STUDENTS' AND TEACHERS' UNDERSTANDINGS

OF MATHEMATICAL FUNCTIONS

A qualifying paper

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Abstract

The goal of the study presented in this paper is to describe students' and teachers' understandings of: (1) the definition of function and (2) the transformations of functions in different representations. This will be achieved through a case study analysis of two ninth grade students, two twelfth grade students, and two mathematics teachers. The six participants were recruited from the same high school in Boston, Massachusetts, which serves a diverse racial, ethnic, and cultural community. The participants were interviewed individually about questions pertaining to the definition of function and the transformations of functions. Analysis of the results revealed that the ninth grade students have a less sophisticated understanding of the concept of function than do twelfth grade students. It was expected that both teachers would have a similar understanding of functions, but it was found that the twelfth grade teacher had a more theoretical and rigid understanding of the concept of function than did the ninth grade teacher, who had a more practical and flexible understanding of the concept of function. In this case, flexible means translating to other representations of functions and/or using different approaches to functions, while rigid means the opposite.

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Introduction

Functions, as proposed by Schwartz (1999) and by Schwartz and Yerushalmy (1992), can be viewed as one of the fundamental objects of mathematics. This position is shared by other researchers, such as Dreyfus and Eisenberg (1982), Eisenberg (1992), Gagatsis and Shiakalli (2004), and Hitt (1998). Functions appear in many college mathematics courses, including calculus courses at all levels and real analysis courses. Functions also appear in almost all high school mathematics courses, beginning with algebra and ending with pre-calculus or calculus. Functions even appear–implicitly and explicitly–in the elementary school curriculum, where students are asked to identify patterns or are presented with data tables or graphs for analysis. Therefore, the concept of function is a core concept in the teaching and learning of mathematics, appearing at all levels of the mathematics curriculum.

Research under the umbrella of the function concept includes, but is not limited to: (a) theoretical models on the development of the function concept; (b) teaching experiments that apply general theories to the specific concept of function; (c) *students' and teachers' conceptions of functions*; and (d) the use of technology in functions-based mathematics classes (e.g., Dubinsky & Harel, 1992b). Empirical research on *students' and teachers' conceptions of functions* has been of special interest to the following researchers: Ayers, Davis, Dubinsky, and Lewin (1988); Breidenbach, Dubinsky, Hawks, and Nichols (1992); Dreyfus and Eisenberg (1982); Dubinsky and Harel (1992a); Dreyfus and Vinner (1982); Even (1990, 1993, 1998); Gagatsis and Shiakalli (2004); Hitt (1998); Monk (1992); Moschkovich, Schoenfeld, and Arcavi (1993); Norman (1992); Sfard (1989, 1992); Slavit (1997); and Vinner (1983), to name only a few. In these studies, the researchers use different systems to code, categorize, and/or describe the conceptions held by students and teachers in order to highlight the distinctions in their *understandings.* The current study aims to further contribute to this line of research, and aims specifically at describing students' and teachers' *understandings* of:

- the *definition* of a function, and
- the *transformations* of functions in *different representations*.

Even though the study reported here is similar in many ways to those previously mentioned, it is different from them for several reasons. The sample includes high school students, as well as their current mathematics teachers, which is unlike the previous studies mentioned. In addition, the analysis examines the interaction between the representation of a function and one's understandings of the function, and it uses a grounded theory approach (Strauss & Corbin, 1998) to analyze the data.

In order to achieve the aims outlined above, the study carried out is first situated within the relevant literature on the concept of function, conceptions of functions, representations of functions, as well as conceptions and representations of functions. It then presents data obtained through a case study analysis of the interviews of four students and two teachers; the interviews focused on their understandings of the definition of function and the transformations of functions in different representations. Finally, these responses are analyzed ad hoc, as well as using a combination of the already existing coding systems used in other studies.

This study is important to both teachers of mathematics and researchers of mathematics education. For teachers of mathematics, it describes the responses that students have to various questions about functions, and highlights their misconceptions as well. For researchers of mathematics education, it implies possible levels of understandings of the concept of function as influenced by the various representations of functions. These descriptions and implications are necessary given that function is a core concept in the teaching and learning of mathematics. The more we know about how a student understands a concept, the better we will be at teaching it to them and developing appropriate curricula.

Last, but not least, it is acknowledged that the research presented here is a case study analysis and is merely descriptive. Thus, more research needs to be done with a larger sample size in order to make truly conclusive statements regarding students' understandings of functions in various representations. In spite of these limitations, the research discussed in this paper can form the basis of such a research endeavor.

Theoretical Background

Given that the focus of the study is on functions, and ones' understandings of functions in different representations, it is prudent to now examine the relevant literature in this section of the paper. First, the concept of function will be explored. Second, the various types of conceptions of functions will be presented. Third, the types of representations of functions will be described. Finally, the interaction between the latter two will be discussed.

Function Concept

Definition of a function.

A function can be defined as "A function $f : S \rightarrow T$ consists of two sets S and T together with a 'rule' that assigns to each $x \in S$ a specific element of T, denoted $f(x)$" (Marsden & Hoffman, 1993, p. 3). This definition of a function is known as the Dirichlet-Bourbaki definition of a function, as it includes the set-theoretic notions of Bourbaki and the rule-based notions of Dirichlet. It is also considered to be a modern definition of function for these same reasons.

Characteristics of a function.

In order to get to this modern definition of function, the concept of function has undergone an interesting evolutionary process (Burnett-Bradshaw, 2007; Kleiner, 1989; Malik, 1980; Markovits, Eylon, & Bruckheimer, 1986; O'Connor and Robertson, 2005; Sfard, 1992; Sierpinska, 1992), which some believe began with the Babylonians. Functions may have first appeared in tabular representational form and as trigonometric functions (2000 B.C.E. – 1299 C.E.), then successively as a relationship of dependence (1300 C.E. – 1499 C.E.), a relationship between varying quantities (1500 C.E. – 1599 C.E.), and in algebra-symbolic and graphical representational form (1600 C.E. – 1699 C.E.). Next functions were defined from an algebraic perspective by Euler (1700 C.E. – 1799 C.E.), then based on an arbitrary correspondence by Dirichlet (1800 C.E. – 1899 C.E.), and finally considered as an arbitrary correspondence between two sets, which followed the emergence of set theory (1900 C.E. – present; Burnett-Bradshaw, 2007). This modern definition of function, thus, has two distinct characteristics – *arbitrariness* and *univalence* (Freudenthal, 1983) – which are used in this paper to form the framework for analyzing the data in this study.

Arbitrariness.

The arbitrariness of a function refers to "both the relationship between the two sets on which the function is defined and the sets themselves" (Even, 1990, p. 528; 1993, p. 96). In terms of the relationship between the two sets, this means that there does not need to be any specific rule of correspondence, i.e., there does not need to be a specific algebra-symbolic expression, a set pattern in a table of values, or a graph with a specific shape. In terms of the sets themselves, the sets do not need to be defined on any specific set of objects, i.e., the sets do not actually have to contain numbers. In other words, the sets (which are referred to as variables by Freudenthal [1983]) can consist of "numbers, number tuples, points, curves, functions, permutands, elements of arbitrary sets" (Freudenthal, 1983, p. 528).

Univalence.

The univalence characteristic of a function refers to the part of the definition that states that for each element in the domain there is only one element (image) in the range (Even, 1990, 1993). Thus, in terms of a relation between two sets, a relation in which each single element in the domain is mapped to its own single element in the range (i.e., a one-to-one relation) or a relation in which more than one element in the domain is mapped to the same single element in the range (i.e., a many-to-one relation) could be a function. While a relation in which each single element in the domain is mapped to more than one element in the range (i.e., a one-to-many relation) or a relation in which more than one element in the domain is mapped to more than one element in the range (i.e., a many-to-many relation) could not be considered a function.

Empirical research.

The definition of a function, including its characteristics, appears to be straightforward to many mathematicians and scholars. This is not necessarily the case for students of mathematics who are learning about functions or who are not very familiar with the concept, which is one of the motivations for this study, as well as for other studies focused on the understandings of the concept of function. Knowledge of the definition of function helps one to determine if a given relation is a function. If one does not know the definition of a function, then one may not know what characteristics are necessary for a relation to be a function. It is, therefore, pertinent to determine how one defines a function in order to *begin* to describe one's understanding of a function. One's understanding of a function is based not only on their definition of a function, but also on the actions they perform on the function, even though one can perform actions on a function without knowing its definition. The act of transforming a function will be discussed in this paper, in addition to discussing how one defines a function.

One of the major research questions posed to participants in studies that focus on how participants define a function is actually "what is the definition of a function?" Participants in these studies are also asked to give examples of functions and/or to determine whether a given relation is a function or not. For instance, the researchers, Dreyfus and Vinner (1982), Vinner (1983), and Vinner and Dreyfus (1989), examined the concept image[1] of a function among college students and junior high school teachers through two main types of questions posed through the use of a questionnaire. More specifically, Dreyfus and Vinner (1982) asked participants to first determine if the given mathematical correspondences were functions or not, and to offer an explanation for their choice. Second, participants were asked to define a function.

The explanations for the correspondence being a function or not were coded in the following categories:

1. One Value (OV): If a correspondence assigns exactly one value to every element in its domain then it is a function. If not – then it is not.
2. Discontinuity (D): The graph is discontinuous or changes its "character" (2 different straight lines).
3. Split domain (SD): The domain of the function splits or the graph changes its "character" (a straight line and a curved line).
4. Exceptional Point (EP): There is one point of exception (also a ground for rejection and acceptance; Dreyfus & Vinner, 1982, p. 16).

[1] "The concept image consists of all mental pictures and all properties which are associated with a given notion in somebody's mind. The concept image is determined by specific examples and by the accumulated experience with the notion. Quite often there is a gap or even a conflict between the student's concept image and the concept definition as taught by the teacher" (Dreyfus & Vinner, 1982, p. 12).

The data revealed that the junior high school teachers were better able to determine if a given relation was a function. This difference in correct answers was significant.

For the second part of the questionnaire, the definitions of a function were coded in the following categories (which is a refinement of the categories used in Vinner [1983]):

I. The function is *any correspondence* between two sets that assigns to every element in the first exactly one element in the second set (Dirichlet-Bourbaki definition).

II. The function is a *dependence* relation between two variables (y depends on x).

III. The function is a *rule of correspondence* (this conception eliminates the possibility of *arbitrary* correspondences).

IV. The function is a manipulation or an operation (one acts on a given number, generally by means of algebraic operations, in order to get its image).

V. The function is a formula, an algebraic term or an equation.

VI. The function is identified, probably in a meaningless way, with one of its visual or symbolic representations (the graph, the symbols "y=f(x)", etc.; Dreyfus & Vinner, 1982, p. 14).

The data for this part of the questionnaire again revealed that the teachers fared better than the college students, and there was again a significant difference in their results.

In Even (1993), prospective teachers' pedagogical and subject matter knowledge of the concept of function was examined through the use of a questionnaire and an interview. In terms of subject matter knowledge, participants' understanding of the characteristics of a function – arbitrariness and univalance – was analyzed from a qualitative perspective. The participants' responses as they related to arbitrariness were typically found to be:

(1) functions are (or can always be represented as) equations or formulas;

(2) graphs of functions should be "nice"[2]; and

(3) functions are "known" (Even, 1993, p. 104).

The participants' responses as they related to univalence were such that most of them knew of this requirement for a relation to be a function, but many did not know why this was a requirement.

The coding categories used by Dreyfus and Vinner (1982), Even (1993), Vinner (1983), and Vinner and Dreyfus (1989) offer an excellent framework for analysis of the data presented in this paper. This is for two reasons. First, Even (1993) offers an analysis based on the essential characteristics of a function (arbitrariness and univalence). This method of interpreting the data allows for establishing a relationship between one's understandings of functions and the actual definition of a function. Thus, for the purposes of this study, the basic categories for understanding participants' responses are derived from Even (1990), which are the same as the essential characteristics of a function. Second, the work of Dreyfus and Vinner (1982), Vinner (1983), and Vinner and Dreyfus (1989), as well as the highlights from the history of the concept of function, offer a qualitative way for interpreting the data obtained. These are the tools that will be used in describing students' and teachers' *understandings* of the *definition* of a function.

Conceptions of Functions

Distinctions: process vs. object.

Having clarified the definition of a function, it is now necessary to familiarize ourselves with the different types of conceptions of functions that have been offered by various researchers. The most predominant distinction used for describing one's concept of a function is *process* versus *object*, which is grounded in the work of Piaget. *Function*, as a mathematical

[2] Participants in the study expected the graphs of functions to be smooth and continuous, and therefore used the term "nice" to express this.

concept, must go through a cognitive construction process as described by Piaget (1975/1977) regarding the construction of conceptual entities. For Piaget (1975/1977), conceptual entities are first thought of as *content* and are second thought of as *form*, where "content is the whole set of sensorimotor processes which produce action, and the form is the system of concepts used by the subject to become aware of this action; therefore to conceptualize this activating content" (Piaget, 1975/1977, p. 147). The analogous terms used in mathematics education are *process* and *object*, respectively. A *process* conception of a mathematical concept is "a form of understanding of a concept that involves imagining a transformation of mental or physical objects that the subject perceives as relatively internal and totally under her or his control" (Dubinsky & Harel, 1992b, p. 20), and an *object* conception of a mathematical concept is "a form of understanding of a concept that sees it as something to which actions and processes may be applied" (Dubinsky & Harel, 1992b, p. 19). Based on these definitions, a mathematical concept such as *function* is first thought of as a *process* and second as an *object*.

Other distinctions.

Reification.

Researchers have expanded on the process-object distinctions in the construction of the concept of mathematical functions. For instance, Sfard (1992) has expanded on the *process-object* distinction by referring to them as *operational-structural* conceptions in her explanation of the process of *reification*[3]. *Reification* is the term Sfard (1987, 1989, 1991, 1992) uses to explain the cognitive development of the concept of function. Bear in mind that the word conception indicates understanding or knowledge of a particular concept, which in turn refers to a notion or an idea (Sfard, 1991). For Sfard (1992), from an *operational* conception a function can be perceived as a computational process that is dynamic, while from a *structural* conception

[3] Details of this process can be found in Sfard (1987, 1989, 1991, 1992), as well as in Burnett-Bradshaw (2007).

a function can behave like an object-like entity that is static, and not dynamic. An *operational* conception includes computational procedures and a *structural* conception includes aggregates of ordered pairs (Sfard, 1989).

Encapsulation.

Breidenbach, Dubinsky, Hawks, and Nichols (1992), and Dubinsky and Harel (1992a) have also expanded on the *process-object* distinction by adding the terms *pre-function* and *action* to highlight other types of conceptions of function in their explanation of the process of *encapsulation*[4]. *Encapsulation* is the term Ayers, Davis, Dubinsky, and Lewin (1988); Breidenbach, Dubinsky, Hawks, and Nichols (1992); and Dubinsky (1991a, 1991b) use to explain the cognitive development of the concept of function. (Even though encapsulation and reification are used to explain the means to an approximately similar endpoint – an object conception of function — the means are very different; Burnett-Bradshaw, 2007.) Dubinsky and his colleagues provide the following definition for the terms they use:

- *Pre-function* conception: the student does not have enough knowledge of functions to perform mathematical tasks that are related to functions.
- *Action* conception: the student is able to compute the value of a function one step at a time, but is unable to think of it in general terms, and hence the concept is static.
- *Process* conception: the student is able to understand that the transformation of quantities by a function is dynamic, i.e., the transformation is a complete activity where you start with objects, act on the objects, and end with new objects. In addition, the student is able to

[4] Details of this process can be found in Ayers, Davis, Dubinsky, and Lewin (1988); Breidenbach, Dubinsky, Hawks, and Nichols (1992); Dubinsky (1991a, 1991b); and Dubinsky and Harel (1992a); as well as in Burnett-Bradshaw (2007).

understand composite and inverse functions in general terms, as well as the notion of one-to-one[5] and onto[6].

- *Object* conception: the student is able to understand functions such that they can perform actions on them that will transform them.

Pointwise vs. global approach.

The different types of conceptions of function, distinguished in the proposed reification and encapsulation approaches, are difficult to identify in empirical data. This is because researchers and teachers have to use various methods to determine one's understanding of a concept. It therefore makes sense to consider the *approach* a person uses in dealing with questions regarding functions as a method for determining if the said person has a process or an object conception of function, which is the framework used for analyzing the data in this study. In terms of types of approaches, one can have a pointwise or a global approach to a function (Bell & Janvier, 1981; Janvier, 1978). Even (1998) gives a clear distinction between these two types of approaches:

> To deal with functions pointwise means to plot, read or deal with discrete points of a function either because we are interested in some specific points only, or because the function is defined on a discrete set. Reading values from a given graph, or finding the discrete density of a discrete random variable, are examples of a pointwise approach to functions. There are also times when we need to consider the function in a global way, and look at its behavior; for example, when we want to sketch the graph of a function given in symbolic form, or when we want to find an extremum of a function which is

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[5] "A function $f : S \rightarrow T$ is called *one-to-one* or an *injection* if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$" (Marsden & Hoffman, 1993, p. 4).

[6] "We say that $f : S \rightarrow T$ is *onto*, or is a *surjection*, when for every $y \in T$, there is an $x \in S$ such that $f(x) = y$, in other words, when the range equals the target" (Marsden & Hoffman, 1993, p. 4).

> defined on the real numbers. Flexibility in moving from a representation to another is intertwined in flexibility in using different approaches to functions. (p. 109)

Based on this distinction, a pointwise approach to a function is indicative of a process conception of a function, while a global approach to a function is indicative of an object conception of a function. This is because a process conception indicates thinking of a function in discrete terms, which is akin to using a pointwise approach, while an object conception indicates thinking of a function as an entity, which is akin to using a global approach. Moreover, the ability to translate between representations is associated with the ability to transition from one approach to the other as necessary.

Empirical research.

Some of the empirical research on the various process-object distinctions, previously mentioned, has been developed within the theoretical models of reification and encapsulation. The empirical research on reification has examined the order of operational and structural conceptions of functions (Sfard, 1987), and has found that one develops an operational conception of function prior to developing a structural conception, suggesting that the mathematics curriculum should be designed to match this process. Empirical research has also examined the phenomenon of reification (Sfard, 1989), finding that not many students develop a structural conception of function, many students have a conception of function that is closer to operational than to structural, and some students developed quasi-structural conceptions of functions.

The empirical research on encapsulation has examined at least three specific areas. First, it has focused on how to foster reflective abstractions (Ayers, Davis, Dubinsky, & Lewin, 1988), finding that the computer operating system Unix is useful in initiating the types of reflective abstractions needed to understand the composition of functions. Second, it has examined how to improve students' conceptions of functions (Breidenbach, Dubinsky, Hawks, & Nichols, 1992), finding that the administered intervention, which consisted of a computer environment, was useful in helping students to change their conception of function from action to process. Third, it has examined the action and process conceptions of functions (Dubinsky & Harel, 1992a), finding that students tend to go back and forth between both conceptions for various reasons.

Empirical studies have also investigated a pointwise versus a global approach to function, determining that there is a need for flexibility between the two approaches (Even 1990, 1998). The research has also indicated that each approach can be powerful within its own right (Even 1998), and one needs to be able to choose the most appropriate approach for any given problem (Bell & Janvier, 1981).

For the purposes of this study, the process-object distinction offered by Dubinsky and Harel (1992b) is the definition that will be used in explaining a participant's level of understanding of function. This definition was chosen because it is the definition that seems to capture the definitions of the other researchers previously mentioned, and it is not biased towards any explanation for the cognitive development of functions such as reification or encapsulation. And, in terms of analyzing the data obtained, the pointwise-global approach will be used in determining whether a participant has a process or an object conception of function. This is because one's approach to functions, or one's interaction with functions, is indicative of one's internal representation of functions, which is in turn indicative of one's understanding of functions (Goldin & Kaput, 1996; Goldin & Shteingold, 2001). This lens was chosen because it is difficult to determine whether one has a process or an object conception of function, as was found in an analysis of reification and encapsulation (Burnett-Bradshaw, 2007), and it is, therefore, necessary to examine how one interacts with and acts upon a function, within each representation, in order to make any conclusions about one's understanding of a function. This then begs the question, what is a representation?

Representations of Functions

Definitions.

A representation can be defined as "a configuration of some kind that, as a whole or part by part, corresponds to, is referentially associated with, stands for, symbolizes, interacts in a special manner with, or otherwise represents something else" (Goldin & Kaput, 1996, p. 398). Representations can be either internal or external (Goldin, 1998; Goldin & Kaput, 1996; Goldin & Shteingold, 2001), where an internal representation can be defined as "possible mental configurations of individuals, such as learners or problem solvers" (Goldin & Kaput, 1996, p. 399), and an external representation can be defined as "physically embodied, observable configurations such as words, graphs, pictures, equations, or computer microworlds" (Goldin & Kaput, 1996, p. 400).

Representations, both internal and external, belong to systems known as "symbol schemes" (Kaput, 1987) or "representational systems" (Goldin, 1987; Lesh, Landau, & Hamilton, 1983). A symbol scheme is "a concretely realizable collection of characters together with more or less explicit rules for identifying and combining them" (Kaput, 1987, p. 162). A representational system is "a collection of elements called (interchangeably) *characters* or *signs*" (Golding, 1987, p. 127) and "there may be specified a set of conditions which describe *permitted configurations of characters* in a representational system" (Goldin, 1987, p. 128). These two definitions are quite similar because they point to elements and a set of stable relations and rules, which are necessary for relating these elements.

For the purposes of the proposed study, the term representation will be used to mean external representations, which is the method most often used to depict a function. This is further explained below. Thus, representations, as used in this paper, are synonymous to notations, which can be iconic or non-iconic (Tolchinsky-Landsmann & Karmiloff-Smith, 1992), and must meet three criteria: "These three criteria – creator, location, and time independence – result in a domain of inquiry that includes the major forms of notations such as drawings, written language, numbers, maps, scale models and pictures" (Lee & Karmiloff-Smith, 1996, p. 187).

Types of representations for functions.

There are various ways to represent a function. For instance, functions can be represented as formulae, Cartesian graphs, arrow diagrams, tables, sets of ordered pairs, and situations from everyday life (Brenner et al., 1997; Coulombe & Berenson, 2001; Even, 1990; Friedlander & Tabach, 2001; Kieran, 1993; Moschkovich, Schoenfeld, & Arcavi, 1993; Yerushalmy & Schwartz, 1993; Yerushalmy & Shternberg, 2001). The study described in this paper, however, focuses only on the algebra-symbolic, graphical, and tabular representation, as these are the forms of representation with which a high school student may be most familiar. The term algebra-symbolic representation, adapted from Brizuela and Schliemann (2004), is used to refer to expressions or equations containing numbers and variables connected by mathematical operations. This term is preferable to the term algebraic or symbolic because the term "algebra" typically refers to equations and less to functions, which is the focus in this paper; and the term "symbol" is used to mean different things by different researchers. The term graphical representation is used to refer to the Cartesian coordinate system, and the term tabular representation is used to refer to a table of values displaying an input and an output.

Empirical research.

The research on the various representations of functions is vast (Janvier, 1987a). Some researchers have examined the impact of technology on representations of functions (Kieran, 1993; Ruthven, 1990; Yerushalmy & Schwartz, 1993; Yerushalmy & Shternberg, 2001). There are other researchers who are interested in translation ability between and within representations (Dufour-Janvier, Bednarz, & Belanger, 1987; Even, 1998; Gagatsis & Shiakalli, 2004; Goldin, 1987; Janvier, 1987b; Lesh, Post, & Behr, 1987; Hitt, 1998; Moschkovich, Schoenfeld, & Arcavi, 1993; Ruthven, 1990), and its relationship to solving mathematical problems (Brenner et al., 1997; Cifarelli, 1998; Coulombe & Berenson, 2001; Dufour-Janvier, Bednarz, & Belanger, 1987; Even, 1998; Friedlander & Tabach, 2001; Gagatsis & Shiakalli, 2004; Hitt, 1998; Janvier, 1987b; Kaput, 1985, 1987; Lesh, Post, & Behr, 1987; Owens & Clements, 1998). The research on representations of functions that is, however, most important to the study presented here is the research that examines the relationship between conceptions of functions and representations of functions (Moschkovich, Schoenfeld, & Arcavi, 1993), to which we will now turn our attention.

Conceptions and Representations of Functions

There is a need to examine the impact of the common representations of functions on the understanding of the concept of functions, as Dubinsky and Harel (1992a) and Moschkovich, Schoenfeld, and Arcavi (1993) have found that students can express a certain understanding of the concept of function in one representation, and express a different understanding of the concept of function in another representation. In fact, Moschkovich, Schoenfeld, and Arcavi (1993) present their framework through the use of a representation-perspective schematic diagram (see Figure 1), and conclude in their study that the ability to move between representations and between perspectives is indicative of an understanding of the concept of function. This schematic characterization of their framework is useful for the study presented here because it considers the three most common representations of mathematical functions (algebra-symbols, tables, and graphs), and it considers the process-object distinction used to explain the cognitive development of functions.

	Representation		
Perspective	Tabular	Algebraic	Graphical
Process			
Object			

Figure 1. Framework for alternate perspectives of functions in typical representations[7].

This particular framework, although useful, needs to be modified for the purposes of this study. This is because, as was previously stated, it is difficult to distinguish between a process conception of function and an object conception of function, i.e., it is far easier to examine the approach – pointwise or global – the participants use in answering the question posed. In addition, the terms used for each type of representation are different. The modified framework that will be used to analyze the data obtained is shown in Figure 2.

	Representation		
Approach	Tabular	Graphical	Algebra-symbolic
Pointwise			
Global			

Figure 2. Framework for examining representations and approaches to functions used in this study.

This is the tool that was used to describe students' and teachers' *understandings* of the *transformations* and *manipulation* of functions in *different representations* in this study.

Research Design

[7]*Note*. Taken from Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (p. 79). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Participants

The participants included two high-performing ninth graders, two high-performing twelfth graders, and two mathematics teachers, all from the same public high school in Boston, Massachusetts, which serves a diverse racial, ethnic, and cultural community. High-performing students were selected to participate in the study as it was simple to distinguish them from their peers, and it was determined that they would be least affected if the interviews overlapped with their class time. The ninth graders (referred to as Student 9-1 and Student 9-2) were currently taking a mathematics class that covered the basic topics in algebra, geometry, and probability and statistics. This class was designed by the teachers at the school and did not use a specific textbook. The twelfth graders (referred to as Student 12-1 and Student 12-2) were taking a combination of pre-calculus and calculus with a focus on various types of functions. This class was also designed by the teachers at the school and did not use a specific textbook. Of the two teachers interviewed, one was the mathematics teacher of the ninth grade participants (referred to as Teacher 9), and the other was the mathematics teacher of the twelfth grade participants (referred to as Teacher 12). The students were selected by their mathematics teachers to participate in the study based on their performance (good grades and high skill level) in their current mathematics class.

Procedure

The participants were asked to complete a profile questionnaire (see Appendices A and B) to determine the amount of exposure each participant had had to the concept of function. Each participant was then individually interviewed about the definition of function, and about the transformations and manipulation of functions in different representations (see Appendix C) by the researcher. The questions were given in the tabular, graphical, or algebra-symbolic representations. The interviews were videotaped, and lasted between 15 and 60 minutes, dependent on the length of time the participants needed to process each question. The interviews were all conducted within a month to reduce the likelihood of any student being exposed to more mathematics instruction than his/her grade level counterpart.

Instruments

Profile questionnaires.

There were two profile questionnaires – one for the students and one for the teacher. The student profile questionnaire (see Appendix A) asked the students information about their previous mathematics class and their current mathematics class, in particular the topics covered and which topics were interesting or difficult. The teacher profile questionnaire (see Appendix B) asked the teacher information about degrees and certification obtained, as well as teaching experience.

Individual interviews.

The individual interview consisted of seven questions (see Appendix C) – the topics addressed by these questions, as well as the representation or format in which they are presented, is summarized in Table 1. The first question and first part of the last question addressed the definition of a function, but the first question was more open-ended and asked the participant to offer their own definition of a function, while the first part of the last question was multiple-choice in nature and asked the participant to choose the best definition of a function from the choices given. The order for the format of the two questions was intentional so that participants could give an initial spontaneous and untainted response to the definition of a function. The second part of the last question asked participants to determine which of two given relations was a function.

Table 1

Summary of Topics Addressed and Representation/Format of Interview Questions

Question Number	Topic	Representation/Format
1a	definition – example	open-ended
1b	definition – non-example	open-ended
1c	definition	open-ended
2	transformation	graphical
3	transformation	graphical
4	transformation	tabular
5	transformation	algebra-symbolic
7a	Definition	multiple choice
7b	definition – univalence	tabular

Questions 2 to 5 addressed the transformations of a function; however, the second and third questions used the graphical representation, the fourth question used the tabular representation, and the fifth question used the algebra-symbolic representation. In these four questions, the participant was asked to state the relationship between two given functions. There was a sixth question administered to the students but not included in this paper's analysis.

Results & Discussion

Profile Questionnaire

Ninth grade students.

The student profile questionnaire revealed that the ninth grade students interviewed (Student 9-1 and Student 9-2) were currently covering basic algebra principles such as order of operations, as well as statistical concepts such as box plots. One of the ninth graders took pre-algebra in the previous year, where the topic of linear equations had been introduced. It is unclear exactly which topics the other ninth grader covered in the previous year, as both students did not attend the same school.

Twelfth grade students.

The student profile questionnaire also revealed that the twelfth grade students interviewed (Student 12-1 and Student 12-2) were taking a combination of pre-calculus and calculus, in which they were currently learning about polynomial functions and their characteristics. In the previous year, both students covered topics in pre-calculus, trigonometry, and statistics.

Teachers.

The teacher profile questionnaire revealed that both of the mathematics teachers interviewed possessed an undergraduate degree in mathematics, a master's degree in education, and teaching certification for high school mathematics. In addition, the ninth grade mathematics teacher (Teacher 9) had one year teaching experience, and had taught algebra I and geometry. The twelfth grade mathematics teacher (Teacher 12) had eight years teaching experience, and had taught the full gamut of high school mathematics courses. This teacher also possessed teaching certification in social studies and in moderate disabilities.

Interviews

Each individual question was initially analyzed ad hoc, and then using a combination of the already existing coding systems used in other studies previously, such as Even's 1993 and 1998 studies and Moschkovich, Schoenfeld, and Arcavi's study (1993). The results of the analyses of each question are presented here, which is followed by a discussion in the following sections. The responses to questions pertaining to participants' understandings of the *definition* of a function are presented first (i.e., responses to Questions 1 and 7), and the responses to questions pertaining to participants' understandings of the *transformations* of functions in different representations (i.e., responses to Questions 2, 3, 4, and 5), are presented second.

Goal 1 – Definition of a function.

Questions 1 and 7 (see Appendix C) focused on the participants' understandings of the *definition* of a function. Question 1 consisted of three parts in which the participants were asked to give an example of a function, an example of a non-function, and the definition of a function, with an accompanying explanation for each. Question 7 consisted of two parts in which the participants were asked to determine which of three options gave the correct definition of a function, and they were also asked to determine which of two relations (given in the tabular representational form) was a function.

The participants' explanations for their responses to Questions 1 and 7 were coded similarly, and were based on the essential characteristics of a function – arbitrariness and univalence. Each explanation was scored with a 1 or a 0 depending on whether it met the following criteria:

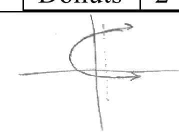
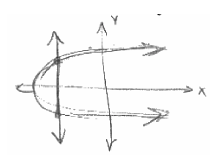
- Arbitrariness (A): The relationship between the two sets comprising the function is arbitrary and/or the members of the sets themselves are arbitrary.
- Univalence (U): The relation between the two sets comprising the function is either one-to-one or many-to-one.

The participants' responses to Questions 1 and 7 are shown in Tables 2 and 3 respectively, and later summarized in Table 4.

It is important to note at this point that even though it seems that emphasis is placed on the participants being able to define a function, it is even more important for the participants to be able to understand the concept of function. Thus, it is more important for a participant to be able to suggest an example of a function or a non-function and to determine if a given relationship is a function, which would be indicative of them understanding the concept of function. It is also hoped that if the participants are able to do these things, then stating the definition of a function will be less challenging for them, and will then become less important in describing their understandings of a function.

Table 2

Results of Question 1

<table>
<tr><th></th><th colspan="3">Question 1(a) – Example of a Function</th><th colspan="3">Question 1(b) – Example of a Non-Function</th><th colspan="3">Question 1(c) – Definition of a Function</th></tr>
<tr><th></th><th></th><th colspan="2">Explanation</th><th></th><th colspan="2">Explanation</th><th></th><th colspan="2">Explanation</th></tr>
<tr><th>Participant</th><th>Function</th><th>A[8]</th><th>U[9]</th><th>Non-function</th><th>A</th><th>U</th><th>Definition</th><th>A</th><th>U</th></tr>
<tr><td>Student 9-1</td><td>$a^2 + b^2 = c^2$</td><td>1</td><td>0</td><td>PEMDAS[10]</td><td>1</td><td>0</td><td>Something that is used to get an answer; a function has a purpose and the purpose is the answer.</td><td>1</td><td>0</td></tr>
<tr><td>Student 9-2</td><td>y = mx + b</td><td>1</td><td>0</td><td><table><tr><td>People</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr><tr><td>Donuts</td><td>2</td><td>4</td><td>6</td><td>8</td><td>10</td></tr></table></td><td>1</td><td>0</td><td>An equation or problem used to solve something.</td><td>1</td><td>0</td></tr>
<tr><td>Student 12-1</td><td>$f(x) = x^3 + 3x^2 – 4x + 3$</td><td>1</td><td>1</td><td></td><td>1</td><td>1</td><td>A function is an equation that will give only one output for one specific input.</td><td>1</td><td>1</td></tr>
<tr><td>Student 12-2</td><td>$f(x) = x^2 + 9x + 3$</td><td>0</td><td>1</td><td></td><td>1</td><td>1</td><td>If you have two different inputs and have two similar outputs then it is not a function. If you have two inputs and get two different outputs then it is a function.</td><td>1</td><td>1</td></tr>
<tr><td>Teacher 9</td><td>$f(x) = x^2$</td><td>1</td><td>0</td><td>y = 5</td><td>1</td><td>0</td><td>It is an equation that dictates a movement or represents a set of data.</td><td>1</td><td>0</td></tr>
<tr><td>Teacher 12</td><td>$y = x^2$</td><td>1</td><td>0</td><td>height vs. shoe size</td><td>1</td><td>1</td><td>Domain; range; rule; every input has only one output.</td><td>1</td><td>1</td></tr>
</table>

[8] A is used to refer to the function characteristic of arbitrariness.
[9] U is used to refer to the function characteristic of univalence.

Table 3

Results of Question 7

Participant	Question 7(a) – Definition of a Function			Question 7(b) – Which one is the function?		
	Definition[11]	Explanation		Function or not?	Explanation	
		A	U		A	U
Student 9-1	i	1	0	neither	1	0
Student 9-2	i, ii, iii	1	0	either	1	0
Student 12-1	iii	1	1	second table	1	1
Student 12-2	i and ii	1	0	second table	1	1
Teacher 9	undecided	1	0	unknown	1	0
Teacher 12	iii	1	1	second table	1	1

[11] Participants were given the following options from which to choose:

i. A function is an algebraic expression in which you can substitute various values for an unknown.
ii. A function is a computational process that produces an output (y) from an input (x).
iii. A function consists of two sets S and T together with a "rule" that assigns to each element of S a specific element of T.

Table 4

Summary of Explanations Given in Questions 1 and 7

Participant	Question 1(a) – Example of a Function		Question 1(b) – Example of a Non-Function		Question 1(c) – Definition of a Function		Question 7(a) – Definition of a Function		Question 7(b) – Which one is the function?	
	Explanation									
	A	U	A	U	A	U	A	U	A	U
Student 9-1	1	0	1	0	1	0	1	0	1	0
Student 9-2	1	0	1	0	1	0	1	0	1	0
Student 12-1	1	1	1	1	1	1	1	1	1	1
Student 12-2	0	1	1	1	1	1	1	0	1	1
Teacher 9	1	0	1	0	1	0	1	0	1	0
Teacher 12	1	0	1	1	1	1	1	1	1	1

Student 9-1.

Question 1(a) – Example of a function.

The first ninth grade student incorrectly stated that what is commonly known as the formula for Pythagoras' Theorem is an example of a function because "it is an equation, a way to find something out." It is correct to say that it is an equation, which is indicative of a *rule* of correspondence, however, the example given by this student was not explicitly defined for two sets, and neither was their mention of the presence or absence of the characteristic of univalence.

Question 1(b) – Example of a non-function.

This student used the popular acronym used to explain order of operations as an example of a non-function because it is "an order in which you do something and it's not a function so it doesn't give you a way to find something out." The student acknowledged that there is no *rule* of correspondence in the example given, but did not recognize that the example is also void of the two sets necessary for a function, as well as the univalence characteristic also necessary for a function.

Question 1(c) – Definition of a function.

This student believed that the definition given was valid because "a function usually like has a purpose, so the purpose would be the answer. So the function is like an equation that you can ultimately get the answer of." The definition and rationale for this response was, however, not based upon the characteristics of arbitrariness and univalence, but rather the need for a specific *rule*.

Question 7(a) – Definition of a function.

In response to Question 7(a), this student selected option (i) as the definition of a function, which stated that, "A function is an algebraic expression in which you can substitute various values for an unknown." The justification for this choice is "Because like what we were doing before, we were substituting variables and we were trying to find out what the unknown was." This definition of a function is grounded in the need for a specific *rule*, as well as there being unknowns instead of variables. This definition does not make correct reference to arbitrariness or to univalence.

Question 7(b) – Definition of a function - Univalence.

This student did not believe that either of the given relations in Question 7(b) were functions because there was no consistent pattern in either of them. Thus, implicit reference is made to the need for a specific *rule*, but this student is unaware of the fact that the *rule* can be arbitrary. There is also no reference to the univalence characteristic of a function.

Summary.

Student 9-1 was consistent in the responses given to Questions 1 and 7. This student believed that a function is an equation such that there would be a specific *rule*, and that there should be unknowns as opposed to variables involved in a function. This understanding is consistently evidenced in the responses given to the questions posed. It should be noted that this student had never been exposed to the definition of a function per se, and that the exposure to examples of functions was limited to linear functions. Therefore, it is understandable that the student was unaware of the need for two arbitrary sets and an arbitrary *rule* of correspondence, as well as the need for univalence in a function. It also indicates that instruction within this area is necessary for one's understanding of the *definition* of a function, as well as one's understandings of various examples of functions, even linear functions.

Student 9-2.

Question 1(a) – Example of a function.

The second ninth grade student stated that what is commonly known as the slope-intercept formula for a linear equation is an example of a function because "it's an equation to help you figure out something else." It is correct to say that it is an equation, which is indicative of a *rule* of correspondence, however, it was not explicitly stated that the sets upon which the function is defined are referred to as *x* and *y*, and neither was their explicit mention of the univalent nature of this particular equation.

Question 1(b) – Example of a non-function.

This student believed that the example given was that of a non-function because it did not match the participant's definition of a function, which is "a function is like an equation to help you solve…to help you solve a problem." In addition, the example given by the student was thought to be "just a set up of data" by the student. The student failed to recognize that the example given did actually meet the criteria necessary for a function, i.e., it consisted of two arbitrary sets, number of people and number of donuts, as well as a *rule* of correspondence, "for every person there's two donuts." In addition, the given example was univalent in nature.

Question 1(c) – Definition of a function.

This student was unsure of the definition of a function and believed that a function was similar to other objects in math. That is, functions are like equations that are used to find something out. This definition gives no mention of arbitrary sets and the characteristic of univalence. Instead, it implies the need for a specific *rule* of correspondence.

Question 7(a) – Definition of a Function.

In response to Question 7(a), this student selected all three of the given options to be the definition of a function. This response is based on the student believing that all three options for the definition of a function were indicative of different types of functions. If this were the case, then it would stand to reason that the student quite possibly believed that a function is in fact an equation, and thus the inability to choose one of the three options given. In addition, the student's explanation made no reference to the need for arbitrariness and univalence.

Question 7(b) – Definition of a function - Univalence.

This student believed that both of the given relations in Question 7(b) were functions because there was no consistent pattern in either of them. Thus, the student showed an awareness of the acceptability of an arbitrary *rule* of correspondence, but was unaware of the univalence characteristic of function.

Summary.

Student 9-2 consistently stated or implied that a function is an equation in the responses given to Questions 1 and 7. This implies a belief in the *need* for a specific *rule* of correspondence. In addition, since there was constant emphasis on the idea of using a function to find something out or to solve a problem, then it is quite likely that the student believed that there are unknowns, and not variables, in a function. The response to Question 7(b) was, however, surprising because the student believed that the *lack* of a *rule* of correspondence in the relations made them functions, which, unbeknown to the student, contradicts the definition repeatedly given by the student. The question probably caused a state of disequilibrium (Beth & Piaget, 1966) in the student's understanding of the definition of function as evidenced in the discrepancy in the student's responses to the questions pertaining to the definition of function. It is possible that the student was never asked questions of this nature before, which caused the student to respond with such contradictory answers.

Finally, it should be noted that this student, much like the first ninth grade student, had never been exposed to the definition of a function per se, and that the exposure to examples of functions was limited to linear functions, as is expected for ninth grade students. Therefore, it is understandable why the student was unaware of the need for two arbitrary sets and an arbitrary *rule* of correspondence, as well as the need for univalence in a function. Again, it seems that instruction within this area and exposure to a variety of types of functions is necessary for one's understanding of the *definition* of a function, as well as one's understandings of various examples of functions, even linear functions which are fairly simple.

Comparison of the responses of Student 9-1 and Student 9-2.

In comparing the responses of Student 9-1 and Student 9-2, both students have a fairly unsophisticated understanding of the concept of function. This is evidenced in their examples and also in their definitions, in which there is a consistent emphasis on the presence of unknowns and solving, as opposed to the presence of a variable. This unsophisticated understanding is further evidenced in Table 4, where we can see that both students have a similar non-understanding of the characteristics of a function. Both students are equally unable to articulate the need for arbitrariness in a function, and both students are equally unaware of the idea of univalence. These results parallel the work of Dreyfus and Vinner (1982), Even (1993), Vinner (1983), and Vinner and Dreyfus (1989), whose research shows that some students are less aware of the univalence characteristic of function than other students. The ninth grade students' responses also clearly revealed the limited variety of functions to which they might have been exposed in their mathematics education, further reinforcing their understandings regarding the concept of function.

Student 12-1.

Question 1(a) – Example of a function.

The first twelfth grade student stated that the quadratic function given is a function because there is an input, being referred to as *x*, and an output. This indicates that the student is aware of the need for two arbitrary sets, but does not explicitly mention anything about the *rule* of correspondence being arbitrary or not. In addition, this student makes reference to the characteristic of univalence by stating that the input "will give you one and only one outcome in the output which makes it a function."

Question 1(b) – Example of a non-function.

In response to the question, the first twelfth grade student stated, before giving an actual example, that, "Something that's not a function is, basically if you put in something for the input and you get two different answers or multiple answers for the output." After being told by the interviewer that any representation would be acceptable, the first twelfth grade student came up with the given graph (shown in Table 2), and then stated that it failed the vertical line test. This participant's response clearly showed an awareness of both characteristics of a function – arbitrariness and univalence.

Question 1(c) – Definition of a function.

This student believed that the definition of a function given was correct because if there is more than one output for each input then the graph would be inaccurate. This explanation is weak, but the definition indicates a strong knowledge of the two characteristics of a function. The definition also implies that the student believes that the *rule* of correspondence should be specific because reference is made to a function being an equation. This reference to a function being an equation could also mean that the student believes that a function must be given in the algebra-symbolic representation.

Question 7(a) – Definition of a function.

In response to Question 7(a), the student selected option (iii) for the definition of a function. The explanation for this choice reflected knowledge of the two characteristics of function – arbitrariness and univalence.

Question 7(b) – Definition of a function - Univalence.

This student chose the second relation as the one representative of a function. Initially, the student thought the answer was dependent on the equation for each relation, which implies the belief in the need for a specific *rule* of correspondence. The student, however, kept examining the relations without any influence from the interviewer, and then exclaimed that the answer had to be the second relation. The explanation that accompanied this exclamation made explicit reference to the characteristic of univalence.

Summary.

In response to every question posed regarding the definition of a function, Student 12-1 was able to identify the two characteristics of function. In fact, the student placed more emphasis on the need for univalence than on the need for arbitrariness, but the emphasis was consistent. The student also showed a solid awareness of the need for two arbitrary sets, but was not consistent in indicating that the *rule* of correspondence could be arbitrary.

Student 12-2.

Question 1(a) – Example of a function.

The second twelfth grade student stated that the quadratic function given was a function based on the fact that it has roots, factors, and it is a parabola. The student then made reference to the univalent characteristic of function by stating that the example given passes the vertical line test. The student, however, did not make any explicit reference to the arbitrariness of the function because there was no mention of the sets upon which the suggested function was defined nor to the *rule* of correspondence.

Question 1(b) – Example of a non-function.

This student was initially unsure about a suitable example of a non-function, and needed to be first reassured by the interviewer that any representation is acceptable for the example. The graphical example shown in Table 1 was then given and was justified as a true example with the existence of the x-axis and y-axis, as well as with the "fails the vertical line test" rule. This participant was, thus, able to make reference to both the arbitrary and univalent nature of functions.

Question 1(c) – Definition of a function.

This student offered a definition of a function, which was believed to be correct because it would not fail the vertical line test. The student was, however, aware of the need for two arbitrary sets and the need for the univalence characteristic; the student was unable to correctly express these characteristics in the definition.

Question 7(a) – Definition of a function.

In response to Question 7(a), this student selected options (i) and (ii) as the definition of a function. The explanation for this answer was that one needed to plug in an input in order to obtain an output. This selection for the definition of a function indicates that the student was aware of the need for two arbitrary sets, but was less aware of the fact that the *rule* of correspondence could be arbitrary. This student also failed to choose the definition that indicated the univalence characteristic of function.

Question 7(b) – Definition of a function - Univalence.

The second twelfth grade student initially stated that neither of the given relations were functions. The interviewer encouraged the student to create a graph of each relation, at which point the student immediately stated that the first relation failed the vertical line test, but that the second did not, and thus, the second relation is the function. This student was able to pick up on the univalence characteristic of functions, but only through the graphical representation. In addition, the student was baffled by the fact that neither relation had a specific *rule* of correspondence.

Summary.

For Student 12-2, there is continuous awareness of the need for two arbitrary sets upon which to define a function. The student also continually expected that the *rule* of correspondence should be specific. In terms of the characteristic of univalence, the student is only able to identify and explain it through the vertical line test, which is a graphical test. This indicates that the student's understanding of this characteristic may be limited.

Comparison of the responses of Student 12-1 and Student 12-2.

In comparing the responses of Student 12-1 and Student 12-2, the first of the two students has a slightly more sophisticated understanding of the concept of function than the second twelfth grade student. This is evidenced in their examples being fairly similar, but their explanations being slightly different. In their explanations, Student 12-1 consistently refers to the arbitrary nature of a function and Student 12-2 does not. Their responses to Question 7a are also different as Student 12-1 chose the option in which the univalent nature of function was prominent, and Student 12-2 did not. This difference in understanding is further emphasized in Table 4, where the slight difference in their understandings is more clearly seen. These results parallel the work of Dreyfus and Vinner (1982), Even (1993), Vinner (1983), and Vinner and

Dreyfus (1989), whose research shows that some students are more aware of the univalence characteristic of function than other students. These results, together with those from Students 9-1 and 9-2, also highlight how univalence remains problematic and challenging for students throughout their high school mathematics education.

Teacher 9.

Question 1(a) – Example of a function.

The ninth grade teacher stated that the common expression of a quadratic function, with intercept zero, is an example of a function because "it's an equation that dictates a movement." It was correct to say that the given example was an equation, which would then be indicative of a *rule* of correspondence. The teacher did not, however, explicitly state that the sets, upon which the function given is defined, are referred to as *x* and *f(x)*. There was also no explicit mention of the characteristic of univalence as it pertains to this particular equation.

Question 1(b) – Example of a non-function.

This teacher was unsure of the example given for a non-function, and stated that "Well, I guess it could be [a function]. But, it's a…a straight line. But, it doesn't take input. That output will always be the same no matter what the input is." The teacher realized the need for the function to be arbitrarily defined upon two sets, an input and an output, and believed the example given to be devoid of such when in fact the input could have been maximized by making it all real numbers and the output was the set containing the number 5. There was also no reference made to the univalent nature of a function. (Note that if the teacher had stated that x = 5 is an example of a non-function, then this would have been correct.)

Question 1(c) – Definition of a function.

The ninth grade teacher believed the definition given was correct because "a function is about…um…putting in information and getting something back out…and…which creates a certain pattern." This explanation makes reference to the arbitrary nature of a function, in terms of there being two arbitrary sets, but believes that the *rule* of correspondence should be specific as opposed to arbitrary. In addition, there is no reference to the univalent nature of functions in this definition.

Question 7(a) – Definition of a function.

In response to Question 7(a), the teacher was undecided as there were different parts of the given options that appealed to the teacher's definition of a function. For instance, the teacher acknowledged the need for two arbitrary sets, and referred to them during the interview as x and y values, the domain and range, and the input and output. The teacher also gravitated to the phrases "algebraic expression," "computational process," and *rule*, and believed these to be necessary in forming the definition of a function. The teacher did indicate that the phrase "computational process" was least used when teaching about functions. This implies that the teacher did not believe in the need for an arbitrary *rule* of correspondence or in the characteristic of univalence.

Question 7(b) – Definition of a function - Univalence.

The teacher was unsure as to which of the two given relations were functions. The teacher first tried to determine the pattern by examining the tables. The teacher then created graphs, and again tried to determine the pattern from the graphs. This led the teacher to conclude that there was not enough information given to determine which of the relations was a function. The fact that the teacher was able to translate between representations showed an awareness of the arbitrary sets upon which a function is defined. The teacher was also constantly looking for a specific *rule* of correspondence, and was unwilling to accept that the *rule* could be arbitrary. In addition, the characteristic of univalence was not alluded to by the teacher.

Summary.

Teacher 9 was fairly consistent in responding to Questions 1 and 7. In response to every question, there was either explicit or implicit mention of the need for two arbitrary sets when thinking about functions. There was also constant mention of the need for a specific *rule* of correspondence often given through an equation or algebraic expression. This implies that the teacher was unaware that the *rule* could be arbitrary. In addition, the teacher never made reference to the characteristic of univalence in any of the responses given. Finally, many of the understandings of Teacher 9 were similar to the understandings of Student 9-1 and Student 9-2, who were current students of Teacher 9. This makes sense, as it is not likely that a student would have a more sophisticated understanding of a given topic than their teacher, but rather an understanding that is less than or equal to that of their teacher (Hill, Rowan, & Ball, 2005).

Teacher 12.

Question 1(a) – Example of a function.

The twelfth grade teacher stated that the common expression of a quadratic function, with intercept zero, is an example of a function because it has a domain or input and a range or output, as well as a *rule*. This explanation addresses the question of arbitrariness; however, no reference to the characteristic of univalence is given in this explanation.

Question 1(b) – Example of a non-function.

The teacher believed that height versus shoe size was an example of a non-function for two reasons. One, there was no set *rule* for the input (height) to obtain the output (shoe size). Second, a given height did not yield exactly one shoe size. In this explanation, there is reference to both the arbitrary and univalence characteristics of functions. It should be noted that despite this teacher's comment, there does not need to be a specific *rule* for a function.

Question 1(c) – Definition of a function.

The twelfth grade teacher's definition was brief and to the point. It made explicit reference to both the arbitrary and univalent nature of functions. It is, however, unclear if the teacher's interpretation of *rule* is specific or arbitrary.

Question 7(a) – Definition of a function.

In response to Question 7(a), the twelfth grade teacher selected option (iii) as the definition of a function. The teacher believed this to be the best definition of a function because it made reference to the univalence characteristic of function, which the teacher deemed to be important. The teacher rejected option (i) because it sounded only like an equation.

Question 7(b) – Definition of a function - Univalence.

This teacher selected the second relation as the function because it matched the definition selected in Question 7(a). That is, the second relation displayed the characteristic of univalence, which distinguished it from the other relation. This indicates that the teacher had a clear understanding of the characteristic of univalence.

Summary.

Teacher 12 displayed an understanding of both characteristics of function. This is displayed in the teacher's responses to Questions 1(b), 1(c), 7(a), and 7(b). The teacher did not mention the characteristic of univalence in the example of a function offered for Question 1(a), but this may be because one does not often think about this characteristic until one is asked to consider a non-function. It is noteworthy that many of the ideas that Teacher 12 had about the concept of function were similar to the understandings of Student 12-1 and Student 12-2, who were also current students of the twelfth grade teacher interviewed. This makes sense, for, as previously stated, it is not likely that a student would have a more sophisticated understanding of a given topic than their teacher, but rather an understanding that is less than or equal to that of their teacher (Hill, Rowan, & Ball, 2005).

Comparison of the responses of Teacher 9 and Teacher 12.

In comparing the responses of the teachers, Teacher 12 had a more theoretical understanding of the concept of function than Teacher 9, who had a more practical understanding of the concept of function. This is evidenced in the differences in their responses and explanations, in which Teacher 12 was consistently more aware of the univalence characteristic of function, and Teacher 9 made no mention of it. Also, Teacher 9 was more willing to accept the different definitions offered by the interviewer, without questioning them, than was Teacher 12, and was more willing to think of a function as an object you put into action, rather than an object upon which you can act. This difference in understanding is more clearly seen in Table 4, and may be explained by the fact that Teacher 9 did not teach the topics addressed in the interview, and was therefore probably less familiar with the details of the topics.

Goal 2 – Transformations of functions.

Questions 2, 3, 4, and 5 addressed the transformations of a function; however, Questions 2 and 3 were presented in the graphical representation, Question 4 was presented in the tabular representation, and Question 5 was presented in the algebra-symbolic representation. In these four questions, the participants were asked to state the relationship between two functions, f(x) and g(x) (see Appendix C). The participants' responses were coded based on the representation in which they answered the question and whether a pointwise or a global approach to answering the question was used.

The coding paradigms used to analyze Questions 2, 3, 4, and 5 are shown in Figures 3, 4, 5, and 6 respectively, and were derived from the analytical framework used by Moschkovich, Schoenfeld, and Arcavi (1993), which was previously shown in Figure 1. The cell in the coding matrices with the "most anticipated" response appears in boldface in each figure. The "most anticipated" response was chosen by the researcher based on two criteria – it was in the representation in which the question was posed and it used a global approach. The "anticipated" response for Question 4 differed from the others because the two functions were presented as discrete data points, and thus a pointwise approach to answering the question was expected. Each cell in each table also describes accurate approaches for each problem for each representation. Each participant could therefore have an "anticipated" response, as well as a few other accurate, non-anticipated responses.

	Representation		
Approach	Tabular (T)	Graphical (G)	Algebra-symbolic (A)
Pointwise	created a table of values for each function by reading each individual point	compared individual input and output values on the graph of each function	computed slope and intercept from given information to find the equation of each function
Global	created a table of values for each function based on deduced slope and intercept	**compared output values on the graph of each function**	deduced slope and intercept from given information to find the equation of each function

Figure 3. Coding Paradigm for Question 2 in the Graphical Representation.

	Representation		
Approach	Tabular (T)	Graphical (G)	Algebra-symbolic (A)
Pointwise	attempted to create a table of values for each function by reading each individual point	compared individual input and output values on the graph of each function	
Global		**compared output values on the graph of each function**	

Figure 4. Coding Paradigm for Question 3 in the Graphical Representation.

	Representation

Approach	Tabular (T)	Graphical (G)	Algebra-symbolic (A)
Pointwise	**compared individual input and output values in the table of each function**	plotted each individual point to form the graph of each function	computed slope and intercept from given information to find the equation of each function
Global		sketched the graph of each function based on deduced slope and intercept	deduced slope and intercept from given information to find the equation of each function

Figure 5. Coding Paradigm for Question 4 in the Tabular Representation.

	Representation		
Approach	Tabular (T)	Graphical (G)	Algebra-symbolic (A)
Pointwise	created a table of values for each function by calculating each individual point	plotted each individual calculated point to form the graph of each function	calculated output values of each function
Global	created a table of values for each function based on deduced slope and intercept	sketched the graph of each function based on qualities obtained from the given equations	**compared the equations of each function**

Figure 6. Coding Paradigm for Question 5 in the algebra-symbolic representation.

The participants' responses to Questions 2, 3, 4, and 5 are shown in Table 5, and were coded with a score of 1 if it corresponded to that which was considered to be the accurate response from the coding paradigms in Figures 3, 4, 5, and 6, and a score of 0 if it was not the response considered to be accurate. The "most anticipated" response cell, based on the coding matrices, appears in boldface for each participant.

The analysis below shows that some participants are unwilling to engage in the use of representations other than that in which the question is posed, while others are willing to do so. This unwillingness is indicative of a rigid, and therefore, not flexible understanding of the concept of function, and can be explained in at least three different ways, which are mutually exclusive of each other. First, the participant takes the question at face value and tries to answer it without translating it into other representations. Second, the participant is *unable* to translate from one representation to another, or is not familiar or comfortable with other representations.

Third, the participant deems it more efficient to respond to the question in the representation shown rather than to waste time translating from one representation to another.

These explanations could imply very different understandings of the concept of function. The former two reasons for not translating a function from one representation to another indicate lack of flexibility, and should be interpreted as a less sophisticated understanding of the concept of function in that representation. The latter reason should not be viewed as negatively as the former two, but rather as a powerful and resourceful approach, though rigid. This approach should be interpreted as a more sophisticated understanding of function within that representation as it utilizes efficiency. Willingness to engage other representations, i.e., a flexibility in using various representations of a given function, should also be interpreted as a more sophisticated understanding of the concept of function (Brenner et al., 1997; Moschkovich, Schoenfeld, & Arcavi, 1993).

Table 5

Results of Questions 2, 3, 4, and 5

Participant	Approach	Question 2			Question 3			Question 4			Question 5		
		Representation											
		T[12]	G[13]	A[14]	T	G	A	T	G	A	T	G	A
Student 9-1	Pointwise	0	1	0	0	1	0	**1**	0	0	0	0	0
	Global	0	**1**	0	0	**1**	0	0	0	1	0	0	**1**
Student 9-2	Pointwise	0	0	0	0	0	0	**1**	0	0	0	0	0
	Global	0	**1**	1	0	**1**	0	0	0	0	0	0	**1**
Student 12-1	Pointwise	0	0	1	0	1	0	**1**	0	0	1	0	0
	Global	0	**0**	1	0	**1**	0	0	0	0	0	0	**1**
Student 12-2	Pointwise	0	0	1	0	0	0	**1**	1	1	0	0	0
	Global	0	**0**	1	0	**1**	0	0	0	1	0	0	**1**
Teacher 9	Pointwise	0	0	0	0	0	0	**0**	0	0	0	0	0
	Global	0	**0**	1	0	**1**	0	0	0	1	0	0	**1**
Teacher 12	Pointwise	0	0	0	0	0	0	**1**	0	0	0	0	0
	Global	0	**1**	0	0	**1**	0	0	0	0	0	0	**1**

[12] T is used to refer to the tabular representation

Student 9-1.

Question 2 – Graphical.

In trying to determine the relationship between the two given functions in the graphical representation, the first ninth grade student initially stated that, "they're both … positive lines, show positive lines of data because they start off at a negative point and then they move on to the positive side of the grid." The student also stated that the two functions are the same length, the first y-coordinate for f(x) and g(x) is -7 and -4 respectively, and they are both at an angle. The interviewer then asked the student, "Suppose you were telling somebody how to plot the graphs. So you gave them the f(x) graph, and you want them to do the g(x) graph, what instructions would you give them?" The student initially responded, "I would tell them that the g(x) graph is, starts 3 units smaller on the y-axis than the f(x) graph … Actually, 3 units higher … Throughout the whole process it's 3 … units higher." This response indicates that the student was able to answer the question in the representation given, and did not translate the function into another representation. Moreover, the student initially used a pointwise approach in answering the question because only the y-intercepts were considered, but concluded with a global approach by considering both functions in their entirety.

Question 3 - Graphical.

In examining the two functions given in Question 3, the student observed that both functions had the same wave pattern and were approximately the same length. The student also noticed that the f(x) graph crossed the y-axis higher than the g(x) graph. The interviewer then asked, "If you were to give someone the f(x) graph and ask them how to plot the g(x) graph what would you say?" The student responded that the f(x) graph is approximately 2 units higher than the g(x) graph, but was pointing at the y-axis. After being prodded, the student then concluded that f(x) is some number greater than g(x). Again, this response indicates that the student was able to answer the question in the representation given, and did not translate the function into another representation. Moreover, the student initially used a pointwise approach in answering the question because only the y-intercepts were considered, but concluded with a global approach by considering both functions in their entirety.

Question 4 - Tabular.

In examining the functions given in Question 4, the student speculated that the f(x) and g(x) functions increase by 8 and 16 respectively, but then realized this is not so. The student continued to study the two functions, and then pointed out that the f(x) and g(x) functions are increasing by 3 and 6 respectively. The student then exclaimed, "Ohhhhh … I see something now … the f(x) graph is half the size of the g(x) graph so that on each, each time it passes through, like each unit has, so like 8 is half of 16, 5 is half of 10, 2 is half of 4, 1 is half of 2, and on and on and on. And so, the f(x) graph is half the size of the g(x) graph." This response indicates that the student needed to translate the function into another representation, i.e., to deduce the slope of each function, in order to begin to determine the relationship. The student also often referred to the function as a graph even though it was given in the tabular representation. This error may be because the previous two questions were given in the graphical representation. The student used a pointwise approach in answering the question, which corresponds to the fact that the function was represented as discrete data points, but only after using a global approach to deduce the slopes of the two functions.

Question 5 – Algebra-symbolic.

In examining the functions given in Question 5, the student initially confused the functions given with those in the previous question. Once this was clarified, the student turned her attention to the functions at hand. The student was doubtful as to the relationship and stated, while pointing at the functions, "Wow. I really do not know how, how you would get that [g(x)] without using that [f(x)]. Like it just seems like it would be simple enough, but you would really have to think about it because right there [f(x)] I could say that obviously you would just take the minus 1. But you would put, put the whole problem in parentheses and you add after the ... 4 times, 4 times minus one, and then put the parentheses and then add two for the squared." The interviewer then guided the student into representing this response in algebra-symbolic form, as the student was only able to verbalize the relationship. Again, this response indicates that the student was able to answer the question in the representation given, and did not translate the function into another representation. Moreover, the student used a global approach by considering both functions in their entirety.

Summary.

Student 9-1, although doubtful of the responses given, answered the questions in the representations in which the questions were posed most of the time. In addition, for questions in the graphical representation, the student initially used a pointwise approach and then a global approach; for questions in the tabular representation, the student used a global approach and then a pointwise approach; and for questions in the algebra-symbolic representation, the student used a global approach. This implies that the student is flexible in approaches to answering the questions posed. Also, the student has a process conception of functions in the graphical and tabular representations, and an object conception of function in the algebra-symbolic representation. This difference in conceptions is consistent with the work of Dubinsky and Harel (1992a) and Moschkovich, Schoenfeld, and Arcavi (1993), who have found that students can express different understandings of the concept of function in different representations, and it is also consistent with the work of Schwartz and Yerushalmy (1992), who state that the graphical and the algebra-symbolic representation, which are object representations, are more likely to evoke an object conception of function.

Student 9-2.

Question 2 - Graphical.

The second ninth grade student was unsure as to what the question was asking and kept referring to the graphs as being representative of information usually seen in a table or as an equation. The student then pointed out that the two lines are parallel, and deduced the slope of each function from their respective graphs, which was found to be the same. The student continued to examine the two functions and pointed out that they both have different x- and y-intercepts, and finally concluded, "They're both the same, I guess. They're both the same lines…just in different positions … One's higher, and one's lower. They're placed…they're the same…the same two lines…just placed different on the axis." This response indicates that the student needed to consider the functions in their algebra-symbolic representation by turning attention to deducing the slope of each function. The student, however, continuously used a global approach in dealing with the two functions.

Question 3 - Graphical.

The student expressed unfamiliarity with the type of graph presented in this question, but eventually pointed out that both functions have the same wave motion or same pattern. The student concluded, "If I was to take a guess on what the lines mean, I guess it would be almost the same as the other graph with the different places on the axes, I guess." This response indicates that the student was able to answer the question in the representation in which the question was posed, and was able to use a global approach to the question by considering both functions in their entirety.

Question 4 - Tabular.

In response to this question, the student first tried to find the difference in the x and f(x) values, in order to determine the rule that connects the two. The interviewer clarified that the ultimate goal is to find the relationship between f(x) and g(x). Once this was clarified, the student says, "Oh! … Okay … Okay…the um…the relationship between f(x) and g(x) is that g(x) is just f(x) multiplied…is just…um…f(x) doubled…it's the same…x is still the same…on the left column. But, both…um…g(x) is just f(x) doubled." This response indicates that the student was able to answer the question in the representation given, and did not translate the function into another representation. Moreover, the student used a pointwise approach by considering the discrete points given for each function.

Question 5 – Algebra-symbolic.

In response to Question 5, the student's first response was, "Okay. All right. You could, um … do f(x) … and then minus one … and then whatever you get for that … square it I guess?" The student was unable to express this answer in algebra-symbolic form, and did not translate the question into any other representation in order to answer it. In addition, the student used a global approach to answer the question, as both functions were considered in their entirety.

Summary.

Student 9-2, who lacked confidence in the responses given and gave many cautious responses, answered the questions in the representations in which the questions were posed most of the time. This is possibly because the student was unwilling to try different strategies due to a lack of confidence. In addition, the student used a pointwise approach when it was warranted and a global approach when it was warranted. This implies that the student was able to exercise good judgment in choosing the most appropriate approach. In addition, the student seems to possess a process conception of functions in the tabular representation, and an object conception of function in the graphical and algebra-symbolic representation. Again, this difference in conceptions is consistent with the work of Dubinsky and Harel (1992a) and Moschkovich, Schoenfeld, and Arcavi (1993), who have found, as stated above, that students can express different understandings of the concept of function in different representations. It is also consistent with the work of Schwartz and Yerushalmy (1992), who state that the graphical and the algebra-symbolic representation, which are object representations, are more likely to stimulate an object conception of function.

Comparison of the responses of Student 9-1 and Student 9-2.

In comparing the responses of the Grade 9 students, both students had the "most anticipated" approach to all four questions. In addition, Student 9-1 engaged in a pointwise graphical approach to Question 2 and 3, and a global algebra-symbolic approach to Question 4, and Student 9-2 engaged a global algebra-symbolic approach to Question 2. This unwillingness to engage in too many other representations is indicative of a rigid understanding of the concept of function, and can be explained in one of two ways. One, the students are taking the question at face value and trying to answer it without translating it into other representations. Or, two, the students are unfamiliar or uncomfortable with other representations, or are unable to translate from one representation to another. These two explanations are surmised from the fact that the two students had only covered basic algebra principles in their current mathematics class, and they therefore imply that the ninth grade students have an unsophisticated understanding of the concept of function.

Student 12-1.

Question 2 - Graphical.

In response to Question 2, the first twelfth grader noticed that both functions were parallel because they have the same slope, which was determined through the use of discrete data points. The y-intercepts, 3 and 6, were deduced from the graph, at which point the student concluded, “When you put, when you make out the equation for the ... two graphs, since they both going to have ... the same slope for them, the only difference between the two ... equations will be the y-intercepts.” This response indicates that the student needed to translate the given function into the algebra-symbolic form. In addition, the student used a pointwise approach to determine the slope and a global approach to determine the y-intercept in order to determine the relationship between the two functions.

Question 3 - Graphical.

In response to Question 3, the first twelfth grader started out by examining the graphs and stating that if the functions were expressed in tabular form, then the x-values would be the same for each function, but the output values would be different by a power or a multiple. The student showed great difficulty in explaining the relationship, but was able to say, “The difference between the y-axis on either graph is going to be the number that you are going to be ... adding or multiplying to [g(x)].” This response indicates that the student was able to answer the question in the representation given, and did not translate the function into another representation on paper, even though the student translated to the tabular representation mentally. Moreover, the student first used a pointwise approach in answering the question because only discrete data points were initially considered, but concluded with a global approach by considering both functions in their entirety.

Question 4 - Tabular.

In response to Question 4, the student very confidently stated, "Alright, the relationship between the two, probably is that, using, looking from what I'm looking at ..., the f(x) and g(x), f(x) is half of what g(x) is, it's ... it's dividing or multiplying it by one half, so the graph of the f(x) would be closer to the x-axis than the g(x) would be." This response indicates that the student was able to answer the question in the representation given and did not translate the function into another representation. Moreover, the student used a pointwise approach in answering the question, which corresponds to the fact that the function was represented as discrete data points.

Question 5 – Algebra-symbolic.

In response to Question 5, the student was initially confused and thought that the solution for x was required. The student then stated that in order to find the g(x) values, then one would need to create an x-y table using a calculator for assistance. The interviewer then explained that there was some relationship between f(x) and g(x) and asked the student to identify it. The student fumbled a bit and then stated, "for g(x), you would be ... subtracting one from the entire equation, and then squaring that entire thing because f(x) is similar to g(x)." This response indicates that the student needed to use other representations of functions to answer the question; however, this may be due to misunderstanding the nature of the question. Once the student understood what was required, the student used a global approach and answered the question in the representation in which it was posed.

Summary.

Student 12-1 consistently answered the questions in the representations in which the questions were posed most of the time. In addition, for questions in the graphical representation, the student initially used a pointwise approach and then a global approach; for questions in the tabular representation, the student used a pointwise approach; and for questions in the algebra-symbolic representation, the student used a pointwise and then a global approach. This implies that the student is flexible in approaches to answering the questions posed. In addition, the student has an object conception of functions in all representations. Once again, this difference in conceptions is consistent with the work of Dubinsky and Harel (1992a) and Moschkovich, Schoenfeld, and Arcavi (1993), who have found that students can express different understandings of the concept of function in different representations. It is also consistent with the work of Schwartz and Yerushalmy (1992), who state that the graphical and the algebra-symbolic representation, which are object representations, are more likely to lead to an object conception of function.

Student 12-2.

Question 2 - Graphical.

In response to Question 2, the second twelfth grader pointed out that both functions are linear and parallel with different positive slopes. The interviewer convinced the student to find the slope of each line, which the student calculated to be ½ through the use of discrete data points. The student deduced the algebra-symbolic representation of f(x) and g(x), and then stated that the functions have the same rates of change, but different y-intercepts. This response indicates that the student needed to use other representations of functions to answer the question. In addition, the student used both a pointwise and a global approach in answering the question.

Question 3 - Graphical.

The student was unsure as to what type of function was displayed in Question 3; however, the student was able to point out, “they’re at the same rate…just in different, like…different sections of the graph." This is interpreted to mean that the graphs are the same, but they are located at different places in the coordinate plane. This response indicates that the student was able to answer the question in the representation given and did not translate the function into another representation. Moreover, the student used a global approach by considering both functions in their entirety.

Question 4 - Tabular.

The student displayed great confidence in responding to Question 4 and stated that, "There's…you can find the difference between the two if you have this, and you say, …g(x)…, …x and y table is, um…in order to get that all you have to do is multiply by two. Then, you would…then you could easily find g(x). And, you could plot it out. And, you could discover the, um…the slope…and the y-intercept. And, you could find out the equation." This response indicates that the student was able to answer the question in the representation shown, using a pointwise approach; however, the student needed to further confirm the answer given by engaging the graphical representation through a pointwise approach, and the algebra-symbolic representation through both a pointwise and a global approach.

Question 5 – Algebra-symbolic.

In response to Question 5, the student stated, "f(x) minus one squared is equal to g(x)." The student, however, experienced some difficulty in writing this in algebra-symbolic form and needed intervention from the interviewer. This response indicates that the student was able to answer the question in the representation given, and used a global approach by considering both functions in their entirety.

Summary.

Student 12-2 tended towards the algebra-symbolic representation and often translated the given information into that form if able. This was achieved by calculating the slope using discrete data points, as well as deducing the y-intercept from the given representation. This is equivalent to using both a pointwise and a global approach to functions. This implies that the student has a preference for the algebra-symbolic representation, and displays this bias in approaches to answering the questions posed. In addition, the student has an object conception of functions in the algebra-symbolic representation. Again, this difference in conceptions is consistent with the work of Dubinsky and Harel (1992a) and Moschkovich, Schoenfeld, and Arcavi (1993), described above.

Comparison of the responses of Student 12-1 and Student 12-2.

In comparing the responses of the Grade 12 students, both students had the "most anticipated" approach to all four questions except for Question 2 which was posed in the graphical representation and in which both students used a pointwise and global algebra-symbolic approach. In addition, Student 12-1 engaged in a pointwise graphical approach to Question 3, and a pointwise tabular approach to Question 5; and Student 12-2 engaged in a pointwise graphical and algebra symbolic and a global algebra-symbolic approach to Question 4. This willingness to engage other representations may be explained by the students using as much of their prior knowledge as is possible to respond to the questions posed, and is indicative of flexibility in using various representations of functions. This flexibility can then be interpreted as a more sophisticated understanding of the concept of function (Brenner et al., 1997; Moschkovich, Schoenfeld, & Arcavi, 1993). Furthermore, the previous analysis regarding students' understandings of the definition of function also implied that the twelfth grade students had a fairly sophisticated understanding of the concept of function.

Teacher 9.

Question 2 - Graphical.

The ninth grade teacher's response to Question 2 is that the two functions are both parallel lines with the same slope, which was deduced from the graphs shown by calculating the slopes using the x-intercepts and y-intercepts. The teacher pointed out other features of the two functions that are characteristic of parallel lines, and concluded that the perpendicular distance between the two lines will always be the same. This response indicates that the teacher was unable to answer the question in the representation given and translated the functions into another representation, algebra-symbolic, in order to determine the relationship. Moreover, the teacher used a global approach in deducing the slopes of the two lines and in determining the relationship between the two functions.

Question 3 - Graphical.

In examining the functions in Question 3, the teacher points out that the two functions have a, "maximum at the same… x …coordinate …however, both of their max[imum]s are at different points on the y … and, it looks like they both have …asymptotes … they're both … concave down." The teacher eventually concludes that, "Well, they look very similar and um…it just…I mean f(x) physically looks like g(x)…just shifted up higher. So, all of the…um…let me think about this…all of the x-coordinates are the same. But, the y-coordinates, um…are going to have a constant added on to them. So, it's shifted up higher." This response indicates that the teacher was able to answer the question in the representation given and did not translate the function into another representation. Moreover, the teacher used a global approach by considering both functions in their entirety.

Question 4 - Tabular.

In response to this question, the teacher translated the given functions into the algebra-symbolic form, by deducing the slopes and the y-intercepts from the tables of values. The teacher then turned attention to the algebra-symbolic representation of each function to determine the relationship. The teacher then concluded that the slope of g(x) is twice that of f(x), and this increases twice as fast. In addition, the two functions will intersect at a point because the functions are not parallel to each other. This response indicates that the teacher was unable to answer the question in the given representation and translated the function into another representation. Moreover, the teacher used a global approach to determine the relationship between the two functions once it was in the algebra-symbolic form.

Question 5 – Algebra-symbolic.

The teacher read through the question twice and then said, "Oh. You can just do, um…So, it'd be $x^3 - 5x^2 + 4x$…right? And, then…subtract one and then square that. So, your…can you compose, um…can you do f(x), um…minus one squared?" This response indicates that the teacher was able to answer the question in the representation given and did not translate the function into another representation. Moreover, the teacher used a global approach by considering both functions in their entirety.

Summary.

Teacher 9, who was nervous about the responses offered, which was evidenced in her nervous laughs, consistently showed preference for functions in the algebra-symbolic form. The teacher's nervousness probably prevented her from engaging other representations in her responses. In answer to questions posed in the graphical and tabular representation, the teacher first translated the function into its algebra-symbolic form before determining the relationship between the two functions. It was only for Question 3 that this did not happen, and this may be because the algebra-symbolic form of the function cannot be determined from the graph shown. In addition, the teacher used a global approach at all times for both the algebra-symbolic and graphical representations, which aligns with Schwartz and Yerushalmy (1992). This implies that the teacher probably has an object conception of functions in the graphical and algebra-symbolic representation. The teacher's conception of functions in the tabular representation could not be determined in this interview. This difference in conceptions is consistent with the work of Dubinsky and Harel (1992a) and Moschkovich, Schoenfeld, and Arcavi (1993), described earlier.

Teacher 12.

Question 2 - Graphical.

In response to Question 2, the twelfth grade teacher stated, "To me…I mean if I'm looking at it and describing, … geometrically, I'd say they're parallel. But, I think in terms of functions, I'd say they're transformations…that, ... g(x) is the transformation of f(x) … That, … if I look at … f(x) … if I looked at moving f(x) over … One, two, three … it looks like that would transform into g(x) … To the left … Or … similarly … moving it up … One, two, three … seems like that would do the same result…actually, moving it up one, two, three…or moving it over one, two, three, four, five, six." This response indicates that the teacher was able to answer the question in the representation given, and used a global approach by considering both functions in their entirety.

Question 3 - Graphical.

In responding to Question 3, the teacher stated, "Again, I feel like this is a transformation…where one …, f(x) has been…depending on where I'm starting from…looks like g(x) has been moved up … to become f(x)…or f(x) has been moved down to become g(x)."

This response indicates that the teacher was able to answer the question in the representation given, and used a global approach by considering both functions in their entirety.

Question 4 - Tabular.

The teacher responded that, “g(x) is twice as much as the output for f(x) … I could say that the output for f(x) is one-half the output for g(x).” This response indicates that the teacher was able to answer the question in the representation given, and used a global approach by considering both functions in their entirety.

Question 5 – Algebra-symbolic.

In response to Question 5, the teacher stated, “take the output for f(x), subtract one, and square it ….” This response indicates that the teacher was able to answer the question in the representation given, and used a global approach by considering both functions in their entirety.

Summary.

Teacher 12 consistently answered the questions in the representations in which the questions were represented. In addition, the teacher consistently used a pointwise approach when deemed appropriate and a global approach when deemed appropriate. This implies that the teacher is flexible in approaches to answering the questions posed. In addition, the teacher appears to have an object conception in all representations.

Comparison of the responses of Teacher 9 and Teacher 12.

In comparing the responses of the teachers, Teacher 9 showed a preference for a global algebra-symbolic approach to all four questions except Question 3 in which a global graphical approach was used, and Teacher 12 had the “most anticipated” approach to all four questions. This willingness of Teacher 9 to engage mainly in the global algebra-symbolic approach may be interpreted as the teacher using as much of her prior knowledge as possible to respond to the questions posed, and therefore as the teacher possessing a flexible approach to the representation of function. Meanwhile, the unwillingness of Teacher 12 to engage in any other representation may be explained by the teacher deeming it more efficient to respond to the question in the representation shown rather than to waste time translating from one representation to another. This rigidity in not translating to other representations can be interpreted as a sophisticated understanding of function due its efficiency. The differences in the understandings of the two teachers is consistent with the differences in the understandings of their students because, as the previous analysis implied, the twelfth grade students had a similar understanding of the concept of function to that of their teacher, Teacher 12 (Hill, Rowan, & Ball, 2005).

Conclusions

Goal 1 – Definition of a Function

The first goal of this paper was to describe students' and teachers' *understandings* of the *definition* of function. In reviewing the results of the questions that focused on such, it was found that the ninth grade students interviewed (Student 9-1 and Student 9-2) had a fairly unsophisticated understanding of the concept of function, which was reflected in their examples of a function and definitions of a function. Their responses to these questions were grounded in the belief that a function is an equation of some sort, consisting of a rule of correspondence and unknowns, but with no emphasis on the arbitrariness or univalence necessary for a function. This level of understanding of the concept of function is not surprising, as these students were not previously exposed to the definition of function, and have only been formally exposed to linear functions at best.

The twelfth grade students interviewed (Student 12-1 and Student 12-2) had a more sophisticated understanding of the concept of function than did the ninth grade students (Student

9-1 and Student 9-2), which was seen in the examples of functions provided, as well as in their definitions. This level of understanding of the concept of function coincides with expectations because both students had been exposed to some of the concepts of pre-calculus and calculus, which focuses on the concept of function. Student 12-1, however, seemed to have a slightly more sophisticated understanding of the concept of function than did Student 12-2. This is evidenced in the differences in their explanations for examples and definition. The differences in their explanations can also be explained by the first student being more articulate than the second, but this seems to be an unlikely hypothesis as the first student was far more consistent in his ideas than the second student.

The teachers interviewed (Teacher 9 and Teacher 12) displayed differences in their understanding of the concept of function. Teacher 9 displayed a more practical understanding of the concept of function, which may be explained by the fact that the teacher was not currently teaching the specific ideas that were discussed in the interview. Thus, it is not that the teacher had an unsophisticated understanding of the concept of function, but rather, the teacher's previous knowledge was not readily available during the interview. Teacher 12, however, displayed a more theoretical understanding of the concept of function, which was evidenced in the teacher's responses to the interview questions. The teacher's familiarity with the ideas is again not surprising as the teacher was currently covering many of the same ideas in his classroom.

These differences in the interviewees' understanding of the concept of function are similar to the differences found in the research by Dreyfus and Vinner (1982), Even (1993), Vinner (1983), and Vinner and Dreyfus (1989) where the participants have varying levels of understanding of the concept of function. It is also similar to the work of Schwartz and

Yerushalmy (1992), who propose that certain representations elucidate certain understandings of functions. The results obtained in this paper can also be interpreted in a different manner. First, the results can help us to formulate possible levels of understanding of the concept of function using a grounded theory approach (Strauss & Corbin, 1998). The results imply that ninth grade students are more likely to be aware of the arbitrariness of functions while twelfth grade students are more likely to be aware of both the arbitrariness and univalence of functions. Teachers of these students, however, will have a more theoretical understanding of the concept of function, and should hence be better able to articulate their understanding of the concept of function, if they are currently teaching that topic.

Second, if such levels can be derived from the data, then there is also evidence to support Piaget and Garcia's (1983/1989) claim that the levels of understanding of the concept of function parallel the historical development of the concept of function. The data indicates that one first understands a function to be a rule of correspondence or an equation, and then there is an understanding of arbitrariness, and finally univalence. This matches the historical development of the concept of function, which was previously discussed (also, see Burnett-Bradshaw, 2007).

Third, given the presence of these levels of understanding, which coincide with a student's grade level, and therefore, a student's exposure to mathematics, it would seem that progression from one level of understanding to the next level of understanding can only be achieved through direct learning experiences of the concept (Duckworth, 1973, 1996; Piaget 1975/1985; Vygotsky, 1978).

Further research using a larger sample size is, however, needed to substantiate these conclusions, i.e., a larger sampling of each student grade level and teachers would be needed to substantiate these claims. Alternatively, a study focused on one particular grade level or on students or teachers only would also further substantiate some of the above claims. In addition, the types of questions asked should be modified based on the results presented here. For instance, the responses to questions pertaining to examples of functions were quite varied, which implies that a question asking one to determine whether a given relation is a function or not might provide more streamlined and useful data.

Goal 2 – Transformations of Functions

The second goal of this paper was to describe students' and teachers' *understandings* of the *transformations* of functions in *different representations*. In reviewing the results of the questions focused on the transformation of functions it was found that the ninth grade students interviewed (Student 9-1 and Student 9-2) typically took the questions at face value and answered them without translating the given information into other representations. This unwillingness to engage other representations is attributed to their limited knowledge of functions rather than to them having an object level conception (global approach as opposed to a pointwise approach) in most representations. This conclusion is further supported by the previous analysis, which indicated that the ninth grade students have a fairly unsophisticated understanding of the concept of function.

The twelfth grade students interviewed (Student 12-1 and Student 12-2) typically answered the questions posed in the representation in which it was presented. These students did, however, engage other representations in the process as well. This willingness to engage in other representations was interpreted as the students using as much of their prior knowledge as possible to respond to the questions posed, which implies that the students have an object conception of function since they are flexible in their approaches (Brenner et al., 1997; Even, 1998; Moschkovich, Schoenfeld, & Arcavi, 1993), i.e., pointwise or global approach. This conclusion is further supported by the previous analysis, which showed that the twelfth grade students have a fairly sophisticated understanding of the concept of function.

The teachers interviewed (Teacher 9 and Teacher 12) showed less similarity in their responses than did the students and their grade level counterparts. Teacher 9 showed a preference for the global algebra-symbolic representation while Teacher 12 always used the "most anticipated" approach to answer the question posed. As stated before, it is not necessarily that Teacher 9 has a less sophisticated conception of function, but rather a more practical and flexible understanding of the concept of function. This conclusion is further supported by the previous analysis in which it was determined that the teacher was not currently teaching the ideas that were focused on during the interview, and was therefore less prepared to answer the questions posed. Teacher 12, on the other hand, seems to have a more theoretical understanding of the concept of function, and therefore a more solid object conception of functions, i.e., a more sophisticated understanding of the concept of function.

The results indicate three things. One, the importance of being able to translate between and within representations when *necessary*, which is similar to other studies (Dufour-Janvier, Bednarz, & Belanger, 1987; Even, 1998; Gagatsis & Shiakalli, 2004; Goldin, 1987; Janvier, 1987b; Lesh, Post, & Behr, 1987; Hitt, 1998; Moschkovich, Schoenfeld, & Arcavi, 1993; Ruthven, 1990). Two, the results also indicate that the differences in the interviewees' understanding of the transformations of functions can help us to formulate possible levels of understanding of the concept of function using a grounded theory approach (Strauss & Corbin, 1998). Three, understandings of the concept of function is dependent on the representation in which the function is shown (Dubinsky & Harel, 1992a; Moschkovich, Schoenfeld, & Arcavi, 1993). The possible levels of understanding are such that ninth grade students are less likely to have an object conception of function, while twelfth graders and teachers, who are currently teaching about functions, are more likely to have an object conception of function. The conception displayed by the student or the teacher is, however, dependent on the representation in which the question is posed (Dubinsky & Harel, 1992a; Moschkovich, Schoenfeld, & Arcavi, 1993).

These conclusions do imply a need for further research using larger sample sizes of each student grade level and teachers, even though the current research provides some perspective on the varying levels of understandings of functions in different representations. Although, a study focused on one particular grade level or on students or teachers only would also further substantiate some of the above claims. Also, the types of questions asked should be modified based on the results presented here. For instance, it may not be necessary to present an interview question in the tabular representation because the tabular representation will only reveal a processing understanding of the concept of function. It may be more important to engage only the algebra-symbolic and graphical representation in order to determine if the participant has an object understanding of the concept of function. Also, given that some participants were unwilling to engage certain representations, it might be necessary to structure the interviews such that both the directions given in the questions and by the interviewer encourage the participants to engage any or all representations. Finally, it might be interesting to observe how participants compare the relationship between three functions instead of two functions, and if the participants are able to perform other types of actions on functions in addition to doing transformations. Last, but not least, the overall results have several implications regarding the teaching of the concept of function. First, the results indicate that the characteristics of functions need to be addressed either implicitly or explicitly once students have been introduced to functions. This will assist in their understanding of what a function is and what it is not. In addition, it will help students to understand which actions can be legitimately performed on a function. Second, the actions which can be performed on a function need to be taught using both a process and object approach, as ultimately the latter approach is the more efficient approach, but understanding of it may not be achieved without understanding of the former (Sfard, 1987). Third, students need to be taught to move fluidly from one representation to the next, as well as understanding the pros and the cons of each representation. This will aid in their ability to choose the representation that is most useful in answering a given question.

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Appendix A: Student Profile Questionnaire

For this year:

1. What mathematics course are you currently taking?

2. What topics have you covered so far?

3. (a) What topic do you find most interesting? Why?

   (b) Give an example of when you may use this topic in everyday life?

4. What topic do you find least interesting? Why?

5. What topic do you find most difficult or challenging? Why?

6. What topic do you feel most confident about? Why?

7. List three main concepts or ideas which you learned in this course.

For last year:

8. What mathematics courses did you take last year?

9. What topics did you cover in that course?

10. (a) What topic did you find most interesting? Why?

    (b) Give an example of when you may use this topic in everyday life?

11. What topic did you find least interesting? Why?

12. What topic did you find most difficult or challenging? Why?

13. What topic did you feel most confident about? Why?

14. List three main concepts or ideas which you learned in this course.

Appendix B: Teacher Profile Questionnaire

Degree(s) Obtained:

Type of Teaching Certification:

Year of Certification:

Background in mathematics:

Courses taken in mathematics (if you can't recall exactly, detail how many years of math you took, what areas of mathematics you covered):

Any other relevant certification or programs completed:

Number of years of mathematics teaching experience:

Mathematics Courses/Topic Taught	Grade Level	Number of Years

Appendix C: Interview Questionnaire

Question 1

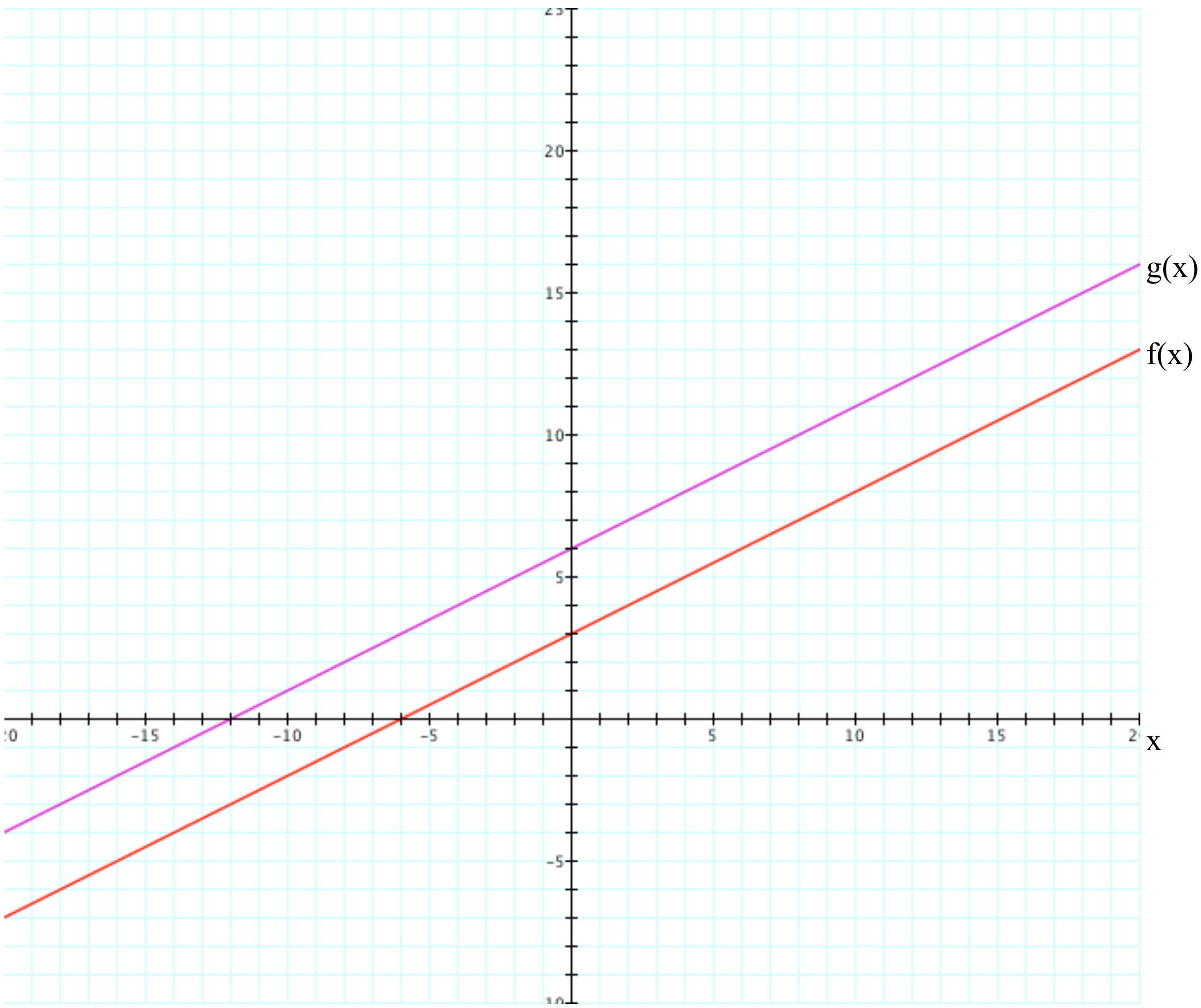
(a) Give me an example of a function. Explain.

(b) Give me an example of something that is not a function. Explain.

(c) What is the definition of a function? Explain.

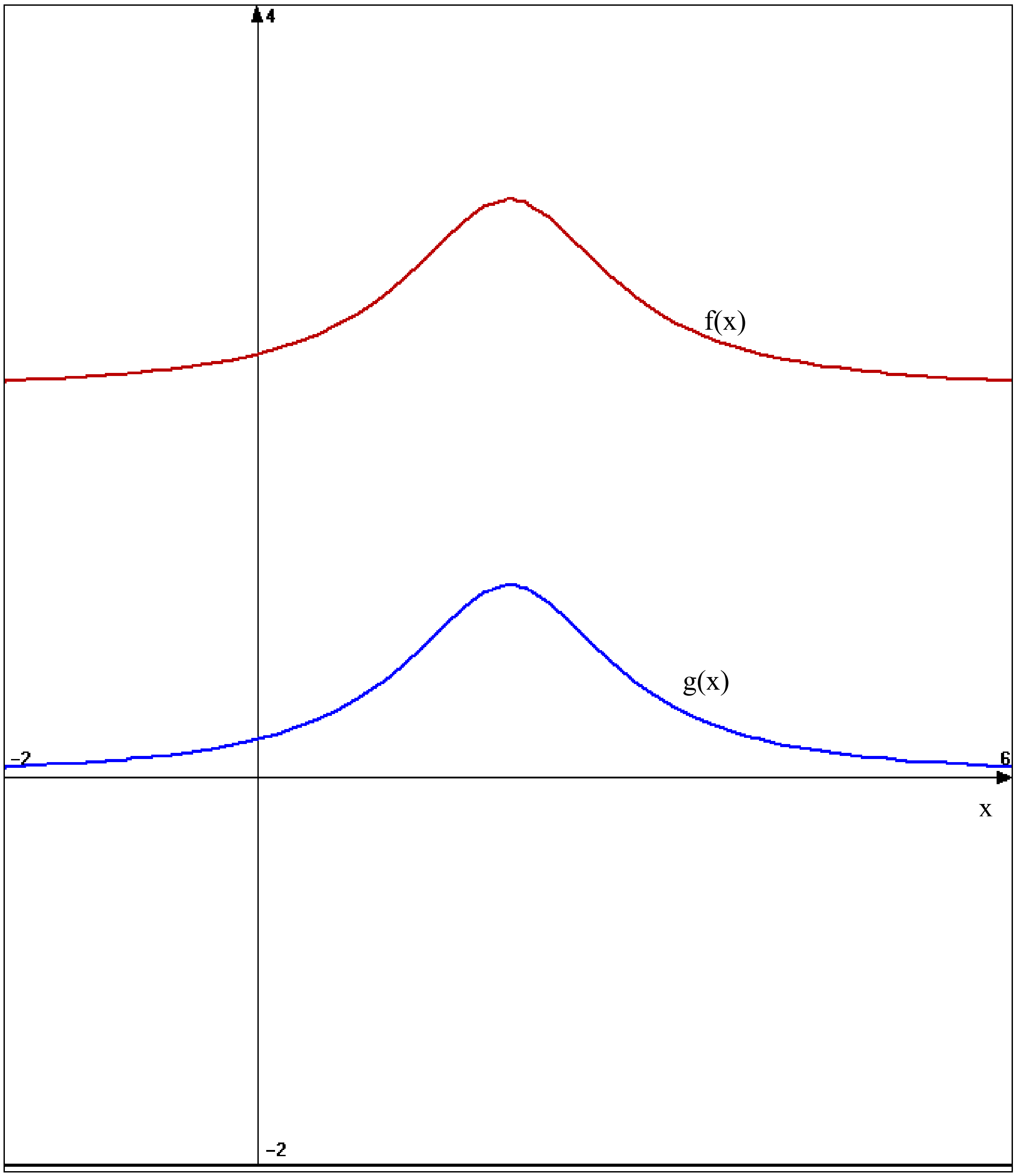
Question 2

Given the following information for the functions *f(x)* and *g(x)*, what could be the relationship between *f(x)* and *g(x)*?



Question 3

Given the following information for the functions *f(x)* and *g(x)*, what could be the relationship between *f(x)* and *g(x)*?



Question 4

Given the following information for the functions *f(x)* and *g(x)*, what could be the relationship between *f(x)* and *g(x)*?

x	*f(x)*
-5	-8
-4	-5
-3	-2
-2	1
-1	4
0	7
1	10
2	13
3	16
4	19
5	22

x	*g(x)*
-5	-16
-4	-10
-3	-4
-2	2
-1	8
0	14
1	20
2	26
3	32
4	38
5	44

Question 5

I have a computer program that is able to generate values for the function

$$f(x) = x^3 - 5x^2 + 4x,$$

but I need the values for the function

$$g(x) = (x^3 - 5x^2 + 4x - 1)^2.$$

I myself do not know how to program a computer. I could learn to program and then write a program that calculates values for $g(x)$. Is there some way for me to be clever and use the $f(x)$-program I have to help me calculate values for $g(x)$?

Question 5 (supplement)

I have a computer program that is able to generate values for the function

$$f(x) = x^3 - 5x^2 + 4x,$$

but I need the values for the function

$$g(x) = (x^3 - 5x^2 + 4x - 1)^2.$$

I myself do not know how to program a computer. I could learn to program and then write a program that calculates values for $g(x)$. Help me to figure out what to do.

(a) When $x = 0$, $f(x) = 0$ and $g(x) = ?$

(b) When $x = 1$, $f(x) = 0$ and $g(x) = ?$

(c) For any value of x, $f(x) = x^3 - 5x^2 + 4x$, and $g(x) = ?$

Question 6

Given the following information, are there any functions in the *f(x)* column that equal a function in the *g(x)* column? Explain.

f(x)	***g(x)***
$3x + 6$	1
$2x + 1$	$3(x + 2)$
x/x	$x + \frac{(x^2 + x)}{x}$

Question 7

(a) Which of the following seems to you to be the best definition of a function?

i. A function is an algebraic expression in which you can substitute various values for an unknown.
ii. A function is a computational process that produces an output (*y*) from an input (*x*).
iii. A function consists of two sets *S* and *T* together with a "rule" that assigns to each element of *S* a specific element of *T*.

(b) The following table of values were computed from a function, but the person who made the table forgot to label one column *x* and the other *f(x)*.

1	9
4	2
4	-2
5	25
9	1
16	-4

Which of the following labeling is correct? How do you know?

x	*f(x)*
1	9
4	2
4	-2
5	25
9	1
16	-4

x	*f(x)*
9	1
2	4
-2	4
25	5
1	9
-4	16