbb{E}[|Z \circ \gamma(t)|^2] &= 2e^t \mathbb{E}[|S(\gamma(t))x_0|^2] \\ &\le 2e^t \mathbb{E}[(e^{-\lambda\gamma(t)}|x_0|)^2] \\ &= 2e^t \mathbb{E}(e^{-2\lambda\gamma(t)})|x_0|^2. \end{aligned}$$

Let $b = 2\lambda$. Notice that if $h(t) = e^t \mathbb{E}[e^{-b\gamma(t)}] \to 0$ as $t \to \infty$, then

$$\mathbb{E}[|X \circ \gamma(t)|^2] \le 2e^t \mathbb{E}(e^{-b\gamma(t)})|x_0|^2 \to 0$$

as $t \to \infty$. In other words, the solution $X \circ \gamma(t)$ exhibits asymptotic stability in $L^2(\Omega, \mathbb{P})$. □

Proof (of (ii)): It is more complicated to prove that if condition (4.10) holds, then $\lim_{t\to\infty} e^t \mathbb{E}[e^{-b\gamma(t)}] = 0$. To achieve this, let $F(s) := \mathcal{L}_t[e^t \mathbb{E}[e^{-b\gamma(t)}]](s)$ where $\mathcal{L}_t$ denotes the Laplace transform. The idea is to use the Final Value Theorem to conclude that $\lim_{s\to 0} sF(s) = \lim_{t\to\infty} e^t \mathbb{E}[e^{-b\gamma(t)}]$. Therefore, rather than showing that $\lim_{t\to\infty} e^t \mathbb{E}[e^{-b\gamma(t)}] = 0$ directly, we can instead show the equivalent result that $\lim_{s\to 0} sF(s) = 0$. A justification of the use of the Final Value Theorem will be given at the end of this section.

That $\lim_{s\to 0} sF(s) = 0$ can be easily deduced under condition (4.10) upon establishing the explicit formula for $sF(s)$ given in the next proposition.

Proposition 4.3.3 *Let* $F(s) = \mathcal{L}_t[e^t\mathbb{E}[e^{-b\gamma(t)}]](s)$. *Then*

$$sF(s) = \frac{s(b+s-1)^{\beta-1} + sa}{(b+s-1)^{\beta} + (s-1)a} = \frac{s(1 + a(b+s-1)^{1-\beta})}{(b+s-1)^{1-\beta}((b+s-1)^{\beta} + (s-1)a)}.$$

Hence, Theorem 4.3.2 will be established once the proposition has been proven.

The idea is to find $sF(s)$ explicitly using properties of the Laplace transform and facts about the 1- and 2-parameter Mittag-Leffler functions, which we now collect here before proving Proposition 4.3.3.

Needed Properties of the Laplace transform of a function. Let f and g be functions which are Laplace transformable.

1. If $j(t) = f(t)$ or $g(t)$, the Laplace transform of the function $j(t)$ is

$$\mathcal{L}_t[j(t)](s) = \int_0^\infty e^{-st} j(t)dt.$$

2. If $f(t) = e^t g(t)$, then

$$\mathcal{L}_t[f(t)](s) = \int_0^\infty e^{-st}e^t g(t)dt = \int_0^\infty e^{-(s-1)t} g(t)dt = \mathcal{L}_t[g(t)](s-1).$$

3. If $f(t) = 1$, $\mathcal{L}_t[f(t)](s) = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}|_0^\infty = \frac{1}{s}$.

4. If $f(t) = \int_0^t g(\tau)d\tau$, then using integration by parts,

$$\begin{aligned}\mathcal{L}_t[f(t)](s) &= \int_0^\infty e^{-st}\int_0^t g(\tau)d\tau dt = -\frac{1}{s}e^{-st}\int_0^t g(\tau)d\tau|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}g(t)dt \\ &= 0 + \frac{1}{s}\int_0^\infty e^{-st}g(t)dt = \frac{1}{s}[\mathcal{L}_t(g(t))(s)].\end{aligned}$$

5. If $f(t) = f_{V(x)}(t)$ is the density function of $V(x)$, then

$$\mathcal{L}_t[f(t)](s) = \int_0^\infty e^{-st}f_{V(x)}(t)dt = \int_0^\infty e^{-sV(x)}d\mathbb{P} = \mathbb{E}[e^{-sV(x)}].$$

Needed Properties of the Mittag-Leffler functions. By definition, the two-parameter Mittag-Leffler function is as follows

$$\mathbf{E}_{\beta,\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta k + \alpha)}, \quad (\beta > 0,\ \alpha > 0,\ x \in \mathbb{C}).$$

As a special case, the one-parameter Mittag-Leffler function is

$$\mathbf{E}_{\beta}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta k + 1)} = \mathbf{E}_{\beta,1}(x), \quad (\beta > 0,\ x \in \mathbb{C}).$$

It then follows that the derivative of the one-parameter Mittag-Leffler function can be written in terms of a two-parameter Mittag-Leffler function in the following way:

Lemma 4.3.4 *For $\beta > 0$ and $x \in \mathbb{C}\backslash\{0\}$,*

$$\frac{d}{dx}\mathbf{E}_{\beta}(\lambda x^{\beta}) = \lambda x^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda x^{\beta}).$$

Proof: Using the series definition of the Mittag-Leffler function and the fact that $\Gamma(z+1) = z\Gamma(z)$,

$$\begin{aligned}
\frac{d}{dx}[\mathbf{E}_{\beta}(\lambda x^{\beta})] &= \frac{d}{dx}\left[\sum_{k=0}^{\infty} \frac{\lambda^k x^{\beta k}}{\Gamma(\beta k + 1)}\right] = \frac{d}{dx}\left[\frac{1}{\Gamma(1)} + \frac{\lambda x^{\beta}}{\Gamma(\beta+1)} + \frac{\lambda^2 x^{2\beta}}{\Gamma(2\beta+1)} + \ldots\right] \\
&= \frac{d}{dx}\left[\frac{\lambda x^{\beta}}{\Gamma(\beta+1)}\right] + \frac{d}{dx}\left[\frac{\lambda^2 x^{2\beta}}{\Gamma(2\beta+1)}\right] + \ldots = \frac{\lambda \beta x^{\beta-1}}{\Gamma(\beta+1)} + \frac{\lambda^2 2\beta x^{2\beta-1}}{\Gamma(2\beta+1)} + \ldots \\
&= x^{-1}\sum_{k=1}^{\infty} \frac{(\lambda x^{\beta})^k}{\Gamma(\beta k)} = x^{-1}\sum_{k=0}^{\infty} \frac{(\lambda x^{\beta})^{k+1}}{\Gamma(\beta(k+1))} \\
&= \lambda x^{\beta-1}\sum_{k=0}^{\infty} \frac{(\lambda x^{\beta})^k}{\Gamma(\beta k + \beta)} = \lambda x^{\beta-1}\mathbf{E}_{\beta,\beta}(\lambda x^{\beta}),
\end{aligned}$$

as desired. □

It is now possible to prove Proposition 4.3.3.

Proof: Notice that

$$\begin{aligned} sF(s) &= s\mathcal{L}_t[e^t\mathbb{E}[e^{-b\gamma(t)}]](s) \\ &= s\mathcal{L}_t[\mathbb{E}[e^{-b\gamma(t)}]](s-1) \quad \text{by property 2 above} \\ &= \mathcal{L}_t\left[\int_0^\infty e^{-bx} s\mu_{\gamma(t)}(dx)\right](s-1), \end{aligned} \tag{4.11}$$

where $\mu_{\gamma(t)}$ is the probability measure induced by $\gamma(t)$.

The goal now is to move the Laplace transform onto $s\mu_{\gamma(t)}$.

$$\begin{aligned} \mathcal{L}_t[s\mathbb{P}(\gamma(t) \le x)](u) &= \mathcal{L}_t[s\mathbb{P}(R(x) \ge t)](u) \quad \text{since } \gamma \text{ and } R \text{ are inverses} \\ &= \mathcal{L}_t[s - s\mathbb{P}(R(x) < t)](u) \\ &= \mathcal{L}_t\left[s - s\int_0^t f_{R(x)}(r)dr\right](u) \quad \text{where } f_{R(x)} \text{ is the density of } R(x) \\ &= \frac{s}{u} - \frac{s}{u}\mathcal{L}_t[f_{R(x)}(t)](u) \quad \text{using properties 3 and 4 above} \\ &= \frac{s}{u} - \frac{s}{u}\mathbb{E}[e^{-uR(x)}] \quad \text{using property 5 above} \\ &= \frac{s}{u} - \frac{s}{u}\mathbb{E}[e^{-u(aE_x+x)}] \quad \text{by the definition of } R(x) \\ &= \frac{s}{u} - \frac{s}{u}e^{-ux}\mathbb{E}[e^{-uaE_x}] \\ &= \frac{s}{u} - \frac{s}{u}e^{-ux}\mathbf{E}_\beta(-uax^\beta) \end{aligned} \tag{4.12}$$

using Theorem 4.3 from [7], which gives the Laplace transform of E_x in terms of $\mathbf{E}_\beta$, the one-parameter Mittag-Leffler function.

Now, since $\mathcal{L}_t[s\mu_{\gamma(t)}(dx)](u) = \mathcal{L}_t[s\mathbb{P}(\gamma(t) \in dx)](u)$, with $u = s-1$, is the quantity of interest, we take the derivative with respect to x of both sides of (4.12) and get

$$\begin{aligned} \mathcal{L}_t[s\mu_{\gamma(t)}(dx)](u) &= \mathcal{L}_t[s\mathbb{P}(\gamma(t) \in dx)](u) \\ &= -\frac{s}{u}[-ue^{-ux}\mathbf{E}_\beta(-uax^\beta) + e^{-ux}(-ua)x^{\beta-1}\mathbf{E}_{\beta,\beta}(-uax^\beta)] \quad (4.13) \\ &= se^{-ux}[\mathbf{E}_\beta(-uax^\beta) + ax^{\beta-1}\mathbf{E}_{\beta,\beta}(-uax^\beta)]. \end{aligned}$$

Plugging (4.13) into (4.11) implies

$$\begin{aligned}
sF(s) &= \mathcal{L}_t\left[\int_0^\infty e^{-bx} s\mu_{\gamma(t)}(dx)\right](s-1) \\
&= s\int_0^\infty e^{-(b+s-1)x}[\mathbf{E}_\beta(-(s-1)ax^\beta) + ax^{\beta-1}\mathbf{E}_{\beta,\beta}(-(s-1)ax^\beta)]dx \\
&= s\int_0^\infty e^{-(b+s-1)x}\mathbf{E}_\beta(-(s-1)ax^\beta)dx \\
&\quad + s\int_0^\infty e^{-(b+s-1)x}ax^{\beta-1}\mathbf{E}_{\beta,\beta}(-(s-1)ax^\beta)]dx \\
&= \frac{s(b+s-1)^{\beta-1}}{(b+s-1)^\beta + (s-1)a} + \frac{sa}{(b+s-1)^\beta + (s-1)a}
\end{aligned}$$

under our assumption that $b+s-1 > |(s-1)a|^{1/\beta}$ for sufficiently small $s \geq 0$, as given in (1.80) in [32].

Therefore,

$$sF(s) = \frac{s(b+s-1)^{\beta-1} + sa}{(b+s-1)^\beta + (s-1)a}.$$

Multiplying the numerator and denominator by $(b+s-1)^{1-\beta}$ gives us the alternate form of $sF(s)$.

This completes the proof of Proposition 4.3.3. □

Thus condition (ii) holds, and Theorem 4.3.2 has now been established. □

Justification for the use of the Final Value Theorem. In order to use the Final Value Theorem, all poles of the function $sF(s)$ must lie in the left-half plane. Thus, we must first extend the domain of the function to the entire complex plane. Notice that the second form of $sF(s)$ given in Proposition 4.3.3 is well-defined for all complex numbers s since it is possible to take positive fractional powers of complex numbers. Thus, it remains to find the poles of the function; in particular, we want to know when the denominator

$$(b+s-1)^{1-\beta}((b+s-1)^\beta + (s-1)a)$$

is equal to 0. This happens when either $b+s-1=0$ or $(b+s-1)^{\beta}=(1-s)a$. In the first case, $s=1-b$, a negative number. In the second case, if we let $x=b+s-1$, we have that $x^{\beta}=-xa+ba$. We want to make sure that $x<b-1$ as that ensures that $s<0$. It is sufficient to show that when $\hat{x}=b-1$, $\hat{x}^{\beta}>-\hat{x}a+ba$. By condition (4.10), $(b-1)^{\beta}>a=(-b+1)a+ba$, as desired. Thus, all poles of $sF(s)$ are in the left-half plane. For more information on this subject, see [12].

4.4 Properties of $R(t)$ and $\gamma(u)$

4.4.1 Properties of $R(t)$

Recall that $R(t)=2L^2E_t+t$, as defined in the previous section. We would like to explore some of the properties of this process. In particular, we would like to see in what ways this process behaves like E_t and how it differs from E_t. The hope is to use this information to gain a better understanding of the inverse of $R(t)$, namely $\gamma(u)$.

Let us begin with the similarities. Like E_t, $R(t)$ lacks stationary and independent increments. This is not terribly surprising, as the addition of the t term in $R(t)$ should not be expected to compensate for the "bad" properties of E_t. The following proofs are mechanically the same as the ones used by Meerschaert and Scheffler in [26] to show that E_t does not have stationary or independent increments.

Lemma 4.4.1 $R(t)$ *does not have stationary increments.*

Proof: Suppose $R(t)$ has stationary increments. Then, for any integer t,

$$\begin{aligned}\mathbb{E}(R(t)) &= \mathbb{E}(R(1)+(R(2)-R(1))+\ldots+(R(t)-R(t-1)))\\ &= t\mathbb{E}(R(1)).\end{aligned}$$

However,

$$\mathbb{E}(R(t))=\mathbb{E}(2L^2E_t+t)$$

$$= \mathbb{E}(2L^2 E_t) + t$$

$$= D(\beta)t^\beta + t,$$

where $D(\beta)$ is a constant.

Since the two quantities are not the same, we have a contradiction. Thus, the process $R(t)$ does not have stationary increments. □

Lemma 4.4.2 $R(t)$ *does not have independent increments.*

Proof: Assume that we have independence of the increments of $R(t)$. Then for t_1, t_2, t_3 such that $0 < t_1 < t_2 < t_3$,

$$\begin{aligned}
\mathbb{E}[(R(t_3) - R(t_2)) \cdot (R(t_2) - R(1))] &= \mathbb{E}[R(t_3) - R(t_2)] \cdot \mathbb{E}[R(t_2) - R(t_1)] \\
&= [\mathbb{E}(R(t_3)) - \mathbb{E}(R(t_2))] \cdot [\mathbb{E}(R(t_2)) - \mathbb{E}(R(t_1))] \\
&= \mathbb{E}(R(t_3)) \cdot \mathbb{E}(R(t_2)) - \mathbb{E}(R(t_2))^2 \\
&\quad - \mathbb{E}(R(t_3)) \cdot \mathbb{E}(R(t_1)) + \mathbb{E}(R(t_2)) \cdot \mathbb{E}(R(t_1)) \\
&= 4L^4 D^2 [(t_3^\beta + t_3)(t_2^\beta + t_2) - (t_2^\beta + t_2)^2 \\
&\quad - (t_3^\beta + t_3)(t_1^\beta + t_1) + (t_2^\beta + t_2)(t_1^\beta + t_1)] \\
&:= K(t_1, t_2, t_3).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbb{E}[(R(t_3) - R(t_2)) \cdot (R(t_2) - R(1))] \\
&\quad = \mathbb{E}[R(t_2)R(t_3)] - \mathbb{E}[R(t_1)R(t_3)] - \mathbb{E}[R(t_2)^2] + \mathbb{E}[R(t_1)R(t_2)] \\
&\quad := L(t_1, t_2, t_3).
\end{aligned}$$

Notice that since the left-hand sides of K and L match, it must be that $K(t_1, t_2, t_3) = L(t_1, t_2, t_3)$.

Meanwhile,

$$\frac{\partial^2 K(t_1,t_2,t_3)}{\partial t_1 \partial t_2} = 4L^4 D^2[(\beta t_2^{\beta-1}+1)(\beta t_1^{\beta-1}+1)]$$

and

$$\frac{\partial^2 L(t_1,t_2,t_3)}{\partial t_1 \partial t_2} = \frac{\partial^2 \mathbb{E}[R(t_1)R(t_2)]}{\partial t_1 \partial t_2}.$$

Well, notice that

$$\begin{aligned}
\frac{\partial^2 \mathbb{E}[R(t_1)R(t_2)]}{\partial t_1 \partial t_2} &= \frac{\partial^2 \mathbb{E}[(2L^2E(t_1)+t_1)(2L^2E(t_2)+t_2)]}{\partial t_1 \partial t_2} \\
&= \frac{\partial^2 \mathbb{E}[4L^4E(t_1)E(t_2)+2L^2t_1E(t_2)+2L^2t_2E(t_1)+t_1t_2)]}{\partial t_1 \partial t_2} \\
&= 4L^4\frac{\partial^2 \mathbb{E}[E(t_1)E(t_2)]}{\partial t_1 \partial t_2} + 2L^2\frac{\partial^2 t_1\mathbb{E}(E(t_2))}{\partial t_1 \partial t_2} + 2L^2\frac{\partial^2 t_2\mathbb{E}(E(t_1))}{\partial t_1 \partial t_2} + \frac{\partial^2 t_1 t_2}{\partial t_1 \partial t_2} \\
&= 4L^4\frac{\partial^2 \mathbb{E}[E(t_1)E(t_2)]}{\partial t_1 \partial t_2} + 2L^2\frac{\partial \mathbb{E}(E(t_2))}{\partial t_2} + 2L^2\frac{\partial \mathbb{E}(E(t_1))}{\partial t_1} + 1 \\
&= 4L^4\frac{\partial^2 \mathbb{E}[E(t_1)E(t_2)]}{\partial t_1 \partial t_2} + 2L^2Dt_2^{\beta-1} + 2L^2Dt_1^{\beta-1} + 1.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
\frac{\partial^2 L(t_1,t_2,t_3)}{\partial t_1 \partial t_2} &= \frac{\partial^2 \mathbb{E}[R(t_1)R(t_2)]}{\partial t_1 \partial t_2} \\
&= 4L^4\frac{\partial^2 \mathbb{E}[E(t_1)E(t_2)]}{\partial t_1 \partial t_2} + 2L^2Dt_2^{\beta-1} + 2L^2Dt_1^{\beta-1} + 1 \\
&= 4L^4\Gamma(\beta)^{-2}(t_1t_2)^{\beta-1}[1-\frac{t_1}{t_2}]^{\beta-1} + 2L^2Dt_2^{\beta-1} + 2L^2Dt_1^{\beta-1} + 1,
\end{aligned}$$

by (3.4) on page 628 of [26].

However, this implies that $K(t_1,t_2,t_3) = L(t_1,t_2,t_3)$ while $\frac{\partial^2 K(t_1,t_2,t_3)}{\partial t_1 \partial t_2} \neq \frac{\partial^2 L(t_1,t_2,t_3)}{\partial t_1 \partial t_2}$, which is a contradiction. Thus, it must follow that $R(t)$ does not have independent increments. □

On the other hand, $R(t)$ and E_t have some notable differences as well. Some of these differences are quite useful. For example, $R(t)$ is strictly increasing while E_t may contain flats. Since $R(t)$ is continuous (because E_t is continuous), this along with the strictly increasing nature of $R(t)$ implies that its inverse $\gamma(u)$ is also continuous and strictly increasing. Thus, in some sense, the behavior of $\gamma(u)$ should be quite different from that of U_t, the inverse of E_t, since U_t is not continuous. (In fact, U_t is a pure jump process.)

Another difference between $R(t)$ and E_t is that $R(t)$ lacks the self-similarity property. In the following proofs, we will first show that E_t inherits the self-similarity property from its inverse U_t, and we will then show that $R(t)$, because of its added t term, does not preserve this property.

Lemma 4.4.3 *The process E_t has the self-similarity property. In other words, $E(t) = t^{\beta} E(1)$.*

Proof: Recall that as a β-stable subordinator, $U(t)$ has the self-similarity property that $U(t) = t^{1/\beta} U(1)$.

Therefore,

$$\begin{aligned}
\mathbb{P}(t^{\beta} E(1) \le x) &= \mathbb{P}(E(1) \le x t^{-\beta}) \\
&= \mathbb{P}(U(x t^{-\beta}) \ge 1) \\
&= \mathbb{P}((t^{-\beta})^{1/\beta} U(x) \ge 1) \\
&= \mathbb{P}(t^{-1} U(x) \ge 1) \\
&= \mathbb{P}(U(x) \ge t) \\
&= \mathbb{P}(E(t) \le x).
\end{aligned}$$

Thus, $E(t) = t^{\beta} E(1)$, as desired. □

Lemma 4.4.4 *The process $R(t)$ is not self-similar.*

Proof: Suppose $R(t)$ is self-similar. Then there exists a function $c(t)$ such that

$$\mathbb{P}(c(t) R(1) \le x) = \mathbb{P}(R(t) \le x). \tag{4.14}$$

However,

$$\mathbb{P}(c(t)R(1) \le x) = \mathbb{P}(c(t)(2L^2E(1)+1) \le x)$$
$$= \mathbb{P}(2L^2c(t)E(1) + c(t)) \le x).$$

Since the left-hand side of (2) equals the right-hand side of (2), it follows that

$$\mathbb{P}(2L^2c(t)E(1) + c(t)) \le x) = \mathbb{P}(2L^2E(t) + t \le x). \tag{4.15}$$

This implies that $2L^2c(t)E(1) = 2L^2E(t)$, which in turn implies that $c(t) = t^\beta$ by the self-similarity property of E_t. However, (3) also implies that $c(t) = t$, which gives us a contradiction. Therefore, $R(t)$ is not self-similar. □

4.4.2 Properties of $\gamma(u)$

Because the process $\gamma(u)$ appears in the Gronwall inequality (1), we are interested in its properties and behavior. One thing we can observe is that it, like its inverse $R(t)$, is not self-similar.

Lemma 4.4.5 *The process $\gamma(u)$ is not self-similar.*

Proof: Suppose $\gamma(u)$ is self-similar. Then there exists a function $d(t)$ such that

$$\gamma(u) = d(u)\gamma(1).$$

Let $c(t) = \frac{1}{d^*(1/t)}$ for $t > 0$ where d^* denotes the inverse of the function $d(t)$. Then it follows that

$$\begin{aligned}
\mathbb{P}(c(t)R(1) \le x) &= \mathbb{P}(R(1) \le xc(t)^{-1}) \\
&= \mathbb{P}(\gamma(xc(t)^{-1}) \ge 1) \\
&= \mathbb{P}(d(c(t)^{-1})\gamma(x) \ge 1) \\
&= \mathbb{P}(\gamma(x) \ge d(c(t)^{-1})^{-1}) \\
&= \mathbb{P}(R[d(c(t)^{-1})^{-1}] \le x)
\end{aligned}$$

$$= \mathbb{P}(R[d(d^*(1/t))^{-1}] \leq x)$$
$$= \mathbb{P}(R(t) \leq x).$$

However, this implies that $R(t)$ is self-similar, which contradicts our previous result. Therefore, the assumption that $\gamma(u)$ is self-similar must be false. □

4.5 A Sufficient Condition for Exponential Stability of a γ-time-changed Solution in $L^2(\Omega, \mathbb{P})$

What if we instead defined R in such a way that it is self-similar and still continuous? Then, γ would also be self-similar and continuous, and we might be able to exploit those properties in order to obtain a better stability result for the composition $X \circ \gamma$.

Let $R(t) := aE_t + t^\beta$ with $a = 2L^2$ and define $\gamma(u)$ as the inverse of $R(t)$, i.e. $\gamma(u) = \inf\{s : R(s) > u\}$.

Lemma 4.5.1 *The processes $R(t)$ and $\gamma(t)$ are both self-similar.*

Proof: Notice that $R(t) = t^\beta R(1)$ in distribution since

$$\begin{aligned}\mathbb{P}(R(t) \leq x) &= \mathbb{P}(aE(t) + t^\beta \leq x) = \mathbb{P}(aE(t) \leq x - t^\beta) \\ &= \mathbb{P}(t^\beta aE(1) \leq x - t^\beta) = \mathbb{P}(t^\beta(aE(1) + 1) \leq x) \\ &= \mathbb{P}(t^\beta R(1) \leq x).\end{aligned}$$

Similarly, $\gamma(t) = t^{1/\beta}\gamma(1)$ in distribution since

$$\begin{aligned}\mathbb{P}(t^{1/\beta}\gamma(1) \leq x) &= \mathbb{P}(\gamma(1) \leq xt^{-1/\beta}) = \mathbb{P}(R(xt^{-1/\beta}) \geq 1) \\ &= \mathbb{P}((t^{-1/\beta})^\beta R(x) \geq 1) = \mathbb{P}(t^{-1}R(x) \geq 1) \\ &= \mathbb{P}(R(x) \geq t) = \mathbb{P}(\gamma(t) \leq x).\end{aligned}$$

This completes the proof. □

Now, we wish to show that $X \circ \gamma(t)$ is not just asymptotically stable in $L^2(\Omega, \mathbb{P})$ but also exponentially stable in $L^2(\Omega, \mathbb{P})$.

Definition 4.5.2 *The solution $X \circ \gamma(t)$ is called exponentially stable in $L^2(\Omega, \mathbb{P})$ if*

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}[|X \circ \gamma(t)|^2] = c$$

for some constant $c < 0$.

Thus, using the newly defined R and γ processes, we have the following theorem.

Theorem 4.5.3 *If $b^\beta > a$, then $X \circ \gamma(t)$ is exponentially stable in $L^2(\Omega, \mathbb{P})$.*

Proof: Using the Gronwall inequality,

$$\mathbb{E}[|X \circ \gamma(t)|^2] \leq 2e^t \mathbb{E}[|Z \circ \gamma(t)|^2] \leq 2e^t \mathbb{E}(e^{-b\gamma(t)})|x_0|^2$$

as before, with $b = 2\lambda$.

Thus, it remains to show that $\lim_{t \to \infty} \frac{1}{t} \log e^t \mathbb{E}[e^{-b\gamma(t)}] < 0$.

Now, by self-similarity of the process $\gamma(t)$,

$$\mathbb{E}[e^{-b\gamma(t)}] = \mathbb{E}[e^{-(bt^{1/\beta})\gamma(1)}]$$

where $\gamma(1)$ is a nonnegative, fixed-time random variable. This form allows us to use an exponential Tauberian theorem that states that the following asymptotic behaviors are equivalent:

1. $-\log \mathbb{P}(\gamma(1) \leq \epsilon) \sim A\epsilon^{-\alpha}$ as $\epsilon \to 0$

2. $-\log \mathbb{E}[e^{-(bt^{1/\beta})\gamma(1)}] \sim (1+\alpha)\alpha^{-\alpha/(1+\alpha)} A^{1/(1+\alpha)} (bt^{1/\beta})^{\alpha/(1+\alpha)}$ as $(bt^{1/\beta}) \to \infty$

where A and α are positive constants.

The trick now is to find the asymptotic behavior of the small ball probability of

$\gamma(1)$ by relating it to that of $U(1)$, our original β-stable subordinator.

$$\mathbb{P}(\gamma(1) \le \epsilon) = \mathbb{P}(R(\epsilon) \ge 1) = \mathbb{P}\left(E(\epsilon) \ge \frac{1-\epsilon^\beta}{a}\right)$$
$$= \mathbb{P}\left(U\left(\frac{1-\epsilon^\beta}{a}\right) \le \epsilon\right) = \mathbb{P}\left(\left(\frac{1-\epsilon^\beta}{a}\right)^{1/\beta} U(1) \le \epsilon\right)$$
$$= \mathbb{P}\left(U(1) \le \epsilon\left(\frac{a}{1-\epsilon^\beta}\right)^{1/\beta}\right) = \mathbb{P}\left(U(1) \le \delta_\epsilon\right)$$

where $\delta_\epsilon = \epsilon\left(\frac{a}{1-\epsilon^\beta}\right)^{1/\beta}$.

Notice that as $\epsilon \to 0$, $\delta_\epsilon \to 0$, and in particular, $\delta_\epsilon \sim a^{1/\beta}\epsilon$. Thus we can use (5.5) from [3] to get the asymptotic behavior of $\mathbb{P}(U(1) \le \delta_\epsilon)$ as δ_ϵ goes to 0. They assert that

$$-\log \mathbb{P}(U(1) \le \delta_\epsilon) \sim (1-\beta)\beta^{\beta/(1-\beta)}\delta_\epsilon^{-\beta/(1-\beta)} \quad \text{as } \delta_\epsilon \to 0.$$

Thus as $\epsilon \to 0$,

$$-\log \mathbb{P}(\gamma(1) \le \epsilon) = -\log \mathbb{P}(U(1) \le \delta_\epsilon)$$
$$\sim (1-\beta)\beta^{\beta/(1-\beta)}(a^{1/\beta}\epsilon)^{-\beta/(1-\beta)}$$
$$= [(1-\beta)\beta^{\beta/(1-\beta)}a^{-1/(1-\beta)}]\epsilon^{-\beta/(1-\beta)}.$$

Thus, by letting $A = [(1-\beta)\beta^{\beta/(1-\beta)}a^{-1/(1-\beta)}]$ and $\alpha = \frac{\beta}{(1-\beta)}$, we can apply the Tauberian theorem that tells us, after simplifying,

$$-\frac{1}{t}\log \mathbb{E}[e^{-(bt^{1/\beta})\gamma(1)}] \sim \frac{1}{t}(1+\alpha)\alpha^{-\alpha/(1+\alpha)}A^{1/(1+\alpha)}(bt^{1/\beta})^{\alpha/(1+\alpha)}$$
$$= \frac{1}{t}\left(\frac{1}{1-\beta}\right)\left(\frac{\beta}{1-\beta}\right)^{-\beta}\left[(1-\beta)\beta^{\beta/(1-\beta)}a^{-1/(1-\beta)}\right]^{(1-\beta)}(bt^{1/\beta})^\beta$$
$$= \left(\frac{1}{1-\beta}\right)^{1-\beta}\beta^{-\beta}[(1-\beta)^{1-\beta}\beta^\beta a^{-1}]b^\beta$$
$$= a^{-1}b^\beta \quad \text{as } bt^{1/\beta} \to \infty.$$

Therefore,

$$\lim_{t\to\infty}\frac{1}{t}\log(e^t\mathbb{E}[e^{-b\gamma(t)}]) = \lim_{t\to\infty}\frac{1}{t}\{t+\log\mathbb{E}[e^{-(bt^{1/\beta}\gamma(1)}]\}$$
$$= \lim_{t\to\infty}(1-a^{-1}b^\beta) < 0 \quad \text{if } b^\beta > a,$$

as desired. □

4.6 Almost Sure Convergence of $X \circ \gamma(t)$

So far, we have looked at the properties of the time-changed solution $X \circ \gamma(t)$. However, we would like to know about the behavior of the mild solution $X(t)$ without any time change. A stepping stone to knowing more about the original mild solution is to look at the almost sure convergence of $X \circ \gamma(t)$. Here we assume that $R(t) := aE_t + t^\beta$ and γ is the inverse of R. The following results were pointed out to me by Kei Kobayashi.

First consider the following lemma.

Lemma 4.6.1 *If $Y_n \to 0$ in L^2 and $\sum_n \mathbb{E}[|Y_n|^2] < \infty$, then $Y_n \to 0$ almost surely.*

Proof: For arbitrary $\epsilon > 0$,

$$\sum_n \mathbb{E}[|Y_n|^2] \geq \sum_n \mathbb{E}[|Y_n|^2 1_{|Y_n|>\epsilon}]$$
$$\geq \epsilon^2 \sum_n \mathbb{P}(|Y_n| > \epsilon).$$

Thus, since $\sum_n \mathbb{E}[|Y_n|^2] < \infty$ by assumption, $\sum_n \mathbb{P}(|Y_n| > \epsilon) < \infty$. By Borel-Cantelli, it follows that $\mathbb{P}(|Y_n| > \epsilon \text{ infinitely often}) = 0$, and it is known that this is equivalent to $Y_n \to 0$ a.s. □

We can now use our previous result about the exponential stability of our time-changed solution $X \circ \gamma(t)$ to get the following theorem.

Theorem 4.6.2 *If $\{t_n\}$ is a sequence with $t_n \geq cn$ for some constant $c > 0$, then $X \circ \gamma(t_n) \to 0$ almost surely, provided that $b^\beta > a$.*

Proof: Recall from Theorem 4.5.3 that $X \circ \gamma(t_n) \to 0$ in L^2. Thus, using Lemma 4.6.1, it suffices to show that $\sum_n \mathbb{E}[|X \circ \gamma(t_n)|^2] < \infty$.

Now, in proving Theorem 4.5.3, we computed the following upper bound

$$\mathbb{E}[|X \circ \gamma(t_n)|^2] \leq 2e^t \mathbb{E}[e^{-b\gamma(t_n)}]|x_0|^2$$

where $-\log \mathbb{E}[e^{-b\gamma(t)}] \sim a^{-1}b^\beta t$ as $t \to \infty$.

So, by definition of $\sim$,

$$\lim_{n\to\infty} \frac{-\log \mathbb{E}[e^{-b\gamma(t_n)}]}{a^{-1}b^\beta t_n} = 1$$

which implies that for arbitrary $0 < \epsilon < 1$, there exists an $N \in \mathbb{N}$ such that

$$1 - \epsilon < \frac{-\log \mathbb{E}[e^{-b\gamma(t_n)}]}{a^{-1}b^\beta t_n} < 1 + \epsilon \implies e^{t_n}\mathbb{E}[e^{-b\gamma(t_n)}] < e^{\{1-(1-\epsilon)a^{-1}b^\beta\}t_n}$$

for all $n \geq N$.

Now, recall our assumption that $b^\beta > a$, which was necessary in showing Theorem 4.5.3. This implies that $a^{-1}b^\beta > 1$, and thus we can choose some $\epsilon_0 \in (0,1)$ such that $a^{-1}b^\beta > \frac{1}{1-\epsilon_0} > 1$. Notice that this implies

$$(1-\epsilon_0)(a^{-1}b^\beta) > 1 \implies e^{(1-\epsilon_0)(a^{-1}b^\beta)-1} > 1 \implies 0 \leq \left(\frac{1}{e^{(1-\epsilon_0)(a^{-1}b^\beta)-1}}\right) < 1.$$

So, for this ϵ_0, there exists an $N_0 \in \mathbb{N}$ such that for $n \geq N$,

$$e^{t_n}\mathbb{E}[e^{-b\gamma(t_n)}] < \left(\frac{1}{e^{(1-\epsilon_0)a^{-1}b^\beta-1}}\right)^{t_n} < \left(\frac{1}{e^{\{(1-\epsilon_0)a^{-1}b^\beta-1\}c}}\right)^n$$

since $t_n \geq cn$. Thus,

$$\sum_{n=N_0}^{\infty} e^{t_n}\mathbb{E}[e^{-b\gamma(t_n)}] < \sum_{n=N_0}^{\infty} \left(\frac{1}{e^{\{(1-\epsilon_0)a^{-1}b^\beta-1\}c}}\right)^n < \infty,$$

completing the proof. □

Chapter 5

Fokker-Planck-Kolmogorov equations associated with time-changed SDEs

Recall from Chapter 2 that the time-changed Q-Wiener process, like a time-changed Brownian motion, is a sub-diffusion process. Rather than having an associated density function, however, it induces a measure since it is a Hilbert-space-valued process. It is this measure that solves a Fokker-Planck-Kolmogorov (FPK) equation.

In finite dimensional spaces, FPK equations are important for two reasons. First, they are deterministic equations that scientists might derive to study a phenomenon. The fact that the deterministic equation is linked with a stochastic process allows the scientists to also use information about the stochastic process to study their phenomenon. Conversely, knowing FPK equations corresponding to a stochastic process is useful for simulation, see [16, 18, 19].

More recently, FPK equations for diffusion processes in infinite dimensional spaces, more precisely on Hilbert spaces, have been analyzed, see [4–6]. We will begin by looking at these cases, where SDEs driven by the Q-Wiener process give rise to FPK equations whose solutions are measures induced by the solutions of the SDEs. We then turn to SDEs driven by the time-changed Q-Wiener process and look at their associated FPK equations. These results were joint work with P. Garmirian and Q. Wu.

Now consider the following classic SDE driven by the Q-Wiener process

$$\begin{cases} dY(t) = [AY(t) + F(t, Y(t))]\mathrm{d}t + C\mathrm{d}W_t \\ \\ Y(0) = x \in H \end{cases} \tag{5.1}$$

where $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $S(t) = e^{tA}$,

$t \geq 0$, in H, and W_t is a K-valued Q-Wiener process on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, \mathbb{P})$ with the filtration $\mathcal{F}_t$ satisfying the usual conditions. Suppose that $F : \Omega \times [0,T] \times H \to H$, $C : \Omega \times K \to H$ and $C \in \Lambda_2(K_Q, H)$. Also assume the initial condition x is an $\mathcal{F}_0$-measurable, H-valued random variable. Let $Y(t)$ be a strong solution of the SDE (5.1), which means $Y(t)$ satisfies the following integral equation

$$Y(t) = x + \int_0^t [AY(s) + F(s, Y(s))]ds + CW_t.$$

The Kolmogorov operator L_0 corresponding to the classic SDE (5.1) is

$$\begin{aligned} L_0\phi(x) = \langle x, A^* D_x\phi(x)\rangle_H + \langle F(t,x), D_x\phi(x)\rangle_H \\ + \frac{1}{2}tr[(CQ^{1/2})(CQ^{1/2})^* D_x^2\phi(x)], \end{aligned} \tag{5.2}$$

where $x \in H$, $t \in [0,T]$, and D_x, D_x^2 denote the first- and second-order Fréchet derivatives in space, respectively. $D(L_0)$ denotes the domain of the operator L_0 and A^* denotes the adjoint of the operator A. For more details on the domain $D(L_0)$, please refer to [4–6]. Let $\mu(\mathrm{d}t, \mathrm{d}x)$ be a product measure on $[0,T] \times H$ of the type

$$\mu(\mathrm{d}t, \mathrm{d}x) = \mu_t(\mathrm{d}x)\mathrm{d}t,$$

where $\mu_t \in \mathbb{P}(H)$ is a Borel probability measure on the Hilbert space H for all $t \in [0,T]$. According to [6], it is possible to define the measure μ_t^Y induced by the solution $Y(t)$ as

$$\mu_t^Y(dy) := (P_t^Y)^*\xi(dy), \tag{5.3}$$

where $\xi \in \mathbb{P}(H)$ is the measure associated with the initial condition x. Also $(P_t^Y)^*$ is the adjoint operator of the transition evolution operator, P_t^Y, defined on the bounded Borel space of H, i.e., $\mathcal{B}_b(H)$, by

$$P_t^Y\phi(x) = \mathbb{E}(\phi(Y(t))|Y(0) = x), \; 0 \leq t \leq T, \; \phi \in \mathcal{B}_b(H), \tag{5.4}$$

which is a semigroup generated by the Markov process $Y(t)$. The induced measure, $\mu_t^Y(dy)$, is interpreted as

$$\int_H \phi(y)\mu_t^Y(dy) = \int_H P_t^Y \phi(y)\xi(dy), \text{ for all } \phi \in \mathcal{B}_b(H). \tag{5.5}$$

Under the assumption

$$\int_{[0,T]\times H} \Big(|\langle y, A^* h\rangle_H| + \|F(t,y)\|_H \Big) \mu(\mathrm{d}t, \mathrm{d}y) < \infty,$$

where $h \in D(A^*)$, the induced measure μ_t^Y satisfies the following FPK equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_H \phi(y)\mu_t^Y(dy) = \int_H L_0\phi(y)\mu_t^Y(dy), \text{ for } dt\text{-a.e., } t \in [0,T], \tag{5.6}$$

where the initial condition is

$$\lim_{t\to 0} \int_H \phi(y)\mu_t^Y(\mathrm{d}y) = \int_H \phi(y)\xi(dy). \tag{5.7}$$

Furthermore, if the domain of the Kolmogorov operator L_0 is comprised of test functions, applying integration by parts yields

$$\frac{\partial}{\partial t}\mu_t^Y = L_0^* \mu_t^Y, \ \mu_0^Y = \xi. \tag{5.8}$$

See [6] for further details of the above discussion. The following lemma is needed to extend the FPK equations (5.6) and (5.7) or (5.8) associated to the solution of the classic SDE (5.1) to the case of the solution to an SDE driven by a time-changed Q-Wiener process.

Lemma 5.0.3 *Let $U_\beta(t)$ be a β-stable subordinator with the cumulative distribution function $F_\tau(t) = \mathbb{P}(U_\beta(\tau) \le t)$ and density function $f_\tau(t)$. Suppose the inverse of $U_\beta(t)$ is E_t with the density function $f_{E_t}(\tau)$. Then, for any integrable function $h(\tau)$*

on $(0,\infty)$*, the function* $q(t)$ *defined by the following integral*

$$q(t) := \int_0^\infty f_{E_t}(\tau)h(\tau)d\tau$$

has Laplace transform

$$\mathcal{L}_{t\to s}\{q(t)\} = s^{\beta-1}[\widetilde{h(\tau)}](s^\beta),$$

where $\widetilde{h(\tau)}(s) = \mathcal{L}_{\tau\to s}\{h(\tau)\}$*.*

Proof: Using the self-similarity property of the β-stable subordinator $U_\beta(t)$, the distribution function $F_{E_t}(\tau)$ associated with the time change E_t is

$$\begin{aligned} F_{E_t}(\tau) &= \mathbb{P}(E_t \le \tau) = \mathbb{P}(U_\beta(\tau) > t) = 1 - \mathbb{P}(\tau^{1/\beta}U_\beta(1) \le t) \\ &= 1 - \mathbb{P}\left(U_\beta(1) \le \frac{t}{\tau^{1/\beta}}\right) = 1 - F_1\left(\frac{t}{\tau^{1/\beta}}\right). \end{aligned} \tag{5.9}$$

By differentiating both sides of (5.9),

$$f_{E_t}(\tau) = -\frac{\partial}{\partial\tau}\left\{\frac{1}{\tau^{1/\beta}}(Jf_1)\left(\frac{t}{\tau^{1/\beta}}\right)\right\},$$

where J is an integral operator defined by

$$(Jf)\left(\frac{t}{a}\right) = \int_0^t f\left(\frac{s}{a}\right)\mathrm{d}s \text{ for all } a > 0. \tag{5.10}$$

Therefore, the Laplace transform

$$\begin{aligned} \mathcal{L}_{t\to s}\left[\frac{1}{a}(Jf_1)\left(\frac{t}{a}\right)\right](s) &= \frac{1}{a}\int_0^\infty (Jf_1)\left(\frac{t}{a}\right)e^{-st}dt = \frac{1}{as}\int_0^\infty f_1\left(\frac{t}{a}\right)e^{-st}dt \\ &= \frac{1}{s}\int_0^\infty f_1(\hat{t})e^{-as\hat{t}}d\hat{t} = \frac{1}{s}\tilde{f}_1(as), \end{aligned} \tag{5.11}$$

where $\widetilde{f_1}(s)$ is the Laplace transform of the function $f(t)$. Now consider

$$q(t) := \int_0^\infty f_{E_t}(\tau)h(\tau)d\tau = -\int_0^\infty \frac{\partial}{\partial\tau}\left\{\frac{1}{\tau^{1/\beta}}(Jf_1)\left(\frac{t}{\tau^{1/\beta}}\right)\right\}h(\tau)d\tau.$$

Applying the Laplace transform of the β-stable subordinator, $U_\beta(t)$, in (2.2) and using (5.11) yields

$$\mathcal{L}_{t\to s}\{q(t)\} = -\int_0^\infty \frac{\partial}{\partial\tau}\left\{\frac{1}{s}e^{-\tau s^\beta}\right\} h(\tau)d\tau = s^{\beta-1}[\widetilde{h(\tau)}](s^\beta).$$

thereby completing the proof. □

Consider the following type of autonomous SDE driven by the time-changed Q-Wiener process on H over the time interval $[0,T]$:

$$\begin{cases} \mathrm{d}X(t) = (AX(t) + F(X(t)))\mathrm{d}E_t + C\mathrm{d}W_{E_t}, \\ \\ X(0) = x \in H, \quad t \geq 0. \end{cases} \tag{5.12}$$

where the coefficients A, F and C are the same as in the classic SDE (5.1). W_{E_t} is a time-changed Q-Wiener process on a complete filtered probability space $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t\leq T}, \mathbb{P})$ with the filtration $\mathcal{G}_t := \tilde{\mathcal{F}}_{E_t}$ satisfying the usual conditions. Suppose that the initial condition x is an $\mathcal{F}_0$-measurable and H-valued random variable. We derive the time-fractional FPK equation associated to the time-changed SDE (5.12) via two different methods: first by applying the time-changed Itô formula and second by using the duality in Theorem 3.3.3. The advantage of the first approach is that it directly reveals the connection between the time-fractional FPK equations in Theorems 5.0.4 and 5.0.5, and the time-changed SDE (5.12). The advantage of the second approach is that it reveals the connection between the time-fractional FPK equations in Theorems 5.0.4 and 5.0.5, and the classic FPK equations in (5.6) and (5.7) or (5.8).

Theorem 5.0.4 *(Chlebak, Garmirian, Wu) Suppose the coefficients A, F and C of the time-changed SDE* (5.12) *satisfy the conditions in Theorem 3.3.4. Let $X(t)$ be the solution to* (5.12). *Also suppose that $X(U_\beta(t))$ is independent of E_t. Then the probability kernel $\mu_t^X(\mathrm{d}x)$ induced by the solution $X(t)$ satisfies the following* *fractional integral equation*

$$D_t^{\beta} \int_H \phi(x)\mu_t^X(\mathrm{d}x) = \int_H L_0\phi(x)\mu_t^X(\mathrm{d}x), \tag{5.13}$$

with initial condition $\mu_0^X(\mathrm{d}x) = \xi(\mathrm{d}x)$*, where* $\phi \in D(L_0)$*,* L_0 *is the Kolmogorov operator defined in* (5.2) *and* D_t^{β} *denotes the Caputo fractional derivative operator as defined in Theorem 2.0.5.*

Proof: (Method of applying the time-changed Itô formula) Let $Y(t) := X(U_t)$. Since E_t and W_{E_t} both are constant over $[U_{t-}, U_t]$, the integrals

$$\int_0^t (AX(s) + F(X(s)))\mathrm{d}E_s \quad \text{and} \quad \int_0^t B(X(s))\mathrm{d}W_{E_s}$$

are constant over $[U_{t-}, U_t]$. Due to the SDE (5.12), its solution $X(t)$ is also constant over $[U_{t-}, U_t]$ and satisfies $Y(E_t) = X(U_{E_t}) = X(t)$. So, for a fixed time $t \in [0, T]$, let μ_t^X and μ_t^Y denote the probability measures induced on H by the stochastic processes $X(t)$ and $Y(t)$, respectively. Thus, for $\phi \in \mathrm{D}(L_0)$,

$$\mathbb{E}(\phi(X(t))) = \int_H \phi(x)\mu_t^X(\mathrm{d}x). \tag{5.14}$$

Since $X(t) = Y(E_t)$, taking the expectation of $X(t)$ conditionally on E_t,

$$\mathbb{E}(\phi(X(t))) = \mathbb{E}(\phi(Y(E_t))) = \int_0^\infty \mathbb{E}(\phi(Y_\tau)|E_t = \tau) f_{E_t}(\tau)\mathrm{d}\tau,$$

where $f_{E_t}(\tau)$ is the density function of E_t. By the assumption that $Y(t) = X(U_t)$ is independent of E_t,

$$\mathbb{E}(\phi(X(t))) = \int_0^\infty \int_H \phi(x)\mu_\tau^Y(\mathrm{d}x) f_{E_t}(\tau)\mathrm{d}\tau = \int_H \phi(x) \int_0^\infty \mu_\tau^Y(\mathrm{d}x) f_{E_t}(\tau)\mathrm{d}\tau. \tag{5.15}$$

Since $\phi \in \mathrm{D}(L_0)$ is arbitrary, combining (5.14) and (5.15) yields

$$\mu_t^X(\mathrm{d}x) = \int_0^\infty \mu_\tau^Y(\mathrm{d}x) f_{E_t}(\tau)\mathrm{d}\tau. \tag{5.16}$$

Again, the solution $X(t)$ to (5.12) is constant on every interval $[U_{r-}, U_r]$, i.e., $X(U_{r-}) = X(U_r) = Y(r)$. Thus, by the time-changed Itô formula,

$$\begin{aligned}\phi(X(t)) - \phi(x_0) &= \int_0^{E_t} L_0\phi(X(U_{r-}))\mathrm{d}r + \int_0^{E_t} \langle \phi_x(X(U_{r-})), C\mathrm{d}W_r\rangle \\ &= \int_0^{E_t} L_0\phi(Y(r))\mathrm{d}r + \int_0^{E_t} \langle \phi_x(Y(r)), C\mathrm{d}W_r\rangle.\end{aligned} \tag{5.17}$$

Again since $\phi \in \mathrm{D}(L_0)$, the integral

$$M(\tau) = \int_0^{\tau} \langle \phi_x(Y(r)), C\mathrm{d}W_r\rangle_H$$

is a square integrable $\mathcal{F}_\tau$-martingale. Taking expectations on both sides of (5.17) and conditioning on E_t gives

$$\begin{aligned}&\mathbb{E}[\phi(X(t))|X(0) = x_0] - \phi(x_0) \\ &\quad= \int_0^{\infty} \mathbb{E}[\int_0^{\tau} L_0\phi(Y(r))\mathrm{d}r + M(\tau)|E_t = \tau, X(0) = x_0] f_{E_t}(\tau)\mathrm{d}\tau \\ &\quad= \int_0^{\infty}\int_0^{\tau} \mathbb{E}[L_0\phi(Y(r))|X(0) = x_0]\mathrm{d}r f_{E_t}(\tau)\mathrm{d}\tau \\ &\quad= \int_0^{\infty}\int_0^{\tau}\int_H L_0\phi(x)\mu_r^Y(\mathrm{d}x)\mathrm{d}r f_{E_t}(\tau)\mathrm{d}\tau \\ &\quad= \int_0^{\infty} \Big(JP^Y(\tau)\Big) f_{E_t}(\tau)\mathrm{d}\tau,\end{aligned} \tag{5.18}$$

where J is the integral operator as defined in (5.10) and $P^Y(r)$ is defined by

$$P^Y(r) = \int_H L_0\phi(x)\mu_r^Y(\mathrm{d}x).$$

On the other hand, computing the left side of (5.18) yields

$$\begin{aligned}&\mathbb{E}[\phi(X(t))|X(0) = x_0] - \mathbb{E}[\phi(x_0)|X(0) = x_0] \\ &\qquad\qquad= \int_H \phi(x)\mu_t^X(\mathrm{d}x) - \int_H \phi(x)\xi(\mathrm{d}x),\end{aligned} \tag{5.19}$$

which also implies the initial condition $\mu_0^X(\mathrm{d}x) = \xi(\mathrm{d}x)$. Using (5.18) and (5.19) together with applying Lemma 5.0.3 yield

$$\mathcal{L}_{t\to s}\left\{\int_H \phi(x)\mu_t^X(\mathrm{d}x)\right\} - \frac{1}{s}\int_H \phi(x)\xi(\mathrm{d}x) \\ = s^{\beta-1}[J_t\widetilde{P^Y(\tau)}](s^\beta) = \frac{s^{\beta-1}}{s^\beta}[\widetilde{P^Y(\tau)}](s^\beta),$$

which indicates

$$s^\beta\mathcal{L}_{t\to s}\left\{\int_H \phi(x)\mu_t^X(\mathrm{d}x)\right\} - s^{\beta-1}\int_H \phi(x)\xi(\mathrm{d}x) = s^{\beta-1}[\widetilde{P^Y(\tau)}](s^\beta). \tag{5.20}$$

Using (5.16) and Fubini's Theorem yield

$$\begin{aligned} \int_H L_0\phi(x)\mu_t^X(\mathrm{d}x) &= \int_H L_0\phi(x)\int_0^\infty \mu_\tau^Y(\mathrm{d}x)f_{E_t}(\tau)\mathrm{d}\tau \\ &= \int_0^\infty\int_H L_0\phi(x)\mu_\tau^Y(\mathrm{d}x)f_{E_t}(\tau)\mathrm{d}\tau \\ &= \int_0^\infty P^Y(\tau)f_{E_t}(\tau)\mathrm{d}\tau. \end{aligned} \tag{5.21}$$

Take Laplace transforms on both sides of (5.21) to yield

$$\begin{aligned} \mathcal{L}_{t\to s}\left\{\int_H L_0\phi(x)\mu_t^X(\mathrm{d}x)\right\} &= \mathcal{L}_{t\to s}\left\{\int_0^\infty P^Y(\tau)f_{E_t}(\tau)\mathrm{d}\tau\right\} \\ &= s^{\beta-1}[\widetilde{P^Y(\tau)}](s^\beta). \end{aligned} \tag{5.22}$$

Recall that the Laplace transform for the Caputo fractional derivative is

$$\mathcal{L}_{t\to s}\{D_t^\beta f(t)\} = s^\beta\mathcal{L}_{t\to s}\{f(t)\} - s^{\beta-1}f(0), \tag{5.23}$$

where $f(t)$ is a real-valued function on $t \geq 0$. Therefore, combining (5.20) and (5.22) yields

$$s^\beta\mathcal{L}_{t\to s}\left\{\int_H \phi(x)\mu_t^X(\mathrm{d}x)\right\} - s^{\beta-1}\int_H \phi(x)\xi(\mathrm{d}x) = \mathcal{L}_{t\to s}\left\{\int_H L_0\phi(x)\mu_t^X(\mathrm{d}x)\right\},$$

which together with (5.23) implies that

$$D_t^\beta\int_H \phi(x)\mu_t^X(\mathrm{d}x) = \int_H L_0\phi(x)\mu_t^X(\mathrm{d}x),$$

thereby completing the proof. □

The next theorem gives the familiar differential form of the FPK equation for the solution to the time-changed SDE (5.12).

Theorem 5.0.5 *(Chlebak, Garmirian, Wu) Suppose the conditions in Theorem 5.0.4 hold. If the domain of the operator L_0 defined in* (5.2) *is a set of test functions, then the probability measure μ_t^X induced by the solution $X(t)$ satisfies the following time-fractional PDE*

$$D_t^\beta \mu_t^X = L_0^* \mu_t^X, \tag{5.24}$$

with initial condition $\mu_0^X(\mathrm{d}x) = \xi(\mathrm{d}x)$, where L_0^* is the adjoint of the operator L_0 and D_t^β denotes the Caputo fractional derivative operator.

Proof: From the proof of Theorem 5.0.4,

$$\int_H \phi(x)\mu_t^X(\mathrm{d}x) - \int_H \phi(x)\xi(\mathrm{d}x) = \int_0^\infty \int_0^\tau \int_H L_0\phi(x)\mu_r^Y(\mathrm{d}x)\mathrm{d}r f_{E_t}(\tau)\mathrm{d}\tau.$$

Since $\phi \in D(L_0)$ is a test function, applying the integration by parts operator yields

$$\int_H \phi(x)\mu_t^X(\mathrm{d}x) - \int_H \phi(x)\xi(\mathrm{d}x) = \int_H \phi(x) \int_0^\infty \int_0^\tau L_0^*\mu_r^Y(\mathrm{d}x)\mathrm{d}r f_{E_t}(\tau)\mathrm{d}\tau,$$

which means

$$\mu_t^X(\mathrm{d}x) - \xi(\mathrm{d}x) = \int_0^\infty \int_0^\tau L_0^*\mu_r^Y(\mathrm{d}x)\mathrm{d}r f_{E_t}(\tau)\mathrm{d}\tau. \tag{5.25}$$

For more details on the adjoint operator L_0^*, please see [13]. Also, from the proof of Theorem 5.0.4, the following equality holds

$$\mu_t^X(\mathrm{d}x) = \int_0^\infty \mu_\tau^Y(\mathrm{d}x) f_{E_t}(\tau)\mathrm{d}\tau. \tag{5.26}$$

Similarly, from the Laplace transforms for (5.25) and (5.26),

$$s^{\beta}\mathcal{L}_{t\to s}\Big\{\mu_t^X(\mathrm{d}x)\Big\} - s^{\beta-1}\xi(\mathrm{d}x) = \mathcal{L}_{t\to s}\Big\{L_0^*\mu_t^X(\mathrm{d}x)\Big\},$$

which implies

$$D_t^{\beta}\mu_t^X = L_0^*\mu_t^X,$$

with initial condition $\mu_0^X(\mathrm{d}x) = \xi(\mathrm{d}x)$. This completes the proof. □

We now use the second approach based on duality to derive the FPK equation for the solution to the time-changed SDEs (5.12).

Theorem 5.0.6 *(Chlebak, Garmirian, Wu) Suppose the FPK equations* (5.6) *and* (5.7) *or* (5.8) *for the classic SDE* (5.1) *are known and the conditions in Theorem 5.0.4 hold. Then, the time-fractional FPK equation associated with the time-changed SDE* (5.12) *follows* (5.13)*. Also if the domain of the operator L_0 defined in* (5.2) *is a set of test functions, the time-fractional FPK equation has the form* (5.24).

Proof: (Method of applying the duality result) From the duality theorem, Theorem 3.3.3, the solution of the time-changed SDE (5.12) is

$$X(t) = Y(E_t), \tag{5.27}$$

where $Y(t)$ is the solution to the classic SDE (5.1). Similarly to (5.4) and (5.3), define the transition evolution operator, P_t^X, induced by the solution, $X(t)$, as follows:

$$\begin{aligned} P_t^X\phi(x) &= E(\phi(X(t))|X(0) = x_0) = E(\phi(Y(E_t))|X(0) = x_0) \\ &= \int_0^{\infty} P_\tau^Y\phi(x) f_{E_t}(\tau)\mathrm{d}\tau, \qquad 0 \le t \le T,\ \phi \in \mathcal{B}_b(H). \end{aligned} \tag{5.28}$$

The probability measure, $\mu_t^X(\mathrm{d}x)$, induced by $X(t)$ is

$$\mu_t^X(\mathrm{d}x) := (P_t^X)^*\xi(\mathrm{d}x), \tag{5.29}$$

which means for all $\phi \in \mathcal{B}_b(H)$,

$$\int_H \phi(x)\mu_t^X(dx) := \int_H P_t^X \phi(x)\xi(dx) = \int_H \int_0^\infty P_\tau^Y \phi(x)\xi(\mathrm{d}x) f_{E_t}(\tau)d\tau. \quad (5.30)$$

Therefore, the connection between the probability measures, $\mu_t^X(\mathrm{d}x)$ and $\mu_t^Y(\mathrm{d}y)$, is obtained from (5.30) by applying Fubini's theorem

$$\int_H \phi(x)\mu_t^X(dx) = \int_0^\infty \int_H \phi(x)\mu_\tau^Y(\mathrm{d}x) f_{E_t}(\tau)d\tau, \quad \text{for all } \phi \in \mathcal{B}_b(H), \quad (5.31)$$

or

$$\mu_t^X(dx) = \int_0^\infty \mu_\tau^Y(\mathrm{d}x) f_{E_t}(\tau)d\tau, \quad \text{for all } \phi \in \mathcal{B}_b(H). \quad (5.32)$$

From Lemma 5.0.3, taking Laplace transforms on both sides of (5.31) produces

$$\mathcal{L}_{t\to s}\left\{ \int_H \phi(x)\mu_t^X(dx)\right\} = s^{\beta-1}\mathcal{L}_{\tau\to s}\left\{ \int_H \phi(x)\mu_\tau^Y(dx)\right\}(s^\beta). \quad (5.33)$$

On the other hand, taking Laplace transforms on both sides of (5.6) leads to

$$s\mathcal{L}_{t\to s}\left\{ \int_H \phi(y)\mu_t^Y(dy)\right\} - \int_H \phi(y)\xi(\mathrm{d}y) = \mathcal{L}_{t\to s}\left\{ \int_H L_0\phi(y)\mu_t^Y(dy)\right\}. \quad (5.34)$$

Replace s by s^β in (5.34) to yield

$$\begin{aligned} s^\beta \mathcal{L}_{t\to s}\left\{ \int_H \phi(y)\mu_t^Y(dy)\right\}(s^\beta) - \int_H \phi(y)\xi(\mathrm{d}y) \\ = \mathcal{L}_{t\to s}\left\{ \int_H L_0\phi(y)\mu_t^Y(dy)\right\}(s^\beta). \end{aligned} \quad (5.35)$$

Thus, combining (5.33) and (5.35) gives

$$\begin{aligned} s^\beta \mathcal{L}_{t\to s}\left\{ \int_H \phi(x)\mu_t^X(dx)\right\} - s^{\beta-1}\int_H \phi(y)\xi(\mathrm{d}y) \\ = s^{\beta-1}\mathcal{L}_{t\to s}\left\{ \int_H L_0\phi(y)\mu_t^Y(dy)\right\}(s^\beta), \end{aligned}$$

which implies

$$
\begin{aligned}
D_t^\beta \int_H \phi(x)\mu_t^X(dx) &= \int_0^\infty \int_H L_0\phi(y)\mu_\tau^Y(dy) f_{E_t}(\tau)\mathrm{d}\tau \\
&= \int_H L_0\phi(y) \int_0^\infty \mu_\tau^Y(dy) f_{E_t}(\tau)\mathrm{d}\tau \qquad (5.36) \\
&= \int_H L_0\phi(y)\mu_t^X(dy),
\end{aligned}
$$

thereby deriving the desired result. Furthermore, if the domain of the Kolmogorov operator, L_0, is comprised of test functions, taking Laplace transforms on both sides of (5.8) yields

$$
s\mathcal{L}_{t\to s}\left\{\mu_t^Y\right\} - \mu_0^Y = \mathcal{L}_{t\to s}\left\{L_0^*\mu_t^Y\right\}. \qquad (5.37)
$$

Also, taking Laplace transforms on both sides of (5.32) and applying Lemma 5.0.3 yield

$$
\mathcal{L}_{t\to s}\left\{\mu_t^X\right\} = s^{\beta-1}\mathcal{L}_{\tau\to s}\left\{\mu_\tau^Y\right\}(s^\beta). \qquad (5.38)
$$

Similarly, combining (5.37) and (5.38) yields

$$
D_t^\beta \mu_t^X = L_0^*\mu_t^X,
$$

which completes the proof. □

Chapter 6

Connections to SPDEs

Although some explicit connections have been made between SDEs in Hilbert space and SPDEs (see [11] for cases involving the heat and wave equations), there is still much work to be done, even in the non-time-changed case. Here, we limit ourselves to showing a connection between three different "time-changed" integrals. In particular, the objective of this chapter is to make connections between the three integrals given below for appropriate integrands:

$$\int_0^T \int_{\mathbb{R}^N} g(t,x) M_E(\mathrm{d}t, \mathrm{d}x) = \int_0^T g(t) \mathrm{d}\tilde{W}_{E_t} = \int_0^T \Phi_t^g \circ J^{-1} \mathrm{d}W_{E_t},$$

where M_E is a time-changed version of a worthy martingale measure, $\tilde{W}_{E_t}$ is a time-changed version of a cylindrical Wiener process and W_{E_t} is a time-changed Q-Wiener process. Section 6.1 focuses on background material, namely what happens when there is no time change, whereas Section 6.2 provides the extension to the time-changed case. Again, these are results that arose from joint work with P. Garmirian and Q. Wu.

6.1 Case without a time change

The connections between integrals in the case where there is no time change are made in [11,21]. The general idea is to first define a specific random field F and an associated Hilbert space K. After deriving these objects, the integral with respect to the resulting cylindrical Wiener process and the integral with respect to the developed martingale measure are shown to be the same for a particular class of integrands. Also a connection between the cylindrical Wiener process and a Q-Wiener process is made that leads to an equivalence of their respective integrals. Thus, a combination of those results proves that all three integrals are equal.

Let $\{F(\phi) \mid \phi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^N)\}$ be a family of mean zero Gaussian random variables called a Gaussian random field. Define the covariance of F by

$$\mathbb{E}(F(\phi)F(\psi)) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(s,x)f(x-y)\psi(s,y)\mathrm{d}y\mathrm{d}x\mathrm{d}s, \tag{6.1}$$

where f is a non-negative, non-negative definite continuous function on $\mathbb{R}^N\backslash\{0\}$, which is integrable in a neighborhood of 0 and for all $\phi \in \mathcal{S}(\mathbb{R}^N)$, the space of C^∞ functions which are rapidly decreasing along with all their derivatives, the following conditions hold

1) there exists a measure μ on $\mathbb{R}^N$ and $m \in \mathbb{N}^+$ such that $\int_{\mathbb{R}^N}(1+|\xi|^2)^{-m}\mu(\mathrm{d}\xi) < \infty$, and
2) $\int_{\mathbb{R}^N} f(x)\phi(x)\mathrm{d}x = \int_{\mathbb{R}^N} \mathcal{F}\phi(\xi)\mu(\mathrm{d}\xi)$, where $\mathcal{F}\phi$ is the Fourier transform of ϕ.

The measure satisfying condition 2) above is called a tempered measure. For the remainder of Chapter 6, fix the specific random field F chosen in (6.1). It is now possible to associate a Hilbert space with this fixed random field F. Let K be the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ with the semi-inner product

$$\langle\phi,\psi\rangle_K := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(x)f(x-y)\psi(y)\mathrm{d}y\mathrm{d}x = \int_{\mathbb{R}^N} \mu(\mathrm{d}\xi)\mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)},$$

where $\phi,\psi \in \mathcal{S}(\mathbb{R}^N)$, and associated semi-norm $\|\cdot\|_K$. Then K is a Hilbert space, see [9]. Moreover, after fixing a time interval $[0,T]$, it is possible to consider the set $K_T := L^2([0,T],;K)$ with the norm

$$\|g\|_{K_T}^2 = \int_0^T \|g(s)\|_K^2\mathrm{d}s.$$

It should be noted that although F was originally defined on smooth, compactly supported functions, it is possible to extend F to functions of the form $1_{[0,T]}(\cdot)\varphi(*)$ where $\varphi \in \mathcal{S}(\mathbb{R}^N)$. This follows from F being a random linear functional such that $\phi \mapsto F(\phi)$ is an isometry from $(C_0^\infty([0,T]\times\mathbb{R}^N), \|\cdot\|_{K_T})$ into $L^2(\Omega,\mathcal{F},\mathbb{P})$ with $C_0^\infty([0,T]\times\mathbb{R}^N)$ dense in K_T. For further details, see [11].

Definition 6.1.1 *A cylindrical Wiener process on a Hilbert space K is a family of random variables $\{\tilde{W}_t, t \geq 0\}$ such that:*

1. *for each $h \in K$, $\{\tilde{W}_t(h), t \geq 0\}$ defines a Brownian motion with mean 0 and variance $t\langle h, h\rangle_K$;*

2. *for all $s, t \in \mathbb{R}^+$ and $h, g \in K$, $\mathbb{E}(\tilde{W}_s(h)\tilde{W}_t(g)) = (s \wedge t)\langle h, g\rangle_K$ where $s \wedge t :=$ $\min(s,t)$.*

From [11], the stochastic process $\{\tilde{W}_t, t \geq 0\}$ defined in terms of the fixed random field F with covariance chosen in (6.1) and given as follows

$$\tilde{W}_t(\varphi) := F(1_{[0,t]}(\cdot)\varphi(*)), \text{ for } \varphi \in K, \tag{6.2}$$

is a cylindrical Wiener process on the Hilbert space K. A complete orthonormal basis $\{f_j\}$ can be chosen such that $\{f_j\} \subset \mathcal{S}(\mathbb{R}^N)$ since $\mathcal{S}(\mathbb{R}^N)$ is a dense subspace of K. Consider the space $L^2(\Omega \times [0,T]; K)$ of predictable processes g such that

$$\mathbb{E}\left(\int_0^T \|g(s)\|_K^2 \mathrm{d}s\right) < \infty.$$

For $g \in L^2(\Omega \times [0,T]; K)$, the integral with respect to the cylindrical Wiener process $\tilde{W}_t$ is defined as

$$\int_0^T g(s)\mathrm{d}\tilde{W}_s := \sum_{j=1}^{\infty} \int_0^T \langle g(s), f_j\rangle_K \mathrm{d}\tilde{W}_s(f_j)$$

where $\{f_j\}$ is an orthonormal basis of K. The series is convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, and the sum does not depend on the choice of orthonormal basis. Additionally, the following isometry holds:

$$\mathbb{E}\left(\left(\int_0^T g(s)\mathrm{d}\tilde{W}_s\right)^2\right) = \mathbb{E}\left(\int_0^T \|g(s)\|_K^2 \mathrm{d}s\right).$$

On the other hand, consider M such that

$$M_t(A) := F(1_{[0,t]}(\cdot)1_A(*)), \quad t \in [0,T], A \in \mathcal{B}_b(\mathbb{R}^N), \tag{6.3}$$

where F is the specific random field chosen in (6.1) and $\mathcal{B}_b(\mathbb{R}^N)$ denotes the set of bounded Borel sets of $\mathbb{R}^N$. The covariance of $\{M_t(A)\}$ is given by:

$$\begin{aligned}\mathbb{E}(M_t(A)M_t(B)) &= \int_{\mathbb{R}^+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} 1_{[0,t]}(s)1_A(x)f(x-y)1_B(y)\mathrm{d}y\mathrm{d}x\mathrm{d}s \\ &= t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} 1_A(x)f(x-y)1_B(y)\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Therefore, M_t defined as in (6.3) is a martingale measure in the following sense.

Definition 6.1.2 *([11]) A process $\{M_t(A)\}_{t\geq 0, A\in\mathcal{B}(\mathbb{R}^N)}$ is a martingale measure with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if:*

1. *for all $A \in \mathcal{B}(\mathbb{R}^N)$, $M_0(A) = 0$ a.s.;*
2. *for $t > 0$, M_t is a sigma-finite $L^2(\mathbb{P})$-valued signed measure; and*
3. *for all $A \in \mathcal{B}(\mathbb{R}^N)$, $\{M_t(A)\}_{t\geq 0}$ is a mean-zero martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.*

Furthermore, in order to define stochastic integrals with respect to a martingale measure, the martingale measure needs to be worthy.

Definition 6.1.3 *([11]) A martingale measure M is worthy if there exists a random sigma-finite measure $K(A \times B \times C, \omega)$, where $A, B \in \mathcal{B}(\mathbb{R}^N)$, $C \in \mathcal{B}(\mathbb{R}_+)$, and $\omega \in \Omega$, such that:*

1. *$A \times B \times C \mapsto K(A \times B \times C, \omega)$ is nonnegative definite and symmetric;*
2. *$\{K(A \times B \times (0,t])\}_{t\geq 0}$ is a predictable process for all $A, B \in \mathcal{B}(\mathbb{R}^N)$;*
3. *for all compact sets $A, B \in \mathcal{B}(\mathbb{R}^N)$ and $t > 0$,*

$$\mathbb{E}(K(A \times B \times (0,t])) < \infty;$$

4. *for all $A, B \in \mathcal{B}(\mathbb{R}^N)$ and $t > 0$,*

$$|\mathbb{E}(M_t(A)M_t(B))| \leq \mathbb{K}(A \times B \times (0,t]) \text{ a.s.}$$

The martingale measure needs to be worthy in order to define stochastic integrals with respect to it. In particular, the stochastic integral with respect to a worthy martingale measure is defined in such a way that it is itself a martingale measure. First, consider elementary processes g of the form

$$g(s,x,\omega) = 1_{(a,b]}(s)1_A(x)X(\omega) \tag{6.4}$$

where $0 \le a < b \le T$, X is bounded and $\mathcal{F}_a$-measurable, and $A \in \mathcal{B}(\mathbb{R}^N)$. If g is an elementary process as in (6.4), then define $g \cdot M$ by

$$g \cdot M_t(B) := \int_0^t \int_B g(s,x)M(\mathrm{d}s,\mathrm{d}x) = X(\omega)\left(M_{t\wedge b}(A\cap B) - M_{t\wedge a}(A\cap B)\right).$$

The definition of $g \cdot M$ can be extended to simple processes, which are finite sums of elementary processes, by using linearity. Let $\mathcal{P}_+$ denote the set of predictable processes $(\omega,t,x) \mapsto g(t,x;\omega)$ such that

$$\|g\|_+^2 := \mathbb{E}\left(\int_0^T \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |g(t,x)|\ f(x-y)\ |g(t,y)|\mathrm{d}y\mathrm{d}x\mathrm{d}t\right) < \infty.$$

By taking limits of simple processes, the definition of $g \cdot M$ extends to all $g \in \mathcal{P}_+$. From [21], $g \cdot M$ is a worthy martingale measure if $g \in \mathcal{P}_+$. Therefore, it makes sense to define the stochastic integral with respect to M as a martingale measure in the following way:

$$\int_0^t \int_A g(s,x)M(\mathrm{d}s,\mathrm{d}x) =: g \cdot M_t(A).$$

The following proposition comes from Proposition 2.6 in [11] and a detailed proof can also be found there.

Proposition 6.1.4 *([11]) Suppose $g \in \mathcal{P}_+$. Then $g \in L^2(\Omega\times[0,T];K)$ and*

$$\int_0^T \int_{\mathbb{R}^N} g(t,x)M(\mathrm{d}t,\mathrm{d}x) = \int_0^T g(t)\mathrm{d}\tilde{W}_t,$$

where M is the worthy martingale measure defined in (6.3) *and $\tilde{W}_t$ is the cylindrical Wiener process defined in* (6.2).

Finally, Proposition 6.1.6 below provides conditions under which these integrals coincide with the integral with respect to a Q-Wiener process. Define the operator $J : K \to K$ by

$$J(h) := \sum_{j=1}^{\infty} \lambda_j^{1/2} \langle h, f_j \rangle_K f_j, \quad h \in K, \tag{6.5}$$

such that $\{f_j\}$ is an orthonormal basis in K and $\lambda_j \geq 0$ with $\sum_{j=1}^{\infty} \lambda_j < \infty$.

Lemma 6.1.5 *Let $Q = JJ^* : K \to K$. Then Q has eigenvalues λ_j corresponding to the eigenvectors f_j, i.e. $Qf_j = \lambda_j$.*

Proof: Let f_j be the orthonormal basis as in (6.5), then

$$\begin{aligned} Qf_j = (JJ^*)(f_j) &= \sum_{l=1}^{\infty} \lambda_l^{1/2} \langle J^* f_j, f_l \rangle_K f_l = \sum_{l=1}^{\infty} \lambda_l^{1/2} \langle f, J f_j \rangle_K f_l \\ &= \sum_{l=1}^{\infty} \lambda_l^{1/2} \langle f_j, \sum_{i=1}^{\infty} \lambda_i^{1/2} \langle f_l, f_i \rangle_K f_i \rangle_K f_l \\ &= \sum_{l=1}^{\infty} \lambda_l \langle f_j, f_l \rangle_K f_l = \lambda_j f_j, \end{aligned}$$

thereby completing the proof. □

Additionally, Q is symmetric (self-adjoint), non-negative definite, and $trQ = \sum_{j=1}^{\infty} \lambda_j < \infty$. The operator $J : K \to K_Q$ is an isometry since

$$\|h\|_K = \|Q^{-1/2} J(h)\|_K = \|J(h)\|_{K_Q}, \quad h \in K,$$

where $Q^{-1/2}$ denotes the pseudo-inverse of $Q^{1/2}$. Furthermore, the inverse operator $J^{-1} : K_Q = Q^{1/2} K \to K$ is also an isometry. Therefore, a Q-Wiener process in K as defined in Section 2 can be constructed as

$$W_t := \sum_{j=1}^{\infty} w_j(t) J(f_j) = \sum_{j=1}^{\infty} w_j(t) \lambda_j^{1/2} f_j, \tag{6.6}$$

where $\{w_j(t) := \tilde{W}_t(f_j)\}_{j=1}^{\infty}$ is a family of independent real-valued standard Brownian motions coming from the random field F chosen in (6.1). As seen in Section **??**, a predictable process ϕ will be integrable with respect to the Q-Wiener process W_t if

$$\mathbb{E}\left(\int_0^T \|\phi(t)\|^2_{\mathcal{L}_2(K_Q,H)}\mathrm{d}t\right) < \infty, \tag{6.7}$$

in particular where $H = \mathbb{R}$. Consider $g \in L^2(\Omega \times [0,T]; K)$, and define the operator, $\Phi_s^g : K \to \mathbb{R}$, by

$$\Phi_s^g(\eta) = \langle g(s), \eta\rangle_K, \quad \eta \in K. \tag{6.8}$$

Thus, the following proposition holds from Dalang and Quer-Sardanyons [11], whose proof is included here as it contains information to be used later.

Proposition 6.1.6 *If $g \in \mathcal{P}_+$ and Φ_t^g as defined in* (6.8)*, then $\Phi_t^g \circ J^{-1}$ satisfies the condition* (6.7) *and*

$$\int_0^T \Phi_t^g \circ J^{-1}\mathrm{d}W_t = \int_0^T g(t)\mathrm{d}\tilde{W}_t,$$

where W_t is the Q-Wiener process defined in (6.6)*, J is the operator as defined in* (6.5) *and $\tilde{W}_t$ is the cylindrical Wiener process defined in* (6.2)*.*

Proof: First, to show that $\Phi_t^g \circ J^{-1} \in \mathcal{L}_2(K_Q, \mathbb{R})$,

$$\begin{aligned}\|\Phi_t^g \circ J^{-1}\|^2_{\mathcal{L}^2(K_Q,\mathbb{R})} &= \sum_{j=1}^{\infty}[(\Phi_t^g \circ J^{-1})(\lambda_j^{1/2} f_j)]^2 = \sum_{j=1}^{\infty}[\Phi_t^g(J^{-1}\lambda_j^{1/2} f_j)]^2 \\ &= \sum_{j=1}^{\infty}\langle g(t), J^{-1}Q^{1/2} f_j\rangle_K^2 = \sum_{j=1}^{\infty}\langle g(t), f_j\rangle_K^2 \\ &= \|g(t)\|_K^2.\end{aligned}$$

Therefore,

$$\mathbb{E}\left(\int_0^T \|\Phi_t^g \circ J^{-1}\|^2_{\mathcal{L}_2(K_Q,\mathbb{R})}\mathrm{d}t\right) = \mathbb{E}\left(\int_0^T \|g(t)\|_K^2\mathrm{d}t\right) < \infty,$$

since $g \in \mathcal{P}_+$ implies that $g \in L^2(\Omega\times[0,T]; K)$. Thus, $\Phi_t^g \circ J^{-1}$ satisfies condition (6.7).

Also, from Lemma 3.1.1,

$$\begin{aligned}\int_0^T \Phi_t^g \circ J^{-1} dW_t &= \sum_{j=1}^{\infty} \int_0^T (\Phi_t^g \circ J^{-1})(\lambda_j^{1/2} f_j) \mathrm{d}\langle W_t, \lambda_j^{1/2} f_j\rangle_{K_Q} \\ &= \sum_{j=1}^{\infty} \int_0^T \langle g(t), f_j\rangle_K \mathrm{d}w_j(t) \\ &= \sum_{j=1}^{\infty} \int_0^T \langle g(t), f_j\rangle_K \mathrm{d}\tilde{W}_t(f_j) \\ &= \int_0^T g(t) \mathrm{d}\tilde{W}_t,\end{aligned}$$

thereby completing the proof. □

Therefore, combining Propositions 6.1.4 and 6.1.6 yields the desired integral connections.

Corollary 6.1.7 *For $g \in \mathcal{P}_+$ and Φ_t^g as defined in* (6.8),

$$\int_0^T \int_{\mathbb{R}^N} g(t,x) M(\mathrm{d}t, \mathrm{d}x) = \int_0^T g(t) \mathrm{d}\tilde{W}_t = \int_0^T \Phi_t^g \circ J^{-1} \mathrm{d}W_t,$$

where M is the martingale measure defined in (6.3)*, $\tilde{W}_t$ is the cylindrical Wiener process defined in* (6.2)*, J is the operator defined as* (6.5) *and W_t is the Q-Wiener process defined in* (6.6)*.*

6.2 Case with a time change

The previous section summarized results from [10,11,21] establishing the equivalence of integrals with respect to a martingale measure, a cylindrical Wiener process, and a Q-Wiener process. This section now extends those results to the time-changed case. The procedure for showing the equivalence of the integrals is very similar to that used in the previous section. The same random field F and Hilbert space K are used to define time-changed versions of a cylindrical Wiener process and a martingale measure. Their associated integrals are then shown to be equal. Finally, a connection between the given time-changed cylindrical Wiener process and a time-changed Q-Wiener process leads to the equivalence of all three integrals.

First, recall that the time change, E_t, is the inverse of a β-stable subordinator. Then, the time-changed cylindrical Wiener process is defined as follows:

Definition 6.2.1 *Let K be a separable Hilbert space. A family of random variables $\{\tilde{W}_{E_t}, t \geq 0\}$ is a time-changed cylindrical Wiener process on K if the following conditions hold:*

1. *for any $k \in K$, $\{\tilde{W}_{E_t}(k), t \geq 0\}$ defines a time-changed Brownian motion with mean 0 and variance $\mathbb{E}(E_t)\langle k, k\rangle_K$; and*
2. *for all $s, t \in R_+$ and $k, h \in K$,*

$$\mathbb{E}(\tilde{W}_{E_s}(k)\tilde{W}_{E_t}(h)) = \mathbb{E}(E_{s\wedge t})\langle k, h\rangle_K.$$

Let $g : R^+ \times \Omega \to K$ be any predictable process such that

$$\mathbb{E}\left(\int_0^T \|g(s)\|_K^2 \mathrm{d}E_s\right) < \infty. \tag{6.9}$$

Define the stochastic integral of g with respect to a time-changed cylindrical Wiener process as follows:

$$\int_0^T g(s)\mathrm{d}\tilde{W}_{E_s} := \sum_{j=1}^{\infty} \int_0^T \langle g_s, f_j\rangle_K \mathrm{d}\tilde{W}_{E_s}(f_j).$$

The following proposition establishes convergence of this series in $L^2(\Omega, \mathcal{G}, \mathbb{P})$.

Proposition 6.2.2 *(Chlebak, Garmirian, Wu) Let $g : \mathbb{R}^+ \times \Omega \to K$ be any predictable process satisfying condition (6.9). Then, the series*

$$\int_0^T g(s)\mathrm{d}\tilde{W}_{E_s} := \sum_{j=1}^{\infty} \int_0^T \langle g_s, f_j\rangle_K \mathrm{d}\tilde{W}_{E_s}(f_j) \tag{6.10}$$

converges in $L^2(\Omega, \mathcal{G}, \mathbb{P})$ where $\mathcal{G} = \tilde{\mathcal{F}}_{E_t}$.

Proof: Let

$$Y_n := \sum_{j=1}^{n} \int_0^T \langle g_s, f_j\rangle_K \mathrm{d}\tilde{W}_{E_s}(f_j).$$

In order to show the convergence of the series defined in (6.10), it is sufficient to show $\{Y_n\}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{G}, \mathbb{P})$. For $n > m$,

$$\begin{aligned}
\|Y_n - Y_m\|_2^2 &= \mathbb{E}\bigg(\sum_{j=m+1}^{n} \int_0^T \langle g_s, f_j \rangle_K \mathrm{d}\tilde{W}_{E_s}(f_j) \bigg)^2 \\
&= \mathbb{E}\bigg[\bigg(\sum_{j=m+1}^{n} \int_0^T \langle g_s, f_j \rangle_K \mathrm{d}\tilde{W}_{E_s}(f_j) \bigg) \bigg(\sum_{i=m+1}^{n} \int_0^T \langle g_s, f_i \rangle_K \mathrm{d}\tilde{W}_{E_s}(f_i) \bigg) \bigg] \\
&= \mathbb{E}\bigg(\sum_{j=i=m+1}^{n} \bigg[\int_0^T \langle g_s, f_j \rangle_K \mathrm{d}\tilde{W}_{E_s}(f_j) \bigg]^2 \bigg) \\
&\quad + \mathbb{E}\bigg(\sum_{j \neq i=m+1}^{n} \bigg[\int_0^T \langle g_s, f_j \rangle_K \mathrm{d}\tilde{W}_{E_s}(f_j) \bigg] \bigg[\int_0^T \langle g_s, f_i \rangle_K \mathrm{d}\tilde{W}_{E_s}(f_i) \bigg] \bigg) \\
&=: I + II.
\end{aligned}$$

Since the time-changed Q-Wiener process W_{E_t} defined in (2.4) is a square integrable martingale with respect to the filtration $\mathcal{G}_t = \tilde{\mathcal{F}}_{E_t}$ defined in (2.5), then for $h = \lambda_j^{-1/2} f_j \in K, j = 1, 2, \cdots$, $\langle W_{E_t}, h \rangle_K$ is also a square integrable martingale with respect to the filtration $\mathcal{G}_t$, i.e., for $0 < s < t$,

$$\mathbb{E}(\langle W_{E_t}, h \rangle_K | \mathcal{G}_s) = \langle W_{E_s}, h \rangle_k.$$

This implies that each projection, which is a time-changed Brownian motion, $w_j(E_t)$, $j = 1, 2, \cdots$, is also a square integrable martingale with respect to the same filtration $\mathcal{G}_t$, i.e., for $0 < s < t$, $\mathbb{E}(w_j(E_t)|\mathcal{G}_s) = w_j(E_s)$. Therefore, the following integral

$$\int_0^T \langle g_s, f_j \rangle_K \mathrm{d}w_j(E_s)$$

is also a square integral martingale with respect to the filtration $\mathcal{G}_t$, and it holds from the Itô isometry that

$$\mathbb{E}\bigg[\int_0^T \langle g_s, f_j \rangle_K \mathrm{d}w_j(E_s) \bigg]^2 = \mathbb{E}\bigg[\int_0^T \langle g_s, f_j \rangle_K^2 \mathrm{d}E_s \bigg].$$

Furthermore, a proof similar to that for Theorem 2.0.8 establishes that the product $w_i(E_t) w_j(E_t)$ is a square integrable martingale for $i \neq j$. This means the quadratic covariation process of martingales $w_i(E_t)$ and $w_j(E_t)$ is zero, i.e., $[w_i(E_t), w_j(E_t)] = 0$. Thus, using the martingale property of $w_j(E_s)$ and its associated integral, along with the Cauchy-Schwartz inequality

$$
\begin{aligned}
I &:= \mathbb{E}\Bigg(\sum_{j=i=m+1}^{n} \bigg[\int_0^T \langle g_s, f_j \rangle_K \mathrm{d}\tilde{W}_{E_s}(f_j) \bigg]^2 \Bigg) \\
&= \sum_{j=i=m+1}^{n} \mathbb{E}\bigg[\int_0^T \langle g_s, f_j \rangle_K \mathrm{d}w_j(E_s) \bigg]^2 \\
&= \sum_{j=i=m+1}^{n} \mathbb{E}\bigg[\int_0^T \langle g_s, f_j \rangle_K^2 \mathrm{d}E_s \bigg] \le \sum_{j=i=m+1}^{\infty} \mathbb{E}\bigg[\int_0^T \langle g_s, f_j \rangle_K^2 \mathrm{d}E_s \bigg].
\end{aligned}
\tag{6.11}
$$

By assumption (6.9),

$$
\sum_{j=1}^{\infty} \mathbb{E}\bigg[\int_0^T \langle g_s, f_j \rangle_K^2 \mathrm{d}E_s \bigg] = \mathbb{E}\bigg[\int_0^T \|g_s\|_K^2 \mathrm{d}E_s \bigg] < \infty,
$$

so the above tail of the partial sum in (6.11) converges to 0 as m (hence n) $\to \infty$. Meanwhile,

$$
\begin{aligned}
II &:= \mathbb{E}\Bigg(\sum_{j\neq i=m+1}^{n} \bigg[\int_0^T \langle g_s, f_j \rangle_K \mathrm{d}\tilde{W}_{E_s}(f_j) \bigg] \bigg[\int_0^T \langle g_s, f_i \rangle_K \mathrm{d}\tilde{W}_{E_s}(f_i) \bigg] \Bigg) \\
&= \sum_{j\neq i=m+1}^{n} \mathbb{E}\bigg[\bigg(\int_0^T \langle g_s, f_j \rangle_K \mathrm{d}\tilde{W}_{E_s}(f_j) \bigg) \bigg(\int_0^T \langle g_s, f_i \rangle_K \mathrm{d}\tilde{W}_{E_s}(f_i) \bigg) \bigg] \\
&= \sum_{j\neq i=m+1}^{n} \mathbb{E}\bigg[\bigg(\int_0^T \langle g_s, f_j \rangle_K \mathrm{d}w_j(E_s) \bigg) \bigg(\int_0^T \langle g_s, f_i \rangle_K \mathrm{d}w_i(E_s) \bigg) \bigg] \\
&= \sum_{j\neq i=m+1}^{n} \mathbb{E}\bigg(\int_0^T \langle g_s, f_j \rangle_K \langle g_s, f_i \rangle_K \mathrm{d}[w_j(E_s), w_i(E_s)] \bigg) \\
&= 0.
\end{aligned}
\tag{6.12}
$$

Thus, combining (6.11) and (6.12) yields $\|Y_n - Y_m\|_2^2 = I + II \to 0$ as $n, m \to \infty$. Consequently, $\{Y_n\}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{G}, \mathbb{P})$, which completes the proof.□

The next proposition concerns defining a cylindrical process from a fixed random field F chosen in (6.1) .

Proposition 6.2.3 *(Chlebak, Garmirian, Wu) Let F be the specific random field defined as in* (6.1) *and K be the associated Hilbert space previously described. For*

$t \geq 0$ and $\phi \in K$, set

$$\tilde{W}_{E_t}(\phi) := F(1_{[0,E_t]}(\cdot)\phi(*)). \tag{6.13}$$

Then, the process $\tilde{W}_{E_t}$ is a time-changed cylindrical Wiener process.

Proof: Consider a fixed $\phi \in K$. $\tilde{W}_{E_t}(\phi)$ is a time-changed Brownian motion by construction. Additionally,

$$\begin{aligned}
\mathbb{E}[\tilde{W}_{E_t}(\phi)] = \mathbb{E}[F(1_{[0,E_t]}(\cdot)\phi(*))] &= \int_0^\infty \mathbb{E}[F(1_{[0,\tau]}(\cdot)\phi(*))]f_{E_t}(\tau)\mathrm{d}\tau \\
&= \int_0^\infty 0 \cdot f_{E_t}(\tau)\mathrm{d}\tau = 0,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[\tilde{W}_{E_t}(\phi)\tilde{W}_{E_t}(\phi)] &= \mathbb{E}[F(1_{[0,E_t]}(\cdot)\phi(*))F(1_{[0,E_t]}(\cdot)\phi(*))] \\
&= \mathbb{E}\left[\int_{\mathbb{R}^+} 1_{[0,E_t]}(s)1_{[0,E_t]}(s) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(x)f(x-y)\phi(y)\mathrm{d}y\mathrm{d}x\mathrm{d}s\right] \\
&= \mathbb{E}\left[\langle\phi,\phi\rangle_K \int_{\mathbb{R}^+} 1_{[0,E_t]}(s)\mathrm{d}s\right] = \mathbb{E}(E_t)\langle\phi,\phi\rangle_K.
\end{aligned}$$

Furthermore, for fixed $s, t \in \mathbb{R}^+$ and $\phi, \psi \in K$,

$$\begin{aligned}
\mathbb{E}[\tilde{W}_{E_t}(\phi)\tilde{W}_{E_s}(\psi)] &= \mathbb{E}[F(1_{[0,E_t]}(\cdot)\phi(*))F(1_{[0,E_s]}(\cdot)\psi(*))] \\
&= \mathbb{E}\int_{\mathbb{R}^+} 1_{[0,E_t]}(r)1_{[0,E_s]}(r) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(x)f(x-y)\psi(y)dydxdr \\
&= \mathbb{E}\int_{\mathbb{R}^+} \langle\phi,\psi\rangle_K 1_{[0,E_t\wedge E_s]}(r)dr = \mathbb{E}(E_{t\wedge s})\langle\phi,\psi\rangle_K.
\end{aligned}$$

Therefore, according to the Definition 6.2.1, $\tilde{W}_{E_t}$ is a time-changed cylindrical Wiener process. □

Moreover, it follows from Proposition 6.2.2 that the integral with respect to the process $\tilde{W}_{E_t}$ defined in (6.13) is well-defined.

On the other hand, E_t is independent of the martingale measure $M_t(A)$. So, for each $\omega \in \Omega$, $t \in [0,T]$, define a time-changed version of $\{M_t(A)\}$ by

$$M_{E_t}(A) = M(A \times [0,E_t]) := F(1_{[0,E_t]}(\cdot)1_A(*)), \tag{6.14}$$

where F is again the random field chosen in (6.1). By conditioning on the time change, the covariance for $M_{E_t}(A)$ is

$$\begin{aligned}\mathbb{E}(M_{E_t}(A)M_{E_t}(B)) &= \int_0^\infty \mathbb{E}(M_\tau(A)M_\tau(B))f_{E_t}(\tau)\mathrm{d}\tau \\ &= \int_0^\infty \tau\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} 1_A(x)f(x-y)1_B(y)\mathrm{d}y\mathrm{d}x\right)f_{E_t}(\tau)\mathrm{d}\tau \\ &= \mathbb{E}(E_t)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} 1_A(x)f(x-y)1_B(y)\mathrm{d}y\mathrm{d}x,\end{aligned}$$

where f_{E_t} is the density function of E_t. Also

$$\mathbb{E}[M_{E_t}(A)]^2 = \mathbb{E}(E_t)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} 1_A(x)f(x-y)1_A(y)\mathrm{d}y\mathrm{d}x < \infty$$

for all $A \in \mathcal{B}_b(\mathbb{R}^N)$. So $M_{E_t}(A)$ has a finite second moment for all $A \in \mathcal{B}_b(\mathbb{R}^N)$. Then the following theorem shows that $\{M_{E_t}(A)\}$ defined in (6.14) is still a martingale measure.

Theorem 6.2.4 *(Chlebak, Garmirian, Wu)* $\{M_{E_t}(A)\}_{t\geq 0, A\in\mathcal{B}(\mathbb{R}^N)}$ *is a martingale measure with respect to the filtration* $\{\tilde{\mathcal{F}}_{E_t}\}_{t\geq 0}$*, where* $\tilde{\mathcal{F}}_t$ *in* (2.5) *is generated by the time change* E_t *and independent Brownian motions* $\tilde{W}_t(f_j), j = 1, 2, \cdots$*, coming from the fixed random field* F *in* (6.1).

Proof: To show $M_{E_t}(A)$ is a martingale measure, it is sufficient to check the conditions in Definition 6.1.2. First, since $E_0 = 0$ a.s., $M_{E_0}(A) = M_0(A) = 0$ a.s. because $M_t(A)$ is a martingale measure.

Second, let A, $B \in \mathcal{B}(\mathbb{R}^N)$ such that $A \cap B = \varnothing$. Then, for fixed τ, $M_\tau(A \cup B)$ and $M_\tau(A) + M_\tau(B)$ are mean zero Gaussian random variables and

$$\begin{aligned}\mathrm{Var}(M_\tau(A\cup B)) &= \iint_{(A\cup B)\times(A\cup B)} f(x-y)\mathrm{d}y\mathrm{d}x \\ &= \iint_{A\times A} f(x-y)dydx + \iint_{B\times B} f(x-y)dydx + 2\iint_{A\times B} f(x-y)\mathrm{d}y\mathrm{d}x \\ &= \mathrm{Var}(M_\tau(A)) + \mathrm{Var}(M_\tau(B)) + 2\mathbb{E}(M_\tau(A)M_\tau(B)) \\ &= \mathrm{Var}\Big(M_\tau(A) + M_\tau(B)\Big).\end{aligned}$$

Also note that

$$M_\tau(A \cup B) = M_\tau(A) + M_\tau(B) \text{ a.s.}$$

Thus, conditioning on the time change yields

$$\begin{aligned}
&\mathbb{P}(M_{E_t}(A \cup B) = M_{E_t}(A) + M_{E_t}(B)) \\
&= \int_0^\infty \mathbb{P}(M_\tau(A \cup B) = M_\tau(A) + M_\tau(B)) f_{E_t}(\tau)\mathrm{d}\tau \\
&= \int_0^\infty f_{E_t}(\tau)\mathrm{d}\tau = 1,
\end{aligned}$$

which means $M_{E_t}(\cdot)$ is additive a.s. Furthermore, assume $A_1 \supset A_2 \supset \ldots$ such that $\cap_n A_n = \varnothing$,

$$\mathbb{E}[M_{E_t}(A_n)]^2 = \mathbb{E}(E_t) \int\int_{A_n \times A_n} f(x-y)\mathrm{d}y\mathrm{d}x \to 0$$

as $n \to \infty$, and so $M_{E_t}(A_n) \to 0$ in $L^2(\mathbb{P})$. This completes the countable additivity of $M_{E_t}(\cdot)$.

Finally, since $\{M_{E_t}(A)\}$ has a finite second moment, a similar argument to the proof of Theorem 2.0.8 shows that $\{M_{E_t}(A)\}$ is a martingale for all $A \in \mathcal{B}(\mathbb{R}^N)$. Therefore, $\{M_{E_t}(A)\}$ is a martingale measure with respect to the filtration $\mathcal{G}_t = \tilde{\mathcal{F}}_{E_t}$. □

Define the dominating measure K by

$$\begin{aligned}
&K(A \times B \times C) \\
&:= \mathbb{E}(\lambda(\{E_s(\omega) : s \in C\})) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} 1_A(x) f(x-y) 1_B(y)\mathrm{d}y\mathrm{d}x,
\end{aligned} \tag{6.15}$$

where $A, B \in \mathcal{B}(\mathbb{R}^N)$ and λ is the Lebesgue measure on $C \in \mathcal{B}(\mathbb{R}_+)$. Then, the following theorem shows that the martingale measure $\{M_{E_t}(A)\}$ is worthy.

Theorem 6.2.5 *(Chlebak, Garmirian, Wu) The martingale measure* $\{M_{E_t}(A)\}_{t\geq 0, A\in\mathcal{B}(\mathbb{R}^N)}$ *is worthy with respect to the filtration* $\{\tilde{\mathcal{F}}_{E_t}\}_{t\geq 0}$*, i.e. the same one as defined in Theorem 6.2.4.*

Proof: To show that the martingale measure $\{M_{E_t}(A)\}$ is worthy, it suffices to show that the dominating measure K defined in (6.15) satisfies the conditions of Definition 6.1.3.

1. For all $C \in \mathcal{B}(\mathbb{R}^+)$, since f is an even function,

$$\begin{aligned} K(A \times B \times C) &= \mathbb{E}(\lambda(\{E_s(\omega) : s \in C\})) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} 1_A(x) f(x-y) 1_B(y) \mathrm{d}y \mathrm{d}x \\ &= \mathbb{E}(\lambda(\{E_s(\omega) : s \in C\})) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} 1_B(x) f(x-y) 1_A(y) \mathrm{d}y \mathrm{d}x \\ &= K(B \times A \times C). \end{aligned}$$

   Additionally, for all $A, B \in \mathcal{B}(\mathbb{R}^N)$, $C \in \mathcal{B}(\mathbb{R}^+)$, since f is non-negative definite,

$$K(A \times A \times C) = \mathbb{E}(\lambda(\{E_s(\omega) : s \in C\})) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} 1_A(x) f(x-y) 1_A(y) \mathrm{d}y \mathrm{d}x \geq 0.$$

2. For all $A, B \in \mathcal{B}(\mathbb{R}^N)$, $t > 0$,

$$K(A \times B \times (0,t]) = \mathbb{E}(E_t) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} 1_A(x) f(x-y) 1_B(y) \mathrm{d}y \mathrm{d}x$$

   is $\tilde{\mathcal{F}}_{E_t}$-measurable.

3. For all compact sets $A, B \in \mathcal{B}(\mathbb{R}^N)$ and $t > 0$,

$$\mathbb{E}|K(A \times B \times (0,t])| = \mathbb{E}(E_t) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} 1_A(x) f(x-y) 1_B(y) \mathrm{d}y \mathrm{d}x < \infty.$$

4. For all $A, B \in \mathcal{B}(\mathbb{R}^N)$ and $t > 0$,

$$\begin{aligned} |\mathbb{E}(M_{E_t}(A) M_{E_t}(B))| &= \mathbb{E}(E_t) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} 1_A(x) f(x-y) 1_B(y) \mathrm{d}y \mathrm{d}x \\ &= K(A \times B \times (0,t]). \end{aligned}$$

Thus, the martingale measure $\{M_{E_t}(A)\}$ is worthy. □

As shown in the non-time-changed case, the stochastic integral with respect to a worthy time-changed martingale measure is also a worthy martingale measure. First, consider elementary processes g of the form

$$g(s,x,\omega) = 1_{(a,b]}(s)1_A(x)X(\omega) \tag{6.16}$$

where $0 \le a < b \le T$, $A \in \mathcal{B}(\mathbb{R}^N)$ and X is both bounded and $\mathcal{G}_a := \tilde{\mathcal{F}}_{E_a}$-measurable. Then define $g \cdot M_E$ by

$$\begin{aligned} g \cdot M_{E_t}(B) &:= X(\omega)(M_{E_t \wedge E_b}(A \cap B) - M_{E_t \wedge E_a}(A \cap B)) \\ &= \int_0^t \int_{\mathbb{R}^N} g(s,x) M_E(\mathrm{d}s, \mathrm{d}x), \end{aligned}$$

where $M_{E_t \wedge E_r} = M_{E_{t \wedge r}}$ for any $t, r \in [0,T]$ and integration with respect to the martingale measure in both s and x is denoted by $M_E(\mathrm{d}s, \mathrm{d}x)$. As usual, this definition of $g \cdot M_E$ can be extended to finite sums of elementary processes and finally to predictable processes g such that

$$\|g\|_\dagger^2 := \mathbb{E}\left(\int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |g(t,x)| f(x-y) |g(t,y)| \mathrm{d}y \mathrm{d}x \mathrm{d}E_t\right) < \infty. \tag{6.17}$$

Let $\mathcal{P}_\dagger$ denote the set of predictable processes $(\omega, t, x) \mapsto g(t, x; \omega)$ such that (6.17) holds. For $g \in \mathcal{P}_\dagger$, $g \cdot M_E$ is a worthy martingale measure and the stochastic integral with respect to M_E is defined by

$$\int_0^t \int_A g(s,x) M_E(ds, dx) =: g \cdot M_{E_t}(A).$$

The following theorem connects an integral with respect to a time-changed martingale measure with an integral with respect to a time-changed cylindrical Wiener process.

Theorem 6.2.6 *(Chlebak, Garmirian, Wu) Let M_{E_t} be the time-changed martingale measure defined in* (6.14) *and $\tilde{W}_{E_t}$ be the time-changed cylindrical process defined in* (6.13). *Suppose that $g \in \mathcal{P}_\dagger$, then,*

$$\int_0^T \int_{\mathbb{R}^N} g(t,x) M_E(\mathrm{d}t, \mathrm{d}x) = \int_0^T g(t) \mathrm{d}\tilde{W}_{E_t}.$$

Proof: First notice that since $g \in \mathcal{P}_\dagger$,

$$\mathbb{E}\left(\int_0^T \|g(s)\|_K^2 dE_s\right) = \mathbb{E}\left(\int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g(s,x) f(x-y) g(s,y) \mathrm{d}y \mathrm{d}x \mathrm{d}E_s\right)$$
$$\leq \|g\|_\dagger^2 < \infty,$$

which means g satisfies the condition (6.9). Furthermore, since the set of elementary processes is dense in $\mathcal{P}_\dagger$, it is sufficient to check that the integrals coincide for elementary processes of the form

$$g(s,x,\omega) = 1_{(a,b]}(s) 1_A(x) X(\omega)$$

where $0 \leq a < b \leq T$, $A \in \mathcal{B}(\mathbb{R}^N)$ and $X(\omega)$ is both bounded and $\tilde{\mathcal{F}}_{E_a}$-measurable. Then,

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^N} g(t,x) M_E(\mathrm{d}t, \mathrm{d}x) &= X(M_{E_T \wedge E_b}(A) - M_{E_T \wedge E_a}(A)) \\
&= X(M_{E_b}(A) - M_{E_a}(A)) \\
&= X(F(1_{[0,E_b]}(\cdot) 1_A(*)) - F(1_{[0,E_a]}(\cdot) 1_A(*))) \\
&= X(F(1_{(E_a,E_b]}(\cdot) 1_A(*))).
\end{aligned}$$

On the other hand, using the linearity of F

$$\begin{aligned}
\int_0^T g(t) d\tilde{W}_{E_t} &= \sum_{j=1}^{\infty} \int_a^b X \langle 1_A, f_j \rangle_K d\tilde{W}_{E_t}(f_j) \\
&= X \sum_{j=1}^{\infty} \langle 1_A, f_j \rangle_K (\tilde{W}_{E_b}(f_j) - \tilde{W}_{E_a}(f_j)) \\
&= X \sum_{j=1}^{\infty} \langle 1_A, f_j \rangle_K [F(1_{[0,E_b]}(\cdot) f_j) - F(1_{[0,E_a]}(\cdot) f_j)] \\
&= X \sum_{j=1}^{\infty} \langle 1_A, f_j \rangle_K [F(1_{(E_a,E_b]}(\cdot) f_j)] \\
&= X[F(1_{(E_a,E_b]}(\cdot) \sum_{j=1}^{\infty} \langle 1_A, f_j \rangle_K f_j] \\
&= X[F(1_{(E_a,E_b]}(\cdot) 1_A(*))],
\end{aligned}$$

thereby completing the proof. □

Now, a connection between the time-changed cylindrical Wiener process and the time-changed Q-Wiener process needs to be established. Define

$$W_{E_t} := \sum_{j=1}^{\infty} w_j(E_t) J(f_j) \tag{6.18}$$

where $w_j(E_t) := \tilde{W}_{E_t}(f_j)$ are time-changed Brownian motions coming from the time change E_t and the fixed random field F, and the operator $J(f_j) = \lambda_j^{1/2} f_j$ is defined as in Proposition 6.1.5. Also for $g \in L^2(\Omega \times [0,T], K)$, define the operator Φ_s^g by

$$\Phi_s^g(\eta) := \langle g(s), \eta \rangle_K \ \eta \in K. \tag{6.19}$$

A predictable process ϕ will be integrable with respect to W_{E_t} if

$$\mathbb{E}\left(\int_0^T \|\phi(t)\|^2_{\mathcal{L}_2(K_Q,H)} dE_t \right) < \infty. \tag{6.20}$$

The connection between the integral with respect to the time-changed cylindrical Wiener process and the time-changed Q-Wiener process is given in the following theorem:

Theorem 6.2.7 *(Chlebak, Garmirian, Wu) Let $\tilde{W}_{E_t}$ be a time-changed cylindrical Wiener process as defined by* (6.13) *and let W_{E_t} be a time-changed Q-Wiener process as defined by* (6.18)*. Let $g \in \mathcal{P}_\dagger$ and let the operator Φ_s^g be defined by* (6.19)*. Then, $\Phi_s^g \circ J^{-1}$ satisfies condition* (6.20) *and*

$$\int_0^T \Phi_s^g \circ J^{-1} dW_{E_s} = \int_0^T g(s) d\tilde{W}_{E_s}.$$

Proof: First, from the proof of Proposition 6.1.6, $\|\Phi_s^g \circ J^{-1}\|^2_{\mathcal{L}_2(K_Q,\mathbb{R})} = \|g(s)\|^2_K$. Thus,

$$\mathbb{E}\left(\int_0^T \|\Phi_t^g \circ J^{-1}\|^2_{\mathcal{L}_2(K_Q,\mathbb{R})} \mathrm{d}E_t \right) = \mathbb{E}\left(\int_0^T \|g(t)\|^2_K \mathrm{d}E_t \right) < \infty,$$

since it was shown in Theorem 6.2.6 that $g \in \mathcal{P}_\dagger$ implies that condition (6.9) holds.

So, $\Phi_t^g \circ J^{-1}$ satisfies the condition (6.20). Also from Definition 3.1.3,

$$\begin{aligned}\int_0^T \Phi_s^g \circ J^{-1} dW_{E_s} &= \sum_{j=1}^{\infty} \int_0^T \Phi_s^g \circ J^{-1}(\lambda_j^{1/2} f_j) \mathrm{d}\langle W_s, \lambda_j^{1/2} f_j \rangle_{K_Q} \\ &= \sum_{j=1}^{\infty} \int_0^T \langle g(s), f_j \rangle_K \mathrm{d}w_j(E_s) \\ &= \sum_{j=1}^{\infty} \int_0^T \langle g(s), f_j \rangle_K \mathrm{d}\tilde{W}_{E_s}(f_j) \\ &= \int_0^T g(s) \mathrm{d}\tilde{W}_{E_s},\end{aligned}$$

thereby completing the proof. □

Finally, combining the results of Theorems 6.2.6 and 6.2.7 yields the desired correspondence of integrals with respect to time-changed processes.

Corollary 6.2.8 *For* $g \in \mathcal{P}_\dagger$ *and* Φ_t^g *as defined in* (6.19)*,*

$$\int_0^T \int_{\mathbb{R}^N} g(t,x) M_E(dt,dx) = \int_0^T g(t) \mathrm{d}\tilde{W}_{E_t} = \int_0^T \Phi_t^g \circ J^{-1} dW_{E_t},$$

where M_{E_t} *is the time-changed martingale measure as defined in* (6.14)*,* $\tilde{W}_{E_t}$ *is the time-changed cylindrical process defined in* (6.13)*,* J *is the operator defined in* (6.5) *and* W_{E_t} *is the time-changed* Q*-Wiener process defined in* (6.18)*.*

Chapter 7

Closing Remarks

There remain ample opportunities to improve the results in this thesis, particularly in Chapters 4 and 6. Recall that the stability results in Chapter 4 were for a time-changed solution $X \circ \gamma(t)$ in one dimension. Many of the arguments made should be applicable to the infinite-dimensional case; however, time constraints did not permit a more rigorous study of this extension. One difficulty might be in finding an appropriate infinite-dimensional version of the change of variable formula given in [23]. Regardless, there is still more work to be done in order to show stability of $X(t)$ rather than $X \circ \gamma(t)$. The hope is to exploit the almost sure convergence of $X \circ \gamma(t)$ and the relationship between $\gamma(t)$ and t to arrive at a satisfactory result. Finally, it would be very interesting to make a more explicit connection to stochastic partial differential equations than was established in Chapter 6. It might be beneficial to limit the scope of this connection to the heat equation, for example, and try to find a correspondence between mild solutions in a similar fashion to [11]. Even this may be quite an undertaking.

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