On the equivariant cohomology of homogeneous spaces

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This dissertation is dedicated to my parents and brother,

who have continued to believe in me

through it all.

Abstract

The first part of this dissertation develops foundational material on the rational cohomology of Lie groups, their classifying spaces, and homogeneous spaces. In parallel, it develops the basics of Borel equivariant cohomology, with an aim to understanding equivariant cohomology of isotropy actions of K on compact homogeneous spaces G/K.

In the last few chapters, we establish several original results on such actions. Briefly, this work essentially reduces the question of when such an action is equivariantly formal to the case the isotropy subgroup K is a torus and the transitively acting group G is simply-connected, then completely classifies the possibilities in the event K further is a circle.

The appendices include an exposition of Borel's original proof of a theorem of Chevalley providing a framework for computing the cohomology of principal bundles; the lengthy original proof of an original result on circles inverted by an inner automorphism of a containing Lie group, which was superseded but the author still wanted to see published somewhere; and some applications (the original motivation for this work) of the Berline–Vergne/Atiyah–Bott localization theorem to classical (pre-1941) results in topology.

A more detailed account of the content, including a delineation of what is original to this work and what is expository, can be found in the introduction.

Acknowledgments

The completion of this degree owes to the generous efforts of many, many people, including all my math teachers throughout my life, but especially Tufts faculty.

Tufts faculty

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Moon Duchin spent many hours, out of sheer goodwill, advising me on every non-mathematical aspect of getting a job in academia (and taking me to non-mathematical talks on campus), and talking to me about politics. It's very largely due to her efforts that I have somewhere to go after I graduate.

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Jason DeVito answered numerous of my questions on the Math StackExchange, becoming my go-to person for questions on homogeneous spaces, and in particular is the first person to have told me about the "Borel–Cartan machine" of Section 8.1.1 that forms a critical part of this work.

Most of the main protagonists in the Eilenberg–Moore spectral sequence era of the study of $H^*(G/K)$ have also come to my aid either over email or in person. Joel Wolf and especially Larry Smith have been very helpful over email, and Paul Baum discussed his thesis with me in person at the Joint Math Meetings in January (introducing me to over a dozen mathematicians during the process, some more than once). He also helpfully informed me what it was like be scolded for insolence by Armand Borel himself.

Marek Mitros suggested I use GAP and suggested products of commuting reflections might suffice to account for the reflected tori in E_6. My fellow Tufts student Christopher O'Donnell, on the other hand, helped teach me the GAP and Python I needed to confirm these elements of W_{E_6} sufficed, picked apart my code, and generally let me distract him from his own work for nearly a week. Jay Taylor told me on MathOverflow that the longest word sufficed to determine reflected circles, in the process providing many details of the method that allows me to replace this lengthy, computer-assisted proof with a much more succinct and uniform one.

Chi-Kwong Fok is responsible for my including the representation-theoretic interpretation of reflectibility in terms of self-dual representations, for pointing out (first, chronologically) that the longest word sufficed, although I didn't really internalize it at the time. I also thank him for many educational emails about the relation of my work to equivariant K-theory.

Mathew Wolak mentioned, at a critical moment, that the Killing form is invariant and definite, allowing me to reduce the sprawling atrocity of case-by-case analysis that was initially my proof of Proposition 8.5.1 to the simple application of the Borel–Cartan machine it was later revealed to have been. Michael Atiyah took the time to answer a historical question regarding the origin of the "Borel localization formula" for higher-dimensional tori (see Historical remarks 9.2.7).

Svjetlana Terzić directed me to the proof in Onishchik's book [Oni94, Thm. 12.2, p. 211] of the equivalent "regular sequence" condition in Theorem 8.4.7, which until then I had only suspected might be the case.

Christopher Allday and Andreas Arvanitoyeorgos also made valuable editorial suggestions regarding the work in Chapter 11. Jun-Hou Fung provided sample code for constructing spectral sequence diagrams; he has a nice REU paper computing the cohomology with field coefficients of simple Lie groups [Fun12].

Bob Cornell lent me math books, including Maor [Mao91] and Hofstadter [Hof79] my junior year of high school. Ed Siegfried told me about quaternions and Tor_1 my senior year, and supervised me in my project to calculate the Carmichael function [Wik15] (which I felt was my own invention and which I named ק *koph*). James Cummings took an interest in me and led my independent study courses in undergrad at CMU.

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Burns Healy talked metal with me, advised me reliably which beer to order, did more than anyone else to finally deplete the Spirit Fund, and perhaps appreciated more than anyone else the proof of the five lemma I added to Wikipedia as an undergraduate.

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Family

I've already dedicated this work to my immediate family, who have somehow never given up on me.

I also wanted to thank, but can't, my grandmother Elisabeth Kasik who loved me unconditionally and shaped me in a lot of ways as a person. I remember her in her little house in Denver, ready with Hungarian pastry and stories and a million neighbors and odd friends who loved her, and I can't believe that her and her house are gone, because in all my memories over the years, that place is the one that is somehow the most *home*. I miss you, Grandma Betty.

And since I've already hit maximum lachrymosity, I have to thank Mocha and Simon too. You were lovely little babies and I miss you so much.

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Chapter 1

Introduction

My goal in this document is to explore the satisfiability and consequences of a technical condition on equivariant cohomology called *equivariant formality*. This notion had already been alighted upon by Borel in Chapter XII of his Seminar [BBF+60], but was not formally defined until the work of Goresky, Kottwitz, and MacPherson [GKM98] in 1997. These three used equivariant formality to build an edifice now called *GKM-theory* which among other things computes the equivariant cohomology $H_T^*(X)$ in the event T is a complex torus acting equivariantly formally on a complex algebraic variety X in such a way that X contains only finitely many T-orbits of dimensions zero and one. This machinery has been applied to Hamiltonian actions on symplectic manifolds, and the action of a compact Lie group K on a homogenous space G/K with $\operatorname{rk} K = \operatorname{rk} G$, and is part of a by now well-understood story which has been intensively applied in parts of symplectic topology and algebraic geometry. The impetus behind this research was a suggestion from my advisor Loring W. Tu that it might be profitable to understand what happens in the much less studied case that $\operatorname{rk} K < \operatorname{rk} G$.

My initial work in this regard was applying the powerful equivariant localization theorems of Berline–Vergne and Atiyah–Bott to characteristic classes, and in doing so, I was able to recover a number of classical (pre-1950) results essentially for free. The metaphor I used when describing orally my exploits was that of wandering around classical algebraic topology with a gigantic hammer, looking for things to hit. I regale the reader with tales of these deeds in Appendix C.

The problem I kept encountering, toting this hammer around, was the one you would expect: I was not sure could actually be hit with it. So a second program arose, which became the more substantial part of this project: that of finding out when the natural action of K on G/K is equivariantly formal, for compact Lie groups G and K. This is taken up in Chapter 11.

The tools involved in this determination are essentially classical, involving the cohomology of homogeneous spaces and invariant theory. The former was a major topic of research for decades, starting with work of Čech and Hopf in the 1930s on cohomology of Lie groups (leading to the discovery of Hopf algebras), and coming into focus in the late 1940s with the discovery of the Leray spectral sequence and the Cartan algebra of principal bundles. The Serre spectral sequence of the Borel fibration $G \to (ES \times G)/S \to BS$, which is the same as the Leray sequence of the map $(ES \times G)/S \to BS$, plays a key role in our work.

After Eilenberg and Moore discovered their spectral sequence, the Eilenberg–Moore spectral sequence of the bundle $G/S \to BS \to BG$ overtook the Serre spectral sequence as the primary tool for studying cohomology of homogeneous spaces [BS67]. It was found that this spectral sequence usually collapsed at E_2. The work of Hans Munkholm [Mun74] and Joel Wolf [Wol77] was apparently considered to be the final nail in this project; they found, roughly, what had already been expected: if k is a field of high enough characteristic, then $H^*(G/K;k) \cong \mathrm{Tor}^{\bullet}_{H^*(BG;k)}\big(k, H^*(BK;k)\big)$ as rings, and under weaker hypotheses, as $H^*(BG;k)$-modules. This reduced the problem to an algebraic one, resolvable algorithmically in any given individual case given the map $H^*(BG;k) \longrightarrow H^*(BK;k)$. At this point, the cohomology of homogeneous spaces seemingly was decided to be a solved problem, and algebraic topologists collectively moved on. While computations of the cohomology of individual homogeneous spaces of interest have been published since, nothing substantial about the general problem seems to have been written since 1977.

Unfortunately, the theoretical conclusion to the computation of $H^*(G/K;k)$ does not immediately translate into transparent formulae for the ring structure or even—what is more relevant to us—for the total Betti number $h^\bullet(G/K) = \dim_k H^*(G/K;k)$, and I resorted to the earlier Borel–Cartan description to recover this data in cases of interest. I take the opportunity in Chapter 8 to recount this deserving and largely forgotten story, which at the same time motivates the Cartan model for equivariant cohomology.[1] It is this development which I consider to be the main feature of this account, and which I spent the greatest time developing. The core insight is that most everything can be thought of as arising as a consequence of Serre spectral sequence of a bundle and the Koszul complex, a primordial acyclic chain complex tied to the cohomology of a universal bundle.

This exposition is largely original, in that the proofs are recalled from the my own memories or created anew except where otherwise noted, and this material is not typically developed in the manner and in this order I do here. Results that here are seen as a consequence of a rational Koszul complex are usually developed as a consequence of connections, Lie algebra cohomology, and the Weil algebra, important analytically-flavored tools which, however, it is actually possible to circumvent almost entirely. Part of my motivation for this lengthy exposition was a mild dissatisfaction with existing texts covering the material of Chapter 8, which tend to rely heavily on unexplained notation without providing recapitulation of its meaning, to require iterative callbacks to earlier material, and in one instance to spend several hundred pages developing in great generality algebraic preliminaries which are then only ever applied to principal bundles of compact Lie groups anyway.

My goal throughout is to require only basic commutative algebra, Lie theory, and algebraic

[1] Already in 1977 [Sul77], Dennis Sullivan describes the Borel–Cartan technology as "current in 1950 and ignored later in topology," and then uses it to motivate Sullivan models; see Hess [Hes99].

topology, so that this dissertation will in principle be readable to a second-year graduate student. To this end, much standard material is briefly recapitulated without narrative in the first two "background" appendices, which are definitively not, however, the right place to first encounter this material.

Parts of this journey will be somewhat impressionistic, some background results will quoted without proof, and the history will not necessarily be entirely correct. Still, I hope at the end the narrative has some worth besides merely as background to my own work.

A note on style: As the reader might already have guessed from the absolute overkill in the acknowledgments, this document hews but imperfectly to the conventions standard in research work.

I initially put in extra background operating off the (spurious) assumption that committee members unfamiliar with equivariant cohomology would appreciate having all the background in one document; but when I realized how dramatically incorrect that was, rather than dialing it back and producing a more modestly-scaled and tightly-targeted document, I just cranked it up to 11 continued regardless—I felt I was committed at that poin,t and really wanted to get out what I had to say in my own way. Compounding this commitment, I came a bit later to the realizations that

1. given that this dissertation is essentially subject only to the approval of my advisor, I can essentially go on about whatever I want, for as long as I want, and
2. this is probably the only published document I will ever have such complete creative control over.

So the authorial voice is very present, the intermittent grandiloquence and weird phrasings I find amusing unchecked, and really, why not? Hopefully the resulting text isn't unreadably irritating for these excesses.

A note on attributions: In the course of this work, I rediscovered many known results, including results from 1940 and 1946–7 which do not seem to be well-known, through what could be considered either an admirable DIY ethos and spirit of adventure or else a pathological inability to make myself familiar with the relevant literature. Some results and proofs I discovered seemed too obvious not to be known, and yet do not appear elsewhere in papers or books (yet!) known to me. Given this history, it seems dangerous to claim a number of proofs presented here as original: it may simply be that the correct citation is contained in some part of the sixty-odd years of relevant literature I've failed to uncover. To handle such instances, I consistently use the coded phrase "the author does not know of a citation" as a hopefully not overly crass way of claiming originality of *effort*, if not of outcome, for a given proof or result.

That said, Section 10.1 and Chapter 11 comprise unequivocally original material. The mild extensions Corollary 10.3.11 and Theorem 10.3.16 of the Shiga–Takahashi criterion are also original.

The following results and proofs, the author does not know a reference for:

- the included proof of the counting lemma Corollary 2.4.5,
- the Serre spectral sequence proof of the bundle lemma Theorem 4.4.1,
- the proof in Section 8.5—or any published proof for that matter—of the Leray–Koszul theorems on $H^*(G/S^1;\mathbb{Q})$,
- the low-tech, low-dimensional topology proof of the standard result Proposition 6.3.4 that the standard rotation action of S^1 on S^2 is equivariantly formal,
- the "natural isomorphism" verions of the statements in Section 6.4, which, though trivial extensions of standard results, allowed the author to prove Theorem 10.1.4,

- the adaptation into modern language and rational coefficients of Borel's proof of Chevalley's and Cartan's theorems in Section 8.1.2 (though this result is subsumed by now-standard techniques of rational homotopy theory),
- the adaptation to sheaves, rather than *couvertures*, of Borel's original proof of Chevalley's theorem, as exposited in Appendix D,
- the cohomological lifting result Proposition B.3.4,
- the equivariant proof of the Hopf–Samelson result in Proposition C.3.1.

The following results reflect only an originality of effort, in that the results and the proofs were already known (if not to the author at the time):

- the tensor decomposition results of Section 6.2,
- the equivariant proof of the fixed point result Proposition C.3.2 and attendant Proposition C.3.3.

Chapter 2

Bundles, actions, and orbits

The main body of this work will be in understanding functors from group actions to abelian groups, so in this chapter we recount some material on continuous group actions that, though standard, is not covered in a typical graduate algebra, topology, or Lie theory course and so seemed to deserve formal demarcation from the algebraic and topological background relegated to the appendices.

2.1. The category of G-actions

Let G be a topological group. The category G-Top of G-actions is specified as follows:

- Objects are pairs $(X, A) \in$ Top of topological spaces, such that X is equipped with a continuous G-action that restricts to an action on A. One says X (as well as A) is a ***G-space***.
- Morphisms are ***G-equivariant maps*** (briefly ***G-maps***): continuous maps of pairs $f\colon (X, A) \longrightarrow (Y, B)$ such that $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$.

The objects are really actions, but we will abusively refer to them as spaces whenever convenient. As with Top, we will consider the class of individual G-spaces and G-maps between such as a full subcategory via the canonical inclusion $X \longmapsto (X, \varnothing)$.

A right G-space and a left are essentially the same: the anti-isomorphism $g \mapsto g^{-1} \colon G \to G$ converts a right action to a left action (and vice versa) via $gx := xg^{-1}$. We will only worry about the difference in the event a space admits both a left and a right action.

There are two canonical functors to Top, the forgetful functor that takes an action $G \curvearrowright (X, A)$ and returns the pair $(X, A) \in \mathsf{Top}$, disregarding the action, and the ***orbit-space functor*** $(X, A) \mapsto (X/G, A/G)$.

The product and coproduct in G-Top have the topological product and disjoint union as underlying spaces. The action on the coproduct is the union of the actions; the intended action on the product $X \times Y$ is the ***diagonal action*** given by $g \cdot (x, y) := (gx, gy)$, so that the projections $X \leftarrow X \times Y \to Y$ are also G-maps. The orbit space of the diagonal action is the ***mixing space***

$$X \underset{G}{\times} Y := X \times Y \big/ (g^{-1}x, y) \sim (x, gy).$$

If we write the action on X instead as a right action, one has the more aesthetically pleasing identification $(xg, y) \sim (x, gy)$.[1] The application of the orbit-space functor $-/G$ to the projection diagram $X \leftarrow X \times Y \to Y$ yields a ***mixing diagram***

$$\begin{tikzcd}
X \arrow[d] & X \times Y \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d] \\
X/G & X \underset{G}{\times} Y \arrow[l] \arrow[r] & Y/G.
\end{tikzcd}$$

We always let G act trivially on the closed unit interval $I = [0, 1]$. A ***G-homotopy*** between two G-maps $f, f' \colon X \longrightarrow Y$ is a G-map $F \colon X \times I \longrightarrow Y$ such that $F(-, 0) = f$ and $F(-, 1) = f'$. This is the proper notion of homotopy within the category G-Top, and has the important feature that application of the orbit-space functor takes a G-homotopy $X \times I \longrightarrow Y$ of G-maps to a homotopy

[1] One could say g commutes with comma.

$(X/G) \times I \longrightarrow Y/G$. In particular, a pair $(f\colon X \rightleftarrows Y\colon f')$ of G-maps such that $f' \circ f$ and $f' \circ f$ are G-homotopic to id_X and id_Y (respectively) descend to a homotopy equivalence $X/G \simeq Y/G$.

We end this section with the observation that mixing with the translation action of G on itself changes nothing.

Lemma 2.1.1 ([Bre72, Prop. II.2.2, p. 73])**.** *Let Γ be a topological group and X a right Γ-space. Then $X \underset{\Gamma}{\times} \Gamma \approx X$.*

Proof. The composite $X \xrightarrow{\approx} X \times \{1\} \hookrightarrow X \times \Gamma \twoheadrightarrow (X \times \Gamma)/\Gamma$ is clearly continuous. It is surjective because the diagonal Γ-orbit of any point $(x, \gamma) \in X \times \Gamma$ contains the point $(x\gamma^{-1}, 1)$, and injective because $(x, 1) = (y, 1) \cdot \gamma = (y\gamma, \gamma)$ only if $\gamma = 1$ and $x = y$. To see its inverse is continuous, consider the composition

$$\begin{aligned} X \times \Gamma &\longrightarrow X \times \Gamma \longrightarrow X, \\ (x, \gamma) &\longmapsto (x, \gamma^{-1}) \longmapsto x\gamma^{-1}. \end{aligned}$$

This map is constant on diagonal Γ-orbits, since $(x\delta, \gamma\delta) \longmapsto x\delta \cdot \delta^{-1}\gamma = x\gamma^{-1}$, so it descends to a continuous map $(X \times \Gamma)/\Gamma \longrightarrow X$, the inverse to the map $X \longrightarrow (X \times \Gamma)/\Gamma$ described above. □

2.1.1. Freedom and efficacy

We will have to deal with fixed point sets of isotropy actions starting in Section 10.2. If a group G acts on a topological space X, we denote the fixed point set of the action by X^G.

A G-action $G \times X \longrightarrow X$ is said to be ***effective*** (or *faithful*) if only the neutral element $1 \in G$ acts trivially: that is, one has the implication

$$\forall x \in X\ (gx = x) \implies g = 1.$$

Equivalently, an action is effective just if it curries to an *injective* homomorphism $G \longrightarrow \mathrm{Homeo}\, X$. A G-action $G \times X \longrightarrow X$ is said to be ***free*** if only the neutral element $1 \in G$ fixes any point: that is, one has the implication

$$\exists x \in X\ (gx = x) \implies g = 1.$$

These conditions can be pleasingly restated in terms of stabilizers thus:

- an action is *effective* when $\bigcap_{x \in X} G_x = \{1\}$;
- an action is *free* when $\bigcup_{x \in X} G_x = \{1\}$.

We will write G-Free for the full subcategory of G-Top whose objects are free G-actions. Contained in G-Free is the full subcategory of principal G-bundles and G-maps (automatically bundle maps), as discussed in Appendix B.1.3.

In Section 4.2.3, we will define a construction that from any G-action produces a free G-action, and use it to define an equivariant cohomology theory.

2.2. The category of G-orbits

The action on a G-space X induces an ***orbit decomposition***: one has

$$X = \coprod_{Gx \in X/G} Gx$$

as sets where the ***orbit*** Gx is defined as $\{gx \colon x \in X\}$ and the inclusion $Gx \hookrightarrow X$ is continuous. The orbits have a fairly tightly constrained structure: the pointwise ***stabilizers*** or ***isotropy subgroups*** $\mathrm{Stab}(x) = G_x := \{g \in G : gx = x\}$ of points $x \in X$ are closed subgroups if X is T_1,[2] and the

[2] It will be in all cases we care about. The proof is that the inverse image of $\{x\}$ under the action $G \times X \longrightarrow X$ and the set $G \times \{x\}$ are both closed if $\{x\}$ is.

orbit–stabilizer theorem induces homeomorphisms

$$G/\operatorname{Stab}(x) \xrightarrow{\sim} Gx:$$

$$g\operatorname{Stab}(x) \longmapsto gx,$$

where $G/\operatorname{Stab}(x)$ is the left coset space with the quotient topology. Such a space is called a ***homogeneous space*** for G. These maps are clearly continuous, and will certainly be open as well if G is compact and X Hausdorff, which they are in our cases of interest. We mark this proposition for later use.

Proposition 2.2.1. *Let G be a locally compact, Hausdorff, second countable topological group acting continuously on a locally compact Hausdorff space X, and* $\operatorname{Stab}(x)$ *the stabilizer in G of $x \in X$. Then the group action induces a homeomorphism*

$$G/\operatorname{Stab}(x) \xrightarrow{\approx} Gx.$$

Proof. The argument above suffices for a compact Lie group acting on a Hausdorff space. See Garrett [Gar10, Prop. 6.0.2] for a proof of the statement with the weaker hypotheses. □

It follows that an understanding of the full subcategory G-Orbit of right quotients of G by closed subgroups is an important component of understanding G-Top. Write $\mathsf{Sub}(G)$ for the category of closed subgroups of G, with morphisms inclusions and isomorphisms $H \xrightarrow{\sim} gHg^{-1}$ induced by inner automorphisms of G. Then the correspondence $K \longmapsto G/K$ induces an equivalence $\mathsf{Sub}(G)^{\mathrm{op}} \xrightarrow{\sim} G$-$\mathsf{Orbit}$ of categories: an inclusion $H \hookrightarrow K$ corresponds to a G-equivariant quotient map $G/K \twoheadrightarrow G/H$; and if $H = gKg^{-1}$, then $r_g \colon x \longmapsto xg$ descends to a G-homeomorphism

$$G/H \xrightarrow{\sim} G/K :$$
$$xH \longmapsto xgK.$$

2.3. Tubes, slices, and G–CW complexes

Given the orbit decomposition, one can ask how much of the structure of G-Top is determined by G-Orbit (or equivalently by $\mathsf{Sub}(G)$), and the answer, in a sense, is all of it, at least if G is a compact Lie group. We reproduce here the topological consequences of this analytic structure, suppressing most actual analysis. Most of this material can be found in the useful Appendix B to the book by Ginzburg *et al.* [GGK02]. Much of the exposition in this section is inspired by a manuscript textbook of Raoul Bott and Loring Tu [BTar].

Let G be a compact Lie group acting smoothly on a manifold M, and $x \in M$. Then $K = \mathrm{Stab}(x)$ is a closed subgroup of G, hence a Lie group, and the coset space G/K is a manifold as noted in Theorem B.4.3, diffeomorphic to the orbit Gx. Moreover, orbits admit equivariant tubular neighborhoods [Kos53].

Theorem 2.3.1 (Equivariant tubular neighborhood theorem (Koszul))**.** *Let G be a compact Lie group, M a smooth G-manifold, and $x \in M$ a point. Then there exists a G-invariant open neighborhood of Gx in M.*

Sketch of proof. Given a Riemannian metric $\langle -,-\rangle'$ on M and Haar measure dg on G, the average $\langle v,w\rangle := \int_G \langle g_* v, g_* w\rangle' \, dg$ defines G-invariant Riemannian metric on M. This induces a G-invariant metric d on M. The neighborhood $B_\varepsilon(Gx) = \{y \in M : d(Gx,y) < \varepsilon\}$ is G-invariant. $\square$

This statement can and should be strengthened.

Proposition 2.3.2 (Koszul)**.** *For small enough ε, the tubular neighborhood of Theorem 2.3.1 is G-isomorphic to the normal bundle.*

Sketch of proof. The G-invariant metric defines a norm $\| - \|$ on TM which allows us to canonically embed the normal bundle ν to Gx as the orthogonal complement of $T(Gx)$ in $(TM)|_{Gx}$. It is a well-known feature of the exponential map that each point $gx \in Gx$, there exists $\varepsilon > 0$ such that $\exp_{gx}$ is defined on the ball $B_\varepsilon(0) \subsetneq T_xM$. Since Gx is compact, there then exists ε small enough that for all x and all $v \in (TM)|_{Gx}$ with $\|v\| < \varepsilon$, the map $\exp\colon (v \in T_{gx}M) \longmapsto \exp_{gx} v$ is defined. The exponential of M, when restricted to $T(Gx)$, is the exponential for Gx and so maps into Gx, so to obtain a potential homeomorphism from some bundle to a tubular neighborhood of Gx, we should restrict our exponential to the orthogonal complement ν to $T(Gx)$, which also has the correct total dimension $\dim\big[(TM)|_{Gx}\big] - \dim Gx = \dim M$. If we decrease ε sufficiently, then by the nonequivariant tubular neighborhood theorem, exp is a G-equivariant diffeomorphism from $\nu_\varepsilon := \{v \in \nu : \|v\| < \varepsilon\}$ to $B_\varepsilon(Gx)$. ☐

Finally, if we write ν_x for the disk $T_xM \cap \nu_\varepsilon$, we can rewrite ν_ε as a disk bundle.

Proposition 2.3.3. *In this notation, there is a G-equivariant diffeomorphism $G \underset{G_x}{\times} \nu_x \xrightarrow{\sim} \nu_\varepsilon$.*

Proof. Under the action

$$\mu\colon G \times \nu_x \longrightarrow \nu_\varepsilon,$$

$$(g, v) \longmapsto g_* v,$$

each $g \in G$ takes ν_x diffeomorphically onto $(\nu_\varepsilon)_{gx}$, since $g \in \operatorname{Diff} M$ is itself smooth and invertible and we constructed the norm on ν to be G-invariant. The fiber of μ over a fixed $v_0 \in (\nu_\varepsilon)_{y_0}$ is the collection of pairs (g, v), (g', v'), etc., with $g_* v = v_0$. We must have $gx = g'x = y_0$ for any two such pairs, so $k = g^{-1}g' \in G_x$, or $g' = gk$ and then

$$g_* v_* = v_0 = g'_* v' = g_* k_* v'$$

implies $(g', v') = (gk, k_*^{-1}v)$ by bijectivity of g_*. So each fiber is an orbit of the diagonal action $G_x \curvearrowright G \times \nu_x$ given by $k \cdot (g, v) = \big(g_*(k_*)^{-1}, k_* v\big)$ of G_x. Quotienting out this action, μ descends to a diffeomorphism $\varphi\colon G \times_{G_x} \nu_x \xrightarrow{\sim} \nu_\varepsilon$, which is left G-equivariant: $\varphi[g'g, v] = g'_* g_* v = g'_* \varphi[g, v]$. $\square$

Given a point $x \in M$, if there is a G_x-invariant superset $X \ni x$ such that GX is an open neighborhood of x satisfying $GX \approx G \times_{G_x} X$, then X is called a ***slice***. We have just seen that a small normal disk ν_x is such a slice, so the result is also traditionally called the *slice theorem*. It is a subtle theorem of Sören Illman that the in addition to these slices, smooth G-manifolds admit another very nice kind of G-equivariant decomposition.

Definition 2.3.4 (Matumoto)**.** A ***G–CW complex*** is a topological space X admitting a filtration (X^n) as follows:

- $X^0 = \coprod_{K \in \mathsf{Sub}(G)} \coprod_\alpha (G/K) \times D^0_\alpha$
- Given X^n, there are G-equivariant maps $\varphi_{\alpha,K}\colon (G/K) \times S^n_\alpha \longrightarrow X^n$, where K runs over $\mathsf{Sub}(G)$ and α over some arbitrary index set, compiling into a map

  $$\varphi = \coprod_{\alpha,K} \varphi_{\alpha,K}\colon \coprod_{\alpha,K} (G/K) \times S^n_\alpha \longrightarrow X^n$$

  such that

  $$X^{n+1} = {X^n \amalg \coprod_{\alpha,K} (G/K) \times D^{n+1}_\alpha} \Big/ {\varphi(s) \sim s}.$$

- $X = \bigcup X^n$ with the weak topology.

Write e^n for the interior of a disk D^n. Then the various pieces $G/K \times e^n$ making up X, called ***G-cells***, are comprised of open n-balls $gK \times e^n$, called the ***n-cells*** of the complex. We write $\mathsf{Cell}_n(X)$ for the set of n-cells of X. A ***G–CW pair*** (X, A) is a G–CW complex X and subcomplex A.

Note that G acts *cellularly* on a G–CW complex X in the sense that if $\sigma \in \mathsf{Cell}_n(X)$, then so also is $g\sigma$ in $\mathsf{Cell}_n(X)$, and given such a cell σ, every point $x \in \sigma$ has the same stabilizer $\mathrm{Stab}(x) =: \mathrm{Stab}\,\sigma$ under the defining G-action. Setting $G = 1$ to be the trivial group in the definition recovers CW-complexes in the traditional Whitehead sense; because an orbit is a "point" in the equivariant sense, this definition is a natural generalization. The following is then plausible.

Proposition 2.3.5 (Matumoto [Mat71b])**.** *Let X be a G–CW complex. Then the orbit space X/G inherits a canonical CW structure whose n-cells are $\{\sigma/G : \sigma \in \mathsf{Cell}_n^G(A)\}$.*

Theorem 2.3.6 (Illman [Ill83])**.** *Let G be a compact Lie group and M a smooth manifold, possibly with boundary. Then M admits a G–CW structure.*

Thus G–CW complexes are a natural class of topological group actions to consider from the point of view of homotopy theory.

Historical remarks 2.3.7. The result Theorem 2.3.6 has an interesting history. It was Takao Matumoto [Mat71a, Prop. 4.4] who first considered G–CW structures and observed that they existed for compact Lie groups G and smooth, closed G-manifolds. His proof relies on an earlier result of Chung-Tao Yang [Yan63], which in turn relies on a result of Stewart Cairns [Cai41] whose proof was never published, and which unfortunately contradicts a valid 1970 result of Larry Siebenmann [Sie77, p. 312]. The contradiction was apparently discovered by John Mather around 1976 and relayed to Sören Illman by Katsuo Kawakubo; Illman reported this error in 1978 [Ill78], and Andrei Verona was able to repair Yang's proof in 1979 [Ver79]. Illman was then able to regain Matumoto's result in 1983 [Ill83, Thm. 7.1], twenty years after Yang's proof and a full forty-two after Cairns's statement.

2.4. Fixed point sets of actions on homogeneous spaces

In order to localize integrals over and study cohomology rings of fixed point sets of actions on homogeneous spaces, as we shall do in Chapter 10 and Appendix C, it helps to understand what these sets are.

We initially focus on describing these fixed point sets in terms of normalizers. Several of these lemmas are due to Oliver Goertsches and Sam Noshari, while the author does not know citations for the others. The proofs (admittedly trivial) are of the author's own design unless noted otherwise.

Lemma 2.4.1. *Let G be a group, H and K subgroups acting respectively by left and right multiplication. The fixed point set $(G/K)^H$ is $\{gK \in G/K : g^{-1}Hg \leqslant K\}$.*

Proof. We have the chain of equivalences of set containments

$$gK \in (G/K)^H \iff HgK = gK \iff g^{-1}HgK = K \iff g^{-1}Hg \leqslant K \iff H \leqslant gKg^{-1}.$$

The last condition is clearly independent of the representative g of $gK \in G/K$. □

This set is easier to describe in the event of a torus action.

Lemma 2.4.2 ([Goe12, Lem. 4.3])**.** *Let G be a compact, connected Lie group, K a closed, connected subgroup, and S the maximal torus of K. The fixed point set $(G/K)^S$ is $N_G(S)K/K$, which is homeomorphic to the coset space $N_G(S)/N_K(S)$.*

Proof (after Goertsches). From Lemma 2.4.2 we know

$$gK \in (G/K)^S \iff g^{-1}Sg \leqslant K.$$

By Theorem B.4.9, the latter condition holds if and only if there is $k \in K$ such that $k^{-1}g^{-1}Sgk = S$, in which case $gk \in N_G(S)$ and $g \in N_G(S)K$. Thus $(G/K)^S = N_G(S)K/K \leqslant G/K$.

Now, there is a natural continuous map $N_G(S) \rightarrowtail N_G(S)K \twoheadrightarrow N_G(S)K/K$, under which an element $n \in N_G(S)$ is sent to K if and only if $n \in K$, so the "kernel" is $N_G(S) \cap K$ and one has a continuous bijection

$$N_G(S)\big/N_G(S) \cap K \longleftrightarrow N_G(S)K\big/K.$$

Since these are compact Hausdorff spaces, this is a homeomorphism. But the denominator on the left is $N_K(S)$ by definition. □

This is even simpler if the isotropy subgroup is a torus.

Corollary 2.4.3. *Let G be a compact, connected Lie group containing a torus S. Then*

$$(G/S)^S = N_G(S)/S$$

Proof. Take $K = S$ in Lemma 2.4.2. □

The Weyl group also comes up in this context as well, for the following reason.

Lemma 2.4.4. *Let G be a compact Lie group and K a closed subgroup containing a maximal torus T. The set of fixed points of the left multiplication action of T on the right coset space G/K is in natural bijection with the coset space W_G/W_K, where $W_G = N_G(T)/T$ is the Weyl group of G. In particular, W_G is the fixed point set of the left action of T on G/T.*

Proof. By Lemma 2.4.2, we have $(G/K)^T \approx N_G(S)/N_K(S)$, so by what is essentially the third isomorphism theorem,

$$(G/K)^T \approx \frac{N_G(T)}{N_K(T)} \longleftrightarrow \frac{W_G}{W_K}. \qquad \square$$

Lemma 2.4.1 obviously implies many fixed point sets are empty.

Corollary 2.4.5. *Let G be a compact Lie group and K and H subgroups. The set of fixed points of the left multiplication action of H on the right coset space G/K is empty unless H is conjugate in G to a subgroup of K. In particular, if $\operatorname{rk} H > \operatorname{rk} K$, then $(G/K)^H = \varnothing$.*

Now that we have at least a philosophical understanding of these fixed point sets, we try to develop as well an understanding of the number and homeomorphism types of their components. These components will turn out to be quotients of centralizers, and their cohomology rings computable by techniques we will develop in Section 8.3.1. One needs the following useful counting lemma.

Proposition 2.4.6 (Noshari [GN15, Proposition 2.3]). *Let Γ be compact Lie group and H a closed subgroup. Then the number of components of Γ/H is given by*

$$\left|\pi_0(\Gamma/H)\right| = \frac{|\pi_0\Gamma| \cdot \left|\pi_0(H \cap \Gamma_0)\right|}{|\pi_0 H|},$$

where Γ_0 is the identity component of Γ.

Proof. Consider the intermediate group $\Gamma_0 H$, which is some union of components of Γ. One has a bundle

$$\Gamma_0 H/H \longrightarrow \Gamma/H \longrightarrow \Gamma/\Gamma_0 H,$$

the fiber of which has only one component, since Γ_0 is connected, so to find $\left|\pi_0(\Gamma/H)\right|$ is to find the index $[\Gamma : \Gamma_0 H] = \left|\pi_0(\Gamma/\Gamma_0 H)\right|$, which is finite because $[\Gamma : \Gamma_0]$ already is, Γ being compact.

Moreover, since indices are transitive,

$$|\pi_0\Gamma| = [\Gamma : \Gamma_0] = [\Gamma : \Gamma_0 H][\Gamma_0 H : \Gamma_0] = \big|\pi_0(\Gamma/H)\big| \cdot [\Gamma_0 H : \Gamma_0].$$

For $\pi_0 H$, one has an analogous decomposition

$$|\pi_0 H| = [H : H \cap \Gamma_0]\,\big|\pi_0(H \cap \Gamma_0)\big|,$$

linked to the previous one by the "second isomorphism theorem" bijection $[\Gamma_0 H : \Gamma_0] = [H : H \cap \Gamma_0]$. Rearranging, one finds

$$\frac{|\pi_0\Gamma|}{\big|\pi_0(\Gamma/H)\big|} = [\Gamma_0 H : \Gamma_0] = [H : H \cap \Gamma_0] = \frac{|\pi_0 H|}{\big|\pi_0(H \cap \Gamma_0)\big|},$$

and isolating $\big|\pi_0(\Gamma/H)\big|$ on one side yields the result. □

In what follows, let G be a compact, connected Lie group, K a closed, connected subgroup, and S a maximal torus of K. Write $N = N_G(S)$ and write $Z = Z_G(S)$ for the centralizer. Recall that $\pi_0 N$ denotes the component group of N. Then $(G/K)^S \approx N/N_K(S)$ is naturally expressed in terms of other known objects.

Lemma 2.4.7. *Conjugation induces maps*

$$N \longrightarrow \operatorname{Aut} S;$$

$$\pi_0 N \rightarrowtail \operatorname{Aut} S.$$

The kernel of the first is the centralizer Z, which is consequently the identity component of the normalizer N.

Proof. Conjugation yields a continuous natural homomorphism

$$\begin{aligned} N &\longrightarrow \operatorname{Aut} S, \\ n &\longmapsto \left(x \longmapsto nxn^{-1}\right). \end{aligned}$$

The kernel of this map is definitionally Z, for conjugation by $n \in N$ fixes S pointwise if and only if n commutes with all elements of S. By the first isomorphism theorem, we obtain an injection $N/Z \rightarrowtail \operatorname{Aut} S$.

Since G is connected, the centralizer Z of S is connected, being the union of those maximal tori of G that contain S. As Z is connected and $\operatorname{Aut} S$ discrete, it follows Z is the connected component of the identity in N. □

Corollary 2.4.8 (Noshari)**.** *Let G be a compact, connected Lie group, K a closed, connected subgroup, and S the maximal torus of K. Then the fixed point set $(G/K)^S \approx N_G(S)/N_K(S)$ has $|\pi_0 N|/|W_K|$ connected components, each homeomorphic to Z/S.*

Proof. To get the number of components, apply Proposition 2.4.6 with $\Gamma = N_G(S)$ and $H = N_K(S)$, noting $N_0 = Z$ by Lemma 2.4.7 and that $N_K(S) \cap Z_G(S) = Z_K(S) = S$ is path-connected because S is a maximal torus of K.

Since $N_G(S)$ acts transitively, the components will be mutually homeomorphic, and the identity component is

$$Z_G(S)N_K(S)/N_K(S) \approx Z_G(S)/Z_G(S) \cap N_K(S) = Z/S$$

by the second isomorphism theorem. □

Corollary 2.4.9. *Let G be a compact, connected Lie group and S a torus in G. Then the fixed point set $(G/S)^S \approx N/S$ has $|\pi_0 N|$ connected components, each homeomorphic to Z/S.*

Chapter 3

Classifying spaces

In this section, we carry out the construction of the ***universal principal G-bundle $EG \to BG$***, which we use essentially as a tool to convert actions into closely related *free actions*. The existence of this bundle is more important than the details of its construction in almost everything that follows, but we will at one key point (Definition 10.1.3) use the fact that EG admits commuting right and left actions of G, and to justify this fact we provide the construction.

3.1. The weak contractibility of EG

The original purpose of the universal principal G-bundle $EG \to BG$ was to be a principal G-bundle such that all others $G \to E \to B$ arose as pullbacks. For that reason, the resulting map $B \longrightarrow BG$ of base spaces classifies the bundle $E \to B$, and is called the ***classifying map*** of the bundle, and BG is called the ***classifying space*** of a principal G-bundle.

The fact that EG is weakly contractible—which is much of why we care about the universal bundle—turns out to be a consequence of that demand. In this subsection, we explain the relevance of this demand. It will simplify the argument to know that all maps of principal G-bundles are pullbacks.

Proposition 3.1.1. *Consider a principal G-bundle map*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r, "f"] & B. \end{tikzcd}$$

The pullback bundle $f^*E \to X$ is isomorphic to $P \to X$ as a principal G-bundle.

Proof. Recall from earlier that the total space $f^*E = X$ is the pullback in Top of the diagram $X \to B \leftarrow E$. Since P also admits a map to such a diagram, there is a continuous map $P \to f^*E$ commutatively filling in

$$\begin{tikzcd} P \arrow[r, dotted] \arrow[rr, bend left] \arrow[dr] & f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ & X \arrow[r, "f"] & B. \end{tikzcd}$$

For any $x \in X$, by assumption, the maps of fibers $P|_x \to E|_{f(x)} \leftarrow (f^*E)|_x$ are G-equivariant homeomorphisms, so $P \to f^*E$ is a bijective G-map. To see its inverse is continuous, it is enough to restrict attention to an open $U \subseteq X$ trivializing both P and f^*E, so we need only show the inverse of a continuous G-bijection φ filling in the diagram

$$\begin{tikzcd} U \times G \arrow[rr, "\varphi"] \arrow[dr] & & U \times G \arrow[dl] \\ & U & \end{tikzcd}$$

is continuous. By commutativity, we may write $\varphi(x, 1) = \big(x, \psi(x)\big)$ for a continuous $\psi\colon U \to G$, so that $\varphi(x, g) = \big(x, \psi(x)g\big)$ by equivariance. Then $\varphi^{-1}(x, g) = \big(x, \psi(x)^{-1}g\big)$, and since ψ and $g \mapsto g^{-1}$ are continuous, so is φ^{-1}. $\square$

Thus the $EG \to BG$ we seek just needs to be a final object in the category of principal G-bundles. Recall that Top admits CW approximations, so that up to homotopy, we may assume the base space of our principal G-bundle $P \to X$ is a CW complex. Then X is built one level at a time from a discrete set X^0 of vertices by gluing disks D^{n+1}_α to the n-skeleton X^n along attaching maps $\varphi_\alpha\colon \partial D^{n+1}_\alpha \approx S^n \longrightarrow X^n$, so we can view P as being constructed inductively from principal G-bundles over these attached cells.

We require one intuitively plausible lemma, which we will not prove.

Lemma 3.1.2 ([Ste51, Cor. 11.6, p. 53])**.** *Let B be a contractible, paracompact Hausdorff space and $E \to B$ an F-bundle for some fiber F. Then E is isomorphic as an F-bundle to $B \times F$.*

By the lemma, principal G-bundles over disks are trivial, so $P|_{X^{n+1}}$ is the identification space of $P|_{X^n}$ with some bundles $D^{n+1}_\alpha \times G \to D^{n+1}_\alpha$, the identifications given by G-maps $S^n_\alpha \times G \longrightarrow P|_{X^n}$.

The task of constructing a G-map $P \longrightarrow EG$ can now be undertaken one cell at a time. To start, $P|_{X^0}$ is a disjoint union of copies of G, and any homeomorphic map of these to fibers of $EG \to BG$ will work. Suppose inductively that a G-map $P|_{X^n} \longrightarrow EG$ has been built, and we want to extend this to the space $P|_{X^n} \cup (D^{n+1} \times G)$, where $D^{n+1} \times G$ is attached by a G-map $S^n \times G \longrightarrow P|_{X^n}$. We can do anything we want over the *interior* of D^{n+1}, and we know what must happen over $P|_{X^n}$, so our only constraint is the composition of the preexisting G-map and the attaching map,

$$\psi\colon S^n \times G \longrightarrow P|_{X^n} \longrightarrow EG.$$

Thus the task is really to extend an arbitrary G-map $S^n \times G \longrightarrow EG$ over the interior of $D^{n+1} \times G$:

$$\begin{tikzcd} D^{n+1} \times G \arrow[dr, dotted] & \\ S^n \times G \arrow[u, hook] \arrow[r, "\psi"'] & EG. \end{tikzcd}$$

But a G-map $\widetilde{\psi}\colon D^{n+1} \times G \longrightarrow EG$ is uniquely determined by its restriction to the standard section

$D^{n+1} \times \{1\}$ since $\widetilde{\psi}(x,g) = \widetilde{\psi}(x,1)g$, so it is necessary and sufficient to extend the restriction $S^n \longrightarrow EG$ to a map $D^{n+1} \longrightarrow EG$. If it is possible to do so, then restrictions of the latter map to concentric spheres of decreasing radius form a nullhomotopy of the map $S^n \longrightarrow EG$, so the condition finally turns out to be that $\pi_n(EG) = 0$.

Proposition 3.1.3. *A principal G-bundle $EG \longrightarrow BG$ is universal just if $\pi_*(EG) = 0$.*

Thus the collapse $EG \longrightarrow \mathrm{pt}$ of the total space is a weak homotopy equivalence, and so if EG is a CW complex, then it is actually contractible by Whitehead's theorem B.2.7. Now seems as good a time as any to derive a corollary we will use repeatedly later.

Corollary 3.1.4. *If G is a path-connected group, then BG is simply-connected.*

Proof. The long exact homotopy sequence Theorem B.2.4 of $G \to EG \to BG$ contains subsequences

$$0 = \pi_{n+1}(EG) \longrightarrow \pi_{n+1}(BG) \longrightarrow \pi_n(G) \longrightarrow \pi_n(EG) = 0,$$

yielding isomorphisms $\pi_{n+1}(BG) \cong \pi_n(G)$ for all n, and in particular for $n = 0$. □

3.2. A construction of EG for G a compact Lie group

Example 3.2.1. Embedding $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ as $\mathbb{C}^n \times \{0\}$, the direct union is the countable direct sum $\mathbb{C}^\infty = \bigoplus_{\mathbb{N}} \mathbb{C}$, which can be seen as the subspace of the countable direct product $\prod_{\mathbb{N}} \mathbb{C}$ such that all but finitely many coordinates are 0. Within $\mathbb{C}^\infty$ lies the ***unit ∞-sphere***

$$S^\infty := \{\vec{z} \in \mathbb{C}^\infty : \sum z_j^2 = 1\}.$$

Write $\mathbb{C}^\infty_\times := \mathbb{C}^\infty \backslash \{0\}$. The scalar multiplication of $\mathbb{C}$ on $\mathbb{C}^\infty$ restricts to a free action of $\mathbb{C}^\times$ on $\mathbb{C}^\infty_\times$ and of S^1 on S^∞, with the same orbit space

$$\mathbb{C}\mathrm{P}^\infty := \mathbb{C}^\infty_\times/\mathbb{C}^\times \approx S^\infty/S^1,$$

called ***infinite complex projective space***. The fiber space $S^\infty \to \mathbb{C}\mathrm{P}^\infty$ can be seen as the increasing union of restrictions $S^{2n-1} \to \mathbb{C}\mathrm{P}^{n-1}$, where we conceive S^{2n-1} as $S^\infty \cap \mathbb{C}^n$. Each $\mathbb{C}\mathrm{P}^n$ admits an open cover by contractible affines, so these restrictions are all principal S^1-bundles, and $S^\infty \to \mathbb{C}\mathrm{P}^\infty$ is as well.

We claim this bundle satisfies the requirements to be $ES^1 \to BS^1$. Because S^∞ is the union of the unit spheres $S^{2n-1} \subsetneq \mathbb{C}^n$, by a compactness argument, any map $S^m \longrightarrow S^\infty$ must lie inside some sufficiently large S^n, and $\pi_m S^n = 0$ for $m < n$. Thus S^∞ is weakly contractible. There is a natural CW structure on S^∞ where two hemispheres D^n attach to each S^{n-1} to form S^n, so we know from Whitehead's theorem S^∞ is contractible, but in fact, it is possible to see so directly as well.

Proposition 3.2.2. *The unit ∞-sphere S^∞ is contractible.*

Proof. There is a homotopy taking the subspace $S' := S^\infty \cap \big(\{0\} \times \mathbb{C}^\infty\big) \approx S^\infty$ with first coordinate zero to the point $e_1 = (1, \vec{0})$, given by

$$h_t(\vec{z}) := (\sin t)e_1 + (\cos t)\vec{z};$$

this is just a renormalization of the straight-line homotopy. Now it will be enough to find a homotopy from S^∞ to S'. Write $s\colon \vec{z} \longmapsto (0, \vec{z})$ for the shift homeomorphism. One's first inclination is to take

$$f_t(\vec{z}) = (1-t)\vec{z} + t \cdot s(\vec{z}).$$

If we can show $f_t(S^\infty)$ avoids $\vec{0} \in \mathbb{C}^\infty$, then the renormalization $\hat{f}_t := f_t/|f_t|$ will suit our purposes. Now note any $\vec{z} \in \mathbb{C}^\infty$ has a last nonzero coordinate z_n, so the n^{th} and $(n+1)^{\text{st}}$ coordinates $\big((1-t)z_n, tz_n\big)$ of $f_t(\vec{z})$ will never simultaneously be zero, and the linear maps $f_t \in \operatorname{End}_{\mathbb{C}} \mathbb{C}^\infty$ are injective. Thus $\hat{f}_t$ is an isotopy. $\square$

Example 3.2.3. Replacing $\mathbb{C}$ with the quaternions $\mathbb{H}$ (respectively, the reals $\mathbb{R}$) and S^1 with $\mathrm{Sp}(1) \approx S^3$ (resp., $\mathrm{O}(1) \approx S^0 \cong \mathbb{Z}/2$), one finds a universal $\mathrm{Sp}(1)$-bundle $E\mathrm{Sp}(1) \to B\mathrm{Sp}(1)$ is

$$S^3 \longrightarrow S^\infty \longrightarrow \mathbb{HP}^\infty$$

and a universal $\mathrm{O}(1)$-bundle $E\mathrm{O}(1) \to B\mathrm{O}(1)$ is

$$S^0 \longrightarrow S^\infty \longrightarrow \mathbb{RP}^\infty.$$

Any closed subgroup $K \leqslant G$ acts freely on EG by a restriction of the G-action, so one has a natural map $EG \longrightarrow EG/K$ with fiber K. It is intuitively plausible that this is also a fiber bundle, and this is actually the case in the event G is a Lie group: by Theorem B.4.3, $G \twoheadrightarrow G/K$ is a principal K-bundle, and the local trivializations $\phi\colon (EG)|_U \xrightarrow{\approx} U \times G$ of $EG \to BG$ and $G|_V \xrightarrow{\approx} V \times K$ of $G \to G/K$ combine to yield local trivializations $\phi^{-1}(U \times G|_V) \longrightarrow U \times V \times K$ making $EG \to EG/K$ a principal K-bundle, so that EG can serve as EK and EG/K as BK.

To make use of this observation, we can use the classic result Theorem B.4.7, due to Peter and Weyl, that every compact Lie group has a faithful finite-dimensional unitary representation. Thus, if we can find $E\mathrm{U}(n)$, we will have bundles $EG \to BG$ for all compact Lie groups G. Here is one construction.

Example 3.2.4. The infinity-sphere S^∞ can be seen as the collection of orthonormal 1-frames in $\mathbb{C}^\infty$ and $\mathbb{CP}^\infty$ as the space of 1-dimensional vector subspaces of $\mathbb{C}^\infty$. Analogously, one can form the infinite complex ***Stiefel manifolds*** $V_n(\mathbb{C}^\infty)$ of orthonormal n-frames in $\mathbb{C}^\infty$, which is to say, sequences $(v_1, \ldots, v_n)$ of n mutually orthogonal vectors of length one, topologized as a subspace of $\prod_n S^\infty$, and the infinite complex ***Grassmannian*** $G_n(\mathbb{C}^\infty)$ of n-planes in $\mathbb{C}^\infty$. Just as S^∞ projects onto $\mathbb{CP}^\infty$, so does each $V_n(\mathbb{C}^\infty)$ project onto $G_n(\mathbb{C}^\infty)$ through the span map $(v_1, \ldots, v_n) \longmapsto \sum \mathbb{C}v_j$. The unitary group $\mathrm{U}(n)$ acts freely on $V_n(\mathbb{C}^\infty)$; if one considers an element of S^∞ as an infinite vertical vector, or a $\infty \times 1$ matrix, then an element of $V_n(\mathbb{C}^\infty)$ can be seen as an $\infty \times n$ matrix, and right multiplication by an $n \times n$ matrix in $\mathrm{U}(n)$ produces another $\infty \times n$ matrix spanning the same column space, so that the fiber of the span map $V_n(\mathbb{C}^\infty) \longrightarrow G_n(\mathbb{C}^\infty)$ is homeomorphic to $\mathrm{U}(n)$. With a little work, it can be seen that $\mathrm{U}(n) \to V_n(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$ is a fiber bundle.

Moreover, an analogue of the contraction of S^∞ in Example 3.2.1 shows $V_n(\mathbb{C}^\infty)$ to be contractible: the idea is to first conduct the isotopy $\hat{f}_t$ of S^∞ consecutively n times, taking S^∞ into $\{0\}^n \times S^\infty$ and hence $V_n(\mathbb{C}^\infty)$ into $V_n\big(\{0\}^n \times \mathbb{C}^\infty\big)$, and then use a renormalized straight-line homotopy generalizing h_t to take $V_n\big(\{0\}^n \times \mathbb{C}^\infty\big)$ to the identity matrix $I_n \in \mathbb{C}^{n \times n} \subsetneq \mathbb{C}^{\infty \times n}$, representing the standard basis of the subspace $\mathbb{C}^n < \mathbb{C}^\infty$. Write g_t for the resulting homotopy $V_n(\mathbb{C}^\infty) \times I \longrightarrow \mathbb{C}^{\infty \times n}$. In the same way that our first guess for S^∞ failed to have image strictly unit-length, this map g_t, while it preserves linear independence, does not preserve orthogonality. But if we postcompose to g_t the Gram–Schmidt orthonormalization procedure, which is a well-defined projection

$$\{n\text{-tuples of linearly independent vectors in } \mathbb{C}^\infty\} \longrightarrow V_n(\mathbb{C}^\infty),$$

we achieve the desired homotopy.

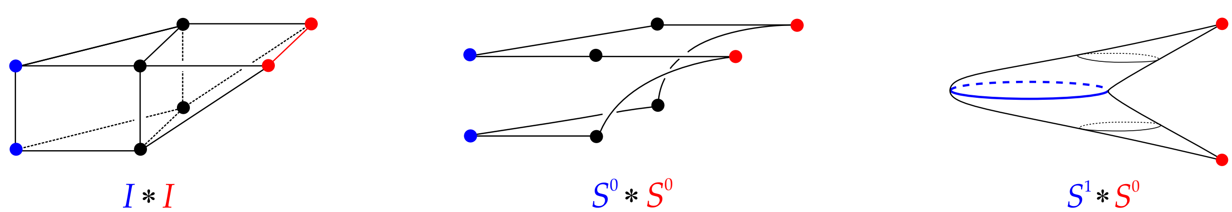
One analogously finds that $V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$ and $V_n(\mathbb{H}^\infty) \to G_n(\mathbb{H}^\infty)$ respectively satisfy the hypotheses for $EO(n) \to BO(n)$ and $ESp(n) \to BSp(n)$. The double cover $V_n(\mathbb{R}^\infty)/SO(n) =: \widetilde{G}_n(\mathbb{R}^\infty)$ of $G_n(\mathbb{R}^\infty)$, the ***oriented Grassmannian*** consisting of all *oriented* n-planes in $\mathbb{R}^\infty$, is a $BSO(n)$.

3.3. Milnor's functorial construction of EG

These constructions, though pleasing, are ad hoc. In 1955, John Milnor [Mil56] found a functorial construction of $EG \to BG$ that works for any topological group G, not even assumed Hausdorff, which we will briefly summarize.

To lay the groundwork, the ***join*** $X * Y$ of two topological spaces X and Y is the quotient of the product $X \times Y \times I$ with an interval by identifications $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$ for all $x, x' \in X$ and all $y, y' \in Y$. We may think of this as an $(X \times Y)$-bundle over I that has been collapsed to X over 0 and to Y over 1, and consider X and Y to be included as these particular end-subspaces.

Figure 3.3.1: Some low-dimensional joins



Examples 3.3.2. The join $I * I$ of two intervals is a 3-simplex Δ^3, the join $S^0 * S^0$ is a circle S^1, and the join $S^1 * S^0$ is a 2-sphere S^2.

It is not hard to see that generally $X * \mathrm{pt}$ is the cone CX on X and, as in the examples above, $X * S^0$ is the suspension SX of X, so the process of iteratively joining points generates the simplices Δ^n and that of iteratively joining copies of S^0 yields spheres S^n.

In particular, the join $S^1 * S^1$ is S^3, which can also be seen more geometrically. If one views S^3 as the unit sphere in $\mathbb{C}^2$, it is "foliated" by

$$T_r := \{(z\cos r, w\sin r) : z, w \in S^1\},$$

for $r \in [0, \pi/2]$, which are tori $S^1 \times S^1$ for $r \in (0, \pi/2)$ and circles for $r \in \{0, \pi/2\}$: the S^1 factor corresponding to the w-coordinate collapses at $r = 0$ and the S^1 corresponding to the z-coordinate collapses at $r = \pi/2$. Further foliating the toral T_r into circles with constant ratio $z/w \in S^1$ yields the Hopf fibration $S^1 \to S^3 \to S^2$.

The key feature of joins is that they are *more connected* than their factor spaces, as one already sees in the sphere examples above.

Lemma 3.3.3. *Let $X, Y \neq \varnothing$ be nonempty spaces. Then the join $X * Y$ is path-connected.*

Proof. Fix a basepoint $x_0 = [x_0, y, 0] \in X \subsetneq X * Y$. From a given $[x, y, t] \in X * Y$, we may trace a path to $y \in Y$ simply by increasing t to 1, then follow the path $t \mapsto [x_0, y, (1-t)]$ back to x_0. □

Lemma 3.3.4. *Let X be a path-connected space and Y nonempty. Then the join $X * Y$ is simply-connected.*

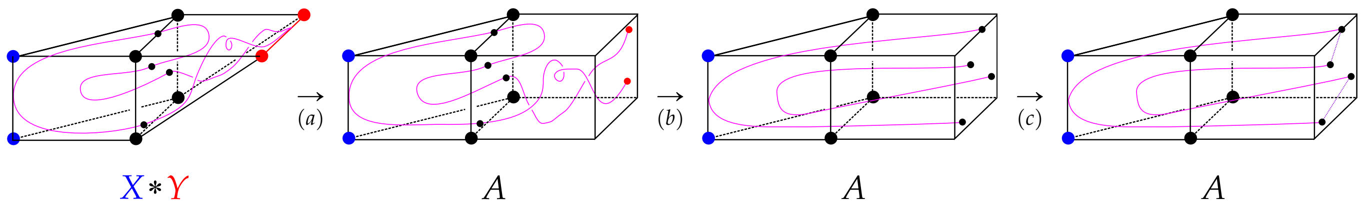
Proof. The projection $t \colon X \times Y \times I \twoheadrightarrow I$ induces a projection $X * Y \twoheadrightarrow I$ we also call t. The subset $V := t^{-1}[0, 1)$ deformation retracts onto X by uniformly decreasing t to 0, and X in turn contracts, within $X * Y$, to any fixed $y_0 \in Y$ via $f_t(x) = [x, y_0, t]$, so V is contractible in $X * Y$.

To show a loop γ in $X * Y$ is nullhomotopic, it therefore suffices to homotope it into V. We could always do so if $X * Y$ deformation retracted into V, but it does not. The partial quotient

$$A := X \times Y \times I \big/ (x, y, 0) \sim (x, y', 0),$$

where the X factor over $t = 1$ is not quotiented out, *does* however deformation retract into $\widetilde{V} = t^{-1}[0,1) \subsetneq A$, so if γ can be lifted to A, it can be homotoped into V.

Figure 3.3.5: Nullhomotoping a curve γ in $X * Y$ to lift into A



Such an attempt to lift a γ to A is represented by arrow (a) in Figure 3.3.5. If we write γ in terms of the coordinates x, y, t, then $t \circ \gamma$ is everywhere defined, but $x \circ \gamma$ is indeterminate when $t \circ \gamma = 1$ and $y \circ \gamma$ when $t \circ \gamma = 0$. Lifting to γ to A amounts precisely to continuously extending $x \circ \gamma$ to the whole domain S^1. This may initially be impossible: because the X-coordinate is collapsed at $t = 1$, it may be that γ is continuous at a point $z \in (t \circ \gamma)^{-1}\{1\}$ even though the left and right limits $(x \circ \gamma)(z + 0)$ and $(x \circ \gamma)(z - 0)$ are not equal. But we can homotope γ into a curve that *does* lift into A, as follows, and then we will be done.

Let $u_s \colon I \longrightarrow I$ be a nondecreasing homotopy fixing $\{0, 1\}$ and such that $u_1[1/2, 1] = \{1\}$.[1] Then $h_s \colon (x, y, t) \longmapsto \big(x, y, u_s(t)\big)$ is a homotopy of $X * Y$ collapsing $t^{-1}[1/2, 1]$ onto Y, so that γ is homotoped to a loop γ' with $t \circ \gamma' = 1$ on the set $C = \{z \in S^1 \colon (t \circ \gamma)(z) \geqslant 1/2\}$. This transition is represented, in A, by the arrow (b) in Figure 3.3.5.

Thus $x \circ \gamma'$ is a continuous function on $S^1 \backslash C$, and we need to extend it to a continuous function $x' \colon S^1 \longrightarrow X$. We should plainly take $x' = x \circ \gamma$ on $\partial C = \{z \in S^1 \colon (t \circ \gamma)(z) = 1/2\}$, so it remains to define x' on the open set $C \backslash \partial C \subsetneq S^1$, which is a union of open intervals (z_0, z_1). But to extend x' over (z_0, z_1) is to find a path in X connecting $x'(z_0)$ and $x'(z_1)$, which we may always do because X is path-connected. This extension is represented by arrow (c) in Figure 3.3.5. $\square$

[1] Explicitly, we can let $u_s \colon I \longrightarrow I$ be the straight-line homotopy between the identity map u_0 and the map $u_1(t) = \min\{1, 2t\}$.

Definition 3.3.6. A nonempty space X is ***(-1)-connected***, and, for each $n \in \mathbb{N}$, is ***n-connected*** if $\pi_j(X) = 0$ for all $j \leqslant n$.

Lemma 3.3.7. *If X is m-connected and Y is n-connected, then $X * Y$ is $(m+n+2)$-connected.*

Proof. The previous lemmas prove the cases $(m,n) \in \{(-1,-1),(0,-1)\}$; now assume $n \geqslant 0$. Applying the Mayer–Vietoris sequence in singular homology to the expected cover $\{U, V\}$, where $U = t^{-1}[0, 2/3) \simeq X$ and $V = t^{-1}(1/3, 1] \simeq Y$ and $U \cap V = t^{-1}(1/3, 2/3) \simeq X \times Y$, one recovers exact fragments

$$H_{N+1}(X) \oplus H_{N+1}(Y) \longrightarrow H_{N+1}(X * Y) \longrightarrow H_N(X \times Y).$$

The first map is induced by the inclusions $X, Y \longrightarrow X * Y$, both of which are nullhomotopic since X and Y are nonempty, so $H_{N+1}(X * Y)$ injects into $H_N(X \times Y)$.

Recall that the Künneth theorem in homology Theorem B.2.2 yields a group isomorphism

$$H_N(X \times Y) \cong \bigoplus_{0 \leqslant j \leqslant N} \big(H_j(X) \otimes H_{N-j}(Y)\big) \oplus \bigoplus_{0 \leqslant j \leqslant N} \mathrm{Tor}_1^{\mathbb{Z}}\big(H_j(X), H_{N-j-1}(Y)\big).$$

These terms can be nonzero only if $j \geqslant m+1$ and $N - j \geqslant n+1$, so adding inequalities, $H_N(X \times Y)$ can be nonzero if and only if $N \geqslant m+n+2$. Equivalently, $H_N(X \times Y) = 0$ for $N \leqslant m+n+1$.

It follows that $H_j(X * Y) = 0$ for $j \leqslant m+n+2$. Since $\pi_1(X) = 0$ by the previous lemma, $X * Y$ is $(m+n+2)$-connected by the Hurewicz Theorem B.2.6. □

It follows by induction that the n-fold iterated join $*^n X$ is $(n-1)$-connected. Including $*^n X$ as the first factor of $*^{n+1} X = (*^n X) * X$, we can form the direct limit

$$EX := \varinjlim \overset{n}{\circledast} X.$$

Because for all n we have $EX \approx (*^{n+1} X) * EX$, it follows that every $\pi_n(EX) = 0$. Note that the E

construction is functorial: a continuous map $\psi\colon X \longrightarrow Y$ induces a continuous map $E\psi\colon EX \longrightarrow EY$ taking $[\vec{x}_j, \vec{t}] \longmapsto [\overrightarrow{\psi(x_j)}, \vec{t}_j]$.

Now let G be a topological group. To construct a G-action on EG, we first provide a different description of it. For any topological space X, write CX for the ***cone*** on X, the quotient of the product $X \times I$ obtained by pinching $X \times \{0\}$ to a point. Then $X * Y$ can be seen as the subspace of $CX \times CY$ consisting of elements $[x, t_1, y, t_2]$ such that $t_1 + t_2 = 1$ and X as the subspace where $t_2 = 0$. Similarly, the triple join $X * Y * Z$ can be seen as $\big\{[x, t_1, y, t_2, z, t_3] \in CX \times CY \times CZ : t_1 + t_2 + t_3 = 1\big\}$, and $X * Y$ as the subspace where $t_3 = 0$, and the infinite join EG can be seen as

$$\Big\{\big([g_j, t_j]\big)_{j\in\mathbb{N}} \in \prod_{\mathbb{N}} CG : \text{only finitely many } t_j \neq 0 \text{ and } \sum t_j = 1\Big\}.$$

Write these elements briefly as $e = [\vec{g}_j, \vec{t}]$. With this topology, each coordinate function $t_j\colon EG \longrightarrow [0,1]$ and restriction $t_j^{-1}(0,1] \longrightarrow G$ of a "coordinate" g_j is continuous. Then a free, continuous right action of G on EG is given by

$$[\vec{g}_j, \vec{t}] \cdot g \coloneqq [\overrightarrow{g_j g}, \vec{t}].$$

Set $BG \coloneqq EG/G$, with the quotient topology.

We still must show $EG \to BG$ is a fiber bundle. Much like projective space, EG admits an open cover by sets $U_j = t_j^{-1}(0,1]$. On U_j, the g_j-coordinate is well-defined and continuous, so

$$\phi_j = (p, g_j)\colon U_j \longrightarrow p(U_j) \times G$$

is a continuous bijection. Its inverse ϕ_j^{-1} is also continuous since it is given by $\big(p(e), g\big) \longmapsto e g_j^{-1} g$. Where defined, $\phi_i \circ \phi_j^{-1}$ is given by $\big(p(e), g\big) \longmapsto \phi_i(e g_j^{-1} g) = \big(p(e), g_i g_j^{-1} g\big)$. The transition function $g \longmapsto g_i g_j^{-1} g$ is clearly continuous on $U_i \cap U_j$, so $EG \to BG$ is a principal G-bundle.

The classifying space construction B is also functorial, because if $\psi\colon G \longrightarrow H$ is a continuous homomorphism, $E\psi$ is fiber-preserving—

$$E\psi\big([\vec{g}_j, \vec{t}] \cdot g\big) = E\psi[\overrightarrow{g_j g}, \vec{t}] = \big[\overrightarrow{\psi(g_j g)}, \vec{t}\big] = \big[\overrightarrow{\psi(g_j)}, \vec{t}\big] \cdot \psi(g) = E\psi\big([\vec{g}_j, \vec{t}]\big) \cdot \psi(g)$$

—so that $E\psi$ descends to a well-defined continuous map $B\psi\colon BG \longrightarrow BH$.

Remark 3.3.8. The most technically demanding part of this proof, Lemma 3.3.4, can be circumvented if one does not care about a hard bound on the number of joins required to achieve n-connectedness: if X is path-connected, then the middle part $t^{-1}(0,1)$ in $X * X$ is also path-connected, so the Seifert–van Kampen theorem applies, which is a much easier proof. We hope our proof of this lemma is fairly readily apprehended; the Milnor original (pp. 431–432) is essentially the explicit statement of a homotopy coupled with a remark that the formula is easily verified to be well defined and to provide the needed contraction.

Historical remarks 3.3.9. The notation for EG and BG descends from a proud historical precedent. The way to denote a bundle $F \to E \xrightarrow{\pi} B$ equipped with a local trivialization with transition functions taking values in $G \leqslant \mathrm{Homeo}(F)$, as late as the 1960s [Ste51; BH58; BH59; BH60], was a quintuple (E, B, F, p, G), with the last two entries often omitted. This arrowless notation requires one to always remember which object lives in which position, but does have the benefit that if a bundle is named ξ, it has canonically associated with it an entourage of ready-named objects

$$(E_\xi, B_\xi, F_\xi, \pi_\xi, G_\xi) = \xi.$$

The standard name for the universal principal G-bundle under this convention is, naturally enough,

$$(E_G, B_G, G, \pi_G, G).$$

In subsequent decades, perhaps as the functorial nature of $E\colon G \mapsto EG$ and $B\colon G \mapsto BG$ is embraced, one can see the subscripts of E_G and B_G gradually move up until one has the $EG \longrightarrow BG$ of modern day.

Chapter 4

Cohomology theories

It would be an understatement to say singular cohomology has proven useful in topology. Given that all symmetry arises as the manifestation of group actions, it would seem fruitful to determine if a similar, equivariant theory might be constructed that takes as input not topological spaces, but continuous group actions. Such constructions have been undertaken. In this chapter, we expound characteristics of singular cohomology we would like equivariant cohomology to share, and then construct a candidate theory.

4.1. Desiderata

Ordinary cohomology is a functor $H^*\colon \mathsf{Top} \longrightarrow k\text{-}\mathsf{CGA}$ whose underlying functor to graded k-vector spaces decomposes as a product $\prod H^n$ of functors $H^n\colon \mathsf{Top} \longrightarrow k\text{-}\mathsf{Mod}$ satisfying the Eilenberg–Steenrod axioms:

- $H^*(\mathrm{pt}) = H^0(\mathrm{pt}) \cong k$.
- Homotopic maps $f, g\colon (X, A) \longrightarrow (Y, B)$ induce equal ring homomorphisms $H^*f = H^*g$.
- If $(X, A) \in \mathsf{Top}$ and A is a deformation retract of some neighborhood of itself in X, then the map $H^*(X/A, A/A) \longrightarrow H^*(X, A)$ induced by the quotient map $(X, A) \longrightarrow (X/A, A/A) =$

$(X/A, \mathrm{pt})$ is an isomorphism.

- There is a natural transformation $\delta\colon H^*(X,A) \longrightarrow H^{*+1}(A)$ between the graded k-Mod-valued functors $(X,A) \mapsto H^*(X,A)$ and $(X,A) \mapsto H^*(A)$, fitting into the exact triangle

$$\begin{tikzcd} H^*A \arrow[rr] & & H^*X \arrow[dl] \\ & H^*(X,A), \arrow[ul, "\delta"] & \end{tikzcd}$$

where the other maps are the ring maps induced by the inclusions $(A,\varnothing) \hookrightarrow (X,\varnothing) \hookrightarrow (X,A)$.

- Given a collection (X_α, A_α) of pairs and (X,A) their disjoint union, the inclusions $(X_\alpha, A_\alpha) \hookrightarrow (X,A)$ induce an isomorphism

$$H^*(X,A) \xrightarrow{\sim} \prod_\alpha H^*(X_\alpha, A_\alpha).$$

A functor satisfying these axioms is called an ***ordinary multiplicative cohomology theory***; the word *multiplicative* is here because we ask the functor take values in graded rings rather than just graded groups Any space X admits a unique map $X \longrightarrow \mathrm{pt}$, and any nonempty space X retracts onto a point $x \in X$, meaning $\mathrm{pt} \to \{x\} \hookrightarrow X \to \mathrm{pt}$ is the identity. Such a retraction induces algebra maps $k = H^0(\mathrm{pt}) \to H^*X \to k$ such that the composition $k \to k$ is the identity, so that the maps $k \to H^*X$ are injections. Because the map $X \to \mathrm{pt}$ is unique, given any map $X \to Y$, the canonical map $k \to H^*X$ is the composition $k \to H^*Y \to H^*X$, so this map $k \to H^*$ is a bit of extra structure to cohomology; for this reason, we call k the ***coefficient ring***.

It can be shown that all cohomology theories agree on CW-complexes. Since any topological space admits a CW approximation with the same weak homotopy type and weak homotopy equivalences induce isomorphisms in cohomology (Theorem B.2.8), these axioms determine ordinary cohomology so far as homotopy theory cares.

A ***G-equivariant cohomology theory*** E^* with coefficients in a commutative ring k is defined analogously, except the "ordinary" demand $E^*(\mathrm{pt}) = E^0(\mathrm{pt}) = k$ is relaxed and the source category is taken to be G-Top.

Explicitly, we want a functor $E^*\colon G\text{-}\mathsf{Top} \longrightarrow E^*\text{-}\mathsf{CGA}$ satisfying the following axioms:

- G-homotopic maps $f, g\colon (X, A) \longrightarrow (Y, B)$ induce equal ring homomorphisms $E^*f = E^*g$.

- If $(X, A) \in G\text{-}\mathsf{Top}$ and A is a G-equivariant deformation retract of some neighborhood of itself in X, then the map $E^*(X/A, A/A) \longrightarrow E^*(X, A)$ induced by the quotient map $(X, A) \longrightarrow (X/A, A/A) = (X/A, \mathrm{pt})$ is an isomorphism.

- There is a natural transformation $\delta\colon E^*(X, A) \longrightarrow E^{*+1}(A)$ between the functors $(X, A) \mapsto E^*(X, A)$ and $(X, A) \mapsto E^*(A)$ from $G\text{-}\mathsf{Top} \longrightarrow \mathsf{gr}\text{-}k\text{-}\mathsf{Mod}$, fitting into the exact triangle

$$\begin{tikzcd} E^*A \arrow[rr] & & E^*X \arrow[dl] \\ & E^*(X,A). \arrow[ul, "\delta"] & \end{tikzcd}$$

- Given a collection (X_α, A_α) of pairs and (X, A) their disjoint union, the inclusions $(X_\alpha, A_\alpha) \hookrightarrow (X, A)$ induce an isomorphism

$$E^*(X, A) \xrightarrow{\sim} \prod_\alpha E^*(X_\alpha, A_\alpha).$$

4.2. Candidate theories

In this section, we moot some possible equivariant cohomology theories to study before settling on Borel equivariant cohomology for the remainder of the work.

4.2.1. The cohomology of the naive quotient

Regular cohomology already does most of what we want, so a first idea might be to precompose it with the orbit functor $-/G$, yielding

$$E^*\colon G\text{-}\mathsf{Top} \longrightarrow k\text{-}\mathsf{CGA},$$
$$X \longmapsto H^*(X/G).$$

This E^* is a G-equivariant cohomology theory essentially because H^* is a cohomology theory; the satisfaction of the axioms is immediately inherited from H^* and the fact that the maps in the source category are G-equivariant. In addition, one has $E^*(\mathrm{pt}) = H^*(\mathrm{pt}/G) = H^*(\mathrm{pt}) = k$, so this is an *ordinary* G-equivariant cohomology theory.

Unfortunately, this E^* discards much useful information. For one thing, any nonempty G-orbit has trivial G-equivariant cohomology under this theory. For another arguable failing, in the standard example of $G = S^1$ acting on $X = S^2$ by rotation about a fixed axis, one has $X/G = S^2/S^1 \approx [-1, 1] \simeq \mathrm{pt}$, so that $E^*(S^2) = k$. While it is informative that this action has trivial E^*-cohomology, one cannot help but feel some information has been lost in translation.

4.2.2. Bredon cohomology

A more homotopy-theoretically informative theory can be derived as follows.

To motivate the construction, suppose first that we are given a G-equivariant cohomology

E^*. As a G-action decomposes set-wise as a union of orbits $Gx \approx G/\mathsf{Stab}_G(x)$, a first step to understanding E^* might be to understand where it takes the translation actions $G \curvearrowright G/K$ for closed $K \leqslant G$, or in other words, its restriction to the orbit category G-Orbit. This restriction is a contravariant functor G-$\mathsf{Orbit} \longrightarrow k$-$\mathsf{CGA}$ or, alternately, a covariant functor

$$M\colon \mathsf{Sub}(G) \longrightarrow k\text{-}\mathsf{CGA}$$

from the category of closed subgroups of G. Call such a functor a ***system of coefficients***. If X is a G–CW complex, so decomposed into G-cells $D^n \times (G/K)$ with trivial G-action on D^n, which in turn equivariantly deformation retract to G/K, this system of coefficients should in principle completely determine E^*. In fact, the equivariant Mayer–Vietoris sequence, a consequence of the axioms for a G-equivariant cohomology theory, implies have the following theorem.

Proposition 4.2.1 ([AP93, Thm. 1.1.3, p. 8])**.** *Let G be a compact Lie group and let E^* and $'E^*$ be G-equivariant cohomology theories in the sense of Section 4.1, and suppose there exists a natural transformation $\eta\colon E^* \longrightarrow {}'E^*$ which is an isomorphism when restricted to the orbit category G-Orbit. Then η is a natural isomorphism.*

The idea of Bredon cohomology is to *start* with a system of coefficients M concentrated in degree 0 and from it produce a (unique) ***ordinary G-equivariant cohomology theory***, "ordinary" in the sense that $E^*(G/K) = E^0(G/K) = M(G/K)$ for $K \in \mathsf{Sub}(G)$, and unique in the sense of the theorem above. The word "ordinary" is chosen because this is an equivariant version of the dimension axiom: the orbits G/K are the equivariant version of points.

Recall from Section 2.2 that the maps of $\mathsf{Sub}(G)$ are the inclusions and the G-conjugacies, corresponding in the dual orbit category G-Orbit to equivariant quotient maps and to equivariant homeomorphisms $G/gKg^{-1} \longrightarrow G/K$ given by $xgKg^{-1} \longmapsto xgK$, so that besides the groups

$M(G/K)$, the functor M carries the data of maps $M(G/K) \longrightarrow M(G/H)$ for $G \geqslant H \geqslant K$ and maps $M(G/K) \longrightarrow M(G/gKg^{-1})$ which maps we will call, with acceptable ambiguity, $M(c_g)$.

Recall that a G–CW complex X is composed of G-cells $e^n \times (G/K)$ which are themselves unions of cells e^n, the collection of which is called $\mathsf{Cell}_n(X)$. If $\sigma \in \mathsf{Cell}_n(X)$, say $\sigma = e^n \times gK \subsetneq e^n \times (G/K)$, then the isotropy subgroup $K = \operatorname{Stab} x$ is constant across points $x \in \sigma$, and we can call this group $\operatorname{Stab}\sigma$. We define the ***G-equivariant cellular cochain group*** with coefficients in M to be

$$C^n_G(X;M) := \Big\{\varphi\colon \mathsf{Cell}_n(X) \longrightarrow \coprod_{K \in \mathsf{Sub}(G)} M(G/K) \;\Big|\; \varphi(\sigma) \in M(G/\operatorname{Stab}\sigma), \quad \varphi(g\sigma) = M(c_g)\varphi(\sigma)\Big\},$$

functions taking each n-cell to the coefficient group corresponding to the common orbit type of its points, and equivariant with respect to the defining action of G on $\mathsf{Cell}(X)$ and the conjugation action of G on M. The definition makes sense because $\operatorname{Stab} g\sigma = g(\operatorname{Stab}\sigma)g^{-1}$. The relative equivariant chains $C^n_G(X,A;M)$ are the subgroup vanishing on $\mathsf{Cell}_n(A)$.

It follows from the definition that such a cochain φ is determined uniquely by its value at one σ in each orbit $G\sigma$, and in particular if there is one orbit $Ge^0 \approx G/K$, one gets $C^0_G(G/K;M) \cong M(G/K)$ as hoped. The differential is defined in tight analogy with the definition of the nonequivariant cellular coboundary map, but we will stop without getting any more explicit, satisfied that with more work these groups yield an ordinary G-equivariant cohomology theory $H^n_G(X,A;M)$.

Historical remarks 4.2.2. This construction is due to Bredon in the event that G is discrete. The general definition for compact groups is apparently due to Matumoto [Mat73]. Bredon cohomology is the equivariant cohomology theory most commonly used in equivariant homotopy theory. Definitions nowadays tend to involve equivariant spectra and more homotopy-theoretic technology than we use here, and replace the spheres S^n_α in our definition Definition 2.3.4 of G–CW complexes by *representation-spheres*, one-point compactifications of vector spaces carrying representations of G. The author knows very little about this subject and dares not explore it further here, but given that it is the main flavor of cohomology in modern equivariant homotopy theory, it seemed it should at least be mentioned.

4.2.3. Borel equivariant cohomology

Having surveyed equivariant cohomology theories we are not interested in at present, for various reasons, we settle on the variant we will care about.

Recall from Chapter 3 that given any topological group G, there exists a principal G-bundle $G \to EG \to BG$ such that EG is contractible, called a *universal principal G-bundle*. Given a left G-space X, the product space $EG \times X$, equipped with the diagonal action, is another G-space homotopy equivalent to X, but the new action is free since

$$(e, x) = g \cdot (e, x) = (eg^{-1}, gx) \implies e = eg^{-1}$$

and the action $G \curvearrowright EG$ is free. Call $X \longmapsto EG \times X$ the ***freeing functor***; we can view it as a projection $G\text{-Top} \longrightarrow G\text{-Free}$. Now we can consider the orbit space of this new, free action. The mixing diagram (Section 2.1) for EG and X is

$$\begin{tikzcd}
EG \arrow[d] & EG \times X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\
BG & EG \underset{G}{\times} X \arrow[l] \arrow[r] & X/G.
\end{tikzcd}$$

The new space on the bottom,

$$X_G := EG \underset{G}{\times} X = {EG \times X}\Big/{(eg, x) \sim (e, gx)},$$

is called the ***homotopy quotient*** of X by G or the *Borel construction* [BBF+60, Def. IV.3.1, p. 52]. This space can be viewed as a sort of completion of X/G, and its ordinary cohomology will be, at least nominally, the subject of most of our researches. Note

$$\mathrm{pt}_G = EG \underset{G}{\times} \mathrm{pt} \approx EG/G = BG.$$

Definition 4.2.3. (Borel [BBF+60, IV.3.3, p. 53]) The singular cohomology $H^*(X_G;k)$ of the homotopy quotient X_G is the (*Borel*) ***equivariant cohomology*** $H^*_G(X;k)$ of the action of G on X. The diagonal G-orbit of $(x,e) \in X \times EG$ is denoted $[x,e]_G$. The equivariant cohomology of a point is $H^*_G := H^*_G(\mathrm{pt}) = H^*(BG)$.

As suggested by the fact that the freeing functor can be seen as a projection, the homotopy quotient construction does not differ essentially from the regular quotient construction if the original action was free.

Proposition 4.2.4. *Let G act freely on a CW complex M. Then the homotopy quotient M_G is homotopy equvalent to the orbit space M/G.*

Proof. The map $\rho\colon [e,x] \mapsto Gx$ from $M_G \to M/G$ has fiber $EG/\mathrm{Stab}(x)$ in general. If G acts freely on M, then all fibers are EG. Since EG is contractible, the long exact homotopy sequence of the bundle $EG \to M_G \to M/G$ shows ρ is a weak homotopy equivalence. By Whitehead's theorem, ρ is a homotopy equivalence. □

As pointed out in (B.1), given a G-space X, the mixing construction yields an X-bundle $X \to$

$X_G \to BG$ associated to the principal G-bundle $G \to EG \to BG$. Explicitly, the projection is

$$\begin{aligned} X_G &\longrightarrow BG, \\ [e,x]_G &\longmapsto eG. \end{aligned}$$

Definition 4.2.5. The bundle $X \to X_G \to BG$ described above is the ***Borel fibration*** of the action of G on X.

4.3. The Atiyah–Hirzebruch–Leray–Serre spectral sequence

Now that we have discussed fiber bundles and cohomology theories, we discuss the behavior of cohomology theories, including equivariant cohomology, on bundles.

Notation 4.3.1. Where possible, we write the total space of a fiber bundle as E, but in situations when we also have a cohomology theory E^* and a spectral sequence $(E_r^{\bullet,\bullet})$ to deal with, we regretfully name the total space X.

Given a fiber bundle $F \to X \xrightarrow{\pi} B$ with B a CW complex, let B^p be the p-skeleton of B. Then (B^p) is an increasing topological filtration of B and $(X^p) := (\pi^{-1}B^p)$ an increasing topological filtration of X. Define $X^p = B^p = \varnothing$ for $p < 0$. Suppose we are also given a multiplicative cohomology theory E^*. Then associated to each pair (X, X^p) is a long exact sequence $E^*X \to E^*X^p \to E^*(X, X^p)$. Because $X^p \subseteq X^{p+1}$, each map $E^*X \longrightarrow E^*X^p$ factors through E^*X^{p+1}, so the topological filtration (X^p) leads to a *decreasing* algebraic filtration

$$F_pE^*X = \ker(E^*X \longrightarrow E^*X^{p-1})$$

of E^*X. The shift in indices is so that F_0E^*X is all of E^*X. Assume for convenience that $\pi_1 B$

acts trivially on $E^*(F)$. Then turning the crank of the associated filtration spectral sequence of Corollary A.5.3, one arrives at the following.

Theorem 4.3.2 (Atiyah–Hirzebruch–Leray–Serre [AH61, Rmk. 2.2])**.** *Let $F \to X \to B$ be a fiber bundle and E^* a multiplicative cohomology theory such that $\pi_1 B$ acts trivially on $E^*(F)$. There exists a right half-plane spectral sequence (E_r, d_r) with*

$$E_2^{p,q} = H^p\big(B; E^q(F)\big),$$
$$E_\infty^{p,q} = \operatorname{gr}_p E^{p+q}(X).$$

This is a simultaneous generalization of the following spectral sequences.

Theorem 4.3.3 (Atiyah–Hirzebruch [AH61, Thm. 2.1])**.** *Let E^* be a multiplicative cohomology theory and X a topological space. There exists a right half-plane spectral sequence (E_r, d_r) with*

$$E_2^{p,q} = H^p\big(X; E^q(\mathrm{pt})\big),$$
$$E_\infty^{p,q} = \operatorname{gr}_p E^{p+q}(X).$$

Proof. Take $F = \mathrm{pt}$ and $X = B$ in Theorem 4.3.2. □

Theorem 4.3.4 (Serre [McC01, Theorem 5.2, p. 135])**.** *Let $F \to E \to B$ be a fibration such that $\pi_1 B$ acts trivially on $H^*(F)$. There exists a first-quadrant spectral sequence (E_r, d_r) with*

$$E_2^{p,q} = H^p\big(B; H^q(F)\big),$$
$$E_\infty^{p,q} = \operatorname{gr}_p H^{p+q}(E).$$

If $H^*(F)$ is a free k-module (for example, if k is a field), we may also write $E_2 \cong H^*(B) \otimes H^*(F)$. Further, *this construction is functorial in that a map of bundles induces a map of spectral sequences.*

Proof. Take $E^* = H^*$ (and write E for X) in Theorem 4.3.2 and apply Theorem B.2.1. □

Definition 4.3.5. The spectral sequence of Theorem 4.3.4 is the ***Serre spectral sequence***.

Critically, this formulation applies to principal bundles.

Proposition 4.3.6. *If $G \to E \to B$ is a principal G-bundle, then $\pi_1 B$ acts trivially on $H^*(G)$.*

Proof. Because the transition functions are given by right multiplication by elements of G, and this action of G on $H^*(G)$ is trivial since G is path-connected, it follows the action of $\pi_1 B$ on $H^*(G)$ is trivial. □

It is important to us to be able to identify the maps in cohomology induced by fiber inclusion and projection to the base.

Proposition 4.3.7. *Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a fibration such that $\pi_1 B$ acts trivially on $H^*(F)$. The fiber projection $i^* \colon H^*(E) \longrightarrow H^*(F)$ is realized by the left-column map $E_\infty^{\bullet,\bullet} \twoheadrightarrow E_\infty^{0,\bullet} \hookrightarrow E_2^{0,\bullet}$ in Theorem 4.3.4: to wit, we can write*

$$\operatorname{gr}_\bullet H^*(E) \xrightarrow{\sim} E_\infty^{\bullet,\bullet} \longrightarrow\!\!\!\!\!\rightarrow E_\infty^{0,\bullet} \lhook\joinrel\longrightarrow E_2^{0,\bullet} \xrightarrow{\sim} H^*(F).$$

Likewise, the base lift $\pi^* \colon H^*(B) \longrightarrow H^*(E)$ is realized by the bottom-row map $E_2^{\bullet,0} \twoheadrightarrow E_\infty^{\bullet,0} \hookrightarrow E_\infty^{\bullet,\bullet}$:

$$H^*(B) \xrightarrow{\sim} E_2^{\bullet,0} \longrightarrow\!\!\!\!\!\rightarrow E_\infty^{\bullet,0} \lhook\joinrel\longrightarrow E_\infty^{\bullet,\bullet} \xrightarrow{\sim} \operatorname{gr}_\bullet H^*(E).$$

Proof [McC01, p. 147]. We have a commutative square

$$\begin{tikzcd}
F \arrow[r, equal] \arrow[d, equal] & F \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\
F \arrow[r, "i"] \arrow[d] & E \arrow[r, "\pi"] \arrow[d] & B \arrow[d, equal] \\
\mathrm{pt} \arrow[r] & B \arrow[r, equal] & B
\end{tikzcd}$$

where each column (and row) is a fibration, with the original fibration in the middle column, and the maps between columns are fiber-preserving. These maps induce maps of spectral sequences, which we can denote as

$$^{F}E_r \longleftarrow E_r \longleftarrow {}^{B}E_r.$$

The middle spectral sequence is the Serre spectral sequence of the original fibration, while $^{F}E_r$ is that of $F \to F \to \mathrm{pt}$, which collapses at $^{F}E_2 = H^*\big(\mathrm{pt}; H^*(F)\big) = H^*(F)$, and $^{F}E_r$ is that of $\mathrm{pt} \to B \to B$, which also collapses instantly, at $^{B}E_2 = H^*\big(B; H^*(\mathrm{pt})\big) = H^*(B)$. On E_2 pages, the induced maps are $E_2(i^*)\colon E_2 \longrightarrow {}^{F}E_2$, which is the left-column projection $H^*\big(B; H^*(F)\big) \longrightarrow H^0\big(B; H^*(F)\big) \cong H^*(F)$, and $E_2(\pi^*)\colon {}^{B}E_2 \longrightarrow E_2$, which is the bottom-row inclusion $H^*(B) \longrightarrow H^*\big(B; H^0(F)\big)$, the maps we would like to descend to the maps $i^* = \mathrm{gr}_\bullet\, i^*$ and $\pi^* = \mathrm{gr}_\bullet\, \pi^*$ on E_∞ pages. The maps between E_∞ pages are

$$\begin{tikzcd}
H^*(F) & & \\
{}^{F}E_2 \arrow[u, "\wr"] & \mathrm{gr}_\bullet\, H^*(E) \arrow[l, "\mathrm{gr}_\bullet\, i^*"'] & {}^{B}E_2 \arrow[l, "\mathrm{gr}_\bullet\, i^*"'] \\
& & H^*(B), \arrow[u, "\wr"]
\end{tikzcd}$$

by the fact that the isomorphism of final page E_∞ with $\mathrm{gr}_\bullet\, H^*(E)$ is natural. But that shows that these maps descend from the E_2 column and row maps as claimed. $\square$

Remark 4.3.8. In the event the fibration $F \to E \xrightarrow{\pi} B$ is in fact a fiber bundle, as it will be in all cases that actually concern us, the Serre spectral sequence is isomorphic from E_2 on to the *Leray spectral sequence* of the map π, which we will introduce in Appendix D.2 to complete our account of Borel's original 1953 proof of Theorem 8.1.12.

We have stated Serre's theorem for singular simplicial cohomology, but he initially stated it for singular cubical homology and cohomology, and it goes through essentially unchanged for Alexander–Spanier cohomology, Čech cohomology, or cohomology with A_{PL}-cochains as we will use in Theorem 8.1.3.

Corollary 4.3.9. *Let $F \to E \to B$ be a fibration such that the action of $\pi_1 B$ on $H^*(F)$ is trivial and $H^*(F)$ is a free k-module. Then the fiber inclusion $F \hookrightarrow E$ is surjective in cohomology if and only if the spectral sequence of the bundle collapses at E_2.*

Proof. Recall from the last remark that the fiber projection $H^*(E) \longrightarrow H^*(F)$ can be realized as $E_\infty \twoheadrightarrow E_\infty^{0,\bullet} \hookrightarrow E_2^{0,\bullet}$. This map is surjective if and only if $E_\infty^{0,\bullet} = E_2^{0,\bullet}$, which in turn means that $E_2^{0,\bullet} = E_2 \cap \ker d_2 = E_3 \cap \ker d_3$ and so on: all differentials vanish on the left column $H^*(F)$.

But all differentials from the left column vanish if and only if the sequence collapses at E_2. The "if" implication is clear. For the "only if," note that by our assumptions, $E_2 \cong H^*(B) \otimes_k H^*(F)$, and d_2 vanishes on $H^*(B)$ by lacunary considerations, so since d_2 is an antiderivation, $d_2 = 0$. But this means $E_3 \cong H^*(B) \otimes H^*(F)$, and since d_3 vanishes on $H^*(F) = E_3^{0,\bullet}$ and $H^*(B)$, one has $d_3 = 0$. By induction, $E_\infty \cong E_2$. □

Corollary 4.3.10. *Let $F \to E \to B$ be a fibration such that the action of $\pi_1 B$ on $H^*(F)$ is trivial and $H^*(F)$ is a free k-module. Suppose further that F and B are of finite type. Then the Poincaré polynomials satisfy*

$$p(E) \leqslant p(B)p(F),$$

(in the sense that each coefficient of $p(B)p(F) - p(E)$ is nonnegative) with equality if and only if the fiber inclusion $F \hookrightarrow E$ is surjective in cohomology.

Proof. We have $E_2 = H^*(B) \otimes H^*(F)$ in the Serre spectral sequence of $F \to E \to B$, so $p(E_2) = p(B)p(F)$. The rank of each $E_r^{p,q}$, and hence the Poincaré polynomial, can only decrease by E_∞, and it can only *fail* to decrease if $E_2 \cong E_\infty$; that is the case if and only if $H^*(E) \twoheadrightarrow H^*(F)$, by Corollary 4.3.9. □

Corollary 4.3.11. *Let $F \to E \to B$ be a fibration such that the action of $\pi_1 B$ on $H^*(F)$ is trivial and $H^*(B)$ and $H^*(F)$ are both concentrated in even degrees. Then the spectral sequence collapses at E_2.*

Proof. If $H^*(B)$ and $H^*(F)$ are both concentrated in even degrees, then so is the tensor product $E_2 = H^*(B) \otimes H^*(F)$ concentrated in even total degree. Since the differentials d_r increase total degree by 1, mapping from even diagonals to odd and vice versa, they must all be trivial, so the sequence collapses at E_2. □

The Serre spectral sequence allows a vast generalization of the covering result Proposition B.3.5.

Proposition 4.3.12. *Let $F \to E \to B$ be a fiber bundle such that the action of $\pi_1 B$ on $H^*(F)$ is trivial and $h^\bullet(B)$ and $h^\bullet(F)$ are finite. Then the Euler characteristics of these spaces satisfy $\chi(E) = \chi(F)\chi(B)$.*

Proof. Consider $E_2 = H^*(B) \otimes H^*(F)$ as a single complex with $\deg(H^p B \otimes H^q F) = p + q$. With this grading, $\chi(E_2) = \chi(B) \cdot \chi(F)$. By repeated application of Proposition A.2.1, one finds

$$\chi(E_2) = \chi(E_3) = \cdots = \chi(E_\infty) = \chi(E). \qquad \square$$

Finally, the construction that we undertook with G–CW complexes makes available a spectral sequence due to Matumoto that expresses equivariant cohomology in terms of singular cohomology.

Theorem 4.3.13 (Equivariant Atiyah–Hirzebruch [Mat73])**.** *Let G be a compact Lie group and E^* a G-equivariant cohomology theory with values in k-*CGA*, and (X, A) a G–CW pair. Then there exists a spectral sequence of k-CGAs, functorial in (X, A), such that*

- $E_1^{p,q} \cong E^{p+q}(X^p, X^{p-1} \cup A)$,
- $E_2^{\bullet,q} \cong H_G^*(X, A; E^q)$, *the Bredon cohomology with $M = E^q$ coefficients, as discussed in Section 4.2.2,*
- $E_\infty^{p,q} = \mathrm{gr}_p\, E^{p+q}(X, A)$.

Remarks 4.3.14. (*a*) Although we will also have occasion to invoke the spectral sequence of a filtered DGA again in Section 8.1.1, from here on out, "spectral sequence" *simpliciter* will connote the cohomological Serre spectral sequence of a bundle. This construct will be deployed with sufficient frequency that we allow ourselves also to abbreviate it *SSS*.

(*b*) These spectral sequences apply more generally in the instance that $\pi_1 B$ fails to act trivially on $E^*(F)$, with the concession that the coefficients $E^*(F)$ must instead be taken to be a sheaf of groups or, at the most concrete, viewed as a $k[\pi_1 B]$-module.

4.3.1. The transgression in the Serre spectral sequence

We will make extensive use of the transgression in the Serre spectral sequence of a bundle in what follows. On the E_2 level, an edge homomorphism d_r takes (a submodule of) $H^{r-1}(F)$ to (a quotient of) $H^r(B)$, but it seems worth stating more explicitly what this means on the cochain level, so we put forth here two slightly more topologically explicit descriptions.

Proposition 4.3.15. *Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a fiber bundle such that the action of $\pi_1 B$ on $H^*(F)$ is trivial. An element $[z] \in H^{q-1}(F)$ (Definition A.5.12) transgresses to the image of $[b] \in H^q(B)$ in $E_q^{q,0}$ if and only if there exists $c \in C^{q-1}(E)$ in the singular cochain group such that $i^*c = z$ and $\delta c = \pi^* b$. This is the picture:*

$$\begin{tikzcd} & C^{q-1}(E) \arrow[r, "i^*"] \arrow[d, "\delta"'] & Z^{q-1}(F) \arrow[dll, dotted, "\tau"'] \\ Z^q(B) \arrow[r, "\pi^*"'] & Z^q(E), & \end{tikzcd} \qquad \begin{tikzcd} & c \arrow[r, mapsto] \arrow[d, mapsto] & z \arrow[dll, dotted] \\ b \arrow[r, mapsto] & \delta c. & \end{tikzcd}$$

Sketch of proof. Recall that the Serre spectral sequence is the filtration spectral sequence associated to the filtration of the simplicial cochain algebra $C^*(E)$ given by

$$F_p C^*(E) = \ker\left(C^*(E) \longrightarrow C^*(\pi^{-1}B^{p-1})\right),$$

the kernels of the restriction maps to the subbundles $\pi^{-1}B^p$ of E lying over the p-skeleta B^p of the base space B. The elements of the complement $F_pH^*(E)\backslash F_{p-1}H^*(E)$, then, are classes represented by cocycles which vanish over the p-skeleton of B but not over its p-cells. Morally, these classes are *carried* by the portion of E lying over the p-cells of B. Elements of $F_0H^*(E)$ are thus already carried by $\pi^{-1}B^0 \approx B^0 \times F$, and arise from $H^*(F)$, while elements in $F_pC^p(E)\backslash F_{p+1}C^p(E)$, on the other hand, are those p-cycles that are only nontrivial on singular chains with all p "directions" arising from the base B, and thus come from p-cycles in the base. At the E_2 level, classes in $H^*(E)$ are arbitrary, while a class in $F_pH^p(E)$ is represented by something in the square $E_2^{p,0} \cong H^p(F) \otimes H^0(F) \cong H^p(F)$.

Let $z \in Z^{q-1}(F)$ be a cocycle in the fiber. It extends (say by zero) to a cochain $c \in C^q(E)$, which need not be a cocycle. Because the differentials d_r in the Serre spectral sequence arise from the singular coboundary map δ through the filtration, $d_r(c)$ is defined in E_r if and only if δ pushes c forward by r steps in the filtration. Thus in this case δ takes a "fiber" cocycle c to a "base" cocycle.

If δc survives to E_q, then in E_2, it is represented by a class in $F_q\big(H^*(B) \otimes H^*(F)\big) = H^q(B) \otimes H^0(F) \cong H^q(B)$, so δc is in the image of $\pi^* \colon Z^q(B) \longrightarrow Z^q(E)$. $\square$

There is another way of explaining this which may be more illuminating and doesn't require us to work at the cochain level. Recall from Theorem B.2.4 that associated to a bundle $F \xrightarrow{i} E \xrightarrow{\pi} B$ is an exact triangle of homotopy groups

$$\pi_*(F) \longrightarrow \pi_*(E) \longrightarrow \pi_*(B) \xrightarrow{\deg -1} \pi_*(F).$$

Thus there is a degree-shifting map linking the homotopy groups of the base and fiber. Viewing $F = E|_{\mathrm{pt}}$ as a specific fiber over a point $\mathrm{pt} \in B$, this sequence can be seen to arise from a long exact sequence of *relative* homotopy groups associated to the pair (E, F):

$$\pi_*(F) \longrightarrow \pi_*(E) \longrightarrow \pi_*(E, F) \xrightarrow{\deg -1} \pi_*(F).$$

Because E is locally trivial, a map of pairs $(D^n, S^{n-1}) \longrightarrow (B, \mathrm{pt})$ can be lifted to a map $(D^n, S^{n-1}) \longrightarrow (E, F)$, and taking homotopy classes, a bit of work shows $\pi_*(E, F) \cong \pi_*(B, \mathrm{pt})$ in such a way that the two long exact sequences can be identified. The long exact sequence of a pair

$$H^*(F) \xrightarrow{\deg +1} H^*(E, F) \longrightarrow H^*(E) \longrightarrow H^*(F).$$

is one of the Eilenberg–Steenrod axioms, but it no longer will do in general to substitute $\widetilde{H}^*(B) = H^*(B, b_0)$ for $H^*(E, F)$. If it did, we would have the desired degree-shifting cohomological map linking the base and the fiber. Nevertheless, π is a map of pairs $(E, F) \longrightarrow (B, b_0)$, so one has map of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^{q-1}(F) \arrow[r, "\delta"] & H^q(E,F) \arrow[r] & H^q(E) \arrow[r, "i^*"] & H^q(F) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^q(\mathrm{pt}) \arrow[r] \arrow[u] & H^q(B,\mathrm{pt}) \arrow[r, "\sim"] \arrow[u, "\pi^*"] & H^q(B) \arrow[r] \arrow[u, "\pi^*"] & H^q(\mathrm{pt}) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

Proposition 4.3.16. *The transgression is given by the composition $(\pi^*)^{-1} \circ \delta$ where defined.*

Proof. If $z \in Z^{q-1}(F)$ is a cocycle representing $[z] \in H^{q-1}(F)$, then $\delta[z] \in H^q(E,F)$ is by definition the class of δc for any cochain $c \in C^q(E)$ such that $i^*c = z$. Such a class may or may not be the image under π^* of some $[b] \in H^q(B, b_0) = H^q(B)$, but if it is, then the elements (z, c, b) satisfy exactly the specification put forth in Proposition 4.3.15. □

Thus the transgressed classes in $H^{q-1}(F)$ can be seen as the images of the connecting homomorphism $\eta = (\pi^{-1})^* \circ \delta$ in a fictitious long exact sequence

$$H^*(F) \xrightarrow{\eta} H^*(B) \longrightarrow H^*(E) \xrightarrow{i^*} H^*(F)$$

of a bundle corresponding to the long exact sequence of homotopy groups. The transgressive elements are morally those for which such a sequence holds.

Remark 4.3.17. A real proof of this characterization of the transgression in the Serre spectral sequence can be found in McCleary [McC01, Thm. 6.6, p. 186].

There is an analogous Serre spectral sequence of a bundle in *homology*, whose differentials are of degree $(-r, r-1)$, and a (partially defined) transgression $H_r(B) \longrightarrow H_{r-1}(B)$. Dually to our definition in cohomology, the transgressed elements of H_qF are images of transgressive elements of $H_{q+1}B$ under an incompletely-defined map τ_* in a fictitious long exact sequence

$$H_*(B) \xrightarrow{\tau_*} H_*(F) \longrightarrow H_*(E) \longrightarrow H_*(B).$$

Because the Hurewicz homomorphism $\pi_*(X, A) \longrightarrow H_*(X, A)$ from homotopy groups to homology groups discussed in Theorem B.2.6 is natural, it pieces together into a map from the homotopy long exact sequence of a pair (E, F) to the homology long exact sequence of that pair. It follows from the existence of this map of long exact sequences and the long exact homotopy sequence of a bundle (Theorem B.2.4) that everything in the image of the Hurewicz map $\pi_*F \longrightarrow H_*F$ is the image of the transgression in every fibration with fiber F, a fact we will have cause to comment on again in Section 7.6. Moreover, the cohomology transgression $\tau\colon H^{q-1}(F) \longrightarrow H^q(B)$ and the homology transgression $\tau_*\colon H_q(B) \longrightarrow H_{q-1}(F)$ are *dual* [Ral].

4.4. A natural lemma on bundles

In this section, we use the Serre spectral sequence to prove a lemma on cohomology of bundles we will use repeatedly to good effect. It seems analogous to the *Leray–Hirsch theorem* that if $F \to E \to B$ is a bundle such that $H^*(E) \to H^*(F)$ is surjective, then $H^*(E) \cong H^*(B) \otimes H^*(F)$ as an $H^*(B)$-module. There is a proof by Larry Smith [Smi67, Cor. 4.4, p. 88] using the Eilenberg–Moore spectral sequence as well as the SSS, but the author found the result independently through the following proof, so it has pride of place.

Let F be a topological space and $\xi_0\colon E_0 \to B_0$ an F-bundle. From the category of F-bundles and F-bundle maps, we can form a slice category $F\text{-}\mathsf{Bun}/\xi_0$ of F-bundles ***over*** ξ_0 as follows. An object of $F\text{-}\mathsf{Bun}/\xi_0$ is an F-bundle ξ equipped with a bundle map $\xi \to \xi_0$; a morphism between objects $\xi' \to \xi_0$ and $\xi \to \xi_0$ is a bundle map $\xi' \to \xi$ making the expected triangle commute. Such a map entails the following commuting prism:

$$\begin{tikzcd}
E' \arrow[r, "h"'] \arrow[d, "\xi'"] \arrow[rr, bend left, "f'"] & E \arrow[r, "f"'] \arrow[d, "\xi"] & E_0 \arrow[d, "\xi_0"] \\
B' \arrow[r, "\bar{h}"] \arrow[rr, bend right, "\overline{f'}"'] & B \arrow[r, "\bar{f}"] & B_0.
\end{tikzcd} \tag{4.1}$$

Note that the maps between total spaces yield two functors

$$\begin{aligned}
F\text{-}\mathsf{Bun}/\xi_0 &\longrightarrow H^*(E_0)\text{-}\mathsf{CGA}: \\
(E \to B) &\longmapsto H^*(E); \\
(E \to B) &\longmapsto H(B) \otimes_{H^*(B_0)} H^*(E_0).
\end{aligned}$$

If $H^*(E_0) \longrightarrow H^*(F_0)$ is surjective, we claim these functors are naturally isomorphic.

Theorem 4.4.1. *Let $\xi_0 \colon E_0 \to B_0$ be an F-bundle such that the fiber inclusion $F \hookrightarrow E_0$ is H^*-surjective, such that $H^*(F)$ is a free k-module, and such that $\pi_1 B_0$ acts trivially on $H^*(F)$. Then the fiber inclusions of all F-bundles over ξ_0 are H^*-surjective, and there is a natural ring isomorphism*

$$H^*(E) \xleftarrow{\sim} H^*(B) \otimes_{H^*(B_0)} H^*(E_0)$$

of functors F-$\mathsf{Bun}/\xi_0 \longrightarrow H^*(E_0)$-$\mathsf{CGA}$. Diagrammatically, the commutative diagram (4.1) gives rise to

$$\begin{tikzcd}
H^*(E') & & H^*(E) \arrow[ll, "h^*"'] \\
H^*(B') \otimes_{H^*(B_0)} H^*(E_0) \arrow[u, "\wr"] & & H^*(B) \otimes_{H^*(B_0)} H^*(E_0). \arrow[ll, "\bar{h}^* \otimes \mathrm{id}"] \arrow[u, "\wr"']
\end{tikzcd}$$

Verbally, if a fiber inclusion is surjective in cohomology, then cohomology takes pullbacks to pushouts.

Proof. By the definition of a bundle map, the fiber inclusion $F \longhookrightarrow E_0$ factors as $F \hookrightarrow E \to E_0$, so the assumed surjectivity of $H^*(E_0) \to H^*(E) \to H^*(F)$ implies surjectivity of the factor $H^*(E) \longrightarrow H^*(F)$.

Because of these surjections, the spectral sequences of these bundles stabilize at their E_2 pages by Corollary 4.3.9. Applying H^* to the right square of the assemblage (4.1) yields

$$\begin{tikzcd} H^*(E) & H^*(E_0) \arrow[l, "f^*"'] \\ H^*(B) \arrow[u, "\xi^*"] & H^*(B_0), \arrow[l, "\bar{f}^*"] \arrow[u, "\xi_0^*"'] \end{tikzcd} \quad \text{which manifests on the } E_2 \text{ page as} \quad \begin{tikzcd} H^*(B) \otimes H^*(F) & H^*(B_0) \otimes H^*(F) \arrow[l, "\bar{f}^* \otimes \mathrm{id}"'] \\ H^*(B) \arrow[u, "\mathrm{id} \otimes 1"] & H^*(B_0). \arrow[l, "\bar{f}^*"] \arrow[u, "\mathrm{id} \otimes 1"'] \end{tikzcd}$$

The commutativity of the left square means there is an induced map of rings

$$\begin{aligned} H^*(B) \otimes_{H^*(B_0)} H^*(E_0) &\longrightarrow H^*(E), \\ b \otimes x &\longmapsto \xi^*(b) f^*(x), \end{aligned}$$

whose E_2 manifestation is the canonical $H^*(B)$-module isomorphism

$$H^*(B) \otimes_{H^*(B_0)} \big[H^*(B_0) \otimes H^*(F)\big] \xrightarrow{\sim} H^*(B) \otimes H^*(F).$$

Since this E_2 map is a bijection, the ring map is an $H^*(E_0)$-algebra isomorphism.

For naturality, note that the ring map $h^* \colon H^*(E) \longrightarrow H^*(E')$ is completely determined its restrictions to its tensor-factors $H^*(B)$ and $H^*(E_0)$. The left square and top triangle of (4.1) imply the commutativity of the squares

$$
\begin{tikzcd}
H^*(E') & H^*(E) \arrow[l, "h^*"'] \\
H^*(B') \arrow[u, "(\xi')^*"] & H^*(B), \arrow[l, "\bar{h}^*"] \arrow[u, "\xi^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H^*(E') & H^*(E) \arrow[l, "h^*"'] \\
H^*(E_0) \arrow[u, "(f')^*"] \arrow[r, equal] & H^*(E_0), \arrow[u, "f^*"']
\end{tikzcd}
$$

so that these factor maps are respectively $\bar{h}^*\colon H^*(B) \longrightarrow H^*(B')$ and $\mathrm{id}_{H^*(E_0)}$. $\square$

Chapter 5

The cohomology of complete flag manifolds

The algebraic relation between a compact group and its maximal torus informs all discussion of invariant subalgebras going forward, and is epistemologically prior to much of our discussion of the cohomology of homogeneous spaces, being treated with *sui generis* methods that do not apply in the general case.

The quotient G/T of a compact, connected Lie group by its maximal torus T, called a ***complete flag manifold***, was among the first homogeneous spaces other than groups and symmetric spaces whose cohomology was understood. This material will be cited at least in Section 6.2, Section 8.3.2, and Theorem B.4.9, and does not fit well anywhere else, so we propound it now. It is fundamental, and but for the discussion of the Serre spectral sequence in Theorem 4.3.4, could have gone earlier.

5.1. The cohomology of a flag manifold

The cornerstone result is the following.

Theorem 5.1.1. *Let G be a compact, connected Lie group and T a maximal torus in G. Then the cohomology of $H^*(G/T)$ is concentrated in even dimensions.*

Proof sketch 1. Associated to G is a *complexified* Lie group $G^{\mathbb{C}}$ which is a complex manifold, and which contains a *Borel subgroup* B, a complex Lie group containing T and such that

$$G^{\mathbb{C}}/B \approx G/T.$$

Thus G/T admits a complex manifold structure and hence a CW structure with even-dimensional cells. □

We will use consequences of this theorem in such a critical way in Section 6.2, Theorem B.4.9, and Section 8.3.2 that we would feel somewhat guilty only sketching this proof, so we reproduce Borel's original 1950 proof. This argument was first published somewhat telegraphically in Leray's contribution [Ler51] to the 1950 Bruxelles *Colloque*, and is elaborated in Borel's thesis [Bor53]. It invokes two facts we shall not prove about invariant differential forms, which are these.

Proposition 5.1.2. *Suppose a compact Lie group G acts on a manifold M. Then every cohomology class in $H^*(M;\mathbb{R})$ is represented by a G-invariant differential form ω, which is determined uniquely by its value $\omega_x \in \Lambda T^*_x M$, an alternating multlinear form on the tangent space of one point x of M.*

Proposition 5.1.3. *Let G be a compact Lie group and K a closed subgroup. The alternating multlinear form $\omega_x \in \Lambda(\mathfrak{g}/\mathfrak{k})^\vee$ representing a G-invariant form $\omega \in \Omega(G/K)$ is invariant under the action $\mathrm{Ad}^*\,K$ of K induced by the conjugation action on K on G.*

Borel's proof of Theorem 5.1.1. By Theorem B.2.1, we may use $\mathbb{R}$ coefficients. Write $\ell = \operatorname{rk} G$ and $n = \dim G - \operatorname{rk} G$. We prove the result by a double induction on ℓ and n. If $\ell = 0$, then G is discrete, and we are done. Inductively suppose we have proven the result for all groups of rank $\ell - 1$. If $n = 0$, then $\operatorname{rk} G = \dim G$, so $G = T$ is a torus and we are done.

Now suppose inductively we have proven the result for ℓ and $n-1$. Note that without loss of generality, by Theorem B.4.4, G can be taken to be of the form $A \times K$ with A a torus and K simply-connected. Since A is a factor of the maximal torus T of G, one has $G/T = K/(T \cap K)$, and $\operatorname{rk} K = \operatorname{rk} G - \operatorname{rk} A$, so we are done by induction on ℓ unless $\operatorname{rk} A = 0$ and $G = K$ is simply-connected.

So assume G is simply-connected. We claim there exists an element $x \in G$ such that $x \notin Z(G)$ and $x^2 \in Z(G)$. To see this, let $x \in G$, and recall from Proposition B.4.11 that $Z(G)$ is the intersection of all maximal tori in G, so in a given maximal torus T one can always find y with $y^{2^n} = x$, but if these lay in all tori for all n, then $Z(G)$ would fail to be discrete. Let K be the identity component of the centralizer $Z_G(x)$ of x. Because x lies in the maximal torus T of G, we know $\operatorname{rk} K = \operatorname{rk} G$, and because $x \notin Z(G)$, the dimension $\dim Z_G(x) = \dim K$ is strictly less than $\dim G$. Thus $H^*(K/T)$ is evenly graded by the inductive assumption.

The tangent space $\mathfrak{g}/\mathfrak{k} = T_{1K}(G/K)$ to the identity coset $1K$ in G/K can be identified with an orthogonal complement $\mathfrak{k}^\perp$ to $\mathfrak{k}$ in $\mathfrak{g}$ in such a way that the isotropy action of K on $T_{1K}(G/K)$ corresponds to the adjoint action of K on $\mathfrak{k}^\perp$.

By Proposition 5.1.2, each de Rham cohomology class on G/K contains a left G-invariant element, which is then determined by its restriction to $T_{1K}(G/K) \cong \mathfrak{k}^\perp$. Such a restriction is, by Proposition 5.1.3, an alternating $(\operatorname{Ad} K)$-invariant multilinear form on $\mathfrak{k}^\perp$. Because x^2 is central, $\operatorname{Ad}(x) \in \mathrm{GL}(\mathfrak{g})$ is an involution; thus $\mathfrak{g}$ splits as the 1-eigenspace $\mathfrak{k}$ and an orthogonal (-1)-eigenspace, which must be $\mathfrak{k}^\perp$. Since $\operatorname{Ad}(x)$ acts as multiplication by -1 on $\mathfrak{k}^\perp$, a nonzero $\operatorname{Ad}(x)$-invariant alternating form on $\mathfrak{k}^\perp$ can only have even degree. As $x \in K$, it follows we must have $H^*(G/K)$ concentrated in even degree.

Now we can apply the Serre spectral sequence to $K/T \to G/T \to G/K$. Both $H^*(K/T)$ and $H^*(G/K)$ are evenly-graded, so by Theorem 4.3.4, so also is G/T. In fact, by Corollary 4.3.11, the spectral sequence collapses at E_2 and $H^*(G/T) \cong H^*(G/K) \otimes H^*(K/T)$ as an $H^*(K/T)$-module.

□

Corollary 5.1.4. *Let G be a compact, connected Lie group and T a maximal torus in G. Then the Euler characteristic of $\chi(G/T)$ is positive.*

5.2. The acyclicity of $G/N_G(T)$

In this section we prove another result whose importance will not immediately be clear, but which recurs in Section 6.4 and Section 10.1.

Proposition 5.2.1. *Let G be a compact, connected Lie group, T a maximal torus in G, and $N = N_G(T)$ the normalizer. Then* $\dim G/N$ *is even and G/N is $\mathbb{Q}$-acyclic:*

$$H^*(G/N;\mathbb{Q}) = H^0(G/N;\mathbb{Q}) \cong \mathbb{Q}.$$

Proof [MT00, Thm. 3.14, p. 159]. The torus T acts on G/N on the left, fixing the identity coset $1N$ (since $T \leqslant N$); we claim this is the only such fixed point. Indeed, let $t \in T$ be a topoological generator. If an element $gN \in G/N$ is fixed under multiplication by t, it is fixed under multiplication by all powers of t, and thus, by continuity, by all of T, so that $TgN = gN$, or $g^{-1}Tg \leqslant N$. Since T is a connected component of N and $1 = g^{-1}1g \in T$, it then follows $g^{-1}Tg = T$, or $g \in N$, so that $gN = 1N$ is the lone fixed point.

Let $\dim \mathfrak{n}^\perp = m$. Because T fixes $1N$, it induces a T-action on the tangent space $\mathfrak{g}/\mathfrak{n} = T_{1N}(G/N)$ to G/N at the identity coset $1N$, which can also be seen as the normal subspace $\mathfrak{n}^\perp < \mathfrak{g}$ orthogonal to $\mathfrak{n}$ in the tangent space to G at 1. Because T acts by isometries on $\mathfrak{n}^\perp$, it leaves invariant ε-spheres S^{m-1} about the origin $0 \in \mathfrak{n}^\perp$. The exponential $\exp\colon \mathfrak{n}^\perp \longrightarrow G/N$ will map a sufficiently small sphere isometrically and T-equivariantly into G/N, and this T-invariant image sphere S^{m-1} divides G/N into a T-invariant disk D^m and a T-invariant complement M. Since T is path connected, the map ℓ_t is homotopic to the identity, so $\chi(\ell_t) = \chi(\mathrm{id})$ on both S^{m-1} and M. As only $1N \in G/N$ is fixed by multiplication by T, and this point lies in the interior of D^m it follows ℓ_t acts without fixed points on S^{m-1} and M. By the Lefschetz fixed point theorem Theorem B.2.11, then,

$$\chi(M) = \chi(S^{m-1}) = 0.$$

It follows m is even. Note that $H^*(G/N, M) \cong H^*(D^m, S^{m-1}) \cong \widetilde{H}^*(S^m)$, so that the relative Euler characteristic $\chi(G/N, M)$ is $1^m = 1$. The long exact sequence of the pair $(G/N, M)$ then gives

$$\chi(G/N) = \chi(M) + \chi(G/N, M) = 0 + 1 = 1.$$

As $G/T \to G/N$ is a finite cover with fiber $W = N/T$ and $H^{\mathrm{odd}}(G/T) = 0$ by Theorem 5.1.1, it follows from Proposition B.3.1 that

$$H^{\mathrm{odd}}(G/N) \cong H^{\mathrm{odd}}(G/T)^W = 0.$$

Thus $h^\bullet(G/N) = \chi(G/N) = 1$, so it must be that $H^*(G/N) = H^0(G/N) \cong \mathbb{Q}$. □

We have the following useful corollary, which will reemerge much later in Appendix C as a consequence of the Berline–Vergne/Atiyah–Bott localization theorem.

Corollary 5.2.2 (Weil)**.** *Let G be a compact, connected Lie group, T a maximal torus in G, and W the Weyl group of G. Then*

$$\chi(G/T) = |W_G|.$$

Proof. Since $G/T \longrightarrow G/N$ is a $|W|$-sheeted covering and $\chi(G/N) = 1$ by Proposition 5.2.1, it follows from Proposition B.3.5 that

$$\chi(G/T) = \chi(G/N) \cdot |W| = |W|. \qquad \square$$

This means in a homogeneous space G/K, one can for cohomological purposes replace K with the normalizer of its maximal torus.

Corollary 5.2.3. *Let G be a compact, connected Lie group, K a closed, connected subgroup of lesser rank, S a maximal torus of K, and $N = N_K(S)$ the normalizer of this torus in K. Then the natural projection $G/N \longrightarrow G/K$ induces a ring isomorphism*

$$H^*(G/K) \xrightarrow{\sim} H^*(G/N).$$

Proof. There is a fiber bundle $K/N \to G/N \to G/K$, whose fiber K/N is acyclic by Proposition 5.2.1, so $\pi_1(G/K)$ acts trivially on $H^*(K/N) = H^0(K/N) \cong \mathbb{Q}$, and the Serre spectral sequence of this bundle collapses instantly at

$$\operatorname{gr}_\bullet H^*(G/N) = E_2 = H^*(G/K) \otimes \mathbb{Q} \cong H^*(G/K).$$

Because the bigraded algebra $H^*(G/N)$ is concentrated in the bottom row, the associated graded construction leaves it unchanged, so this is a ring isomorphism. $\square$

There is also the following further result, later generalized by Chevalley.

Corollary 5.2.4 (Leray). *The ring $H^*(G/T)$ is isomorphic to the regular representation of the Weyl group W_G.*

Proof. One characterization of the regular representation $W \rightarrowtail \operatorname{Aut}\big(\mathbb{Q}[W]\big)$ of a group W is through the character $w \longmapsto \operatorname{tr} w|_{\mathbb{Q}[W]}$ of the representation: a representation L is W-isomorphic to the regular representation just if

$$\operatorname{tr} w|_L = \begin{cases} |W| & w = 1, \\ 0 & w \neq 1. \end{cases}$$

Now consider the standard right action[1] of $W = N_G(T)$ on G/T given by $gT \cdot nT := gnT$. Since

$$gnT = gT \iff nT = g^{-1}gT = T \iff n \in T,$$

no element of W other than the identity has any fixed points. Now this action induces an representation of W in $H^*(G/T)$. For $w \neq 1$, since there are no w-fixed points, w has Lefschetz number $\chi(w) = 0$; but since $H^*(G/T)$ is evenly graded by Theorem 5.1.1, this means that $\operatorname{tr} w|_{H^*(G/T)} = 0$. On the other hand, $\chi(1) = \chi(G/T) = |W|$ by Proposition C.3.3. □

We also can show that the Euler characteristic of a generic compact homogeneous space is zero.

Corollary 5.2.5. *Let G be a compact, connected Lie group and K a closed, connected subgroup of lesser rank. Then $\chi(G/K) = 0$.*

Proof. Let S be a maximal torus of K and T be a maximal torus of G containing S. Then we have a fiber bundle $T/S \to G/S \to G/T$. Since the base is simply-connected, it follows from Proposition 4.3.12 that

$$\chi(G/S) = \chi(G/T)\chi(T/S) = \chi(G/T) \cdot 0,$$

this last since a torus T/S is a product of circles and $\chi(S^1) = 1 - 1 = 0$. Let $N = N_K(S)$ be

[1] N.B. The proof of this result in [MT00, Prop. VII.3.25, p. 399] is wrong, as it tries to use the left multiplication action.

the normalizer in K of its maximal torus S. Since $N \to S$ is a cover with fiber W_K, so also is $G/S \to G/N$, so by Proposition B.3.5,

$$\chi(G/N) = \chi(G/S)\big/|W_K| = 0.$$

Now by Corollary 5.2.3 we have $\chi(G/K) = \chi(G/N) = 0$. $\square$

Historical remarks 5.2.6. The Euler characteristic dichotomy that $\chi(G/K) > 0$ or $= 0$ depending as $\operatorname{rk} G = \operatorname{rk} K$ or $\operatorname{rk} G > \operatorname{rk} K$ is due to Hopf and Samelson [HS40, p. 248].

Chapter 6

First properties of equivariant cohomology

In this chapter, G is a connected Lie group.

6.1. Values on the orbit category and Mayer–Vietoris

We have shown the importance of the orbit category G-Orbit in understanding G-actions, so the first order of business in understanding equivariant cohomology should be to know the values it takes on this category. It will be enough to understand the cohomology ring $H^*(BK)$ of the classifying space BK, as we will come to in Section 7.6. We will primarily care about the case the coefficient ring $k = \mathbb{Q}$, but for some statements we can get away with $k = \mathbb{Z}$.

Proposition 6.1.1. *Let G be a Lie group and K a closed subgroup. Then for any coefficient ring k there is an isomorphism*

$$H^*_G(G/K;k) = H^*(BK;k).$$

Proof. The proof is topological and nearly formal. The equivariant cohomology $H^*_G(G/K)$ is the singular cohomology of the homotopy quotient $(G/K)_G = EG \times_G G/K$, which is homeomorphic to EG/K, taking $P = EG$ in the following lemma. But $EG/K = BK$ for G Lie and K closed, by Theorem B.4.3. □

Lemma 6.1.2 ([BTar, Prop. 4.5])**.** *Let G be a Lie group and K a closed subgroup, and suppose $P \longrightarrow B$ is a principal bundle. Then there is a homeomorphism*

$$P \times_G G/K \longrightarrow P/K,$$
$$[p, gK] \longmapsto pgK.$$

Proof. The continuity and well-definedness of the map follow from the fact the right action

$$P \times G \longrightarrow P,$$
$$(p, g) \longmapsto pg$$

is continuous and constant on each diagonal orbit $[p, g]_G = \{(px, x^{-1}g) : x \in G\}$. It thus descends, by definition, to a continuous map $\varphi\colon P \times_G G \longrightarrow P$, and since this map is right K-equivariant in that $\varphi[p, gk]_G = \varphi[p, g]_G \cdot k$ for $k \in K$, to a well-defined, continuous map $P \times_G G/K \longrightarrow P/K$. The map is bijective and because an inverse is given by the map

$$P/K \longrightarrow P \underset{G}{\times} G/K,$$
$$pK \longmapsto [p, K]_G.$$

It is not immediate this inverse is continuous, but would be if we could factor it through a continuous map $P/K \longrightarrow P \times \{K\}$ taking $pK \longmapsto (p, 1K)$. Though this is not generally possible, it is enough for continuity to be able to do it locally, and this is equivalent to finding local sections of $P \to P/K$. But this we can do: since P is assumed a G-bundle and G is a K-bundle by Theorem B.4.3, it follows P is an K-bundle, and so locally trivial. □

Now that we know what H^*_G does on the orbit category, one hopes to be able to use the decomposition of a compact G-space as a G–CW complex, a union of disks $D^n \times G/K$ along G-equivariant maps, for K and n varying, to understand what it does to a G-space in general, a sort of "cellular equivariant cohomology." And indeed $H^*_G(D^n \times G/H) = H^*_G(G/H) \cong H^*(BH)$, so these building blocks behave as simply as one could hope for.

This approach has mostly theoretical utility, because G–CW decompositions seem to be hard to come by in the wild, but an understanding of the cohomology of a union is still valuable.

Proposition 6.1.3 (Equivariant Mayer–Vietoris)**.** *Let U and V be G-invariant subsets of a G-space X and suppose $U \cup V = X$. Then there exists a long exact sequence*

$$\cdots \longrightarrow H^{n-1}_G(U \cap V) \longrightarrow H^n_G(X) \longrightarrow H^n_G(V) \oplus H^n_G(V) \longrightarrow H^n_G(U \cap V) \longrightarrow H^{n+1}_G(X) \longrightarrow \cdots .$$

Proof. Since U and V are G-invariant, so is $U \cap V$. The decomposition $U \cup V = X$ translates on taking homotopy quotients to a decomposition $U_G \cup V_G = X_G$ such that $U_G \cap V_G = (U \cap V)_G$, and the result follows on applying the Mayer–Vietoris sequence in singular cohomology. □

6.2. The equivariant Künneth theorem

Recall from Section 2.1 that if G acts on spaces X and Y, there is a natural diagonal product action on $X \times Y$. In ordinary cohomology, at least with coefficients in a field k, one has $H^*(X \times Y) \cong H^*X \otimes H^*Y$, and in less favorable circumstances, one still has at least a Künneth spectral sequence as mentioned in Remark A.5.5. With more severe restrictions, something similar holds in equivariant cohomology.

Theorem 6.2.1 (Equivariant Künneth theorem [KV93])**.** *Suppose a topological group G acts continuously on spaces X and Y in such a way that the fiber restriction $H^*_G(Y;k) \longrightarrow H^*(Y;k)$ arising from the*

Borel fibration is surjective, with $H^*(Y;k)$ a free k-module.[1] *Then, for the same coefficient ring k, there exists an isomorphism*

$$H^*_G(X \times Y) \cong H^*_G(X) \underset{H^*_G}{\otimes} H^*_G(Y).$$

This will follow from some diagram equivalences and the recurrent bundle lemma Theorem 4.4.1.

Lemma 6.2.2. *The following diagram commutes, where every column and row form a bundle, the bundles over BG are Borel fibrations, and the central object $(X \times Y)_G$ is the pullback $X_G \times_{BG} \times Y_G$:*

$$\begin{tikzcd}
 & Y \arrow[r, equal] \arrow[d] & Y \arrow[d] \\
X \arrow[r] \arrow[d, equal] & (X \times Y)_G \arrow[r] \arrow[d] & Y_G \arrow[d] \\
X \arrow[r] & X_G \arrow[r] & BG.
\end{tikzcd} \tag{6.1}$$

Proof. The commutativity of the entire diagram will be clear if we show that we can replace $(X \times Y)_G$ with the pullback $X_G \times_{BG} \times Y_G$ in such a way that the lower-right square becomes a pullback square. Let $\Delta EG := \{(e,e) \in EG \times EG\}$ be the diagonal. Now

$$\begin{aligned}
X_G \underset{BG}{\times} Y_G &= \{([e,x],[e',y]) \in X_G \times Y_G : \exists g \in G\ (eg = e')\} \\
&\approx \{(e,x,eg,y) \in EG \times X \times EG \times Y\} \big/ (eh,x,egh',y) \sim (e,hx,e,gh'y) \\
&\approx \{(e,e,x,y) \in \Delta EG \times X \times Y\} \big/ (eg,eg,x,y) \sim (e,e,gx,gy),
\end{aligned}$$

[1] If this fiber restriction is surjective, it is said that the G-action on Y is *equivariantly formal*; we will have much more to say about this condition in later chapters, predominantly for $k = \mathbb{Q}$.

the second homeomorphism because under the relation imposed on $EG \times X \times EG \times Y$, one can always find a representative (e, x, e, y), and restricted to such representatives, the relation imposed by h and h' is just the relation imposed on $\Delta EG \times X \times Y$ in the third term. On the other hand, since $\Delta EG \times X \times Y$ is G-equivariantly homeomorphic to $EG \times X \times Y$, we finally see $(X \times Y)_G \approx X_G \times_{BG} Y_G$. Explicitly, the homeomorphism is given by

$$[e, x, y] \longmapsto \big([e, x], [e, y]\big).$$

The maps to X_G and Y_G take these elements to $[e, x]$ and $[e, y]$ respectively, so the diagram commutes. □

Corollary 6.2.3. *Let G be a Lie group and G and K closed subgroups. Then one has a homeomorphism*

$$(G/H \times G/K)_G \approx BH \underset{BG}{\times} BK$$

Proof. Putting $X = G/H$ and $Y = G/K$ in Lemma 6.2.2, one finds $(G/H \times G/K)_G \approx (G/H)_G \underset{BG}{\times} (G/K)_G$, but from Lemma 6.1.2 one also knows $(G/H)_G \approx BH$ and $(G/K)_G \approx BK$. □

The bundle lemma Theorem 4.4.1 and Lemma 6.2.2 make the equivariant Künneth theorem immediate.

Proof of Theorem 6.2.1. Taking cohomology of diagram (6.1) yields the commutative diagram

$$\begin{tikzcd}
 & H^*(Y) \arrow[r, equal] & H^*(Y) \\
H^*(X) & H^*_G(X \times Y) \arrow[l] \arrow[u] & H^*_G(Y) \arrow[l] \arrow[u] \\
H^*(X) \arrow[u, equal] & H^*_G(X) \arrow[l] \arrow[u] & H^*_G. \arrow[l] \arrow[u]
\end{tikzcd} \tag{6.2}$$

By assumption, the top-right vertical map $H^*_G(Y) \longrightarrow H^*(Y)$ is surjective so by the bundle lemma Theorem 4.4.1, it follows $H^*_G(X \times Y) \cong H^*_G(X) \otimes H^*_G(Y)$ is surjective. □

We state here a corollary about equivariant cohomology of homogeneous spaces, once we prove another bundle equivalence.

Proposition 6.2.4. *Let G be a Lie group and H and K closed subgroups. Then one has homeomorphisms*

$$(G/H)_K \approx BK \times_{BG} BH \approx (G/K)_H.$$

Proof. By symmetry, it will be enough to prove the first homeomorphism. A candidate map is

$$\begin{aligned} \psi\colon (G/H)_K = EG \times_K G/H &\longrightarrow BK \times_{BG} BH, \\ [e, gH]_K &\longmapsto (eK, egH). \end{aligned}$$

One suspects that this map must in fact yield an isomorphism of G/H-bundles over BK, and indeed it does. To define an inverse, start with the homeomorphism

$$EG \times_{BG} EG \longrightarrow EG \times G\colon$$

$$(e, eg) \longmapsto (e, g),$$

and follow it with the projections $EG \times G \to EG \times G/H \to EG \times_K G/H$ to get

$$\phi\colon EG \times_{BG} EG \longrightarrow EG \times_K G/H = (G/H)_K,$$

$$(e, eg) \longmapsto [e, gH]_K.$$

To see ϕ descends to a well defined inverse $BK \times_{BG} BH \longrightarrow (G/H)_K$, let a point $(eK, egH) \in BK \times_{BG} BH$ be given and consider two points in its preimage $EG \times_{BG} EG$. One natural choice is (e, eg), and the others are all (ek, egh) for some $k \in K$ and $h \in H$. We can rewrite (ek, egh) as $\big(ek, (ek)(k^{-1}gh)\big)$. Now

$$\phi\big(ek, (ek)(k^{-1}g)h\big) = [ek, k^{-1}ghH]_K = [e, gH]_K = \phi(e, eg),$$

so ϕ descends to a continuous map $BK \times_{BG} BH \longrightarrow (G/H)_K$ inverse to ψ. Thus the spaces are homeomorphic (G/H)-bundles over BK. □

Remark 6.2.5. For the sake of completeness, we note the composite homeomorphism $(G/H)_K \xrightarrow{\approx} (G/K)_H$ is given by

$$[e, gH]_K \longmapsto (eK, egH) \longmapsto [eg, g^{-1}K]_H.$$

This map does not preserve any particular bundle structure, but does yield an isomorphism $H^*_K(G/H) \cong H^*_H(G/K)$ for any coefficient ring.

Corollary 6.2.6 ([KV93, Prop. 68, p. 161])**.** *Let G be a compact Lie group, K a subgroup of full rank, and H another closed subgroup. Then*

$$H^*_H(G/K) \cong H^*_K(G/H) \cong H^*_H \underset{H^*_G}{\otimes} H^*_K$$

as rings.

Proof. Take $X = G/H$ and $Y = G/K$ in the diagram (6.2). The diagram becomes

$$\begin{tikzcd}
 & H^*(G/K) \arrow[r, equal] & H^*(G/K) \\
H^*(G/H) \arrow[d, equal] & H^*_G\big(G/H \times G/K\big) \arrow[l] \arrow[u] & H^*_K \arrow[l] \arrow[u] \\
H^*(G/H) & H^*_H \arrow[l] \arrow[u] & H^*_G. \arrow[l] \arrow[u]
\end{tikzcd}$$

From Corollary 6.2.3 and Proposition 6.2.4, we know

$$(G/K)_H \approx (G/H)_K \approx BH \underset{BG}{\otimes} BK \approx (G/H \times G/K)_G.$$

Thus the first two rings in the statement of this corollary are always isomorphic. We will prove in Section 8.3.2 that the map $H^*_K \longrightarrow H^*(G/K)$ is surjective, so Theorem 6.2.1 will apply. □

Corollary 6.2.7. *Let G be a compact Lie group and K a connected, closed subgroup of equal rank. Then*

$$H^*_K(G/K) \cong H^*_K \underset{H^*_G}{\otimes} H^*_K.$$

Corollary 6.2.8 (Leray, 1950)**.** *Let G be a compact Lie group and K a closed, connected subgroup of equal* *rank. Then*

$$H^*(G/K;\mathbb{Q}) \cong \mathbb{Q} \underset{H_G^*}{\otimes} H_K^*$$

Proof. Take $H = 1$ in Corollary 6.2.6. □

Remarks 6.2.9. We seem to have obtained Corollary 6.2.8 without any real effort, but this ease is illusory: it depends on knowing $H_K^* \longrightarrow H(G/K)$ is surjective, an important fact which we will only demonstrate later as Theorem 8.3.11, this time relying directly on Theorem 5.1.1, the fact that the cohomology of a flag manifold G/T is concentrated in even dimensions.

For a more explicit presentation of the computation Corollary 6.2.7 in terms of characteristic classes—which initially inspired the author's rediscovery of the work in this section—see Tu [Tu10]. Subsequent to reading this Tu paper, the author rediscovered Proposition 6.2.4 and Corollary 6.2.6 independently of outside enlightenment, without understanding the equivariant Künneth theorem applied and before realizing the seeming ubiquity of the bundle lemma Theorem 4.4.1. Related statements appear throughout the literature; see for example Prop. 3.9 in Goertsches and Töben [GT10a] and Kumar and Vergne [KV93, Prop. 68, p. 161]. Kumar and Vergne prove one of the isomorphisms in Corollary 6.2.6, using the Cartan model absent bundle-theoretic considerations; it may simply be that they do not care about the other isomorphism.

Wilhelm Singhof [Sin93, Proposition (2.3)] also comes very close to stating Proposition 6.2.4, in a rather different context. Singhof's work was on biquotients: if G is a group with subgroups H and K, then the product group $K \times H$ acts on G by

$$(k,h) \cdot g := hgk^{-1},$$

and the ***biquotient*** $K\backslash G/H$ is the quotient of G by this action. If G is a Lie group and the action of $H \times K$ is free, the biquotient $K\backslash G/H$ is a smooth manifold. The class of diffeomorphism classes of such biquotients is strictly larger than that of homogeneous spaces; for example, the exotic 7-spheres can be shown to arise in this fashion.

In our case, G will be compact. When the action of $H \times K$ is free, $K\backslash G/H$ is homotopy equivalent to our $(G/K)_H \approx (G/H)_K \approx BK \times_{BG} BH$ by Proposition 4.2.4, and in that event $H^*(K\backslash G/H) \cong H^*_K \otimes_{H^*_G} H^*_H$ by our result. Singhof showed that if $K \times H$ acts freely on G, then we have the this isomorphism even in the more general case $\operatorname{rk} K + \operatorname{rk} H = \operatorname{rk} G$, using an induction argument on tori to limit the possibly nonzero regions of the Eilenberg–Moore spectral sequence associated to the homotopy pullback square

$$\begin{tikzcd}
K\backslash G/H \arrow[r] \arrow[d] & BH \arrow[d] \\
BK \arrow[r] & BG;
\end{tikzcd}$$

this spectral sequence is known by the work of many (Singhof cites Hans Munkholm [Mun74]) to in this case collapse at $E_2 = \operatorname{Tor}^{\bullet,\bullet}_{H^*_G}(H^*_K, H^*_H)$, and in our case this is an algebra isomorphism. The calculation here that $H^*_K(G/H) \cong H^*_K \otimes_{H^*_G} H^*_H$, on the other hand, holds regardless of how big H is, and irrespective of whether the action is free.

As Singhof only uses the homotopy commutativity of the pullback, it follows that his proof applies equally well to the calculation of $H^*_K(G/H)$ when $\operatorname{rk} K + \operatorname{rk} H = \operatorname{rk} G$ and the subgroup $K \cdot H$ of G contains a maximal torus of G, independently of whether or not $K \times H$ acts freely on G. The Eilenberg–Moore spectral sequence is really the proper tool for talking about cohomology of pullbacks, but we have resisted including this generalization in this chapter because it seemed excessive to also have to include an exposition of the Eilenberg–Moore spectral sequence at this date.

6.3. The standard example

In this section, we compute the equivariant cohomology of the standard motivating action and show it to be equivariantly formal. This result follows more quickly from the equivariant localization theorem Corollary 9.2.5, but this however requires an account of maps $H^*_T \longrightarrow H^*_S$ between cohomology rings of classifying spaces which we only develop in Section 7.8.1.

Write ρ_1 for the rotation action $S^1 \curvearrowright S^2$ on the standard unit sphere fixing the z-axis, and ρ_q for the action "sped up by a factor of q," so that $e^{i\theta} \in S^1$ acts as rotation by $q\theta$ radians. We write the ρ_q-equivariant cohomology of S^2 as $H^*_{\rho_q}(S^2)$.

Proposition 6.3.1. *One has*

$$H^*_{\rho_1}(S^2) \cong \mathbb{Z}[x,y]/(xy), \quad \deg x = \deg y = 2.$$

Proof. The orbits of each ρ_q are the latitudes C_z for fixed z, which are circles for $z \in (-1,1)$ and the poles for $z = \pm 1$. Thus discs $U = z^{-1}[-1, 1/2)$ and $V = z^{-1}(-1/2, 1]$ are invariant, as is their intersection $U \cap V$ which is the annulus $z^{-1}(-1/2, 1/2)$. Because the deformation retractions of U to the south pole $z = -1$ and V to the north pole $z = 1$ are ρ_q-equivariant, it follows $H^*_{\rho_q}(U) \cong H^*_{\rho_q}(V) \cong H^*_{\rho_q}(\mathrm{pt})$. Similarly, since $U \cap V$ deformation retracts to the equatorial circle S^1_{eq}, we have $H^*_{\rho_q}(U \cap V) \cong H^*_{\rho_q}(S^1_{\mathrm{eq}})$. It follows we should try to understand the homotopy quotients $(S^1_{\mathrm{eq}})_{\rho_q}$ and pt_{ρ_q}.

Identifying S^1_{eq} with the complex unit circle S^1,

$$(S^1_{\mathrm{eq}})_{\rho_q} \approx {S^\infty \times S^1_{\mathrm{eq}}} \Big/ {(e\zeta, s) \sim (e, \zeta^q s)}, \qquad e \in S^\infty,\ \zeta \in S^1,\ s \in S^1_{\mathrm{eq}}.$$

The identification relation $\sim$ on $S^\infty \times S^1$ yields $(e,s) \sim (es^{1/q}, 1)$ for each q^{th} root of s. Because these q^{th} roots of s differ multiplicatively by elements of the group μ_q of q^{th} roots of unity, $(X_q)_z$ is homeomorphic to the quotient space S^∞/μ_q, which is is an ***infinite lens space*** L_q. Note that L_2 is the real projective space $\mathbb{R}P^\infty$, while L_1 is just S^∞ again.

For $z \in \{\pm 1\}$, on the other hand, the orbit C_z is a pole S^1 acts on trivially, and

$$\mathrm{pt}_{\rho_q} = {S^\infty \times \mathrm{pt}}\Big/{(e\zeta, \mathrm{pt}) \sim (e, \mathrm{pt})} \approx S^\infty/S^1 = \mathbb{C}P^\infty.$$

So X_q can be seen as the union

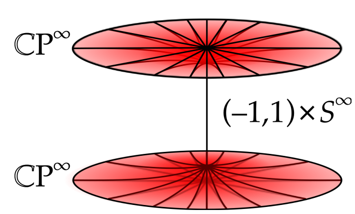
$$\big(\mathbb{C}P^\infty \times \{-1\}\big) \cup \big(L_q \times [-1,1]\big) \cup \big(\mathbb{C}P^\infty \times \{1\}\big)$$

obtained from the disjoint union of these parts by gluing $L_q \times \{1\}$ to $\mathbb{C}P^\infty \times \{1\}$ and $L_q \times \{-1\}$ to $\mathbb{C}P^\infty \times \{-1\}$ along the map

$$\alpha_q\colon L_q \longrightarrow \mathbb{C}P^\infty,$$
$$e\mu_q \longmapsto eS^1.$$

The $q = 1$ case of our adjunction space construction shows that X_1 is obtained from $S^\infty \times [-1,1]$ by gluing on two copies of $\mathbb{C}P^\infty$ to the ends with the canonical projection $S^\infty \longrightarrow \mathbb{C}P^\infty$ as attaching map. Here is a picture of X_1; the middle part is drawn thin because S^∞ is contractible.

Figure 6.3.2: A schematic of $(S^2)_{\rho_1}$



Since S^∞ is contractible, as we demonstrated in Proposition 3.2.2, the attaching map $S^\infty \longrightarrow$

$\mathbb{C}\mathrm{P}^\infty$ is nullhomotopic, so we may homotope X_1 inside itself onto the subspace

$$\mathbb{C}\mathrm{P}^\infty \cup (-1,1) \cup \mathbb{C}\mathrm{P}^\infty.$$

Since $(-1,1)$ is contractible, it follows that X is homotopy equivalent to a wedge $\mathbb{C}\mathrm{P}^\infty \vee \mathbb{C}\mathrm{P}^\infty$. From the fact that $\mathbb{C}\mathrm{P}^\infty = BS^1$, we will show in Section 7.3 as an elementary application of the Serre spectral sequence that $H^*(\mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}[x]$, where $\deg x = 2$. It then follows from the wedge axiom[2] that

$$H^*_{\rho_1}(S^2) = H^*(X_1) \cong H^*(\mathbb{C}\mathrm{P}^\infty \vee \mathbb{C}\mathrm{P}^\infty) = \frac{\mathbb{Z}[x] \times \mathbb{Z}[y]}{(1,0) \sim (0,1)} \cong \mathbb{Z}[x,y]/(xy),$$

where $x, y \in H^2$ are the first Chern classes of the component $\mathbb{C}\mathrm{P}^\infty$'s.

But we promised to see this with the Mayer–Vietoris sequence. Note that $\widetilde{H}^*(S^\infty) = 0$ since S^∞ is contractible. For $n = 0$, since everything is connected, the Mayer–Vietoris sequence yields the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\Delta} \mathbb{Z}^2 \xrightarrow{-} \mathbb{Z} \to 0$, and for $n \geqslant 1$, it gives the fragment

$$0 \to H^n_{\rho_1}(S^2) \longrightarrow H^n(\mathbb{C}\mathrm{P}^\infty) \oplus H^n(\mathbb{C}\mathrm{P}^\infty) \to 0.$$

Thus the map $H^*_{\rho_1}(S^2) \longrightarrow H^*(\mathbb{C}\mathrm{P}^\infty) \times H^*(\mathbb{C}\mathrm{P}^\infty)$ is injective. Because the factor maps $H^*_{\rho_1}(S^2) \longrightarrow H^*(\mathbb{C}\mathrm{P}^\infty)$ are ring homomorphisms induced by the inclusions $U, V \hookrightarrow S^2$, it follows $H^n_{\rho_1}(S^2) \longrightarrow H^*(\mathbb{C}\mathrm{P}^\infty) \times H^*(\mathbb{C}\mathrm{P}^\infty) = \mathbb{Z}[x] \times \mathbb{Z}[y]$ is an injective ring homomorphism, surjective in degrees $\geqslant 0$. In particular, its image contains $(1,1), (x,0), (0,y)$, and a dimension argument shows these are algebra generators for the image. It is not hard to see this ring is isomorphic to $\mathbb{Z}[x,y]/(xy)$. $\square$

[2] The wedge axiom on a cohomology theory, due to Milnor, is that $\widetilde{H}^*(\bigvee_\alpha X_\alpha) \cong \prod_\alpha \widetilde{H}^*(X_\alpha)$. This is a *theorem* under our axioms, but is equivalent to the Eilenberg–Steenrod disjoint union axiom granted the other axioms, and is sometimes substituted for it.

Corollary 6.3.3. *One has*

$$\begin{aligned} H^*_{\rho_q}(S^2) &\cong \mathbb{Z}[s_1, s_2, t]\big/(s_1 s_2, s_1 + s_2 - qt), \\ &\cong \mathbb{Z}[s, t]\big/(s^2 - qst), \qquad\qquad |s_j| = |s| = |t| = 2. \end{aligned}$$

Proof. To determine $H^*_{\rho_q}(S^2)$ for $|q| > 1$ we make use of the map $X_1 \twoheadrightarrow X_q$ which is the identity on the end caps $\mathbb{C}\mathrm{P}^\infty$ and is the quotient map $S^\infty \longrightarrow L_q$ on the slices $z = \text{const}$ arising from latitudes of S^2. Now consider the Mayer–Vietoris sequences of the open covers of X_1 and of X_q given by $z^{-1}[-1, 1/2)$ and $z^{-1}(-1/2, 1]$. These open sets deformation retract respectively onto the end caps $z^{-1}\{-1\}$ and $z^{-1}\{1\}$, which are homeomorphic to $\mathbb{C}\mathrm{P}^\infty$ for both X_1 and X_q. The intersection of these open sets is $z^{-1}(-1/2, 1/2)$, which deformation retracts onto the slice $z^{-1}\{0\}$. This slice is an S^∞ for X_1 and is L_q for X_q. Thus the map of Mayer–Vietoris sequences corresponding to these covers and the map $X_1 \longrightarrow X_1$ is as follows:

$$\begin{tikzcd} H^{2k-1}(S^\infty) \arrow[r] & H^{2k}(X_1) \arrow[r] & H^{2k}(\mathbb{C}\mathrm{P}^\infty)^{\oplus 2} \arrow[r] & H^{2k}(S^\infty) & & \\ H^{2k-1}(L_q) \arrow[r] \arrow[u] & H^{2k}(X_q) \arrow[r] \arrow[u] & H^{2k}(\mathbb{C}\mathrm{P}^\infty)^{\oplus 2} \arrow[r] \arrow[u, equal] & H^{2k}(L_q) \arrow[r] \arrow[u] & H^{2k+1}(X_q) \arrow[r] & H^{2k+1}(\mathbb{C}\mathrm{P}^\infty)^{\oplus 2}. \end{tikzcd}$$

We have already calculated that $H^*(X_1) = \mathbb{Z}[x, y]/(xy)$, and it follows from the Serre spectral sequence of the circle bundle $L_q \twoheadrightarrow \mathbb{C}\mathrm{P}^\infty$ that $H^*(L_q) \cong \mathbb{Z}[u]/(qu)$, where u is the pullback of a generator $x \in H^2(\mathbb{C}\mathrm{P}^\infty)$. Filling in these known values yields the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}\{x^k, y^k\} \arrow[r, equal] & \mathbb{Z}\{x^k, y^k\} \arrow[r] & 0 & & \\ 0 \arrow[r] & H^{2k}(X_q) \arrow[r] \arrow[u] & \mathbb{Z}\{x^k, y^k\} \arrow[r, "f"] \arrow[u, equal] & (\mathbb{Z}/q)u^k \arrow[r] & H^{2k+1}(X_q) \arrow[r] & 0. \end{tikzcd}$$

The map marked f is the difference of the two maps $H^{2k}(\mathbb{C}\mathrm{P}^\infty) \longrightarrow H^{2k}(L_q)$ induced by including $z^{-1}(-1/2, 1/2)$ into $z^{-1}[-1, 1/2)$ and into $z^{-1}(-1/2, 1]$. Since these inclusions are both homotopic to the attaching map $L_q \to \mathbb{C}\mathrm{P}^\infty$ adjoining an endcap $\mathbb{C}\mathrm{P}^\infty$ to X_q, it follows that $f(x^k) = f(-y^k) = u^k$, so $H^{2k}(X_q) = \ker f$ is a free $\mathbb{Z}$-module generated by qx^k, qy^k, and $x^k + y^k$. (One of the generators qx^k and qy^k is redundant, but there is no symmetric way to select just one.)

Since f is surjective, it follows from exactness of the bottom row that $H^{2k+1}(X_q) = 0$, so $H^*(X_q)$ is concentrated in even degree and the ring map $H^*(X_q) \longrightarrow H^*(X_1) = \mathbb{Z}[x,y]/(xy)$ induced by $X_1 \longrightarrow X_q$ is injective. Since $xy = 0$ in $H^*(X_1)$, we have the relations

$$\begin{aligned} (x+y)^j &= x^j + y^j, \\ qx(x+y)^{j-1} &= qx^j, \\ qy(x+y)^{j-1} &= qy^j, \end{aligned}$$

so that the image of $H^*(X_q)$ in $H^*(X_1)$ is the subring generated by qx, qy, $x+y$, or, since the relation $q(x+y) - qx = qy$ holds in $\mathbb{Z}[x,y]/(xy)$, by just $x+y$ and qx. Since $xy = 0$, it is clear that within the image of $H^*(X_q)$, the following relations hold:

$$\begin{aligned} qx \cdot qy &= 0, \\ qx + qy &= q(x+y), \\ (qx)^2 &= q \cdot (qx) \cdot (x+y), \end{aligned}$$

Because each $H^{2k}(X_q)$ is free of rank two, a dimension argument shows there are no other relations. Writing s_1, s_2 for the respective preimages of qx, qy and t for the preimage of $x+y$, the result follows. □

We note in passing that $q = 0$ yields $H^*_{\rho_0}(S^2) = H^*(BS^1 \times S^2) \cong \mathbb{Z}[u] \otimes H^*(S^2)$. We claim that the fiber restriction maps $H^*_{\rho_q}(S^2) \longrightarrow H^*(S^2)$ are surjective for all q.

Proposition 6.3.4. *The actions ρ_q of S^1 on S^2 are equivariantly formal. The fiber projection $H^*_{\rho_q}(S^2) \longrightarrow H^*(S^2)$ can be represented as*

$$\begin{aligned} \mathbb{Z}[s_1, s_2, t]/(s_1 s_2, s_1 + s_2 - qt) &\longrightarrow \Delta[s] \quad (\deg s = 2): \\ t &\longmapsto 0, \\ s_1\, s_2 &\longmapsto s. \end{aligned}$$

First proof. As $S^2 \to X_q \to \mathbb{C}\mathrm{P}^\infty$ is a fiber bundle over a connected space and the cohomology rings of the fiber and base are concentrated in even degree, by Corollary 4.3.11 the Serre spectral sequence collapses at $E_2 \cong H^*(\mathbb{C}\mathrm{P}^\infty) \otimes H^*(S^2)$, so that in particular the map $E_\infty \longrightarrow H^*(S^2)$ is surjective; but this is just the map $H^*(X_q) \longrightarrow H^*(S^2)$ induced by the fiber inclusion, so the action ρ_q is equivariantly formal.

Because $\operatorname{rk} H^2(X_q) = 2$, it is spanned by two generators, one of which maps to the fundamental class in $H^2(S^2)$ and the other of which generates the image of $H^2(\mathbb{C}\mathrm{P}^\infty) \longrightarrow H^2(X_q)$. Because the diagonal S^1-action on $S^\infty \times S^2$ induced by ρ_q is free on the end-cap $S^\infty \times \{z = 1\}$, the projection map $X_q \longrightarrow \mathbb{C}\mathrm{P}^\infty = BS^1$ of the Borel fibration can be seen as a retraction to each $\mathbb{C}\mathrm{P}^\infty$ end-cap. The injection $H^2(X_q) \longrightarrow H^2(\mathbb{C}\mathrm{P}^\infty)^{\oplus 2}$ in the Mayer–Vietoris sequence above is induced by the inclusions of the open sets in the cover, or equivalently by the inclusions of the end-caps. The commutative diagram

$$\begin{tikzcd} & X_q \arrow[dr] & \\ \mathbb{C}\mathrm{P}^\infty \amalg \mathbb{C}\mathrm{P}^\infty \arrow[ur, hook] \arrow[rr] & & \mathbb{C}\mathrm{P}^\infty, \end{tikzcd}$$

where the horizontal map identifies the two copies of $\mathbb{C}\mathrm{P}^\infty$, induces

$$\begin{tikzcd}
 & H^2(X_q) \arrow[dl, "\sim"'] & \\
\mathbb{Z}\{x,y\} = H^2(\mathbb{C}\mathrm{P}^\infty)^2 & & H^2(\mathbb{C}\mathrm{P}^\infty) = \mathbb{Z}\{u\}, \arrow[ul] \arrow[ll, "x+y\ \longleftarrow\ u"]
\end{tikzcd}$$

so the image of $H^*(\mathbb{C}\mathrm{P}^\infty) \longrightarrow H^*(X_q)$ is the subring $\mathbb{Z}[x+y] = \mathbb{Z}[t]$, and the map $H^*(X_q) \longrightarrow H^*(S^2)$ induced by the fiber inclusion $S^2 \hookrightarrow X_q$ must take $t \longmapsto 0$ and $s_1, s_2 \longmapsto [S^2]$. $\square$

Second, geometric proof for $q = 1$. Because each object in the bundle $S^2 \to X_q \to \mathbb{C}\mathrm{P}^\infty$ is simply connected, the Hurewicz theorem gives natural isomorphisms $\pi_2 \xrightarrow{\sim} H_2$ in this sequence, and because H^2 is free and $H_1 = 0$, the sequence

$$H^2(S^2) \longleftarrow H^2(X_q) \longleftarrow H^2(\mathbb{C}\mathrm{P}^\infty)$$

is dual to the sequence

$$\pi_2(S^2) \longrightarrow \pi_2(X_q) \longrightarrow \pi_2(\mathbb{C}\mathrm{P}^\infty).[3]$$

Since $H^2(S^2) \longleftarrow H^2(X_q)$ is dual to $\pi_2(S^2) \longrightarrow \pi_2(X_q)$, it will be enough to understand the homotopy class of a fiber inclusion $\sigma\colon S^2 \longrightarrow X_1$, which factors as

$$S^2 \xrightarrow{\approx} \{e_0\} \times S^2 \hookrightarrow S^\infty \times S^2 \xrightarrow[/S^1]{} X_1.$$

[3] In fact, although we do not need this fact, this last sequence is short exact. For of course $\pi_1(S^2) = 0$. Now $\mathbb{C}\mathrm{P}^\infty$ admits a decomposition as a union of one cell in each even dimension. Its 2-skeleton is S^2, and the attaching map of the 4-cell e^4 is the Hopf map $\partial e^4 = S^3 \longrightarrow S^2$. Any map $S^3 \longrightarrow \mathbb{C}\mathrm{P}^\infty$ can be homotoped by cellularity into the 2-skeleton S^2. Since $\pi_3(S^2)$ is generated by the Hopf map, which is coned off by e^4, it follows $\pi_3(\mathbb{C}\mathrm{P}^\infty) = 0$ as well.

Pick a point $e_0 \in S^1 \subsetneq S^3 \subsetneq S^\infty$; for example, we can set $e_0 = 1 \in \mathbb{C} \subsetneq \mathbb{C}^\infty$. Then

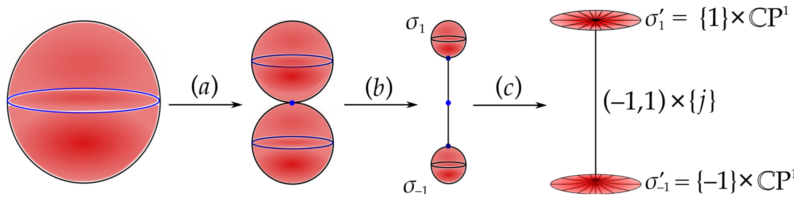
$$\operatorname{im}\sigma \;=\; S^1 \times S^2 \big/ S^1 \;\subsetneq\; S^3 \times S^2 \big/ S^1 \;\subsetneq\; X_1.$$

Over $z = \pm 1$, then, σ_1 is just the inclusion of the poles of S^2 into $\mathbb{C}\mathrm{P}^\infty$ as the class $[1, \vec{0}]$. The other latitudes $\sigma(C_z)$ are taken, under the homeomorphism $(S^\infty \times C_z)/\!\sim\; \longrightarrow S^\infty$ given by $[e, s] \longmapsto es$, to

$$\sigma(C_z) \approx S^1 \subsetneq S^3 \subsetneq S^\infty.$$

We can conceive this S^1 as the unit circle in the first $\mathbb{C}$ factor in $\mathbb{C}^\infty$ and S^3 as the unit sphere in the first two factors $\mathbb{C}^2$. It is harmless to think of the open subset $z^{-1}(-1/2, 1/2)$ as $(-1/2, 1/2) \times S^\infty$. Because $\pi_1(S^3) = 0$, the restriction of σ to a loop $S^1 \approx C_0 \longrightarrow z^{-1}\{0\} \approx S^\infty$ is nullhomotopic in $S^3 \subsetneq S^\infty$. Using such a nullhomotopy in S^3, we can pinch the image under σ of the equator; this is homotopy (a) of Figure 6.3.5. Then we can expand this "contracted segment" in such a way that $\sigma(C_z)$ is a circle for $z \notin [-1/2, 1/2]$, but the image $\sigma(C_z)$ is the point $j = (0, 1) \in S^3$ for $z \in [-1/2, 1/2]$; this is homotopy (b) of Figure 6.3.5. Thus the image of σ is now two spheres σ_{-1} and σ_1 connected by a path.

Figure 6.3.5: Homotoping a representative of $\pi_2 X_1$ into the end-caps $\{\pm 1\} \times \mathbb{C}\mathrm{P}^\infty$



Now that we have done so, we can continuously deform σ to a singular sphere σ' such that the interval of z-values such that $\sigma'(C_z) \approx \mathrm{pt}$ is all of $(-1, 1)$, completing the homotopy (c) in Figure 6.3.5. The image of σ' will be the union of the interval $(-1, 1) \times \{j\}$ and of two spheres, σ'_{-1} and σ'_1 comprising embedded $\mathbb{C}\mathrm{P}^1$'s in the end-caps $\mathbb{C}\mathrm{P}^\infty$.

To see that this singular sphere σ' is what we want, view $\sigma(C_z)$, for each $-1 < z < -1/2$, as a subset of S^3. The process of crushing σ_1 down to $\sigma_1' \approx \mathbb{C}\mathrm{P}^1$, for each fixed z-coordinate circle $\sigma(C_z)$, viewed as a subset of S^3, is just the canonical projection

$$S^3 \twoheadrightarrow \mathbb{C}\mathrm{P}^1,$$
$$(\zeta_1, \zeta_2) \longmapsto [\zeta_1, \zeta_2] = (\zeta_1, \zeta_2) \cdot S^1.$$

The point $\sigma(C_{-1})$ is the class $[1,0] \in \mathbb{C}\mathrm{P}^2 \subsetneq \mathbb{C}\mathrm{P}^\infty$, and the point $\sigma(C_{-1/2})$ is the quaternion $j = (0,1) \in S^3$, so $\sigma'(C_{-1}) = [1,0]$ and $\sigma'(C_{-1/2}) = [0,1]$. The images $\sigma'(C_z) \subsetneq \mathbb{C}\mathrm{P}^1$ for $-1 < z < -1/2$ are circles interpolating between these two poles, so σ'_{-1} ultimately can be seen as a degree-1 map $S^2 \longrightarrow S^2 = \mathbb{C}\mathrm{P}^1 \subsetneq \mathbb{C}\mathrm{P}^\infty$. Similarly, one has $\sigma'(C_{1/2}) = [0,1]$ and $\sigma'(C_1) = [1,0]$, so σ_1' is a degree-(-1) map $S^2 \longrightarrow S^2$.

The Mayer–Vietoris sequence yields an isomorphism $\pi_2(\mathbb{C}\mathrm{P}^\infty)^{\oplus 2} \xrightarrow{\sim} H_2(\mathbb{C}\mathrm{P}^\infty)^{\oplus 2} \xrightarrow{\sim} H_2(X_1)$. If we write $x_*, y_* \in \pi_2(\mathbb{C}\mathrm{P}^\infty)^{\oplus 2}$ for the dual generators to the generators x, y in $H^2(X_1)$, and $v_* \in \pi_2(S^2)$ for the dual to a generator $v \in H^2(S^2)$, then we have just shown $\sigma_* v_* = x_* - y_* \in \pi_2(X_1)$. Thus

$$\sigma^* x v_* = x \sigma_* v_* = 1,$$
$$\sigma^* y v_* = y \sigma_* v_* = -1,$$

so $\sigma^* x = v$ and $\sigma^* y = -v$. $\square$

6.4. Weyl-invariants and the restricted action a maximal torus

In Appendix B.4, we pointed that the maximal torus of a compact, connected Lie group and its Weyl group carry much of its algebraic structure. In this section, we show something similar the same holds for the equivariant cohomology of an action of a compact, connected Lie group K and the restricted action by that group's maximal torus S. To do so, we use two results from Chapter 7. One is Theorem 5.1.1, and the other is Section 7.4, which we will prove later. Later, in Section 10.1, we will use the characterization in this section to obtain Theorem 10.1.4, one of this dissertation's major original results.

To start, we state a natural enhancement of the motivating observation Proposition 4.2.4 about free homotopy quotients.

Lemma 6.4.1. *Let K be a group, S a subgroup, and X and Y free K-spaces admitting a K-equivariant map $X \longrightarrow Y$. Then these diagrams commute:*

$$\begin{tikzcd} X_S \arrow[r] \arrow[d, "\wr"] & X_K \arrow[d, "\wr"] \\ X/S \arrow[r] & X/K, \end{tikzcd} \qquad \begin{tikzcd} X_K \arrow[r, "\simeq"] \arrow[d] & X/K \arrow[d] \\ Y_K \arrow[r, "\simeq"] & Y/K; \end{tikzcd}$$

so up to homotopy, $X_K \longrightarrow Y_K$ is equivalent to $X/K \longrightarrow Y/K$ and $X_S \longrightarrow X_K$ to $X/S \longrightarrow X/K$.

In this statement, the horizontal maps in the first square are the "further quotient" maps $[x,e]_S \longmapsto [x,e]_K \colon \frac{X\times EK}{S} \twoheadrightarrow \frac{X\times EK}{K}$ and $xS \longmapsto xK \colon X/S \twoheadrightarrow X/K$.

Definition 6.4.2. In the rest of this section, we let K be a compact, connected Lie group, S a maximal torus, $N = N_K(S)$ the normalizer of S in K, and $W = N/S$ the Weyl group of K.

Write K-Top for the category of topological spaces with continuous K-actions and K-equivariant continuous maps, K-Free for the full subcategory of free K-actions, $\mathbb{Q}$-CGA for the category of (homomorphisms between) graded commutative $\mathbb{Q}$-algebras, and H_S^*-CGA for subcategory of graded commutative H_S^*-algebras. Recall that the freeing functor $X \mapsto X \times EK$ of Section 4.2.3 taking a right K-action to the diagonal action is a functor K-Top $\to$ K-Free.

Observation 6.4.3. Suppose K acts on the right on a space X. Then W acts on the right on the orbit space X/S by $xS \cdot nS = xnS$, and so on the cohomology $H^*(X/S)$. Given a K-equivariant map $X \longrightarrow Y$, the induced map $X/S \longrightarrow Y/S$ is W-equivariant, so the map $H^*(X/S) \longleftarrow H^*(Y/S)$ is as well.

Lemma 6.4.4. *Suppose a finite group W acts on spaces X and Y and there is a W-equivariant continuous map $X \longrightarrow Y$ inducing a surjection $H^*(X) \underset{\varphi}{\twoheadleftarrow} H^*(Y)$. Then the map $H^*(X)^W \longleftarrow H^*(Y)^W$ is also surjective.*

Proof. The restriction to elements $b \in H^*(Y)^W$ has image in $H^*(X)^W$ by W-equivariance: if $w \cdot b = b$ for all $w \in W$, then $w \cdot \varphi(b) = \varphi(w \cdot b) = \varphi(b)$ is invariant as well.

To see the restriction is surjective, let $a \in H^*(X)^W$. Then it has a preimage $b \in H^*(Y)$, not a priori W-invariant. However, the W-average $\bar{b} = \frac{1}{|W|} \sum_{w \in W} w \cdot b$ certainly is, and by equivariance, $\varphi(\bar{b}) = \bar{a}$. Since a was assumed invariant, this average is just a again. □

Lemma 6.4.5 (Leray, 1950). *There is a natural isomorphism*

$$H^*(X/K) \xrightarrow{\sim} H^*(X/S)^W$$

of functors $(K\text{-Free})^{\mathrm{op}} \longrightarrow \mathbb{Q}\text{-CGA}$.

Proof. The quotient map $X/S \longrightarrow X/K$ factors as

$$X/S \longrightarrow X/N \longrightarrow X/K.$$

The factor $X/S \longrightarrow X/N$ is a regular covering with fiber W, which induces by Proposition B.3.1 an isomorphism $H^*(X/N) \xrightarrow{\sim} H^*(X/S)^W$. The fiber of the factor $X/N \to X/K$ is K/N, and $H^*(K/N) \cong H^*(K/S)$ by Corollary 5.2.3.

Naturality follows because the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X/S \arrow[r] \arrow[d] & X/N \arrow[r] \arrow[d] & X/K \arrow[d] \\ Y \arrow[r] & Y/S \arrow[r] & Y/N \arrow[r] & Y/K \end{tikzcd}$$

commutes and because, by Observation 6.4.3, the map $X/S \longrightarrow Y/S$ is W-equivariant. □

This lemma makes available a natural phrasing of an important, well-known result [Hsi75, Prop. III.1, p. 31].

Corollary 6.4.6. *Let K be a compact, connected Lie group with maximal torus S. Then there are natural isomorphisms of functors $(K\text{-}\mathsf{Top})^{\mathrm{op}} \longrightarrow H_K^*\text{-}\mathsf{CGA}$ and $(K\text{-}\mathsf{Top})^{\mathrm{op}} \longrightarrow H_S^*\text{-}\mathsf{CGA}$ respectively taking*

$$H_K^*(X) \xrightarrow{\sim} H_S^*(X)^W,$$

$$H_S^* \underset{H_K^*}{\otimes} H_K^*(X) \xrightarrow{\sim} H_S^*(X).$$

Proof. Note that X_S is the composition of the freeing functor $X \mapsto EK \times X$ of Definition 6.4.2 and the orbit-space functor $Z \mapsto Z/S$. Since the diagonal action on $EK \times X$ is free, the first isomorphism follows by Lemma 6.4.4.

Also note that $\xi_0\colon BS \longrightarrow BK$ is a (K/S)-bundle. Because $H^*(K/S)$ is evenly-graded by Theorem 5.1.1 and H_S^* is evenly-graded by Section 7.4, the E_2 page of the spectral sequence associated to ξ_0 is concentrated in even rows and columns, meaning it collapses by Corollary 4.3.11 and so the fiber inclusion $K/S \hookrightarrow BS$ is surjective on cohomology by Corollary 4.3.9.

Recall from the beginning of Section 4.4 the category $F\text{-}\mathsf{Bun}/\xi_0$ of bundles over ξ_0. The construction $(-)_{S\hookrightarrow K}\colon X \longmapsto (X_S \to X_K)$ is a functor $K\text{-}\mathsf{Top} \longrightarrow F\text{-}\mathsf{Bun}/\xi_0$; the bundle map $X_S \longrightarrow BS$ comes from the Borel fibration. Now the second isomorphism follows by Theorem 4.4.1. □

Corollary 6.4.7. *Let K be a compact, connected Lie group with maximal torus S and Weyl group W. Then $H^*(BK) \cong H^*(BS)^W$.*

Proof. Take $X = \mathrm{pt}$ in Corollary 6.4.6. □

Remarks 6.4.8. (*a*) The results Lemma 6.4.5 and Corollary 6.4.6 are classical and very well known, except that the naturality of these isomorphisms is never stated. This naturality is the key feature that allowed us to use them to discover our original proof of Theorem 10.1.4, one of the main results of this work; knowing only that these isomorphisms exist abstractly without understanding the relation between them does not suffice to prove the theorem.

(*b*) Lemma 6.4.5 can fail if there exist elements of $H^*(X/S;k)$ annihilated by scalar multiplication by $|W|$. For example, consider the action of $G = \{\pm 1\} \subsetneq \mathbb{R}^\times$ by scalar multiplication on $X = S^\infty \subsetneq \mathbb{R}^\infty$. Then $X/G \approx \mathbb{R}\mathrm{P}^\infty$, and the maximal torus T is trivial, so $W_G = G$, and $X/T = X = S^\infty$ again. With $\mathbb{Z}$ coefficients, one finds

$$H^*(X/G;\mathbb{Z}) \cong \mathbb{Z}[c_1]/(2c_1), \qquad \deg c_1 = 2,$$
$$H^*(X/T;\mathbb{Z})^{W_G} = H^0(S^\infty;\mathbb{Z})^G = \mathbb{Z}.$$

Similarly, with $\mathbb{F}_2$ coefficients,

$$H^*(X/G;\mathbb{F}_2) \cong \mathbb{F}_2[w_1], \qquad \deg w_1 = 1,$$
$$H^*(X/T;\mathbb{F}_2)^{W_G} = H^0(S^\infty;\mathbb{F}_2)^G = \mathbb{F}_2.$$

Historical remarks 6.4.9. Leray had proved a version of Lemma 6.4.5 for classical G [Ler49b] already in 1949, and proved the general version in his *Colloque* paper [Ler51, Thm. 2.2]. The author is indebted to Borel [Bor98] for guiding him to these references.

Chapter 7

The cohomology of Lie groups and classifying spaces

In this chapter, we develop enough of the theory of the cohomology of Lie groups and homogeneous spaces to justify the theorems we use in the last few chapters and the preparations we have made in the preceding ones. This beautiful story seems to be rarely taught nowadays, so we take this opportunity to be a bit more discursive than otherwise we might. We start out with $k = \mathbb{Z}$, retreating shamelessly to $k = \mathbb{Q}$ when torsion rears its head. Importantly, though, if we were willing to deal with such complications, we *would not have to* retreat; this is in contrast with the earlier, Lie-algebraic methods with which this theory is typically developed which rely essentially upon $\mathbb{R}$-algebra structures and destroy torsion off the bat.

The rational cohomology of a compact Lie group G is as simple as anyone has any right to expect, and this simplicity can be seen as caused either by the multiplcation on G or by the existence of invariant differential forms (again a consequence of the multiplication). The Serre spectral sequence will allow us to compute the rational cohomology of the classical groups, a major achievement in the 1930s, in a few pages.

The Serre spectral sequence of $G \to EG \to BG$ will allow us to compute the cohomology of the classifying spaces BG. This computation can be seen (perhaps ahistorically) as inspiring the definition of the Koszul complex, and through it, the definition of Lie algebra cohomology.

Moreover, the Serre spectral sequence of $G \to EG \to BG$ induces a machine, invented by Borel in his thesis, for computing the cohomology of homogeneous spaces G/K. This machine also inspires our definition of the Cartan algebra, another means to compute the cohomology of a homogeneous space which simultaneously is the motivating example behind the Cartan model for equivariant cohomology.

The Cartan algebra was one of the motivating examples behind the definition of minimal models, which developed into a central tool of rational homotopy theory in the late 1960s. We use one tool from rational homotopy theory, the algebra of polynomial differential forms, to update Borel's 1953 proof that the Cartan algebra computes the cohomology of a homogeneous space.

We will cite general references for this material throughout the chapter, and diligently recount historical origins when we know them. Proofs, however, unless explicitly noted otherwise, have been dredged from the author's own memories or created anew.

The innovation in our presentation of this chapter is that we are able to present the Cartan algebra and its application in algebraic terms with essentially no use of the Lie algebra of G, of the Lie derivative, or of connections, and without developing rational homotopy theory. Though many sources cover this material in more or less detail [Colloque; And62; Ras69; GHV76; Oni94], all of them rely on Lie-algebraic methods. Rational homotopy theoretic proofs of Cartan's theorem can be found in texts [FHT01; FOT08], as an application of a much more of a general theory we for lack of space do not develop here. In fact, Cartan's theorem was an early instance of and an inspiration for such methods, as discussed for example by Hess [Hes99].

7.1. The cohomology of the classical groups

Using the Serre spectral sequence, we can recover the main results of the 1930s on the cohomology of Lie groups in a few pages. We start with $\mathbb{Z}$ coefficients, then abscond away to simpler rings when it makes life easier.

7.1.1. Complex and quaternionic unitary groups

Note that $\mathrm{U}(n)$ acts by isometries on $\mathbb{C}^n$, so that it preserves the unit sphere S^{2n-1}. If we view this action as a left action on the space $\mathbb{C}^{n\times 1}$ of column vectors, the first column of an element g of $\mathrm{U}(n)$ determines where it takes the standard first basis vector $e_1 = (1,\vec{0})^\top \in S^{2n-1}$, so the stabilizer of e_1 is the subgroup

$$\begin{bmatrix} 1 & \vec{0} \\ \vec{0}^\top & \mathrm{U}(n-1) \end{bmatrix}$$

of elements with first column e_1, which we will identify with $\mathrm{U}(n-1)$. Since the first vector of $g \in \mathrm{U}(n)$ can be any element of S^{2n-1}, the action of $\mathrm{U}(n)$ on S^{2n-1} is transitive, so the orbit–stabilizer theorem yields a diffeomorphism $\mathrm{U}(n)/\mathrm{U}(n-1) \cong S^{2n-1}$, which is in fact a fiber bundle

$$\mathrm{U}(n-1) \longrightarrow \mathrm{U}(n) \longrightarrow S^{2n-1}.$$

Similarly, the action of $\mathrm{Sp}(n)$ on $\mathbb{H}^n$, preserving the unit sphere S^{4n-1}, gives rise to a fiber bundle

$$\mathrm{Sp}(n-1) \longrightarrow \mathrm{Sp}(n) \longrightarrow S^{4n-1},$$

and the action of $\mathrm{O}(n)$ on $\mathbb{R}^n$, preserving S^{n-1}, gives rise to bundles

$$\mathrm{O}(n-1) \longrightarrow \mathrm{O}(n) \longrightarrow S^{n-1},$$

$$\mathrm{SO}(n-1) \longrightarrow \mathrm{SO}(n) \longrightarrow S^{n-1}.$$

The SSSs of these bundles allow us to recover the cohomology of the classical groups.

Proposition 7.1.1. *The integral cohomology of the unitary group* $\mathrm{U}(n)$ *is given by*

$$H^*\big(\mathrm{U}(n);\mathbb{Z}\big) \cong \Lambda[z_1, z_3, \ldots, z_{2n-1}], \quad \deg z_j = j.$$

This can be seen as saying that in the SSSs of the bundles (right angles down) in the diagram

$$\begin{tikzcd} \mathrm{U}(1) \arrow[r] \arrow[d, "\wr"] & \mathrm{U}(2) \arrow[r] \arrow[d] & \mathrm{U}(3) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \mathrm{U}(n) \arrow[r] \arrow[d] & \mathrm{U}(n+1) \arrow[d] \\ S^1 & S^3 & S^5 & \cdots & S^{2n-1} & S^{2n+1}, \end{tikzcd} \tag{7.1}$$

the simplest possible thing happens, and the cohomology of each object is the tensor product of those of the objects to the left of it and below it.

Proof. The proof starts with the case $\mathrm{U}(1) \cong S^1$, so that $H^*(S^1) \cong \Lambda[z_1]$. Inductively assume $H\big(\mathrm{U}(n)\big) \cong \Lambda[z_1, z_3, \ldots, z_{2n-1}]$ as claimed. We have a fiber bundle

$$\mathrm{U}(n) \longrightarrow \mathrm{U}(n+1) \longrightarrow S^{2n+1},$$

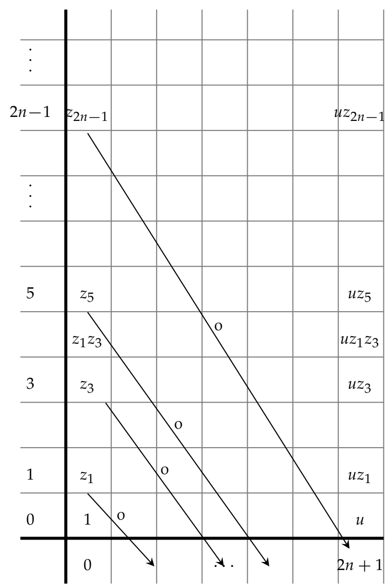
where the cohomology of the fiber and base are known, so the impulse is to use Theorem 4.3.4.

Since the cohomology of the fiber is free abelian by assumption, the E_2 page is given by

$$E_2^{\bullet,0} \otimes E_2^{0,\bullet} = \Lambda[u_{2n+1}] \otimes \Lambda[z_1, z_3, \ldots, z_{2n-1}],$$

and the sequence is concentrated in columns 0 and $2n+1$. Since the bidegree of the differential d_r is $(r, 1-r)$, the only differential that could conceivably be nonzero is $d = d_{2n+1}$, of bidegree $(2n+1, -2n)$.

Figure 7.1.2: The Serre spectral sequence of $\mathrm{U}(n) \to \mathrm{U}(n+1) \to S^{2n+1}$



But this d sends the square $E_{2n+1}^{0,q} = H^q\big(\mathrm{U}(n)\big)$ in the leftmost column into the fourth quadrant, so $dz_j = 0$ for all j. Because d satisfies the product rule and sends all generators of E_{2n+1} into the fourth quadrant, it follows $d = 0$. Thus $E_2 = E_\infty = \Lambda[z_1, z_3, \ldots, z_{2n-1}, u_{2n+1}]$.

A priori, this is only the the associated graded algebra of $H^*\big(\mathrm{U}(n+1)\big)$, but since E_∞ is an exterior algebra, by Proposition A.5.10, there is no extension problem. □

The same proof, applied to the bundles $\mathrm{Sp}(n-1) \to \mathrm{Sp}(n) \to S^{4n-1}$ and starting with $\mathrm{Sp}(1) \approx S^3$, yields the cohomology of the symplectic groups.

Proposition 7.1.3. *The integral cohomology of the symplectic group* $\mathrm{Sp}(n)$ *is given by*

$$H^*\big(\mathrm{Sp}(n);\mathbb{Z}\big) \cong \Lambda[z_3, z_7, \ldots, z_{4n-1}], \quad \deg z_j = j.$$

The diagram associated to this induction is

$$\begin{tikzcd} \mathrm{Sp}(1) \arrow[r] \arrow[d, "\wr"] & \mathrm{Sp}(2) \arrow[r] \arrow[d] & \mathrm{Sp}(3) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \mathrm{Sp}(n) \arrow[r] \arrow[d] & \mathrm{Sp}(n+1) \arrow[d] \\ S^3 & S^7 & S^{11} & \cdots & S^{4n-1} & S^{4n+3}, \end{tikzcd} \tag{7.2}$$

The cohomology of the special unitary groups is closely related to that of the unitary groups.

Proposition 7.1.4. *The integral cohomology of the special unitary group* $\mathrm{SU}(n)$ *is given by*

$$H^*\big(\mathrm{SU}(n);\mathbb{Z}\big) \cong \Lambda[z_3, \ldots, z_{2n-1}], \quad \deg z_j = j.$$

Proof. The determinant map yields a split short exact sequence

$$1 \to \mathrm{SU}(n) \hookrightarrow \mathrm{U}(n) \xrightarrow{\det} S^1 \to 1; \tag{7.3}$$

a splitting is given by $z \longmapsto \mathrm{diag}(z, \vec{1})$. This semidirect product structure means $\mathrm{U}(n)$ is topologically a product $\mathrm{SU}(n) \times S^1$, and it follows from the Künneth theorem B.2.2 that

$$H^*\big(\mathrm{SU}(n)\big) \cong H^*\big(\mathrm{U}(n)\big)/\!/H^*(S^1) = \Lambda[z_1, z_3, \ldots, z_{2n-1}]/(z_1) = \Lambda[z_3, \ldots, z_{2n-1}]. \qquad \square$$

The information we have accumulated makes it easy to cheaply acquire as well the cohomology the complex and quaternionic Stiefel manifolds: the idea is just, in the diagram (7.1), to stop before one gets to U(1).

Proposition 7.1.5. *The integral cohomology of the complex Stiefel manifolds $V_j(\mathbb{C}^n) = \mathrm{U}(n)/\mathrm{U}(n-j)$ is*

$$H^*\big(V_j(\mathbb{C}^n);\mathbb{Z}\big) = \Lambda[z_{2(n-j)+1},\ldots,z_{2n-3},z_{2n-1}].$$

The integral cohomology of the quaternionic Stiefel manifolds $V_j(\mathbb{H}^n) = \mathrm{Sp}(n)/\mathrm{Sp}(n-j)$ is given by

$$H^*\big(V_j(\mathbb{H}^n);\mathbb{Z}\big) = \Lambda[z_{4(n-j)+3},\ldots,z_{4n-5},z_{4n-1}].$$

Proof. The spectral sequences of the bundles (7.1) dealt with in Proposition 7.1.1 all collapsed at the E_2 page, so that in particular the maps $H^*\mathrm{U}(n) \longrightarrow H^*\mathrm{U}(n-1)$ are surjective and the iterated map $H^*\mathrm{U}(n) \longrightarrow H^*\mathrm{U}(n-j)$ is surjective by induction: explicitly, it is the projection

$$\Lambda[z_1,z_3,\ldots,z_{2(n-j)-1}] \otimes \Lambda[z_{2(n-j)+1},\ldots,z_{2n-1}] \twoheadrightarrow \Lambda[z_1,z_3,\ldots,z_{2(n-j)-1}],$$

with kernel $(z_1,z_3,\ldots,z_{2(n-j)-1})$ the extension of the augmentation ideal of the second factor. One has more or less definitionally the fiber bundle

$$\mathrm{U}(n-j) \longrightarrow \mathrm{U}(n) \longrightarrow V_j(\mathbb{C}^n), \tag{7.4}$$

whose SSS collapses at E_2 by Section 8.3.1 since we have just shown the fiber projection is surjective. Thus the base pullback $H^*V_j(\mathbb{C}^n) \longrightarrow H^*\mathrm{U}(n)$ is injective and $H^*V_j(\mathbb{C}^n)$ is an exterior subalgebra of $H^*\mathrm{U}(n)$ whose augmentation ideal extends to the kernel $(z_{2(n-j)+1},\ldots,z_{2n-1})$ of the fiber projection. We see $H^*V_j(\mathbb{C}^n)$ can only be as claimed.

The proof for $H^*V_j(\mathbb{H}^n)$ is entirely analogous. □

7.1.2. Real difficulties

The degeneration of spectral sequences that occurs for unitary and symplectic fails for the orthogonal groups, because in the analogue of the iterated fiber decompostion (7.1) of the orthogonal groups, one encounters spheres of adjacent dimension, which could lead to nontrivial differentials. Indeed, this does lead to rather complicated 2-torsion, so we pass to simpler coefficient rings. Even with this simplification, there seems to be a certain unavoidable difficulty in handling $H^*\mathrm{SO}(n)$, forcing case distinctions and a rather explicit calculation of a map of homotopy groups. The proofs here are, in the author's own opinion, cleaner and more scrutable than those in the source material, but he would not claim they make an easy read. The reader could be forgiven for skipping this section and resuming at Section 7.2, but it seemed right to say what could be explained about $H^*\mathrm{SO}(n)$ with the tools already at hand.

To proceed, we require on a lemma [MT00, Cor. 3.13, p. 121] about the cohomology of a Stiefel manifold $V_2(\mathbb{R}^n)$. The proof here is a hybrid of Mimura and Toda's and that in online notes by Bruner, Catanzaro, and May [BCM]. Recall our notational conventions from Appendix A.3.2.

Lemma 7.1.6. *The real Stiefel manifold $V = V_2(\mathbb{R}^n)$ (for $n \geqslant 4$) has*

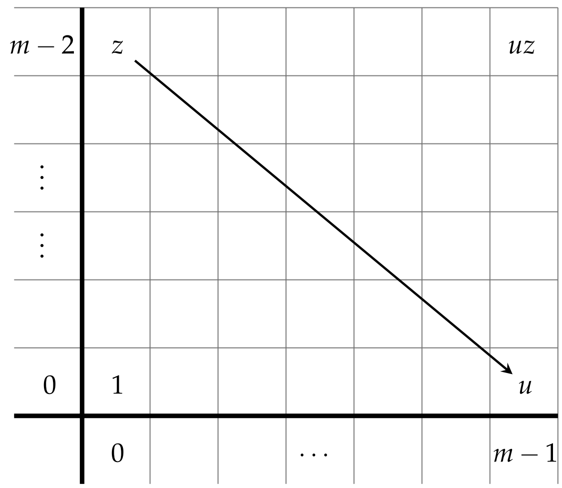
$$H_{n-2}(V) = \begin{cases} \mathbb{Z} & n \text{ even}, \\ \mathbb{Z}/2 & n \text{ odd}. \end{cases}$$

Proof. If we define $V_2(\mathbb{R}^m) := \mathrm{SO}(m)/\mathrm{SO}(m-2)$ as the set of pairs of orthogonal elements of S^{m-1}, or equivalently $n \times 2$ matrices with orthonormal columns, then projection to the first column is a bundle map, yielding

$$S^{m-2} \longrightarrow V_2(\mathbb{R}^m) \longrightarrow S^{m-1}.$$

The associated Serre spectral sequence is as in Figure 7.1.7, and it is clear the lone potentially nonzero differential is $H^{n-2}(S^{n-2}) \xrightarrow{d} H^{n-1}(S^{n-1})$.

Figure 7.1.7: The differential d_{m-1} in the Serre spectral sequence of $S^{m-2} \to V_2(\mathbb{R}^m) \to S^{m-1}$



In particular, we have $H^j(V) = 0$ for $j < n-2$, and $H_j(V)$ as well by the universal coefficient Theorem B.2.1. Since we have assumed $n \geqslant 4$, it follows from the long exact homotopy sequence of the bundle (Theorem B.2.4) that V is simply-connected, so by the Hurewicz Theorem B.2.6, $\pi_{n-2}(V) \cong H_{n-2}(V)$, and we can concern ourselves with this group instead. The long exact homotopy sequence of Theorem B.2.4 contains the subsequence

$$\pi_{m-1}(S^{m-1}) \xrightarrow{\partial} \pi_{m-2}(S^{m-2}) \longrightarrow \pi_{m-2}(V) \longrightarrow \underbrace{\pi_{m-2}(S^{m-1})}_{0},$$

showing $\pi_{n-2}(V) \cong \operatorname{coker} \partial$, so our task is now to identify $\operatorname{im} \partial$.

Note that $V = V_2(\mathbb{R}^m)$ admits a description as $\mathrm{O}(m)/\mathrm{O}(m-2)$ as well as $\mathrm{SO}(m)/\mathrm{SO}(m-2)$.[1] There is a natural map $S^{n-1} \longrightarrow \mathrm{O}(n)$ taking a unit vector v to the reflection r_v in the hyperplane of $\mathbb{R}^n$ orthogonal to v. Evidently $r_v = r_{-v}$, so this map factors through $\mathbb{R}\mathrm{P}^{n-1}$. Recall that we

[1] The point is that we can extend any orthonormal 2-frame $(v, w) \in V_2(\mathbb{R}^m)$ to $g \in \mathrm{O}(n)$, but we can also always alter the last $m-2$ columns to put our representative in $\mathrm{SO}(n)$. If $m = 3$, multiply the last column by -1; if $m - 2 \geqslant 2$, transpose the last two columns.

see $\mathrm{O}(n-1)$ as the stabilizer of the standard first basis vector $e_1 = (1, \vec{0})^\top$ of $\mathbb{R}^n$ under the standard action $(g, v) \mapsto gv$. The evaluation map $p_1\colon g \mapsto ge_1$ is "projection to first column," taking $\mathrm{O}(n) \longrightarrow S^{n-1}$. It factors through "projection to the first two columns," which is a map $p_2\colon \mathrm{O}(n) \longrightarrow V_2(\mathbb{R}^n)$. Concatenating these maps yields a sequence

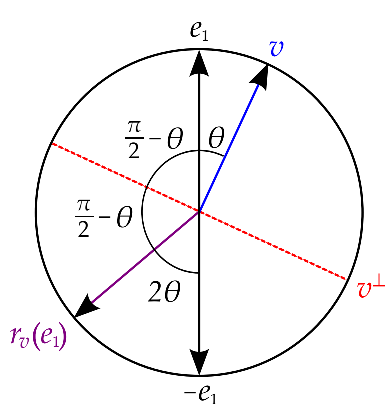
$$\overbrace{S^{n-1} \longrightarrow \mathbb{R}\mathrm{P}^{n-1} \longrightarrow \mathrm{O}(n)}^{r}\overbrace{\xrightarrow[p_2]{} V \xrightarrow[q]{} S^{n-1}}^{p_1}.$$

The composition is given by $p_1 r(v) = p_1(r_v) = r_v(e_1)$.

Let $D^{n-1} = \{v \in S^{n-1} : v \cdot e_1 \geqslant 0\}$ be the northern hemisphere and $S^{n-2} = \partial D^{n-1}$ the equator, those unit vectors perpendicular to e_1. We claim r takes the equator $S^{n-1} \longrightarrow \mathrm{O}(n-1)$ and $p_1 r$ takes the interior $\mathring{D}^{n-1}$ homeomorphically onto $S^{n-1} \backslash \{e_1\}$. For the first claim, if $v \in S^{n-1}$ is perpendicular to e_1, then e_1 in is in the hyperplane fixed by r_v, so $pr(v) = r_v(e_1) = e_1$. That means the first column of r_v is e_1, so by definition $r_v \in \mathrm{O}(n-1)$.

For the second claim, let $v \in \mathring{D}^{n-1}$. If $v = e_1$, then $r_{e_1}(e_1) = -e_1$, and otherwise v and e_1 together span a 2-plane which cuts S^{n-1} in a circle and $v^\perp$ in a line, and $pr(v) = r_v(e_1)$ lies in this plane. See Figure 7.1.8. Since pr preserves these circles, it is be enough to show that the restriction of pr to each open upper semicircle is injective, and this is the case because if $\measuredangle(e_1, v) = \theta \in (-{}^{\pi}\!/\!_2, {}^{\pi}\!/\!_2)$, then $\measuredangle\big(r_v(e_1), -e_1\big) = 2\theta$.

Figure 7.1.8: The reflection of e_1 through $v^\perp$



The restriction of $p_1 r$ to D^{n-1} is then a map of pairs

$$(D^{n-1}, S^{n-2}) \xrightarrow{\iota} (V, S^{n-2}) \xrightarrow{q} (S^{n-1}, \mathrm{pt})$$

which is a homeomorphism on $\mathring{D}^{n-1}$, so the composition $q\iota$ represents a generator of the relative homotopy group $\pi_{n-1}(S^{n-1}, \mathrm{pt}) \cong \pi_{n-1}(S^{n-1})$. Since the map of pairs $q\colon (V, S^{n-2}) \longrightarrow (S^{n-1}, \mathrm{pt})$ induces the isomorphism of relative homotopy groups that translates the long exact homotopy sequence of a pair into that of a bundle, it follows the restriction $\chi = (\iota \restriction S^{n-2})$ represents a generator of $\operatorname{im} \partial$.

We can understand χ as the restriction of r to S^{n-2} followed by projection to the *second* column (since the target $S^{n-2} \subsetneq V$ is the fiber of $V \to S^{n-1}$ over e_1). Because any $v \in S^{n-3}$ is perpendicular to both e_1 and e_2, the reflection r_v will leave the first two coordinates invariant and so be in $\mathrm{O}(n-2)$. Thus $p_2 r(S^3) = \mathrm{pt}$. Since $r_v = r_{-v}$, the same argument as for $p_1 r$ shows that $p_2 r$ takes the interiors of both north and south hemispheres homeomorphically onto $S^{n-2} \backslash \mathrm{pt}$, so restrictions to these hemispheres are maps

$$\tau_\pm\colon (D^{n-2}, S^{n-2}) \longrightarrow (S^{n-2}, \mathrm{pt})$$

representing generators of $\pi_{n-2}(S^{n-2}, \mathrm{pt}) \cong \pi_{n-2}(S^{n-2})$ such that $[\chi] = [\tau_+] + [\tau_-]$. These generators are closely related: $\tau_- = \tau_+ \circ \alpha$, where

$$\begin{aligned} \alpha\colon S^{n-2} &\longrightarrow S^{n-2}, \\ v &\longmapsto -v, \end{aligned}$$

is the antipodal map. Since α is the composition of $n-1$ reflections in $\mathbb{R}^{n-1}$, it is of degree $(-1)^{n-1}$, so that $[\chi] = \partial[\iota]$ is $s_n := \pm\big(1 + (-1)^{n-1}\big)$ times a generator of $\pi_{n-2}(S^{n-2})$. Since $s_{\text{even}} = \pm 2$ and $s_{\text{odd}} = 0$, the group $\pi_{n-2}(V) \cong \mathbb{Z}/s_n\mathbb{Z}$ is as claimed. □

Remark 7.1.9. Since $V_2(\mathbb{R}^n)$ is the set of pairs (v, w) with $v \in S^{n-1}$ and $w \perp v$, it can be seen as the set of unit vectors in the tangent bundle TS^{n-1}. This is a S^{n-2}-bundle associated to a principal $\mathrm{SO}(n-1)$-bundle, and it can be shown that the image of the element 1 of the fiber cohomology group $\mathbb{Z} = H^{n-2}(S^{n-2})$ in the base cohomology group $H^{n-1}(S^{n-1}) = \mathbb{Z}$ is the *Euler class* of this bundle (see Section 7.7); the fact that this number alternates between zero and two can be seen as a reflection of the fact that the Euler characteristics (Appendix A.3.2) of spheres obey the rule $\chi(S^n) = 1 + (-1)^n$.

Corollary 7.1.10. *The nonzero integral cohomology groups of the real Stiefel manifold $V = V_2(\mathbb{R}^n)$ are*

$$H^0(V) \cong H^{2n-3}(V) \cong \mathbb{Z}, \qquad H^{n-2}(V) = \begin{cases} \mathbb{Z} & n \text{ even}, \\ 0 & n \text{ odd}, \end{cases} \qquad H^{n-1}(V) = \begin{cases} \mathbb{Z} & n \text{ even}, \\ \mathbb{Z}/2 & n \text{ odd}. \end{cases}$$

In particular, the differential $H^{n-2}(S^{n-2}) \xrightarrow{d} H^{n-1}(S^{n-1})$ shown in Figure 7.1.7) is zero if n is even and multiplication by 2 if n is odd. The mod 2 cohomology ring of V is

$$H^*(V; \mathbb{F}_2) \cong \Lambda[v_{n-2}, v_{n-1}]$$

Proof. If n is even, we have $\pi_{n-2}(V) = H_{n-2}(V)$ infinite cyclic from Lemma 7.1.6, so by universal coefficients, $H^{n-1}(V)$ is also free abelian, and it follows $d = 0$ and $H^{n-2}(V) \cong \mathbb{Z}$.

If n is odd, we have $\mathbb{Z}/2 \cong \pi_{n-2}(V) = H_{n-2}(V)$, so by universal coefficients, $H^{n-2}(V) = 0$ and $H^{n-1}(V)$ is the sum of $\mathbb{Z}/2$ and a free abelian group. But $H^{n-1}(V)$ is cyclic, since it is $\operatorname{coker} d$, so we have $H^{n-1}(V) \cong \mathbb{Z}/2$.

As for the modulo 2 case, we have $2 \equiv 0 \pmod 2$, so the map d is always zero and the SSS collapses. There is no extension problem simply by a dimension count. □

The main point of this argument, for us, is that the map d is trivial for n even and an isomorphism over $\mathbb{Z}[\frac{1}{2}]$ if n is odd. In the mod 2 case, these differentials are all zero, so we can induct up with spheres rather than $V_2(\mathbb{R}^n)$s.

Corollary 7.1.11. *The mod 2 cohomology ring of $V = V_j(\mathbb{R}^n)$ has a simple system $v_{n-1}, \ldots, v_{n-j}$ of generators (see Definition A.3.4), where* $\deg v_i = i$. *That is,*

$$H^*(V; \mathbb{F}_2) = \Delta[v_{n-1}, v_{n-2}, \ldots, v_{n-j}].$$

Proof. We fix n and prove the result by induction on $j \in [1, n]$. For $j = 1$, the result is just $H^*(S^{n-1}) = \Lambda[v_{n-1}]$. Suppose by induction the result holds for $V_{j-1}(\mathbb{R}^n)$ and the Serre spectral sequence of $S^{n-(j-1)} \to V_{j-1}(\mathbb{R}^n) \to V_{j-2}(\mathbb{R}^n)$ collapses at E_2. Then the E_2 page of the Serre spectral sequence of $S^{n-j} \to V_j(\mathbb{R}^n) \longrightarrow V_{j-1}(\mathbb{R}^n)$ is (additively)

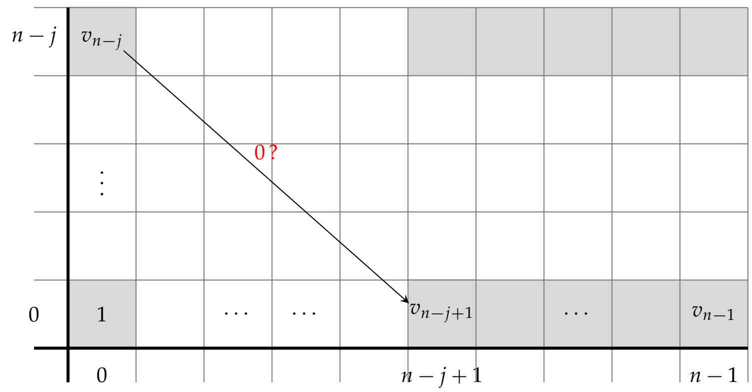
$$E_2 = \Delta[v_{n-1}, \ldots, v_{n-(j-1)}] \otimes \Delta[v_{n-j-1}],$$

so the induction will go through if and only if $E_2 = E_\infty$ in this spectral sequence as well. The only potentially nontrivial differential is $d_{n-(j-1)}$, which vanishes on the base $\Delta[v_{n-1}, \ldots, v_{n-(j-1)}]$ and so is determined by the map

$$H^{n-j-1}(S^{n-j-1}) \xrightarrow{d_{n-j+1}} H^{n-j}\big(V_j(\mathbb{R}^n)\big)$$

indicated in Figure 7.1.12.

Figure 7.1.12: The Serre spectral sequence of $S^{n-j} \to V_j(\mathbb{R}^n) \to V_{j-1}(\mathbb{R}^n)$ over $\mathbb{F}_2$



To see this map is zero, we identify it with the analogous differential in the Serre spectral sequence of $S^{n-j} \to V_2(\mathbb{R}^{n+2-j}) \to S^{n+1-j}$, which we already know to be zero by Corollary 7.1.10. To do that, consider the following commutative diagram:

$$\begin{tikzcd}
S^{n-j} \arrow[r, equal] \arrow[d] & S^{n-j} \arrow[d] & \\
V_2(\mathbb{R}^{n+2-j}) \arrow[r] \arrow[d] & V_j(\mathbb{R}^n) \arrow[r] \arrow[d] & V_{j-2}(\mathbb{R}^n) \arrow[d, equal] \\
S^{n+1-j} \arrow[r] & V_{j-1}(\mathbb{R}^n) \arrow[r] & V_{j-2}(\mathbb{R}^n).
\end{tikzcd}$$

Each row and column is a bundle, and the bundle projections are of the form "consider the first few vectors"; for example, the map $V_j(\mathbb{R}^n) \to V_{j-2}(\mathbb{R}^n)$ simply forgets the last two vectors of a j-frame on $\mathbb{R}^n$, and the fiber over a $(j-2)$-frame is the set of 2-frames orthogonal to those $j-2$ vectors in $\mathbb{R}^n$, and so is a $V_2(\mathbb{R}^{n-j+2})$.

The map of columns induces a map (ψ_r) of spectral sequences from (E_r, d_r) to the spectral sequence $('E_r, 'd_r)$ of the left column, which collapses at $'E_2$. The bottom row is the bundle whose Serre spectral sequence we inductively assumed collapses, so $\psi_{n+1-j}\colon H^{n+1-j}\big(V_{j-1}(\mathbb{R}^n)\big) \longrightarrow$

$H^{n+1-j}(S^{n+1-j})$ is an isomorphism. The relation

$$0 = {}'d_{n+1-j}\psi_{n+1-j} = \psi_{n+1-j}d_{n+1-j}$$

then ensures $d_{n+1-j} = 0$ and we have collapse. □

Taking $j = n - 1$ yields the result we really were after.

Corollary 7.1.13. *The mod 2 cohomology ring of the special orthogonal group* SO(n) *has a simple system* $v_1, \ldots, v_{n-1}$ *of generators:*

$$H^*\big(\mathrm{SO}(m); \mathbb{F}_2\big) = \Delta[v_1, v_2, \ldots, v_{n-1}],$$

where $\mathbb{F}_2\{v_{n-1}\}$ *is the image of* $H^{n-1}(S^{n-1}) \longrightarrow H^{n-1}\big(\mathrm{SO}(n)\big)$.

Remark 7.1.14. We used the induction $S^{n-j} \to V_j(\mathbb{R}^n) \to V_{j-1}(\mathbb{R}^n)$ to pick up the cohomology of the Stiefel manifolds along the way to that of SO(n). We could also have inducted the other way, using

$$\begin{tikzcd}
\mathrm{SO}(2) \arrow[r] \arrow[d, "\wr"] & \mathrm{SO}(3) \arrow[r] \arrow[d] & \mathrm{SO}(4) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \mathrm{SO}(n) \arrow[r] \arrow[d] & \mathrm{SO}(n+1) \arrow[d] \\
S^1 & S^2 & S^3 & \cdots & S^{n-1} & S^n,
\end{tikzcd}$$

in analogy with (7.1). Then the task is to show that the differential $H^{n-1}\big(\mathrm{SO}(n)\big) \longrightarrow H^n(S^n)$ is zero. We can still use the collapse of the Serre spectral sequence of $S^{n-1} \to V_2(\mathbb{R}^{n+1}) \to S^n$ to do this; the relevant bundle map is

$$\begin{tikzcd}
\mathrm{SO}(n) \arrow[r] \arrow[d] & S^{n-1} \arrow[d] \\
\mathrm{SO}(n+1) \arrow[r] \arrow[d] & V_2(\mathbb{R}^{n+1}) \arrow[d] \\
S^n \arrow[r, equal] & S^n.
\end{tikzcd}$$

The induction is substantially subtler over $\mathbb{Z}$ or even over $k = \mathbb{Z}[\frac{1}{2}]$, because the differentials no longer must be trivial. We can use the real Stiefel manifolds $V_2(\mathbb{R}^n) \cong \mathrm{SO}(n)/\mathrm{SO}(n-2)$ as building blocks now, though, the same way we used spheres before:

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{SO}(n-4) \arrow[r] \arrow[d] & \mathrm{SO}(n-2) \arrow[r] \arrow[d] & \mathrm{SO}(n) \arrow[d] \\ & V_2(\mathbb{R}^{n-4}) & V_2(\mathbb{R}^{n-2}) & V_2(\mathbb{R}^n). \end{tikzcd} \tag{7.5}$$

Proposition 7.1.15. *Let $2n+1 \geqslant 3$ be an odd integer and $2j < 2n+1$ an even integer. Then taking coefficients in $k = \mathbb{Z}[\frac{1}{2}]$, we have*

$$H^*\big(\mathrm{SO}(2n+1)\big) \cong \Lambda[z_3, z_7, \ldots, z_{4n-1}], \quad \deg z_{4i-1} = 4i-1.$$
$$H^*\big(V_{2j}(\mathbb{R}^{2n+1})\big) \cong H^*\big(\mathrm{SO}(2n+1)\big)\big/\!\!\big/ H^*\big(\mathrm{SO}(2n-2j+1)\big) \cong \Lambda[z_{4(n-j)+3}, \ldots, z_{4n-1}].$$

Proof. By Corollary 7.1.10, we have $H^*\big(V_2(\mathbb{R}^{2j+1})\big) = \Lambda[z_{4j-1}]$, so the objects in (7.5) have the same cohomology as those in (7.2) which yielded the same structure (over $\mathbb{Z}$) for $H^*\big(\mathrm{Sp}(n)\big)$. The result for $H^*\big(V_j(\mathbb{R}^n)\big)$ follows as in Proposition 7.1.5. □

To recover $V_{2j-1}(\mathbb{R}^{2n})$, consider the map of bundles

$$\begin{tikzcd} V_{2j-2}(\mathbb{R}^{2n-1}) \arrow[r, equal] \arrow[d] & V_{2j-2}(\mathbb{R}^{2n-1}) \arrow[d] & \\ V_{2j-1}(\mathbb{R}^{2n}) \arrow[r] \arrow[d] & V_{2j}(\mathbb{R}^{2n+1}) \arrow[r] \arrow[d] & S^{2n} \arrow[d, equal] \\ S^{2n-1} \arrow[r] & V_2(\mathbb{R}^{2n+1}) \arrow[r] & S^{2n}. \end{tikzcd}$$

The Serre spectral sequence of the middle column collapses at E_2 by an elaboration of our calculation above.[2] Thus we can use the bundle lemma Theorem 4.4.1 to conclude

$$H^*\big(V_{2j-1}(\mathbb{R}^{2n-1})\big) \cong \Lambda[e_{2n-1}] \underset{\Lambda[z_{4n-1}]}{\otimes} \Lambda[z_{4(n-j)+3}, \ldots, z_{4n-1}] = \Lambda[e_{2n-1}] \otimes \Lambda[z_{4(n-j)+3}, \ldots, z_{4n-5}].$$

Taking $n = j$, we recover $H^*\big(\mathrm{SO}(2n)\big)$.

Proposition 7.1.16. *Let $2n \geqslant 2$ be an even integer and $2j-1 < 2n$ odd. Then*

$$H^*\big(V_{2j-1}(\mathbb{R}^{2n})\big) \cong \Lambda[e_{2n-1}] \otimes \Lambda[z_{4(n-j)+3}, \ldots, z_{4n-5}],$$

where $\deg z_i = i$ and $\deg e_{2n-1} = 2n-1$. In particular,

$$H^*\big(\mathrm{SO}(2n)\big) \cong \Lambda[e_{2n-1}] \otimes \Lambda[z_3, \ldots, z_{4n-5}].$$

We can state the result for $\mathrm{SO}(m)$ more uniformly as follows:

Corollary 7.1.17. *Over $k = \mathbb{Z}[\frac{1}{2}]$, the cohomology ring of $\mathrm{SO}(m)$ is*

$$H^*\big(\mathrm{SO}(m); \mathbb{Z}[\tfrac{1}{2}]\big) = \begin{cases} \Lambda[z_3, z_7, \ldots, z_{4n-5}] \otimes \Lambda[e_{2n-1}], & m = 2n, \\ \Lambda[z_3, z_7, \ldots, z_{4n-5}] \otimes \Lambda[z_{4n-1}], & m = 2n+1, \end{cases}$$

where $k \cdot e_{2n-1}$ is the image of $H^{2n-1}(S^{2n-1}) \longrightarrow H^{2n-1}\big(\mathrm{SO}(2n)\big)$.

[2] The relevant bundle map is this:

$$\begin{tikzcd} \mathrm{SO}(2n-2j+1) \arrow[r] \arrow[d, equal] & \mathrm{SO}(2n-1) \arrow[r] \arrow[d] & V_{2j-2}(\mathbb{R}^{2n-1}) \arrow[d] \\ \mathrm{SO}(2n-2j+1) \arrow[r] & \mathrm{SO}(2n+1) \arrow[r] & V_{2j}(\mathbb{R}^{2n+1}). \end{tikzcd}$$

Both rows yield tensor decompositions in cohomology, and the fiber inclusion $\mathrm{SO}(2n-1) \longrightarrow \mathrm{SO}(2n+1)$ is surjective in cohomology with kernel (z_{4n-1}), so the same holds of the right-hand map we are interested in.

To get the cases $V_\ell(\mathbb{R}^m)$ where $\ell \equiv m \pmod 2$, we can use the Serre spectral sequence of

$$S^{m-\ell} \longrightarrow V_\ell(\mathbb{R}^m) \longrightarrow V_{\ell-1}(\mathbb{R}^m).$$

as we did in Corollary 7.1.11. The E_2 page is $H^*\big(V_{\ell-1}(\mathbb{R}^m)\big) \otimes \Delta[s_{m-\ell}]$, and the only potentially nonzero differential is $d_{m-\ell+1}$ which is determined by a map $d\colon H^{m-\ell}(S^{m-\ell}) \longrightarrow H^{m-\ell+1}\big(V_{\ell-1}(\mathbb{R}^m)\big)$. By the last two propositions, the ring $H^*\big(V_{\ell-1}(\mathbb{R}^m)\big)$ is an exterior algebra on generators of degree at least $2m-2\ell+3$ if m is odd, and at least $m-1$ if m is even. In the former case, d is zero by lacunary considerations. In the latter, $\ell \geqslant 2$ since it is of the same parity as m, so we have $m-\ell+1 \leqslant m+1$, with equality if and only if $\ell = 2$. Thus, if $\ell > 2$, then $d = 0$ by lacunary considerations, and if $\ell = 2$, then we showed $d = 0$ in Corollary 7.1.10. So no matter what, the sequence collapses at E_2, so by Proposition A.4.4, we have

$$H^*\big(V_\ell(\mathbb{R}^m)\big) \cong H^*\big(V_{\ell-1}(\mathbb{R}^m)\big) \otimes \Delta[s_{m-\ell}]$$

whenever $m \equiv \ell \pmod 2$.

To compile these cases into one statement, we introduce some notation. Let S be a free k-module or basis thereof and φ a proposition whose truth or falsehood is easily verifiable. We write

$$\Lambda\big[\{S : \varphi\}\big] = \begin{cases} \Lambda[S] & \text{if } \varphi \text{ is true,} \\ k & \text{otherwise.} \end{cases}$$

Then, gathering cases and doing some arithmetic on indices, we arrive at the following.

Proposition 7.1.18 ([BCM, Thm. 2.5]). *The cohomology of the real Stiefel manifold $V_\ell(\mathbb{R}^m)$ with coeffi-*

cients in $k = \mathbb{Z}[\frac{1}{2}]$ is given by

$$H^*\big(V_\ell(\mathbb{R}^m)\big) \;\cong\; \Lambda[z_{4j-1} : 2m-2\ell+1 \leqslant 4j-1 \leqslant 2m-3] \otimes \Lambda[e_{m-1} : m \text{ even}] \otimes \Delta[s_{m-\ell} : m-\ell \text{ even}].$$

Remark 7.1.19. The author found the useful notation for abbreviating case distinctions in Proposition 7.1.18 in the notes by Bruner, Catanzaro, and May [BCM]. Both Mimura and Toda [MT00, Thm. III.3.14, p. 121] and Félix, Oprea, and Tanré [FOT08, Prop. 1.89, p. 84] have misprints in their statements of the result Proposition 7.1.18 where the (even, even) case is omitted and another case repeated twice with different results. For example, Mimura and Toda list two nonisomorphic rings for the case (odd, odd). For those keeping score, the misprint in [FOT08] is also nonisomorphic to the misprint in [MT00].

It is standard to discuss along with $\mathrm{SO}(n)$ its simply-connected double cover $\mathrm{Spin}(n)$.

Proposition 7.1.20. *The cohomology of* $\mathrm{Spin}(n)$ *for $n \geqslant 2$ satisfies*

$$H^*\big(\mathrm{Spin}(n); \mathbb{Z}[\tfrac{1}{2}]\big) \cong H^*\big(\mathrm{SO}(n); \mathbb{Z}[\tfrac{1}{2}]\big).$$

Proof. Since $\pi\colon \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$ is a connected double cover and 2 is invertible, the isomorphism follows immediately from Corollary B.3.2. □

Finally, we will relate without proof the multiplicative structure of $H^*\mathrm{SO}(n)$ and $H^*\mathrm{Spin}(n)$ with $\mathbb{F}_2$ coefficients. The standard proofs invoke Steenrod squares, a technology we have not developed here and which we have made the decision (somewhat arbitrarily, given the length of this thesis) lies too far afield.

Proposition 7.1.21. *The mod 2 cohomology of* $\mathrm{SO}(n)$ *for* $n \geqslant 2$ *is given by*

$$H^*\big(\mathrm{SO}(n);\mathbb{F}_2\big) = \mathbb{F}_2[v_1,\dots,v_{n-1}]/\mathfrak{a},$$

where the ideal $\mathfrak{a}$ *is generated by the relations*

$$v_i^2 \equiv \begin{cases} v_{2i}, & 2i < n, \\ 0, & 2i \geqslant n. \end{cases}$$

Ridding ourselves of excess generators, we can write

$$H^*\big(\mathrm{SO}(n);\mathbb{F}_2\big) = \mathbb{F}_2[v_1,v_3,\dots,v_{\lfloor n/2\rfloor-1}]/\mathfrak{b},$$

where $\mathfrak{b}$ *is the truncation ideal* $(v_i^{\lceil n/i\rceil})$ *generated by the least powers of* v_i *of degree exceeding* $n-1$*.*

The mod 2 cohomology of $\mathrm{Spin}(n)$ *admits a simple system of generators containing an element* z *of degree* $2^{\lceil \log_2 n\rceil}-1$ *and generators* v_j *for each* $j \in [1, n-1]$ *which is not a power of* 2*:*

$$H^*\big(\mathrm{Spin}(n);\mathbb{F}_2\big) = \Delta\big[z_{2^{\lceil\log_2 n\rceil}-1}, v_j \;\;:\;\; 1 \leqslant j < n, \quad j \neq 2^r\big].$$

Historical remarks 7.1.22. The lemma 7.1.6 is due to Eduard Stiefel [Sti41], also the namesake of the Stiefel manifolds and the Stiefel–Whitney classes. A comprehensive account of this material, also including explicit computations for the cohomology of the exceptional groups, can be found in the much recommended book of Mimura and Toda [MT00]. As an indication of the nontriviality of computing $H^*\mathrm{SO}(n)$, even with easier coefficient rings. we point out that while the cohomology ring $H^*\big(\mathrm{SO}(n);k\big)$ for k a field follows immediately from what we have done in this section and extracting the additive structure of the *integral* cohomology is not hard afterward, describing the integral cohomology ring from this data is a nontrivial problem which was seemingly only fully resolved in 1989 [Pit91].

7.2. The rational cohomology of Lie groups

All the cohomology rings of classical Lie groups, over sufficiently simple coefficient rings k, become exterior algebras, and one might wonder whether this holds over Lie groups in general. It has been known since the 1930s that it does, due to work of Heinz Hopf exploiting a natural algebraic structure in the (co)homology of a topological group, a development that essentially reduced the study of Lie group cohomology to obtaining torsion information and collating it back into integral cohomology.

We begin by isolating the essential feature of topological groups for our purposes.

Definition 7.2.1. An ***H-space***[3] is a topological space G equipped with a continuous ***product*** map $\mu\colon G \times G \longrightarrow G$ containing an element $e \in G$ neutral up to homotopy: we demand $g \longmapsto \mu(e,g)$ and $g \longmapsto \mu(g,e)$ be homotopic to id_G.

Such a map induces a ***coproduct*** in cohomology, the composition

$$H^*(G) \xrightarrow{H^*(\mu)} H^*(G \times G) \longrightarrow H^*(G) \otimes H^*(G),$$

where the second map arises through the Künneth theorem. We denote the coproduct by μ^*. Because $H^*(\mu)$ and the Künneth map are maps of graded k-algebras, it follows μ^* is a graded algebra homomorphism, and that if $x \in H^n(G)$, then $\mu^*(x) \in \bigoplus H^j(G) \otimes H^{n-j}(G)$.

Suppose as well that G is connected. We know $\mu(-,e) \simeq \mathrm{id}_G$; diagrammatically, this is the homotopy-commutative triangle below on the left, and taking cohomology whilst being casual

[3] The choice of$H$, due to Serre, is in honor of Heinz Hopf.

about Künneth maps yields the commutative diagram on the right.

$$\begin{tikzcd} G \arrow[r, "\approx"] \arrow[drr, "\mathrm{id}"'] & G \times \{e\} \arrow[r, "i"] & G \times G \arrow[d, "\mu"] \\ & & G \end{tikzcd} \qquad \begin{tikzcd} H^*(G) & H^*(G) \otimes H^0(G) \arrow[l, "\sim"'] & H^*(G) \otimes H^*(G) \arrow[l, "H^*(i)"'] \\ & & H^*(G) \arrow[u, "H^*(\mu)"'] \arrow[ull, "\mathrm{id}"] \end{tikzcd}$$

This means the component of $\mu^*(x)$ lying in $H^n(G) \otimes H^0(G)$ is $x \otimes 1$. The same argument run with the identity $\mu(e, -) \simeq \mathrm{id}_G$ yields the component $1 \otimes x$ in $H^0(G) \otimes H^n(G)$. So

$$\mu^*(x) = 1 \otimes x + x \otimes 1 + \sum(\deg \geqslant 1) \otimes (\deg \geqslant 1).$$

Recall that the cup product $\smile\colon H^*(G) \times H^*(G) \longrightarrow H^*(G)$ is induced in a similar way by the diagonal map $\Delta\colon G \longrightarrow G \times G$ taking $g \longmapsto (g, g)$; to wit, it can be understood as the composition

$$H^*(G) \otimes H^*(G) \longrightarrow H^*(G \times G) \xrightarrow{\Delta^*} H^*(G).$$

As Δ and μ admit some relations on a topological level, we recover some cohomological identities. Trivially but importantly, $\mu \times \mu$ is a map $\prod^4 G \longrightarrow \prod^2 G$ taking the quadruple (x, y, x, y) to the pair $\big(\mu(x, y), \mu(x, y)\big) = (\Delta \circ \mu)(x, y)$. If we write $\tau\colon G \times G \longrightarrow G \times G$ for the transposition switching the first and second coordinates, then $(x, y, x, y) = (\mathrm{id} \times \tau \times \mathrm{id})(x, x, y, y) = (\mathrm{id} \times \tau \times \mathrm{id})(\Delta \times \Delta)(x, y)$, so that

$$\Delta \circ \mu = (\mu \times \mu) \circ (\mathrm{id} \times \tau \times \mathrm{id}) \circ (\Delta \times \Delta). \tag{7.6}$$

Taking the cohomology of (7.6), being casual with Künneth maps again, and recalling the sign conventions for a tensor product of CGAs, one finds that for all homogeneous $a, b \in H^*(G)$,

$$\mu^*(ab) = \mu^*(a)\mu^*(b),$$

so that $\mu^*\colon H^*(G) \longrightarrow H^*(G) \otimes H^*(G)$ is a ring homomorphism. All this inspires the following definition.

Definition 7.2.2. A ***Hopf algebra*** over k is a graded (not necessarily associative) k-algebra A such that $A^0 \cong k$ equipped with an algebra homomorphism $\mu^*\colon A \longrightarrow A \otimes_k A$ such that

$$\mu^*(a) \equiv 1 \otimes a + a \otimes 1 \pmod{\widetilde{A} \underset{k}{\otimes} \widetilde{A}}$$

for each homogeneous $a \in A$. (Here $\widetilde{A} \lhd A$ is the augmentation ideal $\bigoplus_{i \geqslant 1} A^i \cong A/A^0$ of elements of positive degree, as defined in Appendix A.3.)

What we have shown is that, given an H-space G, its cohomology ring $H^*(G)$ is naturally a commutative, associative Hopf algebra. The presence of the coproduct imposes severe constraints on the algebra structure, especially with regard to algebra generators. [Hat02, Prop. 3C.4, p. 285] Here is Hopf's powerful structure theorem.

Theorem 7.2.3 (Hopf, $\operatorname{char} k = 0$: Hopf's theorem [Hop41, Satz I, p. 23] ; Borel, $\operatorname{char} k > 0$)**.** *Let k be a field and A a commutative, associative Hopf algebra over k such that $\dim_k A_n$ is finite for all n. As an algebra,*

- *if $\operatorname{char} k = 0$, then A is a free k-CGA,*
- *if $\operatorname{char} k = 2$, then $A \cong \Lambda V$ for an oddly-graded vector space V,*
- *if $\operatorname{char} k = p > 2$, then A is the tensor product of a free k-CGA and truncated symmetric algebras $k[\alpha]/(\alpha^{p^j})$, where α is even-dimensional.*

Proof [Hat02, p. 285]. We prove the result for $\operatorname{char} k = 0$ by induction on the number n of algebra generators, starting with $n = 0$ so the result is trivial. Inductively suppose we have shown the result for n generators and A is generated by $n+1$. Order these algebra generators $x_1, \ldots, x_n, y$ by weakly increasing degree, and let A' be the subalgebra generated by $x_1, \ldots, x_n$. This is actually a Hopf subalgebra, for $\mu^*(x_j) = 1 \otimes x_j + x_j \otimes 1 + (\deg\ < |y|)$, so the last term cannot involve y, and must lie in A'. Since μ^* is an algebra homomorphism, we must have $\mu^*(A') \leqslant A' \otimes A'$. Because A is a CGA generated by A' and x, there is a surjective k-algebra homomorphism

$$A' \otimes \Lambda[y] \longrightarrow\!\!\!\!\!\rightarrow A \quad \text{if } |y| \text{ is odd,}$$

$$A' \otimes S[y] \longrightarrow\!\!\!\!\!\rightarrow A \quad \text{if } |y| \text{ is even.}$$

To see A is free, it is enough to prove these maps are injective.

If $|y|$ is odd, suppose $a + by = 0$ in A, where $a, b \in A'$. Then $0 = \mu^*(a + by) \in A \otimes A$ projects under $A \otimes A \longrightarrow\!\!\!\!\!\rightarrow A \otimes (A \,/\!\!/\, A')$ to

$$0 = a \otimes 1 + (b \otimes 1)(y \otimes 1 + 1 \otimes y) = \underbrace{(a + by)}_{0} \otimes 1 + b \otimes y = b \otimes y.$$

This can only be zero if b is, but then $0 = a + 0y$, so $a = 0$ and our relation was trivial.

If $|x|$ is even, we instead have to deal with a potential nontrivial relation $\sum a_j y^j = 0$ with $a_j \in A'$. Assume the degree in y of this polynomial is minimal among nontrivial relations, and consider the image of $0 = \mu^* \sum a_j y^j \in A \otimes A$ under the projection to $A \otimes A/(\widetilde{A}', y^2) \cong A \otimes \Delta[y]$. Remembering that the a_j and y^2 map to 0 in the second tensor factor, we see the image is

$$0 = \sum (a_j \otimes 1)(y \otimes 1 + 1 \otimes y)^j = \underbrace{\sum a_j y^j}_{0} \otimes 1 + \sum a_j y^{j-1} \otimes y.$$

We must then have $\sum a_j y^{j-1} = 0$, contradicting the minimality of our relation $\sum a_j y^j = 0$. $\square$

Corollary 7.2.4. *Let G be a compact, connected Lie group. Then $H^*(G; \mathbb{Q})$ is an exterior algebra.*

Proof. We already know generates $H^*(G)$ is a free k-CGA, say on V. If V contained any even-degree elements, then by the theorem, $H^n(G)$ would be nontrivial for arbitrarily large n; but it cannot be, because G is a finite-dimensional CW complex. So V is oddly graded and $H^*(G) \cong \Lambda V$. □

Corollary 7.2.5. *Let G be a Lie group and $G \to E \to B$ a principal G-bundle and suppose $H^*(E) \longrightarrow H^*(G)$ surjects and k is a field of characteristic zero. Then there exists a k-CGA isomorphism*

$$H^*(E) \cong H^*(B) \otimes H^*(E).$$

Proof. By Corollary 4.3.9, one has an $H^*(B)$-module isomorphism $H^*(E) \cong H^*(B) \otimes H^*(G)$. By Corollary 7.2.4, $H^*(G)$ is a free k-CGA, so by Proposition A.4.4, a lifting of $H^*(E) \twoheadrightarrow H^*(G)$ induces a ring isomorphism $H^*(B) \otimes H^*(G) \xrightarrow{\sim} H^*(E)$. □

We can do a bit better in identifying the generators of $H^*(G)$.

Definition 7.2.6. We call x ***primitive*** if $\mu^*(x) = 1 \otimes x + x \otimes 1$. Write

$$PA = \{x \in A : x \text{ is primitive}\}$$

for the ***primitive subspace*** and grade this space by $P^r A = PA \cap A^r$. Note that the only primitive in $A^0 \cong k$ can be the identity so that $P^0 A = 0$ and PA is contained in the augmentation ideal $\widetilde{A}$. If $A = H^*(G)$ is the cohomology ring of an H-space G, we abbreviate $PG := PH^*(G)$. Another way to phrase the definition is to say that PA is the kernel of the k-linear homomorphism

$$\begin{aligned} \psi\colon A &\longrightarrow A \otimes A, \\ x &\overset{\psi}{\longmapsto} \mu^*(x) - (1 \otimes x + x \otimes 1). \end{aligned}$$

There is a natural k-linear composite map

$$P(A) \hookrightarrow \widetilde{A} \twoheadrightarrow \widetilde{A}/\widetilde{A}\widetilde{A} =: Q(A)$$

linking primitives and indecomposables (for which, see Appendix A.3.2), which is an isomorphism in the case we care about.

Proposition 7.2.7 (Milnor–Moore)**.** *Let A be a commutative, cocommutative Hopf algebra finitely generated as an algebra over a field k. Then this canonical map takes $P(A) \xrightarrow{\sim} Q(A)$. In particular, A is generated by primitive elements.*

Proof. The strong statement is more than we need, but we will prove the result in the case A is a coassociative Hopf algebra over a field k of characteristic $\neq 2$ with underlying algebra an exterior algebra, following Mimura and Toda [MT00, p. 369] for injectivity; this weaker version is due to Hopf and Samelson. Write $A = \Lambda V$, for V an oddly-graded vector space. That $V \xrightarrow{\sim} Q(A)$ is clear, so we just need to show V can be chosen such that $P(A) = V$.

Pick a basis X of V. By anticommutativity, a basis of ΛV is given by monomials $y = x_1 x_2 \cdots x_n$ with $x_i \in X$ of weakly increasing degree. If $n > 1$, then we have

$$\mu^*(y) = \prod \mu^*(x_i) = \prod \big(x_1 \otimes 1 + 1 \otimes x_i + (\cdots)\big) = 1 \otimes y + [x_1 \otimes x_2 \cdots x_n] + \sum a \otimes b,$$

where none of the terms $a \otimes b$ have $a \in \mathbb{Q}x_1$. It follows the term $x_1 \otimes x_2 \cdots x_n$ doesn't cancel, and thus $\mu^*(y) \neq y \otimes 1 + 1 \otimes y$, so $P(A) \leqslant V$.

For the other containment, we induct on $\dim V$. Assume the result is proved for n, and that $\dim V = n + 1$. Arrange a homogeneous basis $x_1, \ldots, x_n, y$ of V in weakly increasing degree. By induction, $V' = \mathbb{Q}\{x_1, \ldots, x_n\}$, where we may choose x_j primitive, and it remains to show

y is. Since each x_j is primitive, we have $\mu^*(x_j) \leqslant \Lambda[x_j] \otimes \Lambda[x_j]$ for each j, so the coproduct μ^* descends to a coproduct $\overline{\mu^*}$ on $\Lambda V /\!\!/ \Lambda[x_j]$, and since this is an exterior algebra on n generators, by induction, we have $\overline{\mu^*}(y) = 1 \otimes y + y \otimes 1$ in this quotient, so back in $\Lambda V \otimes \Lambda V$, the difference $\psi(y) := \mu^*(y) - (1 \otimes y + y \otimes 1)$ lies in the ideal $(x_j \otimes 1, 1 \otimes x_j)$. Varying j, we see $\psi(y)$ lies in the the intersection of all these ideals. If we write $x_I := \prod_{i \in I} x_i$, this intersection ideal is that generated by the tensor products $x_I \otimes x_J$ such that $I \amalg J = \{1, \ldots, n\}$ is a partition. In fact, since by definition $\psi(y) \in \widetilde{A} \otimes \widetilde{A}$, it lies in the ideal generated by $x_I \otimes x_J$ with neither I nor J empty. We are then done unless $|y| = \sum_{i=1}^n |x_i|$, so assume this equality holds. Then since $\psi(y)$ is homogeneous and the generating elements $x_I \otimes x_J$ already have the right degree, we can write

$$\psi(y) = \sum_{I \amalg J = \{1,\ldots,n\}} a_{I,J} x^I \otimes x^J$$

for some scalars $a_{I,J} \in k$.

The fact that $(\mu^* \otimes \mathrm{id})\mu^* = (\mathrm{id} \otimes \mu^*)\mu^*$, the coassociativity of A, follows for $H^*(G)$ from the associativity of the multiplication on G. It is not hard to see this is equivalent to the condition $(\psi \otimes \mathrm{id})\psi = (\mathrm{id} \otimes \psi)\psi$. Applying this equation to y we obtain

$$\sum a_{I,J} \psi(x_I) \otimes x_J = \sum a_{I,J} x_I \otimes \psi(x_J),$$

where the sum runs over paritItions $I \amalg J = \{1, \ldots, n\}$ with $I \neq \varnothing \neq J$. These equations expand to

$$\sum a_{I,J} \sum_{I_1, I_2} x_{I_1} \otimes x_{I_2} \otimes x_J = \sum a_{I,J} \sum_{J_1, J_2} x_I \otimes x_{J_1} \otimes x_{J_2},$$

where $I \amalg J = \{1, \ldots, n\}$ as before and in the sums on either side, one has $I_1 \amalg I_2 = I$ and $J_1 \amalg J_2 = J$, and $I, J, I_1, I_2, J_1, J_2 \neq \varnothing$. Fix a partition $I_1 \amalg I_2 \amalg J = \{1, \ldots, n\}$. The coefficients of $x_{I_1} \otimes x_{I_2} \otimes x_J$

on the left-hand side and the right, which must consequently be equal, are $a_{I,J}$ and $a_{I_1,I_2 \amalg J}$. These equalities show all $a_{I,J}$ are equal to some single scalar $a \in k$, so

$$\psi(y) = a \sum_{I,J \neq \varnothing} x_I \otimes x_J = a\psi(x_1 \cdots x_n),$$

or $\psi(y - ax_1 \cdots x_n) = 0$. Thus $x_1, \ldots, x_n, y - ax_1 \cdots x_n$ is a set of primitive generators of A. □

Remark 7.2.8. An analogous result holds in characteristic 2 with the weaker assumption on A that it not necessarily be an exterior algebra, but merely admit a simple system of generators (see Definition A.3.4). The proof is correspondingly much more difficult.

We will later need as well the fact that a map of H-spaces induces a map of primitives in cohomology.

Proposition 7.2.9. *Let $\phi\colon K \longrightarrow G$ be a homomorphism of H-spaces. Then the map $\phi^*\colon H^*(G) \longrightarrow H^*(K)$ in cohomology takes $PG \longrightarrow PK$.*

Proof. To ask a linear homomorphism ϕ be multiplicative is precisely to require $\mu_G \circ (\phi \times \phi) = \phi \circ \mu_K$. In cohomology, then, if $z \in PG$ is primitive, we have

$$\mu_K^* \phi^* z = (\phi^* \otimes \phi^*)\mu_G^* z = (\phi^* \otimes \phi^*)(1 \otimes z + z \otimes 1) = 1 \otimes \phi^* z + \phi^* z \otimes 1. \qquad \square$$

There is a further theorem determining $\dim PG$.

Theorem 7.2.10 (Hopf [Hop40, p. 119])**.** *Let G be a compact, connected Lie group and T a maximal torus. Then the total Betti number $h^\bullet(G) = 2^{\dim T}$.*

Proof [Sam52]. By the preceding theorem, $H^*(G;\mathbb{Q})$ is an exterior algebra, so from Appendix A.3.2 we see $h^\bullet(G) = 2^l$ for some $l \in \mathbb{N}$. To see that $l = \dim T$, consider the squaring map $s\colon g \longmapsto g^2$

on G. Since $s = \mu \circ \Delta$, it follows that for a primitive $a \in H^*(G)$ one has

$$s^*a = \Delta^*\mu^*a = \Delta^*(1 \otimes a + a \otimes 1) = 1 \smile a + a \smile 1 = 2a,$$

so if $[G] \in H^{\dim G}(G)$ is the fundamental class, the product of l independent primitives, one has $s^*[G] = 2^l[G]$. Thus the degree of s is 2^l. On the other hand, restricting to the abelian subgroup $T \cong (\mathbb{R}/\mathbb{Z})^n$, it is easy to see the s-preimage of a generic element of T contains $2^{\dim T}$ points, which, since s is orientation-preserving, should each be counted with multiplicity 1. By a standard theorem on degree [Hat02, Ex. 3.3.8, p. 258] we then know $2^{\dim T} = \deg s = 2^l$, so $l = \dim T$. □

The only reason we cite Samelson's recounting of Hopf's proof is that the report it is taken from is already in English. These results also let us obtain a classical topological fact usually proven through other means.

Corollary 7.2.11 ([BtD85, Prop. V.(5.13), p. 225])**.** *The second homotopy group $\pi_2 G$ of a compact Lie group G is trivial.*

Proof. The universal compact cover $\widetilde{G}$ of G (see Theorem B.4.4) satisfies $\pi_2\widetilde{G} \cong \pi_2 G$ by the long exact homotopy sequence of a bundle Theorem B.2.4, and $\widetilde{G} \cong A \times K$ for A a torus and K simply connected. Using the long exact homotopy sequence of the short exact sequence $0 \to \mathbb{Z}^n \to \mathbb{R}^n \to T^n$, one sees $\pi_2 A = 0$, and since $\pi_1 K = 0$, successively applying the Hurewicz theorem, the universal coefficient theorem, and Hopf's theorem, one finds $\pi_2 K \cong H_2 K \cong H^2 K = 0$, so $\pi_2\widetilde{G} \cong \pi_2 A \times \pi_2 K = 0$. □

Remark 7.2.12. The multiplication on a Lie group G induces a product on $H_*(G;\mathbb{Q})$, the *Pontrjagin product*, making it a Hopf algebra as well, the *homology ring*, which is dual to $H^*(G;\mathbb{Q})$. It is this ring that Hopf originally discovered the structure of, though the way he put it was that the homology ring of G was isomorphic to that of a product $\prod S^{2n_j-1}$ of odd-dimensional spheres. Serre found later [FHT01, p. 216] that this was actually due to a *rational homotopy equivalence*: there is a map $\prod S^{2n_j-1} \longrightarrow G$ inducing isomorphisms

$$\pi_*\Big(\prod S^{2n_j-1}\Big) \otimes \mathbb{Q} \xrightarrow{\sim} \pi_*(G) \otimes \mathbb{Q}$$

on rational homotopy groups. Because the rational Hurewicz map

$$\pi_*\Big(\prod S^{2n_j-1}\Big) \otimes \mathbb{Q} \longrightarrow H_*\Big(\prod S^{2n_j-1}; \mathbb{Q}\Big)$$

is an isomorphism when restricted to the span $\bigoplus \mathbb{Q} \cdot [S^{2n_j-1}]$ of the fundamental classes of the factor spheres, the image of the Hurewicz map $\pi_*(G) \otimes \mathbb{Q} \longrightarrow H_*(G; \mathbb{Q})$ contains the homological primitives $P_*(G) = PH_*(G)$. By Remark 4.3.17, then, these primitives are in the image of the transgression in the homological Serre spectral sequence of any G-bundle.

7.3. The Serre spectral sequence of $S^1 \to ES^1 \to BS^1$

The ideological mainspring of all the spectral sequence calculations we will do in the rest of this document is a sequence that is only two pages page long, the Serre sequence of the universal principal circle bundle $S^1 \to ES^1 \to BS^1$.[4] We use our knowledge of $H^*(S^1)$ and $H^*(ES^1)$ to work out $H^*(BS^1)$.

[4] We earlier, in Section 3.2, identified $S^\infty \to \mathbb{C}\mathrm{P}^\infty$ as a model, but the calculation actually does not require this topological knowledge.

Proposition 7.3.1. *The cohomology of $BS^1 = \mathbb{CP}^\infty$ is given by*

$$H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[u], \quad \deg u = 2.$$

Proof. By Proposition 4.3.6, $\pi_1 BS^1$ acts trivially on $H^*(S^1)$, so we can use untwisted coefficients in Theorem 4.3.4.[5] Thus we can write

$$E_2^{p,q} = H^p\big(BS^1; H^q(S^1;\mathbb{Z})\big)$$

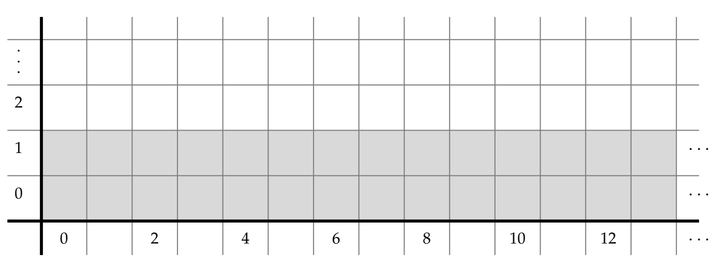
As the total space ES^1 is contractible, its cohomology ring $H^*(ES^1)$ is that of a point, a lone $\mathbb{Z}$ in dimension zero, and the associated graded ring E_∞ again $\mathbb{Z}$ because the filtration is trivial.

The cohomology $H^*(S^1)$ is an exterior algebra $\Lambda[z_1]$, where $z_1 \in H^1(S^1)$ is the fundamental class, so in particular it is a graded free abelian group, and

$$E_2^{p,q} \cong H^p(BS^1) \otimes H^1(S^1).$$

Since the second factor is nonzero only for $q \in \{0, 1\}$, the entire sequence is concentrated in these two rows.

Figure 7.3.2: The potentially nonzero region in the Serre spectral sequence of $S^1 \to ES^1 \to BS^1$



Thus $d = d_2$ is the only differential between nonzero rows, so $E_3 = E_\infty = \mathbb{Z}$ and d *must kill*

[5] In fact, from the homotopy long exact sequence of $S^1 \to ES^1 \to BS^1$, it follows that $\pi_2 BS^1 \cong \mathbb{Z}$ is its only nonzero homotopy group, so $\mathbb{CP}^\infty \simeq BS^1$ is an Eilenberg–Mac Lane space $K(\mathbb{Z}, 2)$. In particular, BS^1 is in particular simply-connected.

everything else in E_2. Because the rows $E_2^{\bullet,q} = 0$ except for $q \in \{0,1\}$ and d decreases q by 1, the complex (E_2, d) breaks, for each $p \in \mathbb{Z}$, into short complexes

$$0 \to E_2^{p,1} \longrightarrow E_2^{p+2,0} \to 0.$$

Because the SSS is concentrated in the first quadrant, all groups in the short complex are definitionally zero for $p < -2$. For $p = -2$, we have the very short complex

$$0 \to E_2^{0,0} \to 0,$$

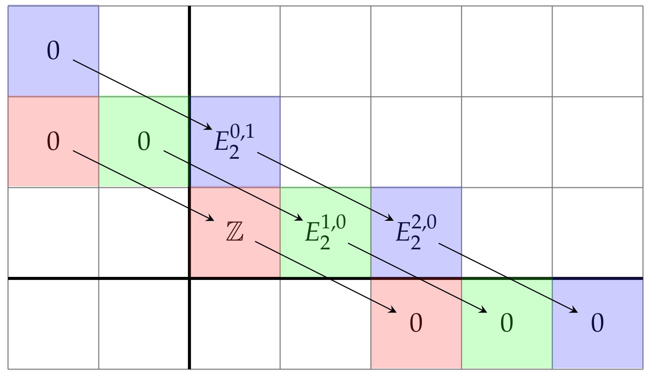
red in Figure 7.3.3, witnessing the apotheosis of $E_2^{0,0} \cong \mathbb{Z}$ to $H^0(ES^1) = E_\infty$. This in fact happens for any SSS where the fiber and base are path-connected, and *must* happen, since $H^0 = \mathbb{Z}$ for all three spaces.

For $p = -1$, we have the very short sequence

$$0 \to E_2^{1,0} \to 0,$$

green in Figure 7.3.3. The middle object must zero because otherwise it would survive to $E_3 = E_\infty$, which would mean $H^1(ES^1) \neq 0$. (Then again, we already knew this because BS^1 is simply-connected and H^0 is always free abelian, so that the universal coefficient theorem B.2.1 yields $H^1(BS^1) \cong H_1(BS^1) \cong \pi_1(BS^1)^{\mathrm{ab}} = 0$.)

Figure 7.3.3: The first few subcomplexes of E_2 in the Serre spectral sequence of $S^1 \to ES^1 \to BS^1$



For $p \geqslant 0$, the total degrees $p+1$ and $p+2$ are positive, so that both groups in the short complex must die in E_3. The only way this can happen is if the d linking them is both injective and surjective, so an isomorphism: that is,

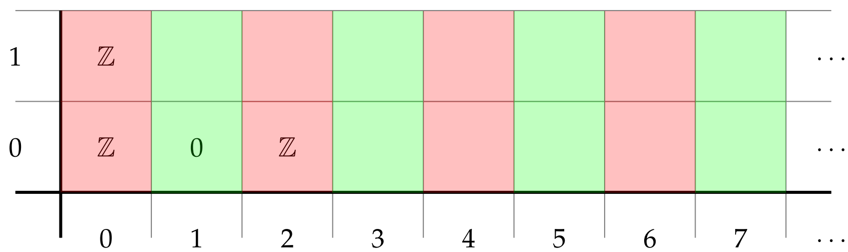
$$E_2^{p,1} \cong E_2^{p+2,0} \quad \text{for all } p \geqslant 0.$$

The first occurrence of this, for $p = 0$, is blue in Figure 7.3.3. On the other hand, the simple fact that $H^0(S^1) \cong \mathbb{Z} \cong H^1(S^1)$ as abstract groups implies, on tensoring with $H^p(BS^1)$, that likewise

$$E_2^{p,0} \cong E_2^{p,1}.$$

Assembling these isomorphisms, all groups in even columns $p = 0, 2, 4, \ldots$ (red in Figure 7.3.4), and all groups in odd columns (green) are isomorphic. The base cases $E_2^{0,0} = H^0(ES^1) = \mathbb{Z}$ and $E_2^{1,0} = \pi_1 BS^1 = 0$ then determine all the other entries: zero in odd columns and $\mathbb{Z}$ in even.

Figure 7.3.4: The partitioning by isomorphism class of groups $E_2^{p,q}$ in the Serre spectral sequence of $S^1 \to ES^1 \to BS^1$



Reading off the bottom row $E_2^{\bullet,0} \cong H^*(BS^1) \otimes H^0(S^1) \cong H^*(BS^1)$, we find the cohomology groups of $BS^1 = \mathbb{C}\mathrm{P}^\infty$ are

$$H^n(\mathbb{C}\mathrm{P}^\infty) = \begin{cases} \mathbb{Z} & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases}$$

Recall that the differential $d = d_2$ was an antiderivation restricting to an isomorphism $H^1(S^1) \xrightarrow{\sim} H^2(BS^2)$. If we write $u = dz \in H^2(BS^2)$ for the image of the fundamental class of S^1, then since $du = 0$, applying the product rule yields

$$d(u^{k+1}z) = (k+1)\underbrace{du}_{0} \cdot u^k z \,+\, u^{k+1} \cdot \underbrace{dz}_{u} = u^{k+2}$$

for $k \geqslant 0$. Since this d is an isomorphism $E_2^{2k,1} \xrightarrow{\sim} E_2^{2k+2,0}$ and z and u are nonzero, it follows by induction that u^k generates $H^{2k}(\mathbb{C}\mathrm{P}^\infty)$ for all k. □

We could more easily have found the graded group structure of $H^*(\mathbb{C}\mathrm{P}^\infty)$ through cellular cohomology after pushing down the increasing union $S^\infty = S^1 \cup S^3 \cup S^5 \cup \cdots$ to a strictly even-dimensional CW structure $\mathbb{C}\mathrm{P}^\infty = e^0 \cup e^2 \cup e^4 \cup \cdots$, but the spectral sequence also makes computing the ring structure almost trivial.

For later reference, note that, topology aside, the calculation we just made is a manifestation of the following algebraic fact. Define B to be the graded ring $\mathbb{Z}[u]$, where $\deg u = 2$, and assign it the trivial differential. Let A be the graded ring $B \otimes \Lambda[z]$, where $\deg z = 1$. Make A a $\mathbb{Z}$-CDGA extending $(B, 0)$ by assigning as differential the unique antiderivation d that vanishes on 0 and satisfies

$$dz = u.$$

Then (A, d) is acyclic: $H^0(A) = \mathbb{Z}$ and $H^n(A) = 0$ for $n > 0$. The reason we were able to deduce $H^*(\mathbb{C}\mathrm{P}^\infty) = \mathbb{Z}[u]$ is that $\mathbb{Z}[u]$ is the unique B that makes an $A = B \otimes \Lambda[z]$ constructed as above acyclic.

7.4. The Serre spectral sequence of $T \to ET \to BT$

The circle is the one-dimensional case of the ***torus*** $T^n = \prod^n S^1$. By the Künneth theorem, one has

$$H^*(T^n) \cong \bigotimes^n H^*(S^1) = \bigotimes^n \Lambda[z] = \Lambda[z_1, \ldots, z_n] = \Lambda H^1(T^n),$$

where z_j is the fundamental class of the j^{th} factor circle and $H^1(T^n) = \mathbb{Z}\{z_1, \ldots, z_n\}$ is the primitive subspace as discussed in Proposition 7.2.7.

To understand $H^*(BT)$, there are at least two options. The first is an analysis analogous to, but more intricate than, that in the last section: one sees easily $d_2 \colon H^1(T) \longrightarrow H^2(BT)$ must be an isomorphism and then puts more work into showing that means d_2 is injective on the entire first column $E_2^{0,\bullet} \cong H^*(T)$ and that $E_3 = E_\infty = \mathbb{Z}$. The second invokes the functoriality of the universal principal bundle construction $G \longmapsto (G \to EG \to BG)$ to make the problem trivial. As the functors E and B preserve products, one has the bundle isomorphism

$$\begin{tikzcd} T \arrow[r, "\sim"] \arrow[d] & \prod S^1 \arrow[d] \\ ET \arrow[r, "\sim"] \arrow[d] & \prod ES^1 \arrow[d] \\ BT \arrow[r, "\sim"] & \prod BS^1, \end{tikzcd}$$

so that $BT = \prod^n \mathbb{C}\mathrm{P}^\infty$ and $H^*(BT) = \bigotimes \mathbb{Z}[u_j] \cong \mathbb{Z}[u_1, \ldots, u_n]$.

The bundle isomorphism in fact induces a Künneth isomorphism of SSSs, so that

$$E_2 = \bigotimes_{j=1}^{n} \big(S[u_j] \otimes \Lambda[z_j]\big) \cong S[\vec{u}] \otimes \Lambda[\vec{z}],$$

with differential d_2 the unique antiderivation annihilating $S[\vec{u}]$ and taking $z_j \mapsto u_j$ for each j. Thus

$$\Big(S[\vec{u}] \otimes \Lambda[\vec{z}], \quad z_j \mapsto u_j\Big)$$

is another example of an acyclic CDGA.

7.5. The Koszul complex

In the spectral sequences of universal bundles $T \to ET \to BT$, the cohomology $H^*(T)$ of the fiber is an exterior algebra and the cohomology H^*_T of the base is a polynomial algebra on the same number of generators, and the algebra generators of fiber and base cancel one another in a one-to-one fashion in the spectral sequence. We claim and will later see that this pattern holds for all compact, connected Lie groups G. If it does, then because the d_r are antiderivations, the edge isomorphisms $d_r\colon E_r^{0,r-1} \xrightarrow{\sim} E_r^{r,0}$ determine the d_r entirely. We have already seen this pattern to obtain for $G = T$ a torus, although this is somewhat evident because the only nontrivial differential is d_2. In order to show this situation holds more generally, we first formalize it.

Because the bijection of generators is in some sense the main feature of these spectral sequences, one could regard the fact these cancellations occur on different pages as an artifact of the filtration, and, regarding all later pages as subpages of E_2 (which makes sense because all pages are free algebras, hence projective) instead consider $d = \sum_r d_r$, on E_2, as the one true differential, letting all cancellation happen at the same time, and one would expect to still have $H_d(E_2) = \mathbb{Z}$. This does not completely make sense, because the edge homomorphisms d_r are maps from subalgebras of E_2 to quotients of it, but it nearly does, and the idea motivates the following definition.

Definition 7.5.1. $V = \bigoplus_{j>0} V_{2j-1}$ be a positively- and oddly-graded free graded k-module. The grading on V induces a grading on ΛV making it a free CGA. Let $\Sigma V = V_{\bullet-1}$ be the ***suspension***, the regrading of V, concentrated in even degree, defined by $(\Sigma V)_j := V_{j-1}$.[6] There is a naturally induced grading on the symmetric algebra $S\Sigma V$, making it a free CGA.

Let $KV := S\Sigma V \otimes \Lambda V$. Because $S^1[\Sigma V] \oplus \Lambda^1[V]$ generates KV as a k-algebra, there is a unique antiderivation ***d*** of degree 1 on KV that restricts on $\Lambda^1 V$ to the defining isomorphism

$$d = \Sigma\colon \Lambda^1[V] \xrightarrow{\sim} V \xrightarrow{\sim} S^1[\Sigma V]$$

of ungraded free k-modules. Consequently, $dS^1[\Sigma V] = 0$ and hence $d(S\Sigma V) = 0$. The complex (KV, d) is the ***Koszul complex*** associated to V. Write $K^n[V] = \bigoplus S^j[\Sigma V] \otimes \Lambda^{n-j}[V]$ for the submodule of KV spanned by products of n generators. This grading of KV, the ***multiplicative grading***, induces a grading of $H_d(KV)$ such that $H^n(KV)$ is the image of the cochains in $K^n[V]$.

[6] The notation is meant to suggest the suspension ΣX of a topological space X, which satisfies $H^n(X) \cong H^{n+1}(\Sigma X)$.

Given a basis (v_j) of V and associated basis (dv_j) of ΣV, the bijections $d\colon kv_j \xrightarrow{\sim} kdv_j$ cancel v_j and dv_j from the cohomology $H(K)$, just like the edge maps in the SSS of $T \to ET \to BT$, so we expect the cohomology to be trivial.

Proposition 7.5.2 (Koszul)**.** *Let k be such that each natural $n \cdot 1$ is invertible.*[7] *Then the Koszul complex is acyclic.*

First proof [Car51, Thm. 1]. The inverse isomorphism $h = d^{-1}\colon S^1[\Sigma V] \xrightarrow{\sim} \Lambda^1[V]$ extends uniquely, just as d does, to an antiderivation of KV of degree -1. We claim it is a chain homotopy of (KV, d).

The composition dh is the projection $K^1[V] \to S^1[V]$ and hd the projection $K^1[V] \to \Lambda^1[V]$, so $hd + dh = \mathrm{id}$ on $K^1[V]$. Inductively assume that also $L = dh + hd = n\,\mathrm{id}$ on $K^n[V]$. Write a decomposable (e.g., basis) element of $K^{n+1}[V]$ as ab, for $a \in K^1[V]$ and $b \in K^n[V]$. Then by the product rule, the base case, and the inductive assumption,

$$L(ab) = (La)b + aL(b) = ab + nab = (n+1)ab,$$

concluding the induction.

For any n-cocycle a we then have $na = (hd + dh)a = dha$, so each d-cocycle is a coboundary for $n \geqslant 1$. Thus $H(KV) = H^0(KV) \cong k$. □

The same argument incidentally also shows the h-cohomology of KV is trivial.

Second proof. Find a k-basis v_j of V, so that $V = \bigoplus kv_j$ and $\Sigma V = \bigoplus kdv_j$. Then we have algebra isomorphisms

$$KV = S\Sigma V \otimes \Lambda V \cong S\left[\bigoplus kdv_j\right] \otimes \Lambda\left[\bigoplus kv_j\right] \cong \bigotimes \left(S[dv_j] \otimes \Lambda[v_j]\right) = \bigotimes K[kv_j],$$

[7] Functionally, this means k contains $\mathbb{Q}$.

and this also holds on the level of CDGAs, because the coproduct differential induced by the Koszul differential on each $K[v_j]$ and the original Koszul differential d on KV are both antiderivations on KV extending the linear isomorphism $\Lambda^1[V] \xrightarrow{\sim} S^1[\Sigma V]$ and such an extension is unique. Because everything in sight is a free k-module, the simplest version of the algebraic Künneth formula Corollary A.3.10 holds, and

$$H^*_d(KV) \cong \bigotimes_j H^*_{d_j}\big(K[v_j]\big) \cong k^{\otimes j} \cong k. \qquad \square$$

We cite here an algebraic lemma for later use.

Proposition 7.5.3. *Let k be a field and $d\colon V \longrightarrow W$ a k-linear map of finite-dimensional k–vector spaces. Extend d uniquely to an antiderivation D on $\Lambda V \otimes SW$ annihilating SW. Then resulting k-CDGA admits a factorization as*

$$(\Lambda V \otimes SW, d) \;\cong\; \big(\Lambda[\ker d], 0\big) \,\otimes\, K[\operatorname{coim} d] \,\otimes\, \big(S[\operatorname{coker} d], 0\big).$$

Proof. Because $V \oplus W$ admits a vector space decomposition

$$(\ker d \oplus \operatorname{coim} d) \oplus (\operatorname{im} d \oplus \operatorname{coker} d),$$

$\Lambda V \otimes SW$ admits an algebra decomposition

$$\Lambda[\ker d] \otimes \big(\Lambda[\operatorname{coim} d] \otimes S[\operatorname{im} d]\big) \otimes S[\operatorname{coker} d]$$

The $\Lambda[\ker d]$ and $S[\operatorname{coker} d]$ factors are annihilated by D and intersect $\operatorname{im} D$ trivially, while by the definition of image and coimage, d induces the linear isomorphism $\operatorname{coim} d \xrightarrow{\sim} \operatorname{im} d$ of the first isomorphism theorem, so that the factor $\Lambda[\operatorname{coim} d] \otimes S[\operatorname{im} d]$ is the Koszul algebra $K[\operatorname{coim} d]$. □

The Koszul complex, which makes its first appearance in thesis work of Koszul dealing with Lie algebra cohomology, which had recently been defined by Chevalley and Eilenberg, was soon discovered to have important uses in commutative algebra. Here is a more general definition.

Definition 7.5.4. Let A be a unital commutative ring over k. Given a sequence $\vec{a} = (a_j)_{j \in J}$ of elements of A, we can form an abstract free k-CGA $\Lambda[z_j]_{j\in J} = \bigotimes_{j \in J} \Lambda[z_j]$ and the tensor algebra

$$K_A\vec{a} := \Lambda[z_j]_{j\in J} \otimes_k A,$$

and make $K_A\vec{a}$ a CDGA by extending the k-linear map $\bigoplus_{j\in J} kz_j \longrightarrow A$ given by $z_j \longmapsto a_j$ to an antiderivation d. We grade $K_A\vec{a}$ by multiplicative degree in the exterior factor, so that

$$K_A^{-n}\vec{a} := \Lambda^n[z_j]_{j\in J} \otimes A,$$

and $\deg d = 1$, and call this grading the ***resolution grading***. The k-CDGA $(K_A\vec{a}, d)$ is the ***Koszul complex*** associated to the sequence $\vec{a}$.

Given an A-module M, the tensor product module

$$K_A(\vec{a}, M) := K_A\vec{a} \underset{A}{\otimes} M = \big(\Lambda[z_j] \underset{k}{\otimes} A\big) \underset{A}{\otimes} M \cong \Lambda^p[z_j] \underset{k}{\otimes} M,$$

inherits a differential, vanishing on M, given by

$$\begin{aligned} d(1 \otimes m) &= 0, & m &\in M, \\ d(z_j \otimes m) &= 1 \otimes a_j m, & j &\in J, \end{aligned}$$

and the resulting chain complex is again called a ***Koszul complex***.

Koszul complexes $K_A(\vec{a}, M)$ being defined by sequence of ring elements, their potential acyclicity is related to properties of this sequence.

Definition 7.5.5. Let A be a unital commutative ring over k. A finite or countable sequence (a_j) of elements of A is called a ***regular sequence*** if for each n, the image of a_n is not a zero-divisor in the quotient ring $A/(a_1, \ldots, a_{n-1})$. Given an A-module M, the same sequence is called ***M-regular*** (or an *M-sequence*) if each a_n annihilates no nonzero elements of the quotient module $M/(a_1, \ldots, a_{n-1})M$. An ideal $\mathfrak{a} \trianglelefteq A$ is called a ***regular ideal*** if it can be generated by a regular sequence.

Regular sequences do not normally remain regular under permutation, but do if all elements lie in the Jacobson radical of A, and in particular if A is a local ring and the elements a_j are non-units [Eis95, Cor. 17.2, p. 426].

Proposition 7.5.6. *Let A be a connected CGA and a_j elements of $\widetilde{A}$; then the sequence (a_j) is regular just if each permutation is.*

Since we really care only about cohomology rings, order in a regular sequence shall never be an issue for us. The connection between Koszul complexes and regular sequences is the following.

Proposition 7.5.7 ([Ser00, IV.A.2, Prop. 3, p. 54])**.** *Given a Noetherian commutative ring A, a sequence $\vec{a}$ of elements of the Jacobson radical of A, and a finitely-generated A-module M, the following conditions are equivalent:*

1. $H^{-n}\big(K_A(\vec{a}, M)\big) = 0$ *for* $n \geqslant 1$*;*
2. $H^{-1}\big(K_A(\vec{a}, M)\big) = 0$*;*
3. *the sequence $\vec{a}$ is M-regular.*

The last relevant fact about Koszul complexes is that they compute Tor.

Proposition 7.5.8. *Let $A = S[\vec{a}]$ be a free commutative k-CGA generated by a sequence $\vec{a}$ of elements of even degree, and let B be an A-CGA. Then the Koszul complex $K_A(\vec{a}, B)$ associated to $\vec{a}$ computes* Tor *in that*

$$H^{-p}(K_A\vec{a} \otimes_A B) \cong \operatorname{Tor}_p^A(k, B), \qquad p \geqslant 0.$$

Proof. The base ring k is an A-algebra in a natural way via $A \longrightarrow\!\!\!\!\rightarrow A/\widetilde{A} \cong k$. Since the generators are independent, by Proposition 7.5.7, the Koszul complex $(K_A\vec{a}, d)$ is acyclic, with

$$H^*(K_A\vec{a}) = H^0(K_A\vec{a}) \cong k[\vec{a}]/(\vec{a}) \cong k.$$

It follows that $K_A^{\bullet}\vec{a}$, with the resolution grading from Definition 7.5.4, is an A-module resolution of k, so that the $-p^{\text{th}}$ cohomology of the sequence

$$\cdots \longrightarrow K_A^{-2}\vec{a} \otimes_A B \longrightarrow K_A^{-1}\vec{a} \otimes_A B \longrightarrow K_A^{0}\vec{a} \otimes_A B \longrightarrow 0$$

computes $\operatorname{Tor}_p^A(k, B)$. $\square$

Note that in fact $\operatorname{Tor}_{\bullet}^A(k, B)$ is a bigraded CGA. The product descends from the product on $\Lambda[z_j] \otimes_k B$, and the second component of the grading from the grading $\bigoplus B^q$ on B. We set

$$\operatorname{Tor}_A^{-p,q}(k, B) = \operatorname{Tor}_p^A(k, B^q) = H^p\big(\Lambda[z_j] \underset{k}{\otimes} B^q\big).$$

Historical remarks 7.5.9. Regular sequences were introduced by Serre in 1955 as *E-sequences* [Bor67, p. 93], and this terminology apparently hung on for quite a while [Bau68, Def. 3.4]. Smith [Smi67, p. 79] uses *ESP-sequence* and calls a graded ideal generated by such a sequence a *Borel ideal*.

7.6. The Serre spectral sequence of $G \to EG \to BG$

"... the behavior of this spectral sequence ... is a bit like an Elizabethan drama, full of action, in which the business of each character is to kill at least one other character, so that at the end of the play one has the stage strewn with corpses and only one actor left alive (namely the one who has to speak the last few lines)."[8] —*J. F. Adams*

We have found $H^*(BT)$ for all tori and by Corollary 6.4.7, we know that $H^*(BG;\mathbb{Q})$ can be viewed as the Weyl-invariant subring $H^*(BT;\mathbb{Q})^W$, so theoretically, we understand $H^*(BG)$ now. In practice, and especially if one wants to understand the situation with $\mathbb{Z}$ coefficients—something we will eventually punt on—there is more work to be done.

In the torus computation, the algebra generators $H^1(T) = PH^*(T)$ of $H^*(T)$ (the primitives, as defined in Section 7.2) and $H^2(BT) \cong QH^*(BT)$ of $H^*(BT)$ (the indecomposables, as defined in Appendix A.3.2) were linked bijectively by nontrivial differentials and were annihilated, and the algebraic repercussions of this bijection sufficed to force $E_\infty = \mathbb{Z}$. To work with merely generators greatly simplifies any computation, so one might hope that such a pattern holds as well for nonabelian groups. The proof of this result is due to Borel in his thesis [Bor53].

Theorem 7.6.1 (Borel [Bor53, Théorème 13.1].). *Let k be a commutative ring, P an oddly-graded free k-module, and ΛP the exterior algebra on P. Suppose (E_r, d_r) is a spectral sequence of bigraded k-algebras such that E_2 admits a tensor decomposition $E_2^{\bullet,0} \otimes E_2^{0,\bullet}$ with $E_2^{0,\bullet} \cong \Lambda P$ and $E_\infty = E_\infty^{0,0} \cong k$. Then $E_2^{\bullet,0} \cong S[\Sigma P]$ is a symmetric algebra on the suspension of P.*

In particular, if we let k be a field, so that $H^*(G;k)$ is an exterior algebra, and apply the result to the Serre spectral sequence of the universal bundle $G \to EG \to BG$, we find $H^*(BG;k)$ is a

[8] This deservedly popular description arises in a description [Ada76] of the behavior of the Adams spectral sequence.

polynomial ring.

Proof [FOT08, p. 39]. The Koszul complex KP associated to P, as an algebra, is the tensor product $S[\Sigma P] \otimes \Lambda P$. The symmetric algebra factor $S[\Sigma P]$ has a natural even grading grading induced by the even grading $(\Sigma P)_p \cong P_{p-1}$ on its generators, and this grading induces a horizontal filtration on KP given by

$$F_p(KP) = \bigoplus_{j \leqslant p} S[\Sigma P]_j \otimes \Lambda P.$$

Because the differentials in KP all have filtration degree at least 2 with respect to this filtration, in the filtration spectral sequence $({}'E_r, {}'d_r)$ associated to KP (see Corollary A.5.4), we have ${}'E_2 = {}'E_1 = {}'E_0 = \mathrm{gr}_\bullet KP \cong KP$.

The left column ${}'E_2^{0,\bullet}$ is given by $\mathrm{gr}_0 KP \cong F_0(KP) \cong \Lambda P$ and the last page ${}'E_\infty = \mathrm{gr}_\bullet H^*(KP) = H^0(KP) = k$ is what we want as well, so by the Zeeman–Moore comparison theorem, to show that $E_2^{\bullet,0} = S[\Sigma P]$ in the original spectral sequence, it will be enough to create a filtration-preserving DGA map $\lambda_2 \colon KP = {}'E_2 \longrightarrow E_2$ inducing an isomorphism on E_∞.

Since ${}'E_2^{0,\bullet} \cong \Lambda P \cong E_2^{0,\bullet}$, the restriction of λ_2 to the left columns should clearly be the identity, and it remains to map the generators ΣP of the bottom row ${}'E_2^{\bullet,0}$ into $E_2^{\bullet,0}$. By construction, each homogeneous subspace $P_{r-1} < \Lambda P$ of algebra generators transgresses to $(\Sigma P)_r$ in $({}'E_r)$, so in order that λ_2 be a chain map, it is necessary and sufficient to show each subspace $P_{r-1} < \Lambda P = E_2^{0,\bullet}$ transgresses in the original spectral sequence. This is a consequence of an induction that simultaneously depends on and proves the following lemma. □

Theorem 7.6.2 (Borel transgression theorem)**.** *With the same hypotheses as in Theorem 7.6.1, all elements of P transgress.*

Proof [MT00, p. 379]. The proof is an induction on maximum degree D of an element of P. The base case $r = 1$ is given by Section 7.4.

Now suppose the theorem holds for $D-2$ odd and that P has maximum degree D. Decompose $P = P_D \oplus P'$, where $\deg P_D = D$ and $\deg P' < D$. By the inductive assumption, the elements of P' transgress. Thus one can apply Theorem 7.6.1 to the spectral subsequence $({}'E_r, {}'d_r) \leqslant (E_r, d_r)$ generated by P' and its transgressions to see that $E_r^{p,q} = {}'E_r^{p,q}$ agree for $(p,q,r) \in [0, D+1] \times [0, D] \times [0, \infty]$. Call this submodule the *determined prism*: all differentials d_r in and out of E_r are already accounted for by $({}'E_r, {}'d_r)$, or else the restrictions of E_r and ${}'E_r$ to the prism would not agree. As $E_\infty^{0,D} = 0$, every element of the space P_D must fail to lie in the kernel of *some* $d_r \colon E_r^{0,D} \longrightarrow E_r^{r,D+1-r}$, but the determined prism contains all these except for $E_{D+1}^{D+1,0}$. It follows P_r must transgress. The same reasoning applied to maps into $E_2^{D+1,0}$ shows, since $E_\infty^{D+1,0} = 0$, that the transgression $P_D \xrightarrow{\tau} E_{D+1}^{D+1,0}$ must be surjective. This concludes the induction. □

Remarks 7.6.3. (*a*) Borel [Bor53, Prop. 16.1] found an additional result over $\mathbb{F}_2$ requiring not that the left column A be an exterior algebra or that its indecomposables be oddly graded, but instead that A admit a simple system x_α of generators (see Definition A.3.4) which transgress. The result then additionally concludes no elements of A other than those in the span of the x_α transgress.

(*b*) Considering the homology Serre spectral sequence of the universal bundle $G \to EG \to BG$, Remark 7.2.12 shows the primitives $PH_*(G) < H_*(G)$ are all in the image of the transgression. Because $H_*(G) \cong H^*(G)$ and $H_*(BG) \cong H^*(BG)$ on the level of graded vector spaces and the homological and cohomological transgressions are dual (Remark 4.3.17), this means all elements of PG transgress in the cohomological Serre spectral sequence. This seems to yield an alternate proof of the Borel transgression Theorem 7.6.2, which would enable us to conclude the proof of Theorem 7.6.1 more simply, without the painful induction, if we so chose.

A profitable rephrasing of Borel's calculation of the SSS $(\widetilde{E}_r, \tilde{d}_r)$ of $G \to EG \to BG$ is to noncanonically lift the edge maps as follows. Writing $PG < H^*(G;\mathbb{Q})$ for the space of primitives and $Q(BG) < H^*(BG;\mathbb{Q})$ for the space of indecomposables, rceall that $H^*(G) \cong \Lambda PG$ and $H^*(BG) \cong S\big[Q(BG)\big]$. The edge isomorphisms descend to maps

$$\tilde{d}_r \colon P^{r-1}H^*(G) \xrightarrow{\sim} Q^r H^*(BG)$$

which can be seen as summing to the isomorphism[9]

$$\tau \colon PG \xrightarrow{\sim} Q(BG).$$

Setting $V = PG$ and constructing the Koszul complex KV, this τ uniquely extends uniquely to the Koszul differential. Because $H^*(BG)$ is free on $Q(BG)$, on the level of CGAs, we recover

$$\widetilde{E}_2 = H^*(BG) \otimes H^*(G) = KV$$

and can consider τ as an antiderivation $\widetilde{E}_2 \longrightarrow \widetilde{E}_2$, which we call a ***choice of transgression***.[10] By construction, it satisfies the following proposition.

Proposition 7.6.4. *The transgression τ lifts the edge homomorphisms $\tilde{d}_r$ in the sense that for each $r \geqslant 0$, the following diagram commutes:*

$$\begin{tikzcd} H^*(G) \arrow[r, "\tau"] & H^*(BG) \arrow[d, two heads] \\ \widetilde{E}_r^{0,r-1} \arrow[u, tail] \arrow[r, "\sim", "\tilde{d}_r"'] & \widetilde{E}_r^{r,0}. \end{tikzcd}$$

We will need a corollary about the original, unlifted transgression to prove Cartan's theorem later in Theorem 8.1.3 and Theorem D.3.1.

[9] I owe this description to Paul Baum's thesis [Bau62, p. 3.3].

[10] The lifting of $Q(BG)$ back to a subspace of $H^*(BG)$ that produces this isomorphism is not unique, but the differences produced by starting with a different lifting turn out to be immaterial, so we will identify $Q(BG)$ with a subalgebra of $H^*(BG)$ at this point, consider it done, and never speak of it again.

Corollary 7.6.5 (Borel)**.** *Let $G \to E \xrightarrow{\pi} B$ be a principal G-bundle classified by $\chi\colon B \longrightarrow BG$. Write τ for the transgression of the universal bundle $G \to EG \to BG$ In the spectral sequence of π, each primitive $z \in PH^*(G)$ transgresses to $\chi^*\tau z$.*

Proof. This follows from the existence of the bundle map from $G \to E \to B$ to $G \to EG \to BG$, which induces a spectral sequence map as in Theorem 4.3.4 intertwining the edge homomorphisms. □

Historical remarks 7.6.6. Borel's proof of this result was more subtle than ours in at least three ways. For one, he did not assume $\mathbb{Q}$ coefficients, but simultaneously worked over $\mathbb{Z}$ and $\mathbb{F}_p$ coefficients for $p > 2$.

Second, the Zeeman–Moore theorem was not available to him, so he did not construct a comparison map, but explicitly, inductively, and through careful bookkeeping ruled out the possibility of $H^*(BG)$ being anything other than a polynomial ring, keeping track at the same time of what elements of ΛP transgressed and ultimately determining them to be only the primitives P themselves.

Thirdly, and most historically remarkably, in determining $H^*(BG)$ Borel did not have access to *BG itself*. In 1952, it was only known in general that *n-universal* principal bundles $E(n,G) \longrightarrow B(n,G)$ existed for each $n \in \mathbb{N}$ with $\pi_i E(n,G) = 0$ for $i \leqslant n$. Borel's $H^*(BG)$ is actually defined as the inverse limit of the rings $H^*\big(B(n,G)\big)$, known cohomology rings of already-existing objects. Resultingly, for numerous topological applications in which I cavalierly deploy BG, Borel must instead invoke $H^*\big(B(n,G)\big)$ for n sufficiently large.

7.7. Characteristic classes

Borel's Theorem 7.6.1, the mod 2 addendum Remarks 7.6.3.(a) and knowledge of the cohomology rings of classical groups from Section 7.1 make instantly available a great deal of algebraic information about classifying spaces.

Corollary 7.7.1. *Let $k = \mathbb{Z}[1/2]$. The cohomology rings of the classifying spaces of the classical groups are*

$$\begin{aligned}
H^*\big(B\mathrm{O}(n);\mathbb{F}_2\big) &\cong \mathbb{F}_2[w_1,\dots,w_n], && \deg w_j = j,\\
H^*\big(B\mathrm{SO}(n);\mathbb{F}_2\big) &\cong \mathbb{F}_2[w_2,\dots,w_n], && \deg w_j = j,\\
H^*\big(B\mathrm{U}(n);\mathbb{Z}\big) &\cong \mathbb{Z}[c_1,\dots,c_n], && \deg c_j = 2j,\\
H^*\big(B\mathrm{SU}(n);\mathbb{Z}\big) &\cong \mathbb{Z}[c_2,\dots,c_n], && \deg c_j = 2j,\\
H^*\big(B\mathrm{Sp}(n);\mathbb{Z}\big) &\cong \mathbb{Z}[q_1,\dots,q_n], && \deg q_j = 4j,\\
H^*\big(B\mathrm{SO}(2n+1);k\big) &\cong k[p_1,\dots,p_{n-1},p_n], && \deg p_j = 4j,\\
H^*\big(B\mathrm{SO}(2n);k\big) &\cong k[p_1,\dots,p_{n-1},e], && \deg p_j = 4j,\ \deg e = 2n.
\end{aligned}$$

Definition 7.7.2. The w_j in the preceding corollary are the ***Stiefel–Whitney classes***, the c_j the ***Chern classes***, the q_j the ***symplectic Pontrjagin classes***, the p_j the ***Pontrjagin classes***, and e the ***Euler class***.

Remark 7.7.3. For $G \in \{\mathrm{U},\mathrm{Sp},\mathrm{SO}\}$, the inclusions $G(n) \hookrightarrow G(n+1)$ preserve objects named c_j, q_j, p_j respectively for $j \leqslant n$ and annihilate $c_{n+1}, q_{n+1}, p_{n+1}$, with the exception that $H^*\big(B\mathrm{SO}(2n+1)\big) \longrightarrow H^*\big(B\mathrm{SO}(2n)\big)$ takes $p_n \longmapsto e^2$.

The Pontrjagin classes and Euler class as described above are actually *integral* in that they are in the image of the canonical map $H^*\big(B\mathrm{SO}(m);\mathbb{Z}\big) \longrightarrow H^*\big(B\mathrm{SO}(m);\mathbb{Z}[1/2]\big)$. These classes carry certain well-known relations. For example, the inclusion $\mathrm{U}(n) \rightarrowtail \mathrm{SO}(2n)$ induces a map $H^{2n}\big(BSO(2n);\mathbb{Z}\big) \longrightarrow H^{2n}\big(BU(n);\mathbb{Z}\big)$ carrying $e \longmapsto c_n$, and mod-2 coefficient reduction $H^n\big(BSO(n);\mathbb{Z}\big) \longrightarrow H^n\big(BSO(n);\mathbb{F}_2\big)$ takes $e \longmapsto w_n$.

All of these rings can also be calculated independently from the result Corollary 6.4.7 that $H^*(BG) \cong H^*(BT)^W$ and an understanding of the Weyl group action on BT. For example, the existence of the Euler class can be seen as a result of the fact that $W_{SO(2n+1)} = \{\pm 1\}^n \rtimes S_n$ and $W_{SO(2n)}$ is the subgroup $S\{\pm 1\}^n \rtimes S_n$, where $S\{\pm 1\}^n < \{\pm 1\}^n$ is the index-two subgroup whose elements contain an even number of -1 entries. The product $e = t_1 \cdots t_n \in \mathbb{Z}[t_1, \ldots, t_n]$ is invariant under $S\{\pm 1\}^n$ but not under all of $\{\pm 1\}^n$, and as a result does not occur in $H^*\big(BSO(2n+1)\big)$; its square $p_n = t_1^2 \cdots t_n^2$ is however invariant under the larger group's action.

The cohomology classes of Definition 7.7.2, elements of a cohomology ring BG only known after 1955 to globally exist, are abstract manifestations of objects associated to vector bundles which were defined in the 1930s and early 1940s by their namesakes.[11]

Definition 7.7.4. Let $E \longrightarrow B$ be a principal G-bundle and $\chi\colon B \longrightarrow BG$ a classifying map. Given $c \in H^*(BG)$, its pullback $\chi^*(c) \in H^*(B)$ is written $c^*(E)$ and called a ***characteristic class*** of $E \longrightarrow B$.

These characteristic classes are functorial invariants of principal bundles: because the universal bundle is terminal, a map of bundles induces a commutative triangle of maps of base spaces.

Proposition 7.7.5. *Let $E \to B$ be a principal G-bundle, let $f\colon B' \to B$ be a continuous map, and let $c \in H^*(BG)$. Then the pullback bundle f^*E satisfies*

$$c(f^*E) = f^*c(E) \in H^*(B).$$

[11] With the obvious exception of the Euler class.

Given a vector bundle $F \to V \xrightarrow{\xi} B$ with transition functions in a linear group G, there is an associated principal G-bundle $G \to P \to B$ as described in Appendix B.1.3, and one can associate to $V \to B$ the characteristic classes of $P \to B$,

$$c(V) := c(P),$$

calling them the ***characteristic classes of the vector bundle*** $V \to B$. For example

- if $\xi\colon V \to B$ is a quaternionic vector bundle it defines symplectic classes $q_j(\xi) \in H^{4j}(B;\mathbb{Z})$,
- if ξ is a complex vector bundle one has Chern classes $c_j(\xi) \in H^{2j}(B;\mathbb{Z})$,
- if ξ is a real vector bundle one has Pontrjagin classes $p_j(\xi) \in H^{4j}(B;\mathbb{Z})$ and Stiefel–Whitney classes $w_j(\xi) \in H^j(B;\mathbb{F}_2)$, and
- if ξ is an *orientable* vector bundle with fiber $F = \mathbb{R}^n$, it has an Euler class $e(\xi) \in H^n(B;\mathbb{Z})$, and the first Stiefel–Whitney class w_1 can be shown to vanish.

A smooth manifold M determines a tangent bundle $TM \to M$, which thus defines a characteristic class

$$c(M) := c(TM) \in H^*(M)$$

for each characteristic class c of the tangent bundle. For example, we can equip TM with a Riemmannian or Hermitian metric to reduce its structure group to $\mathrm{O}(n)$ or $\mathrm{U}(n)$, so all smooth manifolds carry Pontjagin and Stiefel–Whitney classes, orientable smooth manifolds carry an Euler class $e(M) \in H^{\mathrm{top}}(M)$, and almost complex manifolds carry Chern classes.

These classes turn out to be well-defined invariants of the topological manifold underlying M in that they are independent of the chosen metrics and smooth or almost complex structures. To see the metrics are irrelevant, one way to proceed is to note that the Gram–Schmidt construction can be seen as a product decomposition [BT82, Ex. 6.5(a)]

$$\mathrm{SL}(n,\mathbb{R}) = \mathrm{SO}(n)\cdot M,$$

where M is the contractible space of positive-definite symmetric matrices. If we consider $E\mathrm{SO}(n)$ to be $E\mathrm{SL}(n,\mathbb{R})$, which is valid, as discussed in Section 3.2, since $\mathrm{SO}(n)$ and $\mathrm{SL}(n,\mathbb{R})$ are Lie groups, the former closed in the latter, then taking quotients yields the bundle

$$M \longrightarrow B\mathrm{SL}(n,\mathbb{R}) \longrightarrow B\mathrm{SO}(n),$$

with contractible fiber M, so that $B\mathrm{SL}(n,\mathbb{R}) \simeq B\mathrm{SO}(n)$. Similar homotopy equivalences hold for other classifying spaces of linear groups, so one can dispense with the metrics at the negligible cost of viewing the characteristic classes instead as arising in $B\mathrm{GL}(n;\mathbb{F})$ or $B\mathrm{SL}(n;\mathbb{F})$ for $\mathbb{F} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$.

Assume now M is compact and oriented. A characteristic class c in $H^{\mathrm{top}}(M;\mathbb{Z}) \cong \mathbb{Z}$ is then some integer multiple $n\cdot[M]^*$ of the cohomological fundamental class $[M]^*$; alternately, evaluation of c against the homological fundamental class $[M]$ yields an integer n. These integers are called ***characteristic numbers*** of the manifold, and the data given by characteristic numbers for a real manifold can be seen as the composition

$$H^n\big(B\mathrm{SO}(n);\mathbb{Z}\big) \xrightarrow{\chi^*} H^n(M;\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z},$$

where $\chi\colon M \longrightarrow B\mathrm{SO}(n)$ is the characteristic map of the associated principal $\mathrm{SO}(n)$-bundle.

The ***Pontrjagin numbers*** are the images under this composition of the degree-n level of the subring $\mathbb{Z}[p_1, \ldots, p_n]$, and the Euler characteristic can be seen as the image of e:

Theorem 7.7.6. *Let M be a smooth, compact, oriented n-manifold. Then the Euler class $e \in H^n(M;\mathbb{Z})$ and cohomological fundamental class $[M]^* \in H^n(M;\mathbb{Z})$ and the Euler characteristic $\chi(M) \in \mathbb{Z}$ satisfy the relation*

$$e = \chi(M) \cdot [M]^*.$$

This the reason behind the nomenclature *Euler class*. This equivalence also yields an outlandishly complicated way of seeing the Euler characteristic of an odd-dimensional closed manifold is zero.

In Appendix C, we will use equivariant cohomology to describe simple conditions under which characteristic numbers vanish.

7.8. Maps of classifying spaces

The machine for computing $H^*(G/K)$ depends critically on an understanding of the map

$$\rho^* = (Bi)^* \colon H^*(BG) \longrightarrow H^*(BK)$$

induced by the inclusion $i \colon K \hookrightarrow G$; this understanding is also due to Borel [Bor53, §28].

7.8.1. Maps of classifying spaces of tori

To start, let $i\colon S \hookrightarrow T$ be an inclusion of tori. By functoriality, or else by taking $ES = ET$ and representing $BS \longrightarrow BT$ as the "further quotient" map $ET/S \twoheadrightarrow ET/T$, we have a bundle map

$$\begin{tikzcd} S \arrow[r, hook, "i"] \arrow[d] & T \arrow[d] \\ ES \arrow[r, "\simeq"] \arrow[d] & ET \arrow[d] \\ BS \arrow[r, "Bi"] & BT, \end{tikzcd}$$

which induces a map $\big(\psi_r\colon (\widetilde{E}_r, \tilde{d}_r) \longrightarrow (E_r, d_r)\big)$ between the spectral sequences of the bundles. Because these sequences both collapse on the third page, ψ_r is just an isomorphism $H^0(ET) \xrightarrow{\sim} H^0(ES) = \mathbb{Z}$ for $r \geqslant 3$, so we may as well drop page subscripts and consider the lone interesting map, which by Theorem 4.3.4 is

$$\psi = (Bi)^* \otimes i^*\colon H^*(BT) \otimes H^*(T) \longrightarrow H^*(BS) \otimes H^*(S).$$

Because, by the definition of a chain map, we have $d\psi = \psi\tilde{d}$, and, as we have just seen, $d\colon H^1(S) \longrightarrow H^2(BS)$ and $\tilde{d}\colon H^1(T) \longrightarrow H^2(BT)$ are group isomorphisms, we have the commutative square

$$\begin{tikzcd} H^1(S) \arrow[d, "\wr"] & H^1(T) \arrow[l, "i^*"'] \arrow[d, "\wr"] \\ H^2(BS) & H^2(BT). \arrow[l, "(Bi)^*"'] \end{tikzcd} \tag{7.7}$$

Thus $i^*\colon H^1(T) \longrightarrow H^1(S)$ is conjugate to $(Bi)^*\colon H^2(BT) \longrightarrow H^2(BS)$. Since $H^2(BT)$ generates $H^*(BT)$ as an algebra, and $(Bi)^*$ is a ring homomorphism, this means $(Bi)^*$ is determined uniquely by i^*. This i^*, in turn, is described by i in a transparent way. It is dual to the map

$i_*\colon H_1S \longrightarrow H_1T$, or equivalently to the map $\pi_1(i)$.

In the case most critical for us later, S will just be a circle, which we will identify with the standard complex unit circle $S^1 < \mathbb{C}^\times$. Similarly identify T with $(S^1)^n$. Then $i\colon S \hookrightarrow T$ can be written as

$$\begin{aligned} i\colon S^1 &\longrightarrow (S^1)^n, \\ z &\longmapsto (z^{a_1},\ldots,z^{a_n}), \end{aligned}$$

where the exponent vector $\vec{a} \in \mathbb{Z}^n$ is a list of integers with greatest common divisor 1, so that i is injective.[12] If $x_j \in \pi_1(T) = H_1(T)$ is the fundamental class of the j^{th} factor circle and $y \in H_1(S)$ the fundamental class of S, then

$$i_*\colon y \longmapsto \sum a_j x_j.$$

Let (x_j^*) be the dual basis for $H^1(T)$ and $y^* \in H^1(S)$ the cohomological fundamental class. Then the dual map $i^*\colon H^1(T) \twoheadrightarrow H^1(S)$ in cohomology takes $x_j^* \longmapsto a_j y^*$ since

$$(i^* x_j^*)y^* = x_j^*\big(i_* y\big) = x_j^*\Big(\sum a_\ell x_\ell\Big) = a_j.$$

Put another way, the matrix of i^* is the transpose of the matrix of i_*. Write $s = dy \in H^2(BS)$ and $u_j = d_2 x_j^* \in H^2(BT)$ so that $H^*(BS) = \mathbb{Z}[s]$ and $H^*(BT) = \mathbb{Z}[\vec{u}]$. Then, the square above implies that $(Bi)^*(u_j) = a_j s$, so that if $p(\vec{u}) \in \mathbb{Z}[\vec{u}]$ is any polynomial,

$$(Bi)^* p(\vec{u}) = p(a_1 s, \ldots a_n s) = p(a_1,\ldots,a_n)s^{\deg p}.$$

[12] This vector $\vec{a}$ is only well-defined up to the choice of identifications $S \cong S^1$ and $T \cong (S^1)^n$, but will suffice for our later applications.

7.8.2. Maps of classifying spaces of compact, connected Lie groups

Let $K \hookrightarrow G$ be an inclusion of compact, connected Lie groups. If S is a maximal torus of K, then there exists a maximal torus T of G containing S. Through the functoriality of the classifying space functor B and cohomology, this square of inclusions gives rise to two further commutative squares:

$$\begin{tikzcd}
S \arrow[r, hook, "i"] \arrow[d, hook] & T \arrow[d, hook] & & BS \arrow[r, "Bi"] \arrow[d] & BT \arrow[d] & & H^*(BS) & H^*(BT) \arrow[l, "(Bi)^*"'] \\
K \arrow[r, hook] & G & \Longrightarrow & BK \arrow[r, "\rho"'] & BG & \Longrightarrow & H^*(BK) \arrow[u, hook] & H^*(BG). \arrow[l, "\rho^*"] \arrow[u, hook]
\end{tikzcd}$$

The vertical maps in the last square are inclusions by Corollary 6.4.7. Thus ρ^* can be computed as the composition

$$H^*(BG) \xrightarrow{\sim} H^*(BT)^{W_G} \xrightarrow{(Bi)^*} H^*(BS);$$

it follows from the commutativity of the square that the image lies in $H^*(BS)^{W_K} \cong H^*(BK)$.

Example 7.8.1. Let $G = \mathrm{U}(4)$ and $K = \mathrm{Sp}(2)$, identified as a subgroup of G through the injective ring map $\mathbb{H} \rightarrowtail \mathbb{C}^{2\times 2}$ given by $\alpha + j\beta \longmapsto \left[\begin{smallmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{smallmatrix}\right]$. A standard maximal torus for G is given by the subgroup $T = \mathrm{U}(1)^4$ of diagonal unitary matrices, which meets K in the subgroup

$$S = \left\{\mathrm{diag}(z, \bar{z}, w, \bar{w}) \in \mathrm{U}(1)^4 : z, w \in S^1\right\}.$$

With respect to the expected basis of $H_1(T)$ and the fundamental classes of the factor circles $w = 1$ and $z = 1$ of S, and the dual bases in H^1, the maps $H_1(S) \longrightarrow H_1(T)$ and $H^1(S) \longleftarrow H^1(T)$

are given respectively by

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

By the commutative square (7.7), the second matrix also represents $H^2(BS) \longleftarrow H^2(BT)$ with respect to the transgressed bases t_1, t_2, t_3, t_4 of $H^2(BT)$ and s_1, s_2 of $H^2(BS)$.

The Weyl group of $\mathrm{U}(4)$ is the symmetric group S_4 on four letters acting on T and hence BT by permutation of the four coordinates. It follows that when $H^*\big(\mathrm{U}(4)\big)$ is conceived as the invariant subring $H^*(BT)^{S_4}$ of $H^*(BT)$, it is generated by the elementary symmetric polynomials $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ in t_1, t_2, t_3, t_4, lying in respective degrees $2, 4, 6, 8$. These are the first four Chern classes c_j.

The Weyl group of $\mathrm{Sp}(2)$ is the group $\{\pm 1\}^2 \rtimes S_2$, acting on $H^1(S)$ and hence on $H^2(BS) = \mathbb{Q}\{s_1, s_2\}$ by negating and/or switching the two coordinates. It follows the invariant subring $H^*\big(B\mathrm{Sp}(2)\big) \cong H^*(BS)^{W_{\mathrm{Sp}(2)}}$ is generated by $q_1 = s_1^2 + s_2^2$ and $q_2 = (s_1 s_2)^2$. These are the first two symplectic Pontrjagin classes. The generators c_j exhibit the following properties under $H^*(BT)^{S^4} \hookrightarrow H^*(BT) \longrightarrow H^*(BS)$:

$$\begin{aligned} c_1 = t_1 + t_2 + t_3 + t_4 &\longmapsto (s_1 - s_1) + (s_2 - s_2) = 0, \\ c_2 = \frac{1}{2}\big(\sigma_1^2 - \sigma_1(t_1^2, t_2^2, t_3^2, t_4^2)\big) &\longmapsto \frac{1}{2}\big(0 - (s_1^2 + s_1^2 + s_2^2 + s_2^2)\big) = -(s_1^2 + s_2^2) = -q_1, \\ c_3 = (t_1 + t_2)t_3 t_4 + t_1 t_2 (t_3 + t_4) &\longmapsto (0 \cdot -s_2^2) + (-s_1^2 \cdot 0) = 0, \\ c_4 = t_1 t_2 t_3 t_4 &\longmapsto s_1^2 s_2^2 = q_2. \end{aligned}$$

That is, $H^*\big(B\mathrm{U}(4)\big) \longrightarrow H^*\big(B\mathrm{Sp}(2)\big)$ is surjective, a fact we will later be able to see as a consequence of the surjectivity of $H^*\big(\mathrm{U}(4)\big) \longrightarrow H^*\big(\mathrm{Sp}(2)\big)$.

Example 7.8.2. Let $G = \mathrm{Sp}(2)$ and $K = S = \mathrm{SO}(2)$, identified as a subgroup of G through the standard inclusion $\mathbb{R} \longleftrightarrow \mathbb{H}$. One maximal torus T of $\mathrm{Sp}(2)$ containing S is that generated by S and

$$S' = \left\{ \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix} : \theta \in [0, 2\pi] \right\}.$$

With respect to the basis $[S], [S']$ of $H_1(T)$, and the dual bases in H^1, the maps $H_1(S) \longrightarrow H_1(T)$ and $H^1(S) \longleftarrow H^1(T)$ are given respectively by $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\begin{bmatrix}1 & 0\end{bmatrix}$. By (7.7) again the second matrix also represents $H^2(BS) \longleftarrow H^2(BT)$ with respect to the transgressed bases t_1, t_2 of $H^2(BT)$ and t_1 of $H^2(BS)$. Generators for $H\big(B\mathrm{Sp}(2)\big)$ are q_1, q_2 as in Example 7.8.1, and we have

$$q_1 = t_1^2 + t_2^2 \longmapsto t_1^2,$$
$$q_2 = t_1^2 t_2^2 \longmapsto t_1^2 \cdot 0 = 0.$$

This is an instance of a general result about the map ρ^* in the event G is semisimple and S a circle, which we will later use in determining the rings $H^*(G/S^1)$.

Lemma 7.8.3. *Let K be a semisimple Lie group containing a circle S. The image of $H_K^* \longrightarrow H_S^* \cong \mathbb{Q}[s]$ contains $s^2 \in H_S^4$.*

Proof. Let T be a maximal torus of K containing S, so that $H_K^* \longrightarrow H_S^*$ factors as $H_T^W \hookrightarrow H_T^* \to H_S^*$, where W is the Weyl group of K, by the results of Section 7.8.2. Identifiying $H_T^* = \mathbb{Q}[u_1, \ldots, u_n]$ and $H_S^* = \mathbb{Q}[s]$, by Section 7.8.2, the exponents a_j of the inclusion $S^1 \hookrightarrow T \cong (S^1)^n$ determine the map $H_T^4 \longrightarrow H_S^4$ according to the formula $\rho^* q(\vec{u}) = q(\vec{a}) s^2$.

The elements of H_T^4 can be seen as quadratic forms on the vector space $\pi_1 T \otimes \mathbb{Q} \cong H_1(T; \mathbb{Q}) \cong$

$H_2(BT;\mathbb{Q})$. So to show the map $(H^4_T)^W \longrightarrow H^4_S$ is surjective regardless of the choice of S, it suffices to find a W-invariant definite bilinear form on $\pi_1 T \otimes \mathbb{Q}$. But the Killing form $B(-,-)\colon \mathfrak{k} \times \mathfrak{k} \longrightarrow \mathbb{R}$ is Ad-invariant and negative definite by Proposition B.4.12, so the restriction to $\mathfrak{t} \times \mathfrak{t}$ is W-invariant, and the restriction to the diagonal is a W-invariant, negative definite quadratic form B on $\mathfrak{t}$. But $\pi_1 T$ embeds in $\mathfrak{t}$ in a canonical way as the kernel of $\exp\colon \mathfrak{t} \twoheadrightarrow T$, so the restriction of B to $\pi_1 T \otimes \mathbb{Q}$ corresponds to an element of $H^K_4 \cong (H^4_T)^W$ whose $\mathbb{Q}$-span surjects onto H^4_S. $\square$

Historical remarks 7.8.4. The choice of notation ρ^* for this important map follows historical precedent dating back to the heroic era of large tuples described in Historical remarks 3.3.9. Borel and later Hirzebruch canonically assigned the name $\rho(K,G)$ to the map $BK \longrightarrow BG$ induced by an inclusion $K \hookrightarrow G$ and $\rho^*(K,G)$ to the resulting map $H^*(BG) \longrightarrow H^*(BK)$ in cohomology.

Chapter 8

The cohomology of homogeneous spaces

In this chapter, and again starting in Chapter 10, we discuss properties of a specific type of compact homogenous space G/K in terms of the transitively acting group G and the isotropy subgroup K. Now seems like a good time to formalize this setup.

Definition. Let G be a compact, connected Lie group, and K a closed, connected subgroup. In this situation we call (G, K) a ***compact pair*** of Lie groups.

Our discussion will really be about properties of such pairs. Associated to a compact pair (G, K) are three fiber bundles. The first, $K \to G \to G/K$, follows from Theorem B.4.3. The second is the Borel fibration $G \to G_K \to BK$, which is a principal G-bundle. The third is the fibration $G/K \to BK \to BG$, where the projection $Bi\colon BK \longrightarrow BG$ can be seen as the "further quotient" map $EG/K \twoheadrightarrow EG/G$. Thus each three consecutive terms of the sequence

$$K \xrightarrow{i} G \xrightarrow{j} G/K \xrightarrow{\chi} BK \xrightarrow{Bi} BG \tag{8.1}$$

form a fibration up to homotopy. This section is devoted to a general discussion of the implications of this fiber sequence in the resulting cohomology sequence

$$H^*(K) \xleftarrow{i^*} H^*(G) \xleftarrow{j^*} H^*(G/K) \xleftarrow{\chi^*} H^*_K \xleftarrow{\rho^*} H^*_G. \tag{8.2}$$

It is a curious historical coincidence that the study of the cohomology of homogeneous spaces seems to break into three basic periods, the first studying the Leray spectral sequence of the first three terms, the second studying the Serre spectral sequence of the second three terms, and last studying the Eilenberg–Moore spectral sequence of the last three terms. It is the second period characterization that we employ in what follows, but all these maps will have some relevance for us.

8.1. The Borel–Cartan machine

We begin by introducing the device that will carry out all our computations.

8.1.1. The spectral sequence map

The five terms of (8.1) form the labeled subdiagram in the following diagram of bundle maps.

$$\begin{tikzcd} K \arrow[r, hook, "i"] \arrow[d] & G \arrow[r, equal] \arrow[d, "j"] & G \arrow[d] \\ EK \arrow[r] \arrow[d] & G_K \arrow[r] \arrow[d, "\chi"] & EG \arrow[d] \\ BK \arrow[r, equal] & BK \arrow[r, "\rho"] & BG \end{tikzcd} \tag{8.3}$$

Here the first and last columns are universal bundles and the second column is the Borel fibration. As the Borel fibration is a principal G-bundle, the classifying map $\rho\colon BK \longrightarrow BG$ must exist; explicitly, if we take $EK = EG$, then the map on total spaces inducing $\rho = Bi\colon eK \longmapsto eG$ is given by $[e,g]_K \longmapsto eg$. The middle row can be seen as

$$EG \times_K K \hookrightarrow EG \times_K G \twoheadrightarrow EG \times_G G,$$

the outer terms being homeomorphic to EG by Lemma 2.1.1. Upon taking cohomology, the positions of EK and EG in the diagram and commutativity provide another reason the compositions $i^* \circ j^*$ and $\chi^* \circ \rho^*$ are zero.

The Borel approach to understanding the cohomology of $H^*(G/K)$ depends on the G-bundle map between the second two bundles,

$$\begin{tikzcd} G \arrow[r, equal] \arrow[d, "j"'] & G \arrow[d] \\ G_K \arrow[r] \arrow[d, "\chi"'] & EG \arrow[d] \\ BK \arrow[r, "\rho"'] & BG. \end{tikzcd} \tag{8.4}$$

This bundle map induces a map from the spectral sequence $(\widetilde{E}_r, \tilde{d}_r)$ of the universal bundle, which we now completely understand, to the spectral sequence (E_r, d_r) of the Borel fibration, which we do not. As $G_K \simeq G/K$, the latter sequence converges to $H^*(G/K)$. We write

$$(\psi_r)\colon (E_r, d_r) \longleftarrow (\widetilde{E}_r, \tilde{d}_r)$$

for this map of spectral sequences. Recall from Appendix A.5.2 that these maps $\psi_r\colon \widetilde{E}_r \longrightarrow E_r$ are DGA homomorphisms, meaning $d_r \circ \psi_r = \psi_r \circ \tilde{d}_r$, and each descends from that on the previous page, meaning $\psi_{r+1} = H^*(\psi_r)$. The map $\psi_2\colon E_2 \longleftarrow \widetilde{E}_2$ between second pages is

$$\rho^* \otimes \mathrm{id}\colon H^*(BK) \otimes H^*(G) \longleftarrow H^*(BG) \otimes H^*(G),$$

where $\mathrm{id}_{H^*(G)}$ is the isomorphism $\widetilde{E}_2^{0,\bullet} \xrightarrow{\sim} E_2^{0,\bullet}$ of the leftmost columns and $\rho^* = (Bi)^*\colon H^*(BK) \longleftarrow H^*(BG)$ is the map $\widetilde{E}_2^{\bullet,0} \to E_2^{\bullet,0}$ of bottom rows.

It is a consequence of the following lemma that the map ρ^* at least largely determines $H^*(G/K)$.

Proposition 8.1.1. *Let G be a compact, connected Lie group whose primitive subspace $PG < H^*(G)$ is concentrated in degree $\leqslant q-1$. Then if $G \to E \to B$ is a principal G-bundle, its SSS collapses at E_{q+1}.*

Proof. Recall that the spectral sequence $(\widetilde{E}_r, \tilde{d}_r)$ of the universal G-bundle collapses at $\widetilde{E}_{q+1} = \widetilde{E}_\infty = \mathbb{Q}$. Because $G \to E \to B$ is principal, it admits a bundle map to the universal bundle, as in (8.4) inducing a spectral sequence map $(\psi_r)\colon (\widetilde{E}_r, \tilde{d}_r) \longrightarrow (E_r, d_r)$, which is a chain map, meaning $d_r\psi_r = \psi_r\tilde{d}_r$. Thus the edge maps $d_r\colon E_r^{0,r-1} \longrightarrow E_r^{r,0}$ all vanish for $r > q$. Now, the d_r also vanish on the bottom row $E_r^{\bullet,0}$ by lacunary considerations, and are antiderivations with respect to an algebra structure on E_r descending from that of $E_2 = H^*(B) \otimes H^*(G)$, so they vanish entirely for $r > q$. □

In particular, since the edge homomorphisms of the universal bundle spectral sequence $(\widetilde{E}_r, \tilde{d}_r)$ are determined entirely composition by an isomorphism $\tau\colon PG \xrightarrow{\sim} Q(BG)$ restricting the transgression, it follows much of the structure of (E_r, d_r) is determined by the composition $\rho^* \circ \tau$. In fact, in the next subsection we will show that this composition *itself yields* a differential d on E_2, the *Cartan differential*, such that $H^*(E_2, d) \cong H^*(G/K)$ and (E_r, d_r) is the filtration spectral sequence associated to the filtered DGA (E_2, d), equipped with the horizontal filtration induced from H_K^*.

8.1.2. Cartan's theorem

In this subsection, we prove Cartan's theorem that the complex described above actually determines $H^*(G/K)$ completely. To do so, we will have to briefly invoke a cochain-level description of the situation, and rather than use singular cochains, we compute cohomology with Dennis Sullivan's rational algebra A_{PL} of ***polynomial differential forms***. This object plays an essential role in rational homotopy theory [FHT01, p. 121], but for our purposes, all we need to know about A_{PL} is that it is a contravariant functor $\mathsf{Top} \longrightarrow \mathbb{Q}\text{-}\mathsf{CDGA}$ such that $H^*\big(A_{\mathrm{PL}}(X)\big) \cong H^*(X;\mathbb{Q})$ for CW complexes X. That $A_{\mathrm{PL}}(X)$ is itself already a CGA and not merely a CGA up to homotopy is the key feature.

Temporarily taking a step back from homogeneous spaces, consider the universal bundle $G \to EG \to BG$. Lifting indecomposables, which is possible by Proposition A.4.3 since $H^*(BG)$ is a free CGA, the transgression yields a map

$$P(G) \xrightarrow[\tau]{\sim} Q(BG) \hookrightarrow H^*(BG),$$

Since $H^*(BG)$ is also a free CGA, there exists a CGA section $i^*\colon H^*(BG) \longrightarrow A_{\mathrm{PL}}(BG)$, so we can lift τ to $i^*\tau\colon PH^*(G) \longrightarrow A_{\mathrm{PL}}(BG)$.

Now consider a principal G-bundle $G \to E \xrightarrow{\pi} B$ The bundle is classified by some map $\chi\colon B \longrightarrow BG$, inducing a ring map $\chi^*\colon A_{\mathrm{PL}}(BG) \longrightarrow A_{\mathrm{PL}}(B)$, and we can form the composition

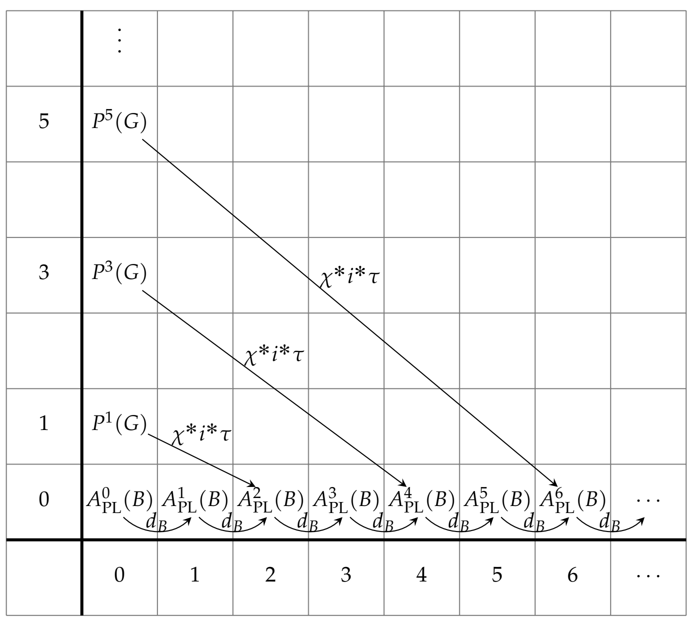
$$\chi^* i^* \tau\colon P(G) \to H^*(BG) \to A_{\mathrm{PL}}(BG) \to A_{\mathrm{PL}}(B)$$

Because $H^*(G) = \Lambda P(G)$ is a free CGA, we can extend this lifted transgression uniquely to an antiderivation on

$$C := A_{\mathrm{PL}}(B) \otimes H^*(G)$$

which we will again call $\chi^* i^* \tau$ and which vanishes on $A_{\mathrm{PL}}(B)$. Similarly, the differential d_B of $A_{\mathrm{PL}}(B)$ extends uniquely to an antiderivation on C annihilating $\mathbb{Q} \otimes H^*(G)$, which we again call d_B. We consider C as a $\mathbb{Q}$-CDGA with respect to the unique differential $d_C := d_B + \chi^* i^* \tau$ extending both d_B and $\chi^* i^* \tau$. See Figure 8.1.2.

Figure 8.1.2: The differential of the algebra $C = A_{\mathrm{PL}}(B) \otimes H^*(G)$ as defined on primitives



Let (z_ℓ) be a basis of $P(G)$ and set $\beta_\ell = (\chi^* i^* \tau) z_\ell$ for each ℓ, so that we have

$$d_C(\alpha \otimes 1) = d_B \alpha \otimes 1, \quad \alpha \in A_{\mathrm{PL}}(B);$$

$$d_C(1 \otimes z_\ell) = \beta_\ell \otimes 1.$$

The chain maps $\big(A_{\mathrm{PL}}(B), d_B\big) \to (C, d_C) \to \big(H^*(G), 0\big)$ induce ring homomorphisms $H^*(B) \to H^*(C) \to H^*(G)$.

Theorem 8.1.3 (Chevalley [Car51][Kos51][Bor53, Thm 24.1, 25.1]). *Let $G \xrightarrow{j} E \xrightarrow{\pi} B$ be a principal G-bundle, and let (C, d_C) and λ be as above. Then there exists an isomorphism $\lambda^*\colon H^*(C, d_C) \xrightarrow{\sim} H^*(E)$ making the following diagram commute:*

$$\begin{tikzcd}
 & H^*(C) \arrow[dd, "\lambda^*"] \arrow[dr] & \\
H^*(B) \arrow[ur] \arrow[dr, "\pi^*"'] & & H^*(G). \\
 & H^*(E) \arrow[ur, "j^*"'] &
\end{tikzcd}$$

Proof. We want to construct a chain map $\lambda\colon C \longrightarrow A_{\mathrm{PL}}(E)$ into the algebra of polynomial differential forms on E (any CDGA calculating $H^*(E)$ would do), which we will then show to be a quasi-isomorphism by showing it induces an isomorphism between later pages of the associated filtration spectral sequences. The spectral sequence corresponding to $H^*\big(A_{\mathrm{PL}}(E)\big) \cong H^*(E)$ will be the SSS (E_r, d_r) of $G \xrightarrow{j} E \xrightarrow{\pi} B$ with respect to A_{PL} cochains.

Note that by construction and by Corollary 7.6.5 a primitive $z_\ell \in H^{r-1}(G)$ transgresses in E_r to $d_r[z_\ell] = [\beta_\ell]$. By Proposition A.5.13, this means there exists an form $\gamma_\ell \in A_{\mathrm{PL}}(E)$ such that $[j^*\gamma_\ell] = z_\ell \in H^*(G)$ and $d_E\gamma_\ell = \pi^*\beta_\ell \in A_{\mathrm{PL}}(E)$. Define λ on algebra generators by

$$\begin{aligned}
\lambda\colon A_{\mathrm{PL}}(B) \otimes H^*(G) &\longrightarrow A_{\mathrm{PL}}(E), \\
\alpha \otimes 1 &\longmapsto \pi^*\alpha, \\
1 \otimes z_\ell &\longmapsto \gamma_\ell.
\end{aligned}$$

Then λ is a chain map by construction, for following through the formulas on generators, we have

$$d_E\lambda(\alpha\otimes 1) = d_E\pi^*\alpha = \pi^* d_B\alpha = \lambda d_C(\alpha\otimes 1);$$

$$d_E\lambda(1\otimes z_\ell) = d_E\gamma_\ell = \pi^*\beta_\ell = \lambda d_C(1\otimes z_\ell).$$

Filter B by its p-skeleta B^p, and E by the preimages $\pi^{-1}B^p$ of these, and filter C and $A_{\mathrm{PL}}(E)$ by

$$F_pC = \bigoplus_{i\geqslant p} A^i_{\mathrm{PL}}(B)\otimes H^*(G), \qquad F_pA_{\mathrm{PL}}(E) = \ker\big(A_{\mathrm{PL}}(E) \longrightarrow A_{\mathrm{PL}}(\pi^{-1}B^{p-1})\big).$$

Then λ preserves filtration degree for elements of $H^*(B)$, which is enough to see that it sends $F_pC \longrightarrow F_pA_{\mathrm{PL}}(E)$.

Write (E_r, d_r) still for the spectral sequence of the filtration on $A_{\mathrm{PL}}(E)$ and $('E_r, 'd_r)$ for that of the filtration on C. The former is just the SSS of $G \to E \to B$ using A_{PL} cochains (Theorem 4.3.4),

$$E_2 = H^*(B)\otimes H^*(G).$$

This form applies because the transition functions of a principal G-bundle are elements of G, which act trivially on the cohomology of the fibers G since G is path-connected, so by the action $\pi_1(B) \longrightarrow \operatorname{Aut} H^*(G)$ is trivial.

On the other hand, following through the reasoning in Corollary A.5.4 in this case, $'E_0$ is the associated graded algebra $\operatorname{gr} C \cong C$, and $'d_0$ is the differential induced by $d_C = d_B + \chi^* i^*\tau$. Since $\chi^* i^*\tau$ is induced by the transgression τ, it has filtration degree $\geqslant 2$ on all elements it fails to annihilate outright, and so vanishes under the associated graded algebra construction, and likewise d_B adds one to the filtration degree, so ${}'d_0 = 0$ and ${}'E_1 = {}'E_0 \cong C$. Thus ${}'d_1 = d_B$ and

$$ {}'E_2 \cong H^*(B) \otimes H^*(G) \cong E_2. $$

Now that we know these pages can both be identified with $H^*(B) \otimes H^*(G)$ in a natural way, it remains to show $\lambda_2 \colon {}'E_2 \longrightarrow E_2$ becomes the identity map under these identifications. But this is also the case by construction: the base elements $\alpha \in A_{\mathrm{PL}}(B) \otimes 1$ and $\lambda(\alpha \otimes 1) = \pi^*\alpha \in A_{\mathrm{PL}}(E)$ both become $[\alpha] \otimes 1$ in ${}'E_2 = E_2$ and the fiber elements $1 \otimes z_\ell \in 1 \otimes H^*(G)$ and $\lambda(1 \otimes z_\ell) = \gamma_\ell \in A_{\mathrm{PL}}(E)$ each become $1 \otimes [j^*\gamma_\ell] = 1 \otimes z_\ell$.

Since λ_2 is a chain isomorphism, it follows every λ_r is for $2 \leqslant r \leqslant \infty$. Since λ is a map of filtered DGAs inducing an isomorphism

$$ {}'E_\infty = \operatorname{gr} H^*\big(A_{\mathrm{PL}}(B) \otimes H^*(G), d_C\big) \xrightarrow{\sim} \operatorname{gr} H^*(E) = E_\infty $$

of associated graded cohomology algebras, it follows from Proposition A.5.1 that the map

$$ H^*(\lambda) \colon H^*\big(A_{\mathrm{PL}}(B) \otimes H^*(G), d_C\big) \longrightarrow H^*(E) $$

is also an isomorphism. □

Remark 8.1.4. We are fairly committed to a classical viewpoint in this work, but those with some grounding in rational homotopy theory might note that $(SQ(BK) \otimes \Lambda PG, d)$ is a *Sullivan model*.

The algebra $C = A_{\mathrm{PL}}(B) \otimes H^*(G)$, although simpler than $A_{\mathrm{PL}}(E)$, still involves the algebra $A_{\mathrm{PL}}(B)$ of polynomial forms on the base B, which though graded-commutative and smaller than the algebra of singular cochains on B, is still typically a very large ring (if B is a CW complex of positive dimension, then $\dim_{\mathbb{Q}} A_{\mathrm{PL}}(B) \geqslant 2^{\aleph_0}$), which we would rather replace with $H^*(B)$.

The E_2 page of the filtration spectral sequence associated to the filtration induced from the "horizontal" grading on $A_{\mathrm{PL}}(B)$ is the algebra we want, namely $H^*(B) \otimes H^*(G)$ equipped with the differential d_2 vanishing on $H^*(B)$ and sending $z \in PG$ to $(\chi^*\tau)z = \big[(\chi^* i^* \tau)z\big] \in H^{|z|+1}(B)$.

Definition 8.1.5. The algebra $C = H^*(B) \otimes H^*(G)$ equipped with the antiderivation d extending

$$P(G) \xrightarrow{\tau} Q(BG) \hookrightarrow H^*(BG) \xrightarrow{\chi^*} H^*(B)$$

is the ***Cartan algebra*** of the principal bundle $G \to E \to B$.

Remark 8.1.6. Observe that the Cartan algebra of a principal bundle $G \to E \to B$ is the Koszul complex (Definition 7.5.4) of a sequence $\vec{a}$ in $H^*(B)$ of images of generators of $H^*(BG)$ under the characteristic map $\chi^* \colon H^*(BG) \longrightarrow H^*(B)$. This follows because indeed $H^*(BG) = S\big[Q(BG)\big]$ by Borel's Theorem 7.6.1 and $\Lambda PG \otimes SQ(BG)$, equipped with $\tau \colon PG \xrightarrow{\sim} Q(BG)$, is the Koszul complex of PG. In particular, one has the following isomorphism.

Proposition 8.1.7. *Let $G \to E \to B$ be a principal bundle and C its Cartan algebra. Then there is an isomorphism*

$$H^*(C) \cong \mathrm{Tor}^{\bullet,\bullet}_{H^*_G}\big(\mathbb{Q}, H^*(B)\big).$$

Proof. By Remark 8.1.6, C is the Koszul complex of the map $\chi^* \colon H^*(BG) \longrightarrow H^*(B)$, and by Proposition 7.5.8, the cohomology of this complex is $\mathrm{Tor}^{\bullet,\bullet}_{H^*_G}\big(\mathbb{Q}, H^*(B)\big)$. □

We would like to find a chain of quasi-isomorphisms linking $\big(A_{\mathrm{PL}}(B) \otimes H^*(G), d_C\big)$ with $C = \big(H^*(B) \otimes H^*(G), d\big)$. One natural sufficient condition is for there to exist a quasi-isomorphism $\big(H^*(B), 0\big) \longrightarrow \big(A_{\mathrm{PL}}(B), d_{A_{\mathrm{PL}}(B)}\big)$. This follows from the following lemma, as applied to the Cartan algebra. Recall from Definition A.4.6 that in this situation the complex $\big(A_{\mathrm{PL}}(B), d_{A_{\mathrm{PL}}(B)}\big)$ is called *formal*.

Lemma 8.1.8. *Let (B, d_B) be a graded formal k-DGA with $\lambda\colon \big(H^*(B), 0\big) \longrightarrow (B, d_B)$ a quasi-isomorphism. Let F be a CGA, and suppose $C = B \otimes F$ is equipped with a differential ξ vanishing on $B \otimes k$ and sending generators of f into the subring of d_B-cocycles $Z(B)$. Extend d_B to an antiderivation vanishing on $k \otimes F$, and define $d_C = d_B + \xi$, and note that ξ descends to a differential $\bar{\bar{\xi}}$ on $H^*(B) \otimes F$. Then*

$$\lambda \otimes \mathrm{id}_F\colon \big(H^*(B) \otimes F, \bar{\bar{\xi}}\big) \longrightarrow (B \otimes F, d_C)$$

is a quasi-isomorphism.

Proof. To see $\lambda \otimes \mathrm{id}$ is a chain map, it is enough to verify on generators from $H^*(B)$ and F Since ξ vanishes on $B \otimes k$, for $b \in B$ we have

$$d_C(\lambda \otimes \mathrm{id})\big([b] \otimes 1\big) = (d_B + \xi)(\lambda b \otimes 1) = (d_B \lambda) b \otimes 1 = 0 = (\lambda \otimes \mathrm{id})\bar{\bar{\xi}}\big([b] \otimes 1\big),$$

and for a generator f of F we have

$$d_C(\lambda \otimes \mathrm{id})(1 \otimes f) = d_C(1 \otimes f) = \xi(f) \otimes 1 = (\lambda \otimes \mathrm{id})\bar{\bar{\xi}}(1 \otimes f),$$

so $\lambda \otimes \mathrm{id}_F$ is a chain map. Equipping both bigraded algebras with the horizontal filtration induced by the grading on B, it is clear ξ increases filtration degree. Form the filtration spectral sequences of each. It is clear the E_2 page for $H^*(B) \otimes F$ is just $H^*(B) \otimes F$ again, and as noted in the lead-up to this lemma, the E_2 for $B \otimes F$ is $H^*(B) \otimes F$ by the argument in the proof of Theorem 8.1.3. The map λ_2 is then the identity, so by Proposition A.5.7, $\lambda \otimes \mathrm{id}$ is a quasi-isomorphism. □

Corollary 8.1.9. *Let B be a generalized symmetric space in the sense of Definition A.4.8 and $G \to E \to B$ a principal G-bundle over B. Then the Cartan algebra calculates $H^*(E)$.*

Proof. By Example A.4.9, a generalized symmetric space is formal, so Lemma 8.1.8 applies. □

Corollary 8.1.10 (Koszul, [Kos51]). *Let (G, K) be a pair such that G/K is a symmetric space. Then the Cartan algebra of $K \to G \to G/K$ calculates $H^*(G)$.*

The case of critical interest to us, of course, is the Borel fibration $G \longrightarrow G_K \longrightarrow BK$.

Definition 8.1.11. The Cartan algebra of the Borel fibration $G \longrightarrow G_K \longrightarrow BK$, given by $C = H^*(BK) \otimes H^*(G)$ equipped with antiderivation d extending $\rho^* \circ \tau\colon P(G) \longrightarrow Q(BG) \longrightarrow H^*(BK)$, is the ***Cartan algebra of the pair*** (G, K).

The key theorem, due to Cartan, is that the Cartan algebra of a compact pair (G, K) does compute $H^*(G/K)$.

Theorem 8.1.12 (Cartan, [Car51, Thm. 5, p. 216][Bor53, Thm. 25.2]). *Given a compact pair (G, K), there is an isomorphism $H^*\big(H^*(BK) \otimes H^*(G)\big) \xrightarrow{\sim} H^*(G/K)$ making the following diagram commute:*

$$\begin{tikzcd}
 & H^*\big(H^*(BK) \otimes H^*(G)\big) \arrow[dr] \arrow[dd] & \\
H^*(BK) \arrow[ur] \arrow[dr, "\chi^*"'] & & H^*(G) \\
 & H^*(G/K) \arrow[ur, "j^*"'] &
\end{tikzcd} \tag{8.5}$$

Proof. Because $H^*(BK) \cong S\big[Q(BG)\big]$ is a free CGA, it is formal and Lemma 8.1.8 applies. $\square$

Remark 8.1.13. One might object that the inclusion of the hypothesis that the base B be formal is heavy-handed. Unfortunately, not all manifolds are formal, and in the instance one is not, the Cartan algebra of a bundle *can fail* to compute the cohomology of the total space. For an example of this phenomenon, see Section 3 of Baum and Smith [BS67, p. 178].

Corollary 8.1.14. *There is an isomorphism*

$$\operatorname{Tor}^{\bullet,\bullet}_{H^*(BG)}\big(\mathbb{Q}, H^*(BK)\big) \cong H^*(G/K).$$

Proof. By Theorem 8.1.12 and Proposition 8.1.7, $H^*(G/K) \cong H^*(C) \cong \operatorname{Tor}^{\bullet,\bullet}_{H^*(BG)}(\mathbb{Q}, H^*(B))$. □

Remark 8.1.15. If we set $K = G$, this statement makes precise our motivating claim in the introduction to Section 7.5 that the differentials in the SSS of the universal bundle $G \to EG \to BG$ filter an antiderivation τ extending the transgression which can be seen as the "one true differential." In the same way, the SSS of the Borel fibration $G \to G_K \to BK$ filters the differential on the Cartan algebra. This does not make this SSS, which we have already exploited to such effect, any less valuable: we will see examples in the next section where the Cartan algebra is unpleasantly complicated and it behooves us to look at the associated graded algebra $E_\infty = \operatorname{gr} H^*(G/K)$ instead. Moreover, in precisely the complement of this "bad" case, the associated graded construction is an isomorphism, so that the SSS of the Borel fibration calculates $H^*(G/K)$ on the algebra level. Rather than one description being more powerful, it is the *equivalence* of these two descriptions that turns out to be critical.

Remark 8.1.16. It is only fair to say at one point why we insist so fervently that K be connected. The main issue is that if K is not connected, then BK will not be simply-connected, and the Serre spectral sequence of the Borel fibration is calculated with local coefficients. One can still say some things, for if $K_0 < K$ is the identity component, then $BK_0 \longrightarrow BK$ and $G/K_0 \longrightarrow G/K$ are finite coverings, so if $|\pi_0 K|$ is invertible in k, one can embed $H^*(G/K)$ as the $\pi_0 K$-invariants of $H^*(G/K_0)$ by Proposition B.3.1 and likewise H^*_K as the $\pi_0 K$-invariants of $H^*_{K_0}$.

That G be connected, on the other hand, is not a real restriction if we insist K be connected, for then K will lie in the identity component G_0 of G and G/K will factor homeomorphically as $\pi_0 G \times G_0/K$, a finite disjoint union of copies of G_0/K.

Historical remarks 8.1.17. The original, unpublished statement of Chevalley's theorem, as best the author can tell, applied to the de Rham cohomology of a smooth principal G-bundle with compact total space. This statement is cited by Cartan and Koszul both (without proof) in the *Colloque* proceedings. Borel's generalization of this result, as proved in his thesis, removes the smoothness hypotheses by relying, instead of on forms, on an object of Leray's creation known as a *couverture*, which was superseded so quickly and so thoroughly by the ring of global sections of a fine sheaf of $\mathbb{R}$-CDGAs that it never acquired an English translation. Borel's statement of the result still requires compactness of the base because it relies on (what is essentially) sheaf cohomology with compact supports and a result of Cartan which in modern terms can be interpreted as saying a resolution of the constant sheaf $\underline{\mathbb{R}}$ on a paracompact Hausdorff space by a fine sheaf of $\mathbb{R}$-CDGAs always exists. Neither the principal bundle $G \to EG \to BG$ nor a $\mathbb{Q}$-CDGA model of cohomology was available to Borel at the time, so in his statement [Bor53, Thm 24.1] of Chevalley's theorem, our $H^*(B)$ is replaced with (essentially, again) a fine resolution $\mathscr{B}$ of the real constant sheaf on B.

As we have noted in Historical remarks 7.6.6, Borel did not actually have BK available, so his proof was slightly complicated by the need to invoke n-universal K-bundles $E(n, K) \longrightarrow B(n, K)$ for n sufficiently large. Borel's proof also incorporated not the Serre spectral sequence as we did, but the more sophisticated *Leray spectral sequence* from which Serre extracted his, applied simultaneously to an early formulation of a sheaf and a *couverture*. We will reproduce a less drastic modernization of Borel's original argument in Appendix D, and delve slightly further there into the meaning of the Leray spectral sequence, fine sheaves, and *couvertures*.

8.2. The structure of the Cartan algebra, I

The Cartan algebra makes a few results on $H^*(G/K)$ easy which would require more sophistication if attacked with the map of spectral sequences that was the subject of Section 8.1.1. We reproduce here the important bundle diagram (8.4), the spectral sequence of which the Cartan algebra encodes.

$$\begin{tikzcd} G \arrow[r, equal] \arrow[d, "j"'] & G \arrow[d] \\ G_K \arrow[r] \arrow[d, "\chi"'] & EG \arrow[d] \\ BK \arrow[r, "\rho"'] & BG. \end{tikzcd}$$

One important subobject of the Cartan algebra is related to the image of the map $j^*\colon H^*(G/K) \longrightarrow H^*(G)$ induced by $j\colon G \twoheadrightarrow G/K \simeq G_K$.

Definition 8.2.1. The image of $j^*\colon H^*(G/K) \longrightarrow H^*(G)$ is called the ***Samelson subring*** of $H^*(G)$. It meets the primitives $PG \leqslant H^*(G)$ in the ***Samelson subspace*** $\widehat{P}$.

The importance of the Samelson subspace is that in fact it generates $\operatorname{im} j^*$.

Proposition 8.2.2. *The Samelson subring is the exterior algebra* $\Lambda\widehat{P}$.

Proof. At the end of the spectral sequence (E_r, d_r) of the Borel fibration $G \to G_K \to BK$, the projection $E_\infty^{\bullet,\bullet} \twoheadrightarrow E_\infty^{0,\bullet}$ to the first column is the "associated graded" map $\operatorname{gr} j^*\colon \operatorname{gr} H^*(G/K) \longrightarrow H^*(G)$ (Proposition 4.3.7), so $\operatorname{im} j^* \cong E_\infty^{0,\bullet}$. To show this is $\Lambda\widehat{P}$, it will be enough to show each $E_r^{0,\bullet}$ is an exterior algebra on a subspace of PG. Note that $E_2^{0,\bullet}$ of the first page is $H^*(G) = \Lambda PG$ and assume inductively that $E_r = \Lambda P_r$ for some $P_r \leqslant PG$.

In the spectral sequence $(\widetilde{E}_r, \tilde{d}_r)$ of the universal bundle, the subspace $P^{r-1}H^*(G) < H^*(G)$ survives to $\widetilde{E}_r$, after which it disappears because of the edge isomorphism

$$\tilde{d}_r\colon P^{r-1}(G) \xrightarrow{\sim} \widetilde{E}_r^{r,0}.$$

From the chain equations

$$\psi_r \circ \tilde{d}_r = d_r \circ \psi_r \tag{8.6}$$

which arise from the bundle map (8.4) to the universal bundle, it follows inductively that $P^{r-1}H^*(G)$ cannot vanish before E_{r+1}, because for each $q < r$ one has

$$d_q z = d_q \psi_q z = \psi_q \tilde{d}_q z = \psi_q 0 = 0.$$

On the other hand, the differential d_r annihilates $P_r \cap E_r^{0,\leqslant r-2}$ because it sends it into the fourth quadrant. If we write $V = P_r \cap E_r^{0,\leqslant r-1}$ and $V^\perp$ for the subspace of P_r spanned by elements of degree other than $r-1$, then since $\Lambda P_r \cong \Lambda V^\perp \otimes \Lambda V$, we need only show that $\ker(d_r \restriction \Lambda V) = \Lambda(V \cap \ker d_r)$. But that follows from Proposition 7.5.3. ☐

Proposition 8.2.3. *If $\widetilde{H}_K$ is the augmentation ideal of H_K^*, then one has $\widehat{P} = d^{-1}\big(\widetilde{H}_K \cdot \operatorname{im} d\big)$.*

Proof. Let $z \in P^{r-1}G$, and suppose $d_r z = 0$ in E_r, so that z survives to $E_\infty^{0,\bullet}$. Then it must be that the lift $dz \in E_2$ lies in the kernel of $E_2^{\bullet,0} \twoheadrightarrow E_r^{\bullet,0}$, which is the ideal generated by the lifts to E_2 of the images of previous edge maps $d_i \colon E_i^{0,i-1} \longrightarrow E_i^{i,0}$. Since these edge differentials lift to those components of d which advance the horizontal filtration by fewer than r steps, it follows $dz \in \widetilde{H}_K \cdot \operatorname{im} d$.

On the other hand, if $dz \in \widetilde{H}_K \cdot \operatorname{im} d$ and $d_r z = 0$ in $\operatorname{im} r$, then the $\operatorname{im} d$ components can only arise from *earlier* differentials, as later differentials d_{r+n} send z past dz in the horizontal filtration. ☐

The Samelson subring is in fact a tensor factor of $H^*(G/K)$.

Definition 8.2.4. Let (G,K) be a compact pair. We write $\check{P} := PG/\widehat{P}$, and call this the ***Samelson complement***; the notation is supposed to indicate its complementarity to $\widehat{P}$.

Proposition 8.2.5. *The Cartan algebra admits a coproduct decomposition*

$$(H_K^* \otimes \Lambda PG, d) \cong \big(H_K^* \otimes \Lambda \check{P}, d\big) \otimes (\Lambda \widehat{P}, 0).$$

The proof is just what one would naively hope; we paraphrase from Greub *et al* [GHV76, 3.15 Thm. V, p. 116].

Proof. Choose some $\mathbb{Q}$-linear section

$$\widehat{P} \longrightarrow \ker d \leqslant H_K^* \otimes \Lambda PG$$

of the column projection $\ker d \twoheadrightarrow H^*(G/K) \xrightarrow{j^*} H^*(G)$. This section extends uniquely to a ring injection $f \colon \Lambda\widehat{P} \longrightarrow \ker d$ which we can extend further to a ring map

$$(H_K^* \otimes \Lambda\check{P}) \otimes \Lambda\widehat{P} \longrightarrow H_K^* \otimes \Lambda PG$$
$$(a \otimes \check{z}) \otimes \hat{z} \longmapsto (a \otimes \check{z}) \cdot f(\hat{z}).$$

This ring map is also a chain map, since it is the identity on the first tensor-factor of its domain and since for $\hat{z} \in \Lambda\widehat{P}$ we have $0 = d(f\hat{z}) = f\big(0(\hat{z})\big)$.

It remains to see f is bijective. Note that f is the identity on $HK \otimes \Lambda\check{P}$ and that given an element $z \in \widehat{P}$, since f is defined to be a section of the projection to the leftmost column, we have $f(z) \equiv 1 \otimes z \pmod{\widetilde{H}_K^*}$. Thus f preserves the the horizontal filtration induced by the filtration $F_p H_K^* = \bigoplus_{i \geqslant p} H_K^p$ on the base H_K^* and induces an isomorphism $\mathrm{gr}_\bullet f$ on associated graded algebras. By Proposition A.5.7, f is an isomorphism. □

Corollary 8.2.6. *Let (G, K) be a compact pair. Then there exists a tensor decomposition*

$$H^*(G/K) \cong H^*(H_K^* \otimes \Lambda\check{P}, d) \otimes \Lambda\widehat{P},$$

where the subring $\Lambda\widehat{P} = \operatorname{im} j^* \leqslant H^*(G)$ is induced from the projection $j \colon G \longrightarrow G/K$.

We write the first factor as J.

Corollary 8.2.7. *The factor J satisfies Poincaré duality.*

Proof. Since G/K is a compact manifold, $H^*(G/K)$ is a PDA by Theorem A.3.12, and the exterior algebra $\Lambda\widehat{P}$ is a PDA, so by Proposition A.3.14, so also must be the remaining factor J. □

The same way that $\operatorname{im} j^*$ admits a description as the leftmost column of E_∞ for the SSS of $G \to G_K \to BK$, so also the image of χ^* admits a description as the bottom row $E_\infty^{\bullet,0}$.

Definition 8.2.8. The map $\chi^*\colon H_K^* \longrightarrow H^*(G/K)$ is traditionally called the ***characteristic map*** and $\operatorname{im}\chi^* \cong H_K^* /\!/ H_G^*$ is the ***characteristic subring*** of the pair (G,K). The factor $J = H^*\big(H_K^* \otimes \Lambda\check{P}, d\big)$ of $H^*(G/K)$ in the decomposition Corollary 8.2.6 is called the ***characteristic factor***.

The name *characteristic subring* arises because, up to homotopy, the classifiying map $G/K \longrightarrow BK$ of the principal K-bundle $K \to G \to G/K$ is the projection $\chi\colon G_K \longrightarrow BK$ of the Borel fibration (see (8.1)), and the characteristic classes of the former K-bundle bundle lie in $\operatorname{im}\chi^*$. The *characteristic factor* is so called because $H_K^* \hookrightarrow H_K^* \otimes H^*(G)$ factors through $H_K^* \otimes \Lambda\check{P}$, making clear the following containment.

Proposition 8.2.9. *The characteristic ring* $\operatorname{im}\chi^*$ *is contained in the characteristic factor J.*

The cohomology sequence (8.2) is coexact at H_K^*, yielding the following pleasing description of the characteristic subring.

Proposition 8.2.10. *The characteristic subring is given by* $\operatorname{im}\chi^* \cong H_K^* /\!/ H_G^*$.

Proof. The bottom row H_K^* lies in the kernel of the Cartan differential, and meets its image in the ideal $\mathfrak{j}$ generated by $\rho^*(\operatorname{im}\tau)$. Since $\tau\colon P(G) \xrightarrow{\sim} Q(BG)$ surjects onto generators of H_G^*, it follows that the ideal $\mathfrak{j}$ which is the kernel of $H_K^* \longrightarrow H^*\big(H_K^* \otimes H^*(G)\big)$ is generated by the image $\rho^*\widetilde{H}_G$ of the augmentation ideal, so this image is $H_K^*/(\rho^*\widetilde{H}_G) =: H_K^* /\!/ H_G^*$, the ring-theoretic cokernel. By the commutativity of the diagram (8.5), this image subalgebra corresponds to $\operatorname{im}\chi^*$ in $H^*(G/K)$. □

This information is already enough to compute $H^*(G/K)$ in many cases of interest.

8.3. Cohomology computations, I

Lest we miss the trees for the forest in fleshing out our general description of the Cartan algebra, we take a detour to describe the cohomology of two popular classes of homogeneous spaces G/K, namely those for which $H^*(G) \to H^*(K)$ is surjective and those for which $\operatorname{rk} G = \operatorname{rk} K$.

8.3.1. Cohomology-surjective pairs

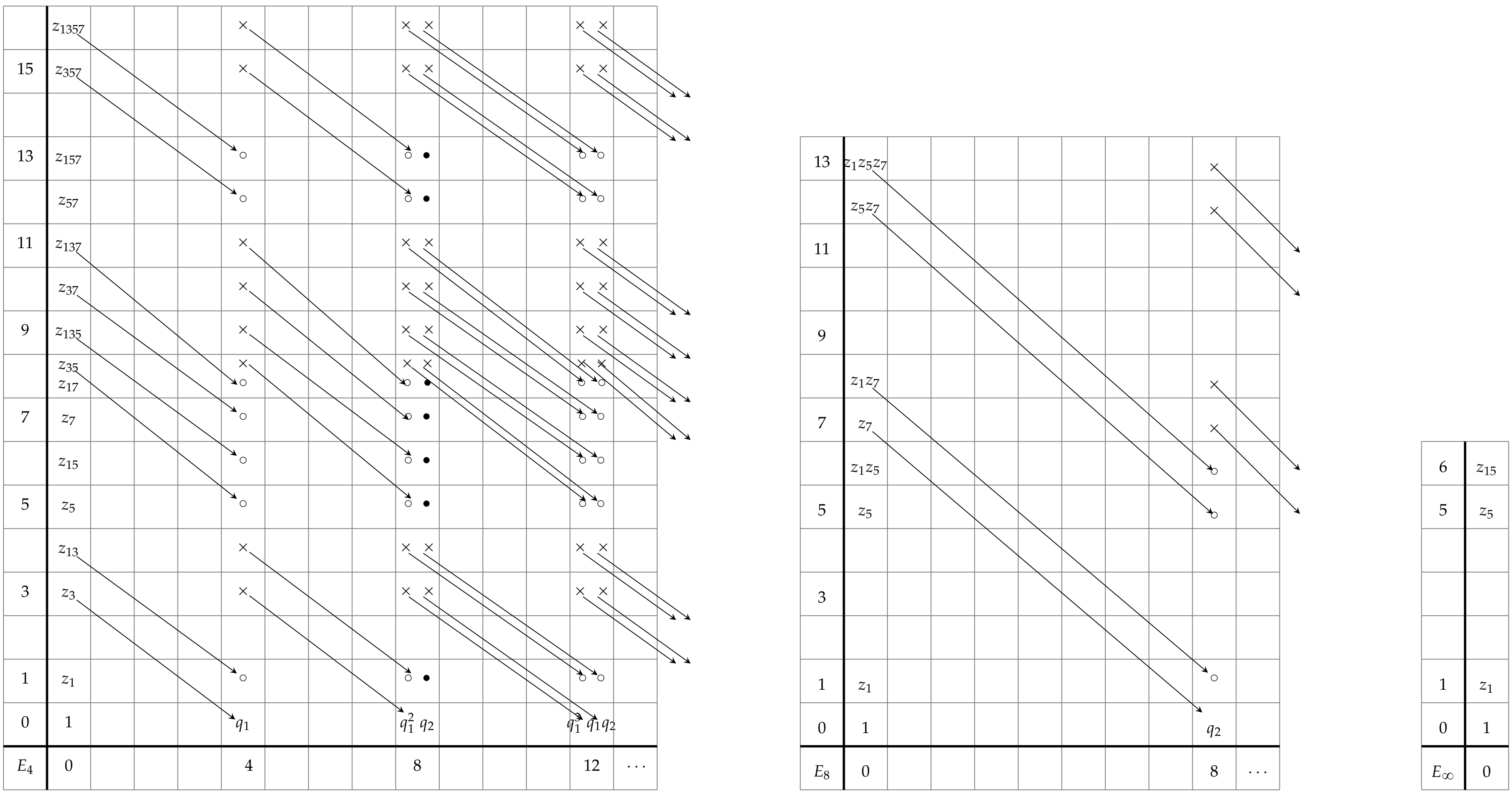
The map (8.4) of spectral sequences lets us easily reobtain Hans Samelson's classic theorem that $H^*(G) \cong H^*(K) \otimes H^*(G/K)$ whenever $H^*(G) \twoheadrightarrow H^*(K)$. Pictorially, this means the Serre spectral sequence of $G \to G_K \to BK$ looks like that of $\mathrm{U}(4) \to \mathrm{U}(4)_{\mathrm{Sp}(2)} \to B\mathrm{Sp}(2)$, as pictured in Figure 8.3.3; for now, just look at the E_∞ page, on the right.

Definition 8.3.1. If (G, K) is a compact pair such that $K \hookrightarrow G$ induces a surjection $H^*(G) \longrightarrow H^*(K)$ in cohomology, we call (G, K) a ***cohomology-surjective pair***.

Theorem 8.3.2 (Samelson [Sam41, Satz VI(b), p. 1134])**.** *Suppose that (G, K) is a cohomology-surjective pair. Then*

1. *$\rho^*\colon H_G^* \longrightarrow H_K^*$ is surjective,*
2. *$\chi^*\colon H_K^* \longrightarrow H^*(G/K)$ is trivial,*
3. *the Samelson subspace $\widehat{P}$ is complementary to $P(K)$ in $P(G)$,*
4. *$H^*(G/K)$ is the exterior algebra $\Lambda\widehat{P} \cong \Lambda P(G) /\!\!/ \Lambda P(K)$, and*

Figure 8.3.3: The Serre spectral sequence of $\mathrm{U}(4) \to \mathrm{U}(4)_{\mathrm{Sp}(2)} \to B\mathrm{Sp}(2)$; nonzero differentials (shown) send $\times \mapsto \circ$, whereas $\bullet$s survive to the next page



5. $H^*(G) \cong H^*(K) \otimes H^*(G/K)$.

6. *If the Poincaré polynomials of PG and PK are respectively* $p(PG) = \sum_{j=1}^{n} t^{d_j}$ *and* $p(PK) = \sum_{j=1}^{\ell} t^{d_j}$, *then* $p(G/K) = \prod_{j=\ell+1}^{n}(1 + t^{d_j})$.

Proof. By Proposition 7.2.9 the fact $i\colon K \hookrightarrow G$ is a group homomorphism implies $i^*\colon H^*(G) \twoheadrightarrow H^*(K)$ takes the primitives $P(G) \longrightarrow P(K)$. Because we have assumed i^* surjective, it follows $i^*P(G) = P(K)$ and because i^* is a ring homomorphism that $\ker i^* \cong \Lambda\big[P(G)/P(K)\big]$.

The outer columns of (8.3) are a bundle map between the universal principal K- and G-bundles, inducing a map of Serre sequences interleaving the transgressions. Restricting to primitives, one has the commutative diagram

$$\begin{tikzcd} P(K) \arrow[d, "\tau_K", "\wr"'] & P(G) \arrow[l, two heads, "i^*"'] \arrow[d, "\tau_G"', "\wr"] \\ Q(BK) & Q(BG), \arrow[l, "Q\rho^*"] \end{tikzcd} \tag{8.7}$$

which implies that $Q(\rho^*)Q(BG) = Q(BK)$ and hence that $\rho^*\colon H^*(BG) \longrightarrow H^*(BK)$ is also surjective. It follows from the triviality of $\chi^* \circ \rho^*$ that the characteristic subring $\operatorname{im}\big(\chi*\colon H_K^* \longrightarrow H^*(G/K)\big)$ is $\mathbb{Q}$.

If we embed $P(K) \rightarrowtail P(G)$ by taking a section of i^*, we see from the transgression square (8.7) that the complement of $P(K)$ is annihilated by $\rho^* \circ \tau_G$, so that the Samelson subspace $\widehat{P} \leqslant P(G)$ is a complement to $P(K)$, or $\widehat{P} \cong P(G)/P(K)$.

Because $\rho^* \circ \tau$ ends $P(K)$ onto $Q(BK)$ and annihilates $\widehat{P}$, we have a ring factorization of $E_2 \cong H^*(BK) \otimes H^*(G)$ as

$$\big[H^*(BK) \otimes H^*(K)\big] \otimes \Lambda\widehat{P},$$

which respects the transgression in that all differentials are trivial on $\widehat{P}$, and the left tensor factor is the beginning of the filtration spectral sequence corresponding to the Koszul complex on $Q(BK)$ (cf. Proposition 7.6.4). It follows $E_\infty = E_\infty^{0,\bullet} \cong \Lambda\widehat{P}$. Thus we can identify the short coexact sequence $H^*(K) \underset{i^*}{\leftarrow} H^*(G) \underset{j^*}{\leftarrow} H^*(G/K)$ with

$$0 \leftarrow \Lambda P(K) \longleftarrow \Lambda\big[P(K) \oplus \widehat{P}\big] \longleftarrow \Lambda\widehat{P} \leftarrow 0;$$

the tensor factorization is valid simply because by Proposition A.4.3 the free CGA $\Lambda P(K)$ is projective.

The result on Poincaré polynomials follows from the statements in Appendix A.3.2, since $p(\Lambda PG) = \prod_{j=1}^{n}(1+t^{d_i})$ and $p(\Lambda PK) = \prod_{j=1}^{\ell}(1+t^{d_i})$. □

Remarks 8.3.4. (*a*) With the benefit of hindsight, our calculations of the cohomology rings of $\mathrm{SU}(n)$ in Proposition 7.1.4 and of $V_j(\mathbb{C}^n)$ and $V_j(\mathbb{H}^n)$ in Proposition 7.1.5 can all be seen to be of this form.

(*b*) The Samelson isomorphism $H^*(G) \cong H^*(G/K) \otimes H^*(K)$ also follows directly from Corollary 7.2.5 independent of any consideration of classifying spaces.

Proposition 8.3.5 ([Car51, 1°, p. 69][Bor53, Corollaire, p. 179])**.** *Let $i\colon K \longhookrightarrow G$ be an inclusion of compact, connected Lie groups. Then $\rho^*\colon H_G^* \longrightarrow H_K^*$ is surjective if and only if $i^*\colon H^*(G) \longrightarrow H^*(K)$ is.*

Proof. This follows immediately from the commutative square (8.7) in the proof of Theorem 8.3.2 since the vertical maps are isomorphisms. □

Most of these conditions are clearly equivalent. In fact, a weaker dimension condition on $H^*(G/K)$ is equivalent to cohomology-surjectivity.

Proposition 8.3.6 ([GHV76, Thm. 10.19.X,(6) p. 466])**.** *Let (G,K) be a compact pair. One has*

$$h^\bullet(G) \leqslant h^\bullet(G/K) \cdot h^\bullet(K),$$

with equality if and only if (G,K) is cohomology-surjective.

Proof [GHV76, Cor. to Thm. 3.18.V, p. 125]. This follows from Corollary 4.3.10 as applied to the Serre spectral sequence of $K \to G \to G/K$, □

Example 8.3.7. Recall from Example 7.8.1 that $H^*\big(B\mathrm{U}(4)\big) \longrightarrow H^*\big(B\mathrm{Sp}(2)\big)$ is surjective. From Proposition 8.3.5, we see as well that $H^*\big(\mathrm{U}(4)\big) \longrightarrow H^*\big(\mathrm{Sp}(2)\big)$, as promised. We had

$$\begin{aligned} c_1 &\longmapsto 0, \\ c_2 &\longmapsto -q_1, \\ c_3 &\longmapsto 0, \\ c_4 &\longmapsto q_2, \end{aligned}$$

so in the primitive subspace $P\mathrm{U}(4) = \mathbb{Q}\{z_1, z_3, z_5, z_7\}$ we have $P\mathrm{Sp}(2) = \mathbb{Q}z_3 \oplus \mathbb{Q}z_7$ and $\widehat{P} = \mathbb{Q}z_1 \oplus \mathbb{Q}z_5$. It follows from Section 8.3.1 that

$$H^*\big(\mathrm{U}(4)/\mathrm{Sp}(2)\big) \cong \Lambda[z_1, z_5], \quad \deg z_j = j.$$

The resulting spectral sequence, Figure 8.3.3, appears complicated, but this complexity is only apparent. Staring closely at the picture, one sees that $\Lambda\widehat{P} = \Lambda[z_1, z_5]$ is a tensor-factor, to which nothing ever happens, and the massive simplifications after the 4^{th} and 8^{th} pages just witness that the Koszul complexes $K[z_3]$ and $K[z_7]$ are other tensor-factors.

Alternately, not bothering with the picture, the transgression in the universal principal $\mathrm{U}(4)$-bundle takes $z_1 \longmapsto c_1$ and $z_5 \longmapsto c_3$, this means that $\Lambda\widehat{P} = \Lambda[z_1, z_5]$ splits off in the Cartan algebra immediately, and $S[q_1, q_2] \otimes \Lambda[z_3, z_7]$ is a Koszul complex, so acyclic.

A little more work shows that $H^*_{\mathrm{U}(2n)} \longrightarrow H^*_{\mathrm{Sp}(n)}$ is surjective for all n with kernel the odd Chern classes, and it follows

$$H^*\big(\mathrm{U}(2n)/\mathrm{Sp}(n)\big) \cong \Lambda[z_1, \ldots, z_{4n-3}], \quad \deg z_j = j.$$

As an example application of Samelson's theorem, we prove a result which will be of use to us later in investigating equivariant formality of isotropy actions.

Lemma 8.3.8. *Let S be a torus in a compact, connected Lie group G and $Z = Z_G(S)$ its centralizer in Z. The cohomology of Z decomposes as*

$$H^*(Z) \cong H^*(S) \otimes H^*(Z/S).$$

Consequently, $H^*(Z/S)$ is an exterior algebra on $\operatorname{rk} G - \operatorname{rk} S$ generators and $h^\bullet(Z/S) = 2^{\operatorname{rk} G - \operatorname{rk} S}$.

Proof. By Theorem 8.3.2, it will be enough to show the inclusion $S \hookrightarrow Z$ surjects in cohomology. Since S is normal in Z, the quotient Z/S is another Lie group, so $\pi_2(Z/S) = 0$ by Corollary 7.2.11 and in the long exact homotopy sequence (Theorem B.2.4) of the bundle $S \to Z \to Z/S$ we find the fragment $0 = \pi_2(Z/S) \to \pi_1 S \to \pi_1 Z$. Since S and Z are topological groups, their fundamental groups are abelian by Proposition B.4.2 and hence isomorphic to their first homology groups by Proposition B.2.5, so $H_1(S; \mathbb{Z}) \longrightarrow H_1(Z; \mathbb{Z})$ is injective. It follows from Theorem B.2.1 that $H_1(S; \mathbb{Q}) \longrightarrow H_1(Z; \mathbb{Q})$ is injective, and, dualizing, that $H^1(Z; \mathbb{Q}) \longrightarrow H^1(S; \mathbb{Q})$ is surjective. Since $H^1(S)$ generates $H^*(S)$, it must be that $H^*(Z) \longrightarrow H^*(S)$ is surjective as well.

The statement on Betti number follows because Z must have the same rank as G, since S is contained in some maximal torus of G by Theorem B.4.9. □

Historical remarks 8.3.9. Proposition 8.3.5 was first proven by Cartan [Car51, 1°, p. 69][Bor53, Corollaire, p. 179].

A surjection $H^*(G) \longrightarrow H^*(K)$ in cohomology corresponds dually to an injection $H_*(K) \longrightarrow H_*(G)$ in homology, and it was this condition Hans Samelson researched in the work in which the tensor decomposition (5.) above was first proven [Sam41]. It has since been said that K is *totally nonhomologous to zero* in G. Samelson said the *Isotropiegruppe U nicht homolog in der Gruppe G ist* or $U \nsim 0$, the letter U for *Untergruppe* (our K), and showed if the fundamental class $[K] \in H_*(K)$ did not become zero in $H_*(G)$, then $H_*(K) \longrightarrow H_*(G)$: the fundamental class $[K] \in H_*(K;\mathbb{Q}) \cong \Lambda(PK)^*$ is the product of a set of algebra generators, so if $\rho_*[K] \neq 0$ in $H_*(G)$, then ρ_* is injective. The "totally" is redundant and sometimes dropped for this reason.

When the cohomology ring rather than the homology ring became the primary actor, later expositors (e.g. [GHV76]) named the condition, by analogy, *totally noncohomologous to zero*, even though that name taken literally would imply the surjection $H^*(G) \twoheadrightarrow H^*(K)$ should be injective. These conditions have been abbreviated variously *TNHZ*, *TNCZ*, and *n.c.z.* For safety's sake, in dealing with this situation we will always simply say a map surjects in cohomology.

8.3.2. Pairs of equal rank

We recast some of the results from Chapter 5 in this framework.

Definition 8.3.10. A compact (G, K) is an ***equal-rank pair*** if $\operatorname{rk} G = \operatorname{rk} K$.

Theorem 8.3.11 (Leray)**.** *Let (G, K) be an equal-rank pair. Then*

1. $\rho^* \colon H_G^* \longrightarrow H_K^*$ *is injective,*

2. *$\chi^*\colon H_K^* \longrightarrow H^*(G/K)$ is surjective,*

3. *the Samelson subspace $\widehat{P}$ is trivial,*

4. *$H^*(G/K) \cong E_\infty^{\bullet,0}$ is $H_K^* /\!\!/ H_G^* \cong H_T^{W_K} /\!\!/ H_T^{W_G}$.*

5. *If the Poincaré polynomials of $P(G)$ and $P(K)$ are respectively $p(PG) = \sum_{j=1}^n t_j^{2g_j-1}$ and $p(PG) = \sum_{j=1}^n t_j^{2k_j-1}$, then the Poincaré polynomial of G/K is*

$$p(G/K) = \frac{p(BK)}{p(BG)} = \prod_{j=1}^n \frac{1-t^{2k_j}}{1-t^{2g_j}}. \tag{8.8}$$

Proof. The inclusion $K \hookrightarrow G$ induces an injection of Weyl groups $W_K \rightarrowtail W_G$. and in turn an inclusion $H_T^{W_G} \hookrightarrow H_T^{W_K} \hookrightarrow H_T^*$ of Weyl invariants. Recalling from Corollary 6.4.7 that $H_G^* \cong H_T^{W_G}$, this means $H_G^* \rightarrowtail H_K^* \rightarrowtail H_T^*$ and in particular $\rho^*\colon H_G^* \longrightarrow H_K^*$ is injective.[1] Since the transgression $\tau\colon PG \xrightarrow{\sim} Q(BG)$ is also injective, the composition $\rho^* \circ \tau\colon PG \longrightarrow H_K^*$ is as well, so its kernel $\widehat{P}$ is 0. The injectivity of ρ^* combined with the fact $\operatorname{im}\chi^* \cong H_K^* /\!\!/ H_G^*$ means $H_K^* \cong H_G^* \otimes \operatorname{im}\chi^*$ as an H_G^*-module, so the Cartan algebra $H^*(BK) \otimes H^*(G)$ factors as

$$(\operatorname{im}\chi^*, 0) \otimes \big(H_G^* \otimes H^*(G), d\big).$$

Since the second term is a Koszul complex, which has trivial cohomology by Proposition 7.5.2, we have $H^*(G/K) \cong \operatorname{im}\chi^* = H_K^* /\!\!/ H_G^*$ by the Künneth theorem.

As far as Poincaré polynomials are concerned, note first that we are implicitly assuming the

[1] We have proved this from abstract results about invariants, but these maps arise from the cohomology of the base spaces in the sequence

$$\begin{tikzcd} G_T \arrow[r] \arrow[d] & G_K \arrow[r] \arrow[d] & EG \arrow[d] \\ BT \arrow[r] & BK \arrow[r] & BG, \end{tikzcd}$$

of principal G-bundles maps, where the maps of total spaces can be conceived as "further quotient" maps among quotients of $EG \times G$.

results of Section 7.2 that $H^*(G)$ and $H^*(K)$ are exterior algebras in order to conclude $Q(BG) \cong \Sigma PG$ is spanned by generators of degree $2g_j$, and $H^*(BG) = S\big[Q(BG)\big]$ is a polynomial ring on these generators. so by the results of Appendix A.3.2, we have

$$p(BG) = \prod_j \frac{1}{1-t^{2g_j}} \qquad \text{and} \qquad p(BK) = \prod_j \frac{1}{1-t^{2k_j}}.$$

The H_G^*-module isomorphism $H_K^* \cong H_G^* \otimes H^*(G/K)$ reduces on the the level of graded vector spaces to

$$p(BK) = p(BG) \cdot p(G/K).$$

Multiplying through by $p(BG)^{-1} = \prod_j (1 - t^{2g_j})$ yields the claimed formula. □

Corollary 8.3.12 (Leray, [Bor53, Prop. 29.2, p. 201])**.** *Let G be a compact, connected Lie group and T a maximal torus. Then the characteristic map $\chi^* \colon H^*(BT) \longrightarrow H^*(G/T)$ is surjective, and if the Poincaré polynomial of $P(G)$ is $p(PG) = \sum_{j=1}^n t_j^{2g_j - 1}$, then*

$$p(G/T) = \prod_{j=1}^n \frac{1-t^2}{1-t^{2g_j}}.$$

We have also a converse.

Proposition 8.3.13. *If $H^*(G/K)$ is concentrated in even degrees, then K and G are of equal rank.*

Proof. If $H^*(G/K)$ is concentrated in even degrees, then the Euler characteristic $\chi(G/K) > 0$. Thus the result follows from Corollary 5.2.5; if we had $\operatorname{rk} K < \operatorname{rk} G$, then we would also have $\chi(G/K) = 0$. □

This result also admits a purely algebraic proof involving commutative algebra and the Samelson subspace.

Corollary 8.3.14 (Borel [Bor53, Corollaire, p. 168]). *Suppose (G,K) is a pair of compact, connected Lie groups such that the characteristic homomorphism $\chi^*\colon H_K^* \longrightarrow H^*(G/K)$ is surjective. Then for every principal G-bundle $G \to E \to B$, the fiber inclusion of the quotient bundle $G/K \longrightarrow E/K \to B$ is surjective in cohomology.*

Proof. The principal bundle $G \to E \to B$ is classified by a map $B \longrightarrow BG$, inducing a bundle map to the universal bundle $G \to EG \to BG$. Taking the right quotient of the total spaces of both bundles by K yields a bundle map

$$\begin{tikzcd} G/K \arrow[r, equal] \arrow[d, "f"'] & G/K \arrow[d, "\chi"] \\ E/K \arrow[r, "h"] \arrow[d] & BK \arrow[d] \\ B \arrow[r] & BG. \end{tikzcd}$$

But the existence of this map puts us in the situation of By Theorem 4.4.1, one has $H^*(E/K) \longrightarrow H^*(G/K)$ surjective; moreover,

$$H^*(E/K) \cong H^*(B) \otimes_{H_G^*} H_K^*. \qquad \square$$

Theorem II, sec. 10.7, Theorem VII, sec. 3.17, and the Corollary to Proposition V, sec. 3.18.

Example 8.3.15. Consider the pair $\big(\mathrm{U}(n), T^n\big)$. The Weyl group $W_{\mathrm{U}(n)}$ is the symmetric group S_n acting on $H_T^* = \mathbb{Q}[t_1, \ldots, t_n]$ by permuting the generators $t_j \in H^2(BT)$, so $H^*_{\mathrm{U}(n)} = \mathbb{Q}[c_1, \ldots, c_n]$ is generated by the elementary symmetric polynomials $c_j = \sigma_j(\vec{t})$. It follows that the cohomology of the complex flag manifold $\mathrm{U}(n)/T^n$ is

$$H^*\big(\mathrm{U}(n)/T^n\big) \cong \mathbb{Q}[t_1, \ldots, t_n]/(c_1, \ldots, c_n),$$

with Poincaré polynomial

$$p\big(\mathrm{U}(n)/T^n\big) \;=\; {(1-t)^n}\Big/{\prod_{j=1}^n (1-t)^j} \;=\; 1(1+t)(1+t+t^2)\cdots(1+t+t^2+\cdots+t^{n-1}),$$

which, evaluated at $t=1$, yields rational dimension $n! = |S_n| = |W_{\mathrm{U}(n)}|$. We will see this is no coincidence.

If we take $n=2$, then

$$\mathrm{U}(2)/T^2 \;=\; {\mathrm{U}(2)}\Big/{\mathrm{U}(1)\times\mathrm{U}(1)} \;\approx\; G(1,\mathbb{C}^2) = \mathbb{C}\mathrm{P}^1 \approx S^2,$$

so we know what to expect. Indeed, $c_1 = t_1+t_2$ and $c_2 = t_1t_2$ in $H_T^* = \mathbb{Q}[t_1,t_2]$, so

$$H^*\big(\mathrm{U}(2)/T^2\big) \;\cong\; H^*_{T^2} /\!\!/ H^*_{\mathrm{U}(2)} \;=\; {\mathbb{Q}[t_1,t_2]}\Big/{(t_1+t_2, t_1t_2)} \;\cong\; \mathbb{Q}[t_1]/(t_1^2)$$

as predicted.

For a less trivial example, take $n=3$, so that $c_1 = t_1+t_2+t_3$ and $c_2 = t_1t_2+(t_1+t_2)t_3$ and $c_3 = t_1t_2t_3$. Since we are setting each $c_j \equiv 0$, we can eliminate out the generator $t_3 \equiv -(t_1+t_2)$ and know $0 \equiv c_2 \equiv t_1t_2 - (t_1+t_2)^2 = -(t_1^2+t_2^2+t_1t_2)$. Simplifying $c_3 \equiv 0$ yields $t_1t_2^2 + t_1^2t_2 \equiv 0$, so

$$H^*\big(\mathrm{U}(3)/T^3\big) \;\cong\; {\mathbb{Q}[t_1,t_2]}\Big/{(t_1^2+t_2^2+t_1t_2,\; t_1^2t_2+t_1t_2^2)}.$$

See Figure 8.3.16.

Figure 8.3.16: A basis for the E_∞ page of the Serre spectral sequence of $\mathrm{U}(3) \to \mathrm{U}(3)_{T^3} \to BT^3$

0	1		t_1, t_2		t_1^2, t_2^2		$t_1^3+t_2^3$
	0		2		4		6

Example 8.3.17. Consider the pair $\big(\mathrm{Sp}(n), \mathrm{Sp}(k)\times\mathrm{Sp}(n-k)\big)$, yielding the quotient $G(k,\mathbb{H}^n)$. The Weyl group $W_{\mathrm{Sp}(n)}$ is the semidirect product symmetric group $\{\pm1\}^n \rtimes S_n$, where S_n acts by permuting the entries of $\{\pm1\}^n$, and $W_{\mathrm{Sp}(n)}$ acts on $H_T^* = \mathbb{Q}[t_1,\ldots,t_n]$ by permuting and negating the generators $t_j \in H^2(BT)$, so $H^*_{\mathrm{Sp}(n)} = \mathbb{Q}[q_1,\ldots,q_n]$ is generated by the elementary symmetric polynomials $q_j = \sigma_j(t_1^2,\ldots,t_n^2)$ in the squares $t_j^2 \in H^4(BT)$. The Weyl group $W_{\mathrm{Sp}(k)\times\mathrm{Sp}(n-k)} = W_{\mathrm{Sp}(k)} \times W_{\mathrm{Sp}(n-k)}$ permutes the subrings $\mathbb{Q}[t_1,\ldots,t_k]$ and $\mathbb{Q}[t_{k+1},\ldots,t_n]$ separately, so

$$H^*\big(G(k,\mathbb{H}^n)\big) \;\cong\; \mathbb{Q}[t_1,\ldots,t_k]^{W_{\mathrm{Sp}(k)}} \otimes \mathbb{Q}[t_{k+1},\ldots,t_n]^{W_{\mathrm{Sp}(n-k)}} \Big/ (q_1,\ldots,q_n)$$

We will calculate explicitly what happens if $n=5$ and $k=3$. For convenience, set $u_j = t_j^2$. The numerator ring $H^*_{\mathrm{Sp}(3)} \otimes H^*_{\mathrm{Sp}(2)}$ is the polynomial subring $\mathbb{Q}[r_1,r_2,r_3,s_1,s_2]$ of $\mathbb{Q}[u_1,u_2,u_3,u_4,u_5]$ generated by the five generators on the left, and the denominator is the ideal generated by the elements on the right:

$$\begin{aligned}
r_1 &= u_1+u_2+u_3, & q_1 &= r_1+s_1,\\
r_2 &= u_2(u_1,u_2,u_3), & q_2 &= r_1s_1+r_2+s_2,\\
r_3 &= u_1u_2u_3, & q_3 &= r_3+r_2s_1+r_1s_2,\\
s_1 &= u_4+u_5, & q_3 &= r_3s_1+r_2s_2,\\
s_2 &= u_4u_5; & q_5 &= r_3s_2.
\end{aligned}$$

Imposing the congruences generated by setting each $q_j \equiv 0$ and crunching relations a few times yields

$$H^*\big(G(3,\mathbb{H}^5)\big) \;\cong\; \mathbb{Q}[r_1,r_2] \Big/ (r_1^4 - r_1^2r_2 - r_2^2,\, 2r_1^3r_2 + 3r_1r_2^2), \qquad \deg r_1 = 4,\ \deg r_2 = 8.$$

Historical remarks 8.3.18. Leray's determination of $H^*(G/T)$ dates back to 1946 in the event G is a compact, connected, classical simple group [Ler46b]. By 1949, he only requires that the universal compact cover (see Theorem B.4.4) $\widetilde{G}$ of G contain no exceptional factors [Ler49a]. His original statement of Theorem 8.3.11 requires no exceptional group to occur as factors of the universal compact cover $\widetilde{G}$ of G, but allows K to be any closed subgroup, not necessarily connected, of equal rank. His additional requirement on G is removed by the time of his contribution [Ler51] to the 1950 Brussels *Colloque de Topologie*. The formula (8.8) was first conjectured by Guy Hirsch and is hence traditionally called the *Hirsch formula*. According to Dieudonné [Die09, p. 448], Cartan and Koszul obtained this result independently around the same time. The initial proof that $H^*(G/T)$ is the regular representation of W_G also dates to Leray in the Bruxelles conference; he had earlier shown in [Ler49a] the same result holds if G is finitely covered by a product of classical groups.

8.4. The structure of the Cartan algebra, II: formal pairs

Returning to our discussion of homogeneous spaces, let (G, K) be a compact pair and consider the Cartan algebra $H_K^* \otimes H^*(G)$ with differential d induced by $\rho^* \circ \tau$.

Recall that if the Samelson subspace $\widehat{P} \leqslant H^*(G)$ is the subspace of the primitives of G where d vanishes and $\check{P} = PG/\widehat{P}$ is the Samelson complement, we defined the *characteristic factor* to be $J := H^*\big(H_K^* \otimes \Lambda\check{P}, d\big)$ and found a tensor decomposition (Corollary 8.2.6)

$$H^*(G/K) \cong J \otimes \Lambda\widehat{P}.$$

One would like in a similar way to be able to tensor-factor out the characteristic subring $\operatorname{im}\chi^*$ from J, but *this is not generally possible*. The best we are able to do in this regard is the following.

Proposition 8.4.1. *The characteristic ring* $\operatorname{im}\chi^*$ *is simultaneously a subring and quotient ring of the characteristic factor* $J = H^*(H_K^* \otimes \Lambda\check{P})$.

Proof. Since the image of d meets H_K^* in $\rho^* H_G^*$, the composite projection

$$H_K^* \otimes H^*(G) \twoheadrightarrow H_K^* \twoheadrightarrow H_K^* /\!/ H_G^* = \operatorname{im}\chi^*$$

descends in cohomology to a homomorphism $H^*(G/K) \twoheadrightarrow \operatorname{im}\chi^*$ split by the defining inclusion $\operatorname{im}\chi^* \hookrightarrow H^*(G/K)$. □

In this section, we explore the propitious case in which the characteristic subring $\operatorname{im}\chi^*$ *is* the characteristic factor J.

Definition 8.4.2. If $H^*(G/K) \cong \operatorname{im}\chi^* \otimes \Lambda\widehat{P}$, we call (G, K) a ***formal pair*** (traditionally, such a pair is called a *Cartan pair*).

Example 8.4.3. Suppose (G, K) is a cohomology-surjective pair. Then, by Theorem 8.3.2, the characteristic factor J is trivial.

Example 8.4.4. Suppose (G, K) is an equal-rank pair. Then, by Theorem 8.3.11, the Samelson subring $\Lambda\widehat{P}$ is trivial and the characteristic factor J is the characteristic ring $\operatorname{im}\chi^*$.

One can see formal pairs as the smallest class of cases that contains both these extreme examples. Another way of seeing it is this: the first interesting page of the Serre spectral sequence of the Borel fibration $G \to G_K \to BK$ is $E_2 = E_2^{\bullet,0} \otimes E_2^{0,\bullet} \cong H_K^* \otimes H^*(G)$, a coproduct of CGAs, with one tensor-factor each arising from the base and the fiber of the fibration. In our examples in Section 8.3.2 and Section 8.3.1, this tensor-product structure persisted throughout the entire sequence, in that the decomposition $E_r = E_r^{\bullet,0} \otimes E_r^{0,\bullet}$ continued to hold, and

$$E_\infty = E_\infty^{\bullet,0} \otimes E_\infty^{0,\bullet} = \big(H_K^* /\!/ H_G^*\big) \otimes \Lambda\widehat{P}$$

was the tensor product of the characteristic subring $\operatorname{im}\chi^*$ and the Samelson subring $\Lambda\widehat{P}$.[2] For a representative example, see Figure 8.5.4. This is also the optimal situation from a purely numerical perspective, because, in particular, the tensor decomposition yields a factorization

$$p(G/K) = p(E_\infty^{\bullet,0}) \cdot p(E_\infty^{0,\bullet}), \tag{8.9}$$

of Poincaré polynomials and in particular, setting the formal variable $t = 1$, a factorization

$$h^\bullet(G/K) = \dim_{\mathbb{Q}} E_\infty^{\bullet,0} \cdot \dim_{\mathbb{Q}} E_\infty^{0,\bullet}.$$

We will expound a number of properties of and equivalent characterizations of the formal pair condition, in the process justifying the nomenclature. The very fact that there are so many ways of stating this property should be a further argument, were one needed, for the naturality of the concept.

But first we introduce an important bound on the dimension of the Samelson subspace.

Definition 8.4.5. The ***deficiency*** of a compact pair (G, K) is the integer

$$\operatorname{df}(G, K) := \operatorname{rk} G - \operatorname{rk} K - \dim \widehat{P}.$$

Proposition 8.4.6. *The deficiency is a natural number. That is, for any compact pair (G, K), one has*

$$\dim PG - \dim PK \geqslant \dim \widehat{P}.$$

Proof [Bau68, Lem. 3.7, p. 26]. Since $\check{P} \oplus \widehat{P} = PG$ by definition, it is enough to show $\dim \check{P} \geqslant$

[2] We concede that in those examples, it was the tensor product of precisely one of those factors—there are historical reasons why those cases were studied first.

$\dim PK$. This can be shown through Poincaré polynomials. We may view H_K^* as an algebra over the polynomial ring $A = S\big[\tau(\check{P})\big]$ by restricting $\rho^*\colon H_G^* \longrightarrow H_K^*$. If we lift a basis of $H_K^* /\!/ H_G^* = H_K^* /\!/ A$ back to H_K^*, this basis spans H_K^* as an A-module (typically with some redundancy; we do not expect H_K^* to be a free A-module). Thus $p(H_K^* /\!/ H_G^*) \cdot p(A) \geqslant p(H_K^*)$ (in that each coefficient of t^n on the left is at least its counterpart on the right), or dividing through,

$$p(H_K^* /\!/ H_G^*) \geqslant \frac{p(H_K^*)}{p(A)}.$$

Both the numerator and denominator on the right-hand side are products of factors $1 - t^n$, by (A.2). There are $\dim PK$ such factors in the numerator and $\dim \check{P}$ in the denominator, so if we had $\dim PK > \dim \check{P}$, the rational function $p(H_K^*)/p(A)$ would have a pole at $t = 1$, but this is impossible because it is majorized by the polynomial $p(H_K^* /\!/ H_G^*)$. $\square$

Theorem 8.4.7 ([Oni94, Thm. 12.2, p. 211]). *Let (G, K) be a compact pair. The following conditions are equivalent:*

1. *(G, K) is a formal pair.*
2. *The kernel $(\operatorname{im} \widetilde{\rho}^*)$ of the characteristic map $H_K^* \xrightarrow{\chi^*} H^*(G/K)$ is a regular ideal in the sense of Definition 7.5.4.*
3. *The characteristic factor J in the decomposition $H^*(G/K) \cong J \otimes \Lambda\widehat{P}$ is evenly-graded.*
4. *The deficiency $\operatorname{df}(G, K) = \dim PG - \dim PK - \dim \widehat{P}$ is zero.*

Proof. We always have $H^*(G/K) \cong J \otimes \Lambda\widehat{P}$, so the task is to prove the remaining conditions are equivalent to the statement $J = \operatorname{im} \chi^*$.

1 $\iff$ 2. If we singly grade the CDGA $C = H_K^* \otimes \Lambda \check{P}$, by

$$\cdots \longrightarrow H_K^* \otimes \Lambda^2 \check{P} \longrightarrow H_K^* \otimes \Lambda^1 \check{P} \longrightarrow H_K^* \to 0, \tag{8.10}$$

where the differential d vanishes on H_K^* and is induced by

$$\check{P} \hookrightarrow PG \xrightarrow[\tau]{\sim} Q(BG) \xrightarrow{\rho^*} H_K^*$$

then $J = \operatorname{im} \chi^* = H_K^* /\!/ H_G^*$ if and only if $H^*(C) = H^0(C)$. But if we write $\vec{x}$ for a basis of $\tau(\check{P}) \leqslant H_G^*$, then C is the Koszul complex $K_{H_G^*}(\vec{x}, H_K^*)$ of Definition 7.5.4. The commutative algebra result Proposition 7.5.7 then states this Koszul complex is acyclic if and only if the sequence is regular.

1 $\implies$ 3. This is clear since $\operatorname{im} \chi^* = H_K^* /\!/ H_G^*$ inherits an even grading from H_K^*.

3 $\implies$ 2. If J is evenly graded, then H^1 of the Koszul complex C of (8.10) above must be zero because $\check{P} \leqslant PG$ is oddly-graded. But by Proposition 7.5.7, this also means $J = H^*(C) = H^0(C) = H_K^* /\!/ H_G^* = \operatorname{im} \chi^*$.

2 $\iff$ 4 [Oni94, p. 144]. Write $y_1, \ldots, y_n$ for a basis of $Q(BK)$ and $b_1, \ldots, b_\ell$ for a basis of $\tau(\check{P}) \leqslant S[y_i]$. Note that we know that $\operatorname{df}(G, K) \geqslant 0$ in any event by Proposition 8.4.6, and if $\operatorname{df}(G, K) = 0$, then $\dim \check{P} = \dim PK$.

Working over $k = \overline{\mathbb{Q}}$ or $\mathbb{C}$, the ring $k[y_i]/(b_j)$ is finite-dimensional as a k-module, so the variety $V = V(b_1, \ldots, b_\ell) \subseteq k^n$ is zero-dimensional. By a result of algebraic geometry [VA67, Ch. 16], the sequence (b_j) is regular if and only each component of V is $(n - \ell)$-dimensional. Thus (b_j) is regular if and only if $\operatorname{rk} K = n = \ell = \dim \check{P}$. $\square$

The justification for our choice of terminology is the following result:

Theorem 8.4.8 ([Oni94, p. 145][GHV76, Thm. 10.17.VIII])**.** *A pair (G,K) is formal if its Cartan algebra is formal in the sense of Definition A.4.6.*

Proof [GHV76, Thm. 2.19.VIII, Thm. 3.30.XI, Thm. 10.17.VIII]. For the forward direction, one always has an algebra map

$$\begin{aligned} \lambda\colon (H_K^* \otimes \Lambda PG, d) &\longrightarrow \big((H_K^* /\!\!/ H_G^*) \otimes \Lambda \widehat{P}, 0\big), \\ a \otimes 1 &\longmapsto \big(a + \widetilde{(\operatorname{im}\rho^*)}\big) \otimes 1, \\ 1 \otimes z &\longmapsto 1 \otimes \big(z + (\check{P})\big), \end{aligned}$$

which is in fact a DGA homomorphism since $d(1 \otimes \check{P})$ is contained in $\operatorname{im}\rho^*$. If (G,K) is a formal pair, so that $H^*(G/K) \cong (H_K^* /\!\!/ H_G^*) \otimes \Lambda \widehat{P}$, then λ is a quasi-isomorphism, so the Cartan algebra $(H_K^* \otimes \Lambda PG, d)$ is formal.

The other direction, attacked without development of the algebra of Sullivan models, requires more work, but this is the tack we take here. Write $Q = Q(BK)$ and $P = \check{P} \leqslant \Lambda[PG]$, so that the reduced Cartan algebra (C,d), with $(\Lambda\widehat{P}, 0)$ factored out, is $(SQ \otimes \Lambda P, d)$ for some differential d. We will show that if this algebra is formal, then the tensor-factor inclusion $SQ \hookrightarrow SQ \otimes \Lambda P = C$ induces a surjection $SQ \twoheadrightarrow H^*(C) = J$, so $J = H_K^* /\!\!/ H_G^*$. The result will really follow for all oddly-graded spaces P and evenly-graded Q and all antiderivations d.

Consider the desuspended space $\Sigma^{-1}Q$ given by $(\Sigma^{-1}Q)^p := Q^{p+1}$, so that $SQ \otimes \Lambda\Sigma^{-1}Q = K[\Sigma^{-1}Q]$, equipped with the natural differential d_Q extending the degree-1 isomorphism $\Sigma^{-1}Q \xrightarrow{\sim} Q$, is a Koszul complex. We use this Koszul complex to "untwist" our original complex, in a sense which will become clear.

Because (C,d) is formal, there is a zig-zag of quasi-isomorphisms connecting it to $\big(H^*(C), d\big)$ through a sequence of DGAs (B_i, d_i). The Koszul map $\Sigma^{-1}Q \xrightarrow{\sim} Q$ induces maps to C and to

$H^*(C)$ and we may lift this and project this linear map to propagate it to the other B_i, completing a commutative cone

$$\begin{tikzcd}
 & & & \Sigma^{-1}Q \arrow[dlll] \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\
C & B_1 \arrow[l] \arrow[r] & B_2 & \cdots \arrow[l] \arrow[r] & B_{n-1} & B_n \arrow[l] \arrow[r] & H^*(C).
\end{tikzcd}$$

If $\varkappa_i \colon \Sigma^{-1}Q \longrightarrow B_i$ is the i^{th} map, then there is a unique differential D_i on $B_i \otimes \Lambda\Sigma^{-1}Q$ that is d_i on $B_i \otimes \mathbb{Q}$ and is $\varkappa_i$ on $\mathbb{Q} \otimes \Sigma^{-1}Q$, and the DGA maps between (B_i, d_i) naturally extend to DGA maps between the DGAs $(B_i \otimes \Lambda\Sigma^{-1}Q, D_i)$. Filtering these bigraded algebras by the B_i-degree, and applying the filtration spectral sequence of Corollary A.5.4 to these algebras and the DGA maps between them, we find induced isomorphisms between the E_2 pages $H^*(B_i) \otimes \Lambda\Sigma^{-1}Q$. By Theorem A.5.6 and Proposition A.5.7, these DGA maps also induce isomorphisms of E_∞ pages and hence are quasi-isomorphisms.

In particular, $\big(H^*(C) \otimes \Lambda\Sigma^{-1}Q, d_Q\big)$ and $\big(C \otimes \Lambda\Sigma^{-1}Q, d + d_Q\big)$ are quasi-isomorphic, where $d + d_Q$ stands for the unique antiderivation which is d on $S \otimes \mathbb{Q}$ and which takes $\Sigma^{-1}Q \longrightarrow Q \leqslant SQ$. There is also a natural map

$$C \otimes \Lambda\Sigma^{-1}Q = SQ \otimes \Lambda P \otimes \Lambda\Sigma^{-1}Q \longrightarrow \Lambda P$$

which we claim is also a quasi-isomorphism. Indeed, bigrading $SQ \otimes \Lambda P \otimes \Lambda\Sigma^{-1}Q$ by

$$(SQ \otimes \Lambda P \otimes \Lambda\Sigma^{-1}Q)^{p,q} = (SQ \otimes \Lambda\Sigma^{-1}Q)^p \otimes (\Lambda P)^q,$$

and applying the filtration spectral sequence to the p-filtration, one encounters an isomorphism on the E_2 pages because d_1 is the Koszul differential on the $SQ \otimes \Lambda\Sigma^{-1}Q$ factor. Thus by Theorem A.5.6 and Proposition A.5.7 again, we have a quasi-isomorphism $\big(H^*(SQ \otimes \Lambda P) \otimes \Lambda\Sigma^{-1}Q, d_Q\big) \longrightarrow (\Lambda P, 0)$. Moreover, the natural maps $C \longrightarrow C \otimes \Lambda\Sigma^{-1}Q$ and $C \twoheadrightarrow \Lambda P$ fit into a commutative triangle inducing the cohomology triangle

$$\begin{tikzcd} & H^*(SQ \otimes \Lambda P) \arrow[dl] \arrow[dr, "j^*"] & \\ H^*\big(H^*(C) \otimes \Lambda\Sigma^{-1}Q\big) \arrow[rr, "\sim"] & & \Lambda P. \end{tikzcd}$$

The algebra on the left is the cohomology of the Koszul complex which computes $\mathrm{Tor}^*_{S[Q]}\big(\mathbb{Q}, H^*(C)\big)$, by Proposition 7.5.8, so the kernel of the left map

$$H^*(C) \longrightarrow \mathrm{Tor}^0_{S[Q]}\big(\mathbb{Q}, H^*(C)\big) = \mathbb{Q} \otimes_{S[Q]} H^*(C) \; = \; H^*(C)/\!/S[Q]$$

is the ideal of $H^*(C)$ generated by the image of the generators Q. Since the triangle is commutative, this is also the kernel of the right map $j^*\colon H^*(C) \longrightarrow \Lambda P$, or in other words, the sequence

$$SQ \xrightarrow{\chi^*} H^*(SQ \otimes \Lambda P) \xrightarrow{j^*} \Lambda P$$

is coexact. The second map j^* is induced by the projection map $SQ \otimes \Lambda P \longrightarrow \Lambda P$ with kernel $\widetilde{SQ} \otimes \Lambda P$, so its $\ker j^*$ is made up of those cohomology classes in $H^*(C)$ represented in C by elements of $\widetilde{SQ} \otimes \Lambda P$. Write $\mathfrak{a}$ for the kernel of $H^*(C) \twoheadrightarrow \mathrm{im}\,\chi^*$, viewed as the subalgebra represented by cocycles in $SQ \otimes \widetilde{\Lambda P}$. Then we have $H^*(C) = \mathfrak{a} \cdot H^*(C) + \mathrm{im}\,\chi^*$ by the assumption on $\ker j^*$, so by Nakayama's lemma [AM69, Cor. 2.7], $H^*(C) = \mathrm{im}\,\chi^*$. □

Proposition 8.4.9. *Let (G, K) be a formal pair of Lie groups. If the Poincaré polynomials of the Samelson* *subspace $\widehat{P}$, the Samelson complement $\check{P}$, and the primitive space PK are given respectively by*

$$p(\widehat{P}) = \sum_{j=1}^{\operatorname{rk} G - \operatorname{rk} K} t^{d_j}, \qquad p(\check{P}) = \sum_{\ell=1}^{\operatorname{rk} K} t^{c_j}, \qquad p(PK) = \sum_{\ell=1}^{\operatorname{rk} K} t^{k_j},$$

then the Poincaré polynomial of G/K is

$$p(G/K) = p(\Lambda\widehat{P}) \cdot \frac{p(BK)}{p\big(S[\Sigma\check{P}]\big)} = \prod_{j=1}^{\operatorname{rk} G - \operatorname{rk} K} (1 + t^{d_j}) \cdot \prod_{\ell=1}^{\operatorname{rk} K} \frac{1 - t^{c_j+1}}{1 - t^{k_j+1}}$$

and its total Betti number is

$$h^{\bullet}(G/K) = \frac{2^{\operatorname{rk} G}}{2^{\operatorname{rk} K}} \cdot \prod_{\ell=1}^{\operatorname{rk} K} \frac{c_\ell + 1}{k_\ell + 1} = \frac{\big|\pi_0 N_G(K)\big|}{|W_K|} 2^{\operatorname{rk} G - \operatorname{rk} K}$$

Proof. By equation (8.9), given the equations (8.9) and (A.2), all that really remains to be shown is that $p(H_K^* /\!/ H_G^*) = p(BK)/p\big(S[\Sigma\check{P}]\big)$ as claimed. But Theorem 8.4.7, the generators of $\operatorname{im}\rho^*$ form a regular sequence of $\operatorname{rk} K$ elements of H_K^* of degrees $c_j + 1$. These generators are thus algebraically independent and form a polynomial subalgebra $S \cong S[\Sigma\check{P}]$ of H_K^* such that H_K^* is a free S-module. The result then follows from Proposition A.3.8. □

Proposition 8.4.10 ([Oni94, Rmk., p. 212])**.** *Suppose (G, K) is a compact pair and S a maximal torus of K. Then (G, K) is a formal pair if and only if (G, S) is.*

Proof. This follows from Corollary 6.4.6, with $X = G$. Write W for the Weyl group of K. If (G, S) is formal, then $H_S^*(G) = H^*(G/S) \cong (H_S^* /\!/ H_G^*) \otimes \Lambda\widehat{P}$. Since the W-action on $H^*(G)$ descends from the K-action, which is trivial since K is path connected, the action of W on $H_S^*(G)$ affects only the bottom row $H_S^* /\!/ H_G^*$, and we have

$$H^*(G/K) = H_K^*(G) \cong H_S^*(G)^W \cong \Big(H_S^*/\!/H_G^*\Big)^W \otimes \Lambda\widehat{P} \cong \Big((H_S^*)^W/\!/H_G^*\Big) \otimes \Lambda\widehat{P} \cong \big(H_K^*/\!/H_G^*\big) \otimes \Lambda\widehat{P}.$$

On the other hand, if (G, K) is formal, so that $H^*_K(G) \cong (H^*_K /\!/ H^*_G) \otimes \Lambda\widehat{P}$, then

$$H^*(G/S) \cong H^*_S \underset{H^*_K}{\otimes} H^*(G/K) \cong H^*_S \underset{H^*_K}{\otimes} H^*_K /\!/ H^*_G \otimes \Lambda\widehat{P} \cong H^*_S /\!/ H^*_G \otimes \Lambda\widehat{P}. \qquad \square$$

Proposition 8.4.11. *Suppose (G, K) is a compact pair such that G/K is a symmetric space. Then (G, K) is a formal pair.*

Proof. We have already stated in Example A.4.9 that a symmetric space G/K is formal, but here is an actual proof. This venerable argument, essentially due to Élie Cartan, would actually turn into a proof G/K is *geometrically* formal with the mere addition of a proof that the representing forms we find are in fact harmonic.

Recall from Proposition 5.1.2 that elements of $H^*(G/K; \mathbb{R})$ are all represented by G-invariant forms on G/K, which are determined by their values at the identity coset, which are elements of the exterior algebra $\Lambda(\mathfrak{g}/\mathfrak{k})^\vee$, and further from Proposition 5.1.3 that G-invariance on $\Omega(G/K)$ translates to $(\mathrm{Ad}^* K)$-invariance in $\Lambda(\mathfrak{g}/\mathfrak{k})^\vee$. Thus all elements of $H^*(G/K)$ are represented by elements of $\big(\Lambda(\mathfrak{g}/\mathfrak{k})^\vee\big)^K$. Let $\theta \in \operatorname{Aut} G$ be the involution fixing K, so that $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k}$ the Lie algebra of K and $\mathfrak{p}$ the (-1)-eigenspace of $\theta_* \in \operatorname{Aut} \mathfrak{g}$. Then $\mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$, so $H^*(G/K)$ is represented by elements of $\Lambda[\mathfrak{p}^\vee]^K$.

We claim that all these elements correspond to closed differential forms. Indeed, because θ is a Lie group automorphism, the induced map θ^* on $\Omega(G/K)$ commutes with the exterior derivative d, and so the same holds true of the differentials induced on $\Lambda(\mathfrak{g}/\mathfrak{k})^\vee \cong \Lambda[\mathfrak{p}^\vee]$. Now, since θ_* acts as $-\mathrm{id}$ on $\mathfrak{p}$, its dual θ^* acts as $-\mathrm{id}$ on $\mathfrak{p}^\vee$ and so acts as $(-1)^\ell \cdot \mathrm{id}$ on the summand $\Lambda^\ell[\mathfrak{p}^\vee]$ spanned by wedge products of ℓ elements of $\mathfrak{p}^\vee$. Let ω be one such element. Then, since $d \circ \theta^* = \theta^* \circ d$, we have

$$(-1)^{\ell+1} d\omega = \theta^* d\omega = d\theta^*\omega = (-1)^\ell d\omega,$$

so $d\omega = 0$. Thus all elements of $\Lambda[\mathfrak{p}^\vee]^K \rightarrowtail \Omega(G/K)$ are closed, so G/K is a formal space. □

Remarks 8.4.12. Though the formality condition on pairs (G, K) is convenient, is natural, has many equivalent formulations, is guaranteed by several commonly studied sufficient conditions, and is invariant under the act of replacing the isotropy group K with its maximal torus S, there still seems to be no simpler way of determining formality of a randomly given pair (G, K) than carefully examining the image of the map $\rho^*\colon H_G^* \longrightarrow H_S^*$, and our knowledge has arguably not improved in any major way since regular sequences were introduced into commutative algebra in the mid-1950s. Indeed, it seems computing the map ρ^* is an NP-hard problem [Ama13, Sec. 1].

The deficiency first appears in Paul Baum's 1962 doctoral dissertation [Bau62], where it is shown *inter alia* that if $k = \mathbb{Z}$ or k is any field and $H^*(G;k)$ and $H^*(K;k)$ are exterior algebras and the analogue of the deficiency with k coefficients satisfies $\operatorname{df}(G, K) \leqslant 2$, then the Eilenberg–Moore spectral sequence of $G/K \to BK \to BG$ collapses at $E_2 = \operatorname{Tor}_{H_K^*}(k, H_G^*)$. The deficiency thus links our account with the Eilenberg–Moore spectral sequence analysis of the cohomology of homogeneous spaces. This deficiency is actually an invariant of the homogeneous space G/K and not just of the compact pair (G, K) according to a theorem of Arkadi Onishchik; see Onishchik [Oni72].

Historical remarks 8.4.13. What we call a *formal pair* is traditionally called a *Cartan pair* (as seen, e.g., in the standard reference by Greub *et al.* [GHV76, p. 431]). The condition already arises in Cartan's classic transgression paper in the *Colloque* [Car51, (3) on p. 70], so the attribution is just, but the name is made inconvenient by the vast prolificacy of the Cartans: pursuant to the work of Cartan *père* on symmetric spaces, the pair $(\mathfrak{k}, \mathfrak{p})$ of ± 1-eigenspaces of the Lie algebra $\mathfrak{g}$ induced by an involutive Lie group automorphism $\theta\colon G \to G$ is also called a *Cartan decomposition* or a *Cartan pair*. (This writer spent an embarrassingly long time in finally convincing himself these two concepts of "Cartan pair" were entirely unrelated.)

The formal pair condition also appears in the (Russian-language) writings of Doan Kuin', where—at least as the translator would have it—K is said to be *in the normal condition* in G. This locution did not catch on. We hope that despite the existence of standard terminology, the semantic overload placed on the word *regular*, and the possible confusion of *formality* per se with *equivariant formality*, this section has made the case that these terms are natural, justified, and preferable.

8.5. The cohomology of G/S^1

In order to obtain Theorem 11.1.7, arguably the main result of this thesis, we needed a grasp on the cohomology rings $H^*(G/S;\mathbb{Q})$ of homogeneous spaces G/S for G compact connected and S a circle. We found the following dichotomy; these are the only two options because $\dim_{\mathbb{Q}} H^1(S) = 1$.

Proposition 8.5.1. *Let G be a compact, connected Lie group and S a circle subgroup. Then the rational cohomology ring $H^*(G/S)$ has one of the following forms.*

1. *If $H^1(G) \longrightarrow H^1(S)$ is surjective, then there is $z_1 \in H^1(G)$ such that*

$$H^*(G/S) \;\cong\; {}^{H^*(G)}\!/_{(z_1)}.$$

   In terms of total Betti number, $h^\bullet(G) = \frac{1}{2}h^\bullet(G/S)$.

2. *If $H^1(G) \longrightarrow H^1(S)$ is zero, there are $z_3 \in H^3(G)$ and $s \in H^2(G/S)$ such that*

$$H^*(G/S) \cong \frac{H^*(G)}{(z_3)} \otimes \frac{\mathbb{Q}[s]}{(s^2)}.$$

   In terms of total Betti number, $h^\bullet(G) = h^\bullet(G/S)$.

This turns out to be a trivial generalization of long-known results. General statements on the cohomology of a homogeneous space were already available to Jean Leray in 1946, the year after his release from prison [Mil00, sec. 3, item (4)]. In the second of his four *Comptes Rendus* announcements from that year [Ler46a, bottom of p. 1421], he states the following result.[3]

Theorem 8.5.2 (Leray, 1946)**.** *Let G be a compact, simply-connected, Lie group and S a closed, one-parameter subgroup [viz. a circle]. Then there exist an $n \in \mathbb{N}$, a primitive element $z_{2n+1} \in H^{2n+1}(G)$, and a nonzero $s \in H^2(G/S)$ such that*

$$H^*(G/S) \;\cong\; \frac{H^*(G)}{(z_{2n+1})} \otimes \frac{\mathbb{Q}[s]}{(s^{n+1})}$$

The following year, Jean-Louis Koszul published a note [Kos47b, p. 478, display] in the *Comptes Rendus* regarding Poincaré polynomials for these spaces.

Theorem 8.5.3 (Koszul, 1947)**.** *Let G be a semisimple Lie group and S a circular subgroup. Then the Poincaré polynomials (in the indeterminate t) of G/S and G are related by*

$$p(G/S) = p(G)\frac{1+t^2}{1+t^3}.$$

This result implies that in fact $n = 1$ in Leray's theorem. This enhanced version of Leray's result follows from Proposition 8.5.1 simply because $H^1(G) \cong H^2_G = 0$ for semisimple groups. We will rely on the part of the proposition on total Betti numbers later to prove Theorem 11.1.7. The author is unaware of any published proof of the Leray and Koszul results, which is part of the motivation for including a proof of Proposition 8.5.1 here.

Before doing so, we illustrate the result with a representative example. Let S be a circle contained in the first factor $\mathrm{Sp}(1)$ of the product group $G = \mathrm{Sp}(1) \times \mathrm{U}(2)$. The cohomology of G

[3] See also Borel [Bor98, par. 12]; only owing to Borel's summary are we confident "compact Lie group" is the contextually-correct interpretation of Leray's *groupe bicompact*.

is the exterior algebra

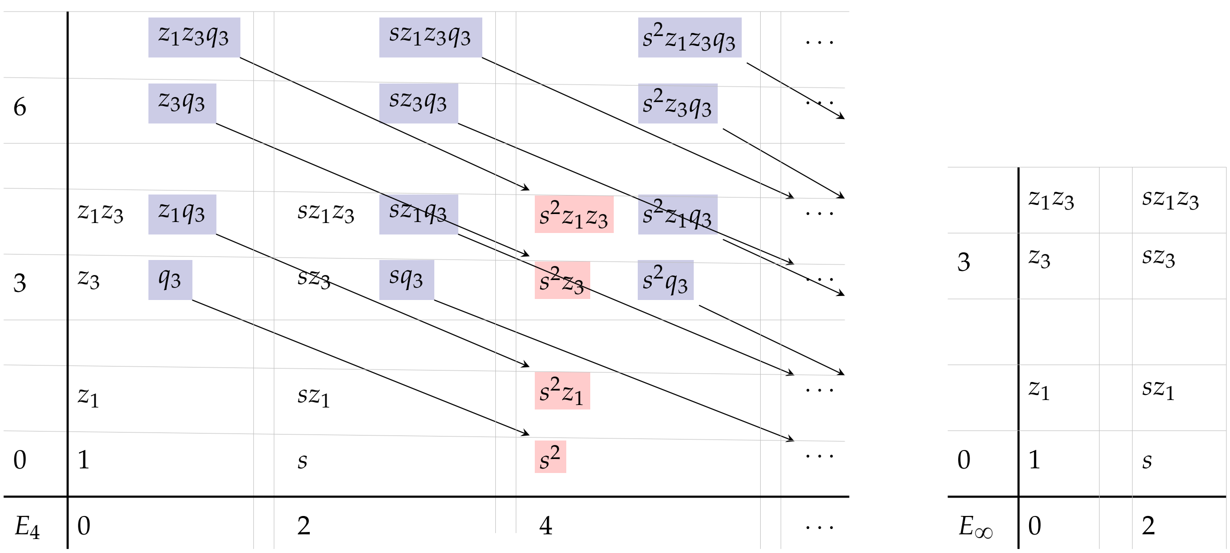
$$H^*(G) = \Lambda[q_3, z_1, z_3], \quad \deg z_1 = 1, \ \deg z_3 = \deg q_3 = 3,$$

and that of BS is

$$H_S^* = \mathbb{Q}[s], \qquad \deg s = 2.$$

The spectral sequence (E_r, d_r) associated to $G \to G_S \to BS$ is as follows. Its E_2 page is the tensor product $H_S^* \otimes H^*(G)$. Because the map $H^1(G) \longrightarrow H^1(S)$ is zero, the differential d_2 is zero, and d_3 is zero for lacunary reasons, so $E_4 = E_2$. The differential d_4 annihilates each of s, z_1, z_3 and takes $q_3 \longmapsto s^2$.

Figure 8.5.4: The Serre spectral sequence of $\mathrm{Sp}(1) \times \mathrm{U}(2) \to \big(\mathrm{Sp}(1) \times \mathrm{U}(2)\big)_S \to BS$



Because d_4 is an antiderivation, its kernel is the subalgebra $\mathbb{Q}[s] \otimes \Lambda[z_1, z_3]$ and its image the ideal (s^2) in that subalgebra. Elements mapped to a nonzero element by d_4 are marked as blue in the diagram and elements in the image in red; the vector space spanned by these elements vanishes in E_5. Thus $E_5 = \Delta[s] \otimes \Lambda[z_1, z_3]$, where $\Delta[s] = \mathbb{Q}[s]/(s^2) \cong H^*S^2$. For lacunary reasons,

$E_5 = E_\infty$. In fact,

$$G/S = \mathrm{Sp}(1)/_S \times \mathrm{U}(2) \approx S^2 \times \mathrm{U}(2),$$

so this tensor decomposition was not unexpected.

This example has all the features of the general case; the pair is always formal, and either it is cohomology-surjective or else d_4 is a nontrivial differential taking some $z_3 \longmapsto s^2 \in H^4_S$, which then collapses the sequence at E_5. If $H^1(G) \neq 0$, then the exterior subalgebra of $H^*(G)$ generated by $H^1(G)$, an isomorphic $H^*(A)$, is in the Samelson subring, and can be split off before running the spectral sequence; the factoring out of this subalgebra is the algebraic analogue of the product decomposition Proposition 11.3.1 of G/S.

Lemma 8.5.5. *A compact pair (G, S^1) is formal.*

Proof. Consider the map $\rho^* \colon H^*_G \longrightarrow H^*_S$ in the sequence

$$H^*_G \xrightarrow{\rho^*} H^*_S \xrightarrow{\chi^*} H^*(G/S).$$

Because ρ^* is a homomorphism of graded rings and $H^*_S \cong \mathbb{Q}[s]$ is a polynomial ring in one variable, the cokernel $(\rho^* \widetilde{H}_S)$ of χ^* is generated by a single homogeneous element and hence is a regular ideal (s^n) for some n. By Theorem 8.4.7, it follows (G, S) is a formal pair. □

Proof of Proposition 8.5.1. If $H^1(G) \twoheadrightarrow H^1(S)$, then Samelson's Corollary 7.2.5 applies and yields the result, so assume instead this map is zero. By Lemma 8.5.5, (G, S) is a formal pair, so

$$H^*(G/S) \cong H^*_S /\!/ H^*_G \otimes \Lambda \widehat{P}$$

with $\dim \widehat{P} = \operatorname{rk} G - \operatorname{rk} S = \operatorname{rk} G - 1$ and $\dim \check{P} = 1$. It follows that $\rho^* \circ \tau$ takes $\check{P} \xrightarrow{\sim} \mathbb{Q}s^\ell$ for some ℓ, yielding Leray's theorem. To obtain Koszul's, it remains to show $\ell = 2$.

By Proposition B.3.4, we may replace G with its universal compact cover $A \times K$, where A is a torus and K simply-connected, and S with the identity component of its lift in this cover. If $H^1(G) \longrightarrow H^1(S)$ is trivial, then because $H^*(A)$ is generated by $H^1(A)$, it follows $H^*(A) \leqslant \Lambda\widehat{P}$ splits out of the Cartan algebra, so we may as well assume $G = K$ is semisimple.

We now return to the map of spectral sequences described in Section 8.1.1. Recall the differentials in the spectral sequence (E_r, d_r) of the Borel fibration $K \to K_S \to BS$ vanish on H_S^* and are otherwise completely determined by by the composition

$$\rho^* \circ \tau \colon PK \longrightarrow H_K^* \longrightarrow H_S^*.$$

Because K is semisimple, $H^1(K) = 0$, so it follows $H_K^2 = 0$ as well by Borel's calculation from Section 7.6 of the spectral sequence of $K \to EK \to BK$. The edge homomorphisms d_2 and d_3 then must be zero, so

$$E_4 = E_2 = H_S^* \otimes H^*(K)$$

and the first potentially nontrivial differential is

$$d_4 \colon H^3(K) \xrightarrow{\sim} H_K^4 \longrightarrow H_S^4.$$

By Lemma 7.8.3, this is surjective, so $dz = \rho^*\tau z = s^2$ for some $z \in P^3(K)$. Thus $(\widetilde{\operatorname{im}\rho^*})$ is generated by s^2 as claimed, concluding the proof. □

Historical remarks 8.5.6. Our presentation in this chapter of Cartan algebra computation of the cohomology ring $H^*(G/K;\mathbb{Q})$ of a homogeneous space G/K introduced what we believe to be the least possible algebraic overhead, very little analysis at all, and only hints of rational homotopy theory. That said, such a presentation is dishonest as an origin story. This work was originally cast in Lie-algebraic terms, with the transgression we have been so casual about explicitly determined by a connection and induced from a CDGA called the *Weil algebra* . The Weil algebra, as an algebra, is the Koszul complex of Section 7.5, but is outfitted with a *different* differential which incorporates the adjoint action of the Lie algebra of G. It does this to emulate the behavior of connection and curvature forms determined by a connection on a principal bundle, and these in turn arise due to a desire to understand the cohomology of the total space of a principal bundle in terms of forms arising from pullback in its base. Thus it is an algebraic model of the cohomology of $EG \to BG$ and the homotopy quotient that predates the general discovery of these objects. The story of understanding the cohomology of the base of a bundle through invariant forms starts with the work of Élie Cartan in the early 1900s and continues through the work of Henri Cartan and his school (Koszul, Borel, and for a time Leray, with major unpublished contributions by Chevalley and Weil) in the late 1940s and early 1950s.

The main and classical source for these developments is the conference proceedings [Colloque] to the 1950 *Colloque de Topologie (espaces fibrés)*, held in Bruxelles, with contributions by Beno Eckmann, Heinz Hopf, Guy Hirsch, Koszul, Leray, and Cartan. The second of the two papers by Cartan in this volume, "La transgression dans un groupe de Lie et dans un espace fibré principal" [Car51], promulgates in Lie-algebraic terms what we have called the Cartan algebra and was directly responsible for the institution of the *Cartan model* of equivariant cohomology, a full ten years before Borel model gained currency. The classic sketched proof of the *equivariant de Rham theorem* showing the equivalence between these two models of equivariant cohomology is also contained in this terse paper.

There is no shortage of secondary sources for the work of this school [And62; Ras69; GHV76; Oni94], especially as it applies to the Cartan model of equivariant cohomology [GS99; GLS96; GGK02], so the author can only hope his in recasting these results in terms of elementary algebra over the rationals is of independent interest to some person other than himself.

Chapter 9

Equivariant formality

9.1. Equivariant formality of group actions

The main motivation of this document is to discuss the *equivariant formality* of certain Lie group actions. It is past time we defined the concept.

Definition 9.1.1 ([GKM98])**.** The action of a topological group G on a space X is said to be ***equivariantly formal*** if the fiber inclusion $X \longhookrightarrow X_G$ in the Borel fibration $X \to X_G \to BG$ surjects in cohomology.

The computational utility of this condition is that it implies the H^*_G-module structure on $H^*_G(X)$ is as simple as one could hope [GGK02, Lemma C.24, p. 208]. Looking back, one can also see it is the condition on Y in Theorem 6.2.1.

Proposition 9.1.2. *Let a topological group G act on a topological space X such that $H^*(X;k)$ is a free k-module of finite type. The following conditions are equivalent:*

1. *The action of G on X is equivariantly formal.*

2. *The SSS of the Borel fibration $X \to X_G \to BG$ collapses at E_2.*

3. *The equivariant cohomology $H^*_G(X)$ is isomorphic to $H^*_G \otimes H^*(X)$ as an H^*_G-module.*

4. *The H^*_G-rank of $H^*_G(X)$ is $h^\bullet(X)$.*

Proof. We prove a cycle of implications.

$1 \implies 2$. If $H^*(X_G) \longrightarrow H^*(X)$ is surjective, then the SSS of the Borel fibration collapses at E_2 by Corollary 4.3.9.

$2 \implies 3$. If the SSS of the Borel fibration collapses at E_2, then

$$\operatorname{gr}_\bullet H^*_G(X) = E_\infty = E_2 = H^*_G \otimes H^*(X)$$

by Theorem 4.3.4, and $H^*_G(X) \cong \operatorname{gr}_\bullet H^*_G(X)$ as an H^*_G-module (regardless of how badly the associated graded construction modifies the multiplication as a whole).

$3 \implies 4$. This is trivial.

$4 \implies 1$. We prove the contrapositive. Suppose that $H^*_G(X) \longrightarrow H^*(X)$ fails to be surjective. Considering the SSS (E_r, d_r) of the Borel fibration, this nonsurjective map can be considered as the composition

$$E_\infty \twoheadrightarrow E_\infty^{0,\bullet} \hookrightarrow E_2^{0,\bullet},$$

so the containment $E_\infty^{0,\bullet} < E_2^{0,\bullet}$ must be strict; there is some $z \in E_2^{0,\bullet} \cong k \otimes H^*(X)$ which is not in E_∞. Since this z generates a cyclic H^*_G-module summand $H^*_G \otimes k \cdot z$ of E_2, it follows then that $\operatorname{rk}_{H^*_G} E_\infty < \operatorname{rk}_{H^*_G} E_2 = h^\bullet(X)$. □

In fact, equivariant formality of an action of a compact, connected Lie group depends only on the restricted action of its maximal torus.

Lemma 9.1.3 ([GGK02, Prop. C.26, p. 207]). *If K is a compact, connected Lie group and S a maximal torus, and K acts on a space X, then the action of K is equivariantly formal if and only if the restricted action of S is.*

Proof. We have an X-bundle map between Borel fibrations,

$$\begin{tikzcd} X_S \arrow[r] \arrow[d] & X_K \arrow[d] \\ BS \arrow[r] & BK, \end{tikzcd}$$

so if the action of K is equivariantly formal, then by Theorem 4.4.1, the fiber inclusion $X \hookrightarrow X_S$ is H^*-surjective as well.

Now suppose that the action of S is equivariantly formal, and consider the homomorphism of spectral sequences induced by the X-bundle homomorphism above. The map $H_K^* \longrightarrow H_S^*$ is an inclusion of Weyl group invariants by Lemma 6.4.5, so the map on E_2 pages is injective, and the differential for the E_2 page of $X \to X_K \to BK$ is the restriction of that for $X \to X_S \to BS$. But the latter differential is zero, by assumption, so the former is as well. By induction on page number, the sequence for $X \to X_K \to BK$ collapses at E_2 as well, so $H^*(X_K) \longrightarrow H^*(X)$ is surjective. □

Remark 9.1.4. In fact, that S is a maximal torus was inconsequential to the "only if" direction of the theorem: any restriction of an equivariantly formal action to a subgroup remains equivariantly formal.

9.2. The Borel localization theorem and equivariant formality

In Section 6.3, we computed the equivariant cohomology of a rotation action on S^2 by embedding this ring into the equivariant cohomology of the fixed point set and showed these actions to be equivariantly formal. The method of our computation turns out to be an instance of a much more general pattern: for torus actions on compact manifolds, much of the structure of equivariant cohomology is determined by the fixed point set. The reduction of the previous section should also count as some evidence that a focus on torus actions might be worthwhile. (In fact, we only neglected to prove this important theorem in Chapter 6 because we had not yet developed the theory of maps $H_T^* \longrightarrow H_S^*$ which we need here.)

9.2.1. The localization theorem

Let T be a torus and M a compact T-manifold. The fixed point set M^T is compact and T-invariant—and so itself a T-manifold—and the inclusion $M^T \hookrightarrow M$ induces a restriction $H_T^*(M) \longrightarrow H_T^*(M^T)$ in equivariant cohomology. This latter ring is the simplest one could hope: it is the singular cohomology of the homotopy quotient $(M^T)_T$, which, since the action $T \curvearrowright M^T$ is trivial, is

$$ET \underset{T}{\times} M^T \;\approx\; \frac{ET}{T} \times M^T \;=\; BT \times M^T.$$

Then the traditional singular Künneth isomorphism, even with $\mathbb{Z}$ coefficients, yields

$$H_T^*(M^T) = H_T^* \otimes H^*(M^T), \tag{9.1}$$

so $H_T^*(M^T)$ is a free H_T^*-module

Remark 9.2.1. This is the "largest" $H_T^*(M)$ could be: the Serre spectral sequence of the Borel fibration $M \to M_T \to BT$ shows that $H_T^*(M)$ is a subquotient of $H_T^* \otimes H^*(M)$ as an H_T^*-module.

Both $H_T^*(M)$ and $H_T^*(M^T)$ are H_T^*-algebras, the latter well-understood, so there is a natural hope one could understand the latter in terms of the former.

Theorem 9.2.2 (Atiyah–Borel [BBF+60, Prop. IV.3.6, p. 54][Hsi75, Thm. III.1, p. 40])**.** *Let T be a torus and M a compact T-manifold. Denote by M^T the fixed point set of the action of T on M. There exists a nonzero element $f \in H_T^*$ such that the H_T^*-algebra homomorphism*

$$H_T^*(M;\mathbb{Z}) \longrightarrow H_T^*(M^T;\mathbb{Z})$$

induced in T-equivariant cohomology by restriction becomes an isomorphism upon inversion of f. That is to say, there exists a nonzero $f \in H_T^*$ for which the induced map

$$H_T^*(M;\mathbb{Z})_f \longrightarrow H_T^*(M^T;\mathbb{Z})_f$$

of algebras over $(H_T^*)_f = H_T^*[f^{-1}]$ is an isomorphism.

Proof. The proof will proceeds induction on a T-invariant open cover, using the equivariant Mayer–Vietoris sequence. By the equivariant tubular neighborhood Theorem 2.3.1, each orbit Tx admits a T-invariant open neighborhood V_x in M. Each $x \in M^T$ is an orbit, so $U := \bigcup_{x\in M^T} V_x$ is a T-invariant open neighborhood of M^T. Since M is compact, U and finitely many other V_x, call them V_{x_j}, suffice to cover M. By the definition of a tubular neighborhood, U equivariantly deformation retracts to M^T and each V_{x_j} to Tx_j, so one has $H_T^*(U) \cong H_T^*(M^T)$ and $H_T^*(V_j) \cong H_T^*(Tx_j)$. We now have only to apply the equivariant Mayer–Vietoris sequence Proposition 6.1.3 to this cover $\{U, V_j\}_j$ to recover $H_T^*(M)$.

We understand the ring $H_T^*(M^T)$ exactly as well as we understand $H^*(M^T)$, so it remains to understand the $H_T^*(Tx_j)$. Fix a j and write $S = S_j = \mathrm{Stab}(x_j) < T$. We know $Tx_j \approx T/S$ from Proposition 2.2.1 and $H_T^*(T/S) \cong H^*(BS)$ from Proposition 6.1.1. As a subgroup of a torus, S is constrained to be of the form $S_0 \times F$, where the torus S_0 is its identity component and F is a finite abelian subgroup of T. Writing $BS = ET/S$, we can view BS_0 as a finite covering space of BS:

$$F \longrightarrow BS_0 \longrightarrow BS.$$

If $m = m_j = |F|$, and we set $k = \mathbb{Z}[\frac{1}{m}]$, then by Corollary B.3.2, the map $H^*(BS;k) \longrightarrow H^*(BS_0;k)$ is an isomorphism. Thus, if we replace $\mathbb{Z}$ with $\mathbb{Z}[\frac{1}{m}]$, we can replace S with S_0. For convenience, we invert *all* the m_j in our coefficient ring now, replacing $\mathbb{Z}$ with $k = \mathbb{Z}[1/m_1, \ldots, m_n]$. The short exact sequence

$$0 \to S \to T \to T/S \to 0$$

of tori splits, by Proposition A.4.2, so that $T \cong S \times T/S$. By the Künneth Theorem B.2.2, then, $H^*(BS)$ is a tensor factor of $H^*(BT)$, so there is a factor projection $H^*(BT) \twoheadrightarrow H^*(BS)$ inducing a H_T^*-algebra structure on $H^*(BS)$. The kernel of the projection is generated by the $\operatorname{rk} T - \operatorname{rk} S > 0$ elements of H_T^2 that generate the polynomial subring $H^*\big(B(T/S)\big)$. Any $g \in H_{T/S}^2$ acts as multiplication by 0 on $H_S^* = H^*(BS)$, so by the discussion in Appendix A.1, if we invert g we get $H_S^*[g^{-1}] = 0$.

Thus, for each x_j, there is an element $g_j \in H_T^*$ annihilating $H_T(V_j) \cong H_T^*(Tx_j) \cong H^*(BS_j)$. We claim the least common multiple of the g_j annihilates $H_T^*\big(\bigcup V_j\big)$. Indeed, suppose inductively for $W = \bigcup_{j=1}^{n-1}$ that a nonzero element $h \in H_T^*$ annihilates $H_T^*(W)$. The restriction map $H_T^*(W) \longrightarrow H_T^*(W \cap V_h)$ is an H_T^*-algebra homomorphism, so h also annihilates $H_T^*(W \cap V_n)$, and $\operatorname{lcm}(h, g_n)$ annihilates this ring, $H_T^*(W)$, *and* $H_T^*(V_n)$. The equivariant Mayer–Vietoris sequence of the pair (W, V_n) contains exact fragments

$$H_T^{*-1}(W \cap V_n) \longrightarrow H_T^*(W \cup V_n) \longrightarrow H_T^*(W) \oplus H_T^*(V_n),$$

and by Lemma A.1.1, since $\operatorname{lcm}(h, g_n)$ annihilates the outer terms, it also annihilates the inner term. Thus if $V = \bigcup_j V_j$, the nonzero element $\operatorname{lcm}(g_j) \in H_T^*$ annihilates $H_T^*(V)$. Now consider the equivariant Mayer–Vietoris sequence of the cover $\{U, V\}$ of M:

$$H_T^{*-1}(U \cap V) \longrightarrow H_T^*(M) \longrightarrow H_T^*(U) \oplus H_T^*(V) \longrightarrow H_T^*(U \cap V).$$

On inverting $g = \operatorname{lcm}(g_j)$, we obtain an exact sequence

$$0 \longrightarrow H_T^*(M)[g^{-1}] \longrightarrow H_T^*(U)[g^{-1}] \oplus 0 \longrightarrow 0$$

where, as noted above, $H_T^*(U) = H_T^*(M^T)$. To get this isomorphism, we inverted $1/m_1 \ldots m_n$ so to obtain the theorem with coefficients in $\mathbb{Z}$, we take $f = \operatorname{lcm}(m_j g_j)$. $\square$

Example 9.2.3. In our examples from Section 6.3, the rotation actions $\rho_q \colon S^1 \curvearrowright S^2$, the fixed point set $(S^2)^{S^1}$ was the doubleton S^0 containing the north and south poles. By Corollary 9.2.5, we have $\operatorname{rk}_{H^*_{S^1}} H^*_{\rho_1}(S^2) = \operatorname{rk}_{\mathbb{Z}} H^*\big(S^0;\mathbb{Z}\big) = 2$.

If we go ahead and all the nonzero elements of H_T^*, we obtain a simpler statement.

Notation 9.2.4. Recall that we write $h^\bullet(X) = \dim_{\mathbb{Q}} H^*(X;\mathbb{Q})$ for the total Betti number. If T is a torus, we write $\widehat{H}_T^* \cong \mathbb{Q}(u_1, \ldots, u_\ell)$ for the field of fractions of $H_T^* \cong \mathbb{Z}[u_1, \ldots, u_\ell]$ and $\widehat{H}_T^*(X)$ for the localization $\widehat{H}_T^* \underset{H_T^*}{\otimes} H_T^*(X)$.

Corollary 9.2.5. *Let T be a torus and M a compact T-manifold. Then*

$$\widehat{H}_T^*(M) \;\cong\; \widehat{H}_T^*(M^T) \;\cong\; \widehat{H}_T^* \underset{\mathbb{Q}}{\otimes} H^*(M^T).$$

Proof. The first isomorphism is immediate from Theorem 9.2.2 on further localization, while the second follows from tensoring (9.1) with $\widehat{H}_T^*$. $\square$

Corollary 9.2.6. *Let T be a torus and M a compact T-manifold. Then*

$$\operatorname{rk}_{H_T^*} H_T^*(M) = h^\bullet(M^T).$$

Proof. The rank of a finitely generated H_T^*-module N is the same as the dimension over $\widehat{H}_T^*$ of $\widehat{H}_T^* \otimes_{H_T^*} N$. Taking $N = H_T^*(M)$, and using Corollary 9.2.5 and the fact $(M^T)_T \approx BT \times M^T$, we get the chain of equations.

$$\mathrm{rk}_{H_T^*} H_T^*(M) = \dim_{\widehat{H}_T^*} \widehat{H}_T^*(M) = \dim_{\widehat{H}_T^*} \widehat{H}_T^*(M^T) = h^\bullet(M^T). \qquad \square$$

Historical remarks 9.2.7. The main result of this section is a result is often called the "Borel localization theorem," though Borel only proved it for $T = S^1$. In Hsiang Wu-Yi's (項武義) standard text [Hsi75, p. 39], it is only called a "localization theorem of Borel–Atiyah–Segal type," so the author asked on online for the origin story [Bee]. The result was apparently first stated by Atiyah in unpublished 1965 Warwick lecture notes which this writer does not know how to obtain. Atiyah (via email) confirms that the result in equivariant cohomology for higher-dimensional tori is probably originally due to him, but warns that history is never straightforward and the published record is only part of the story. The Atiyah–Segal completion theory in equivariant K-theory was inspired by this result.

The theorem in cohomology has been substantially generalized since its discovery, both in replacing T with more general groups (at the cost of replacing M^T with a related set; see for example Pedroza–Tu [PT07]) and in also considering strata of n-dimensional orbits, M^T being the 0-dimensional case. See for instance Goertsches–Töben [GT10b] and Franz–Puppe [FP07].

9.2.2. Equivariant formality of torus actions and fixed points

We can use localization to obtain a simple numerical criterion for equivariant formality of torus actions. We will depend on the results of Appendix B.3 and Section 9.2 to such an extent that uniform statements would be impossible in the rest of this document if we did not, as we always will *from here on out, take coefficient ring* $k = \mathbb{Q}$ unless explicitly stated otherwise. We denote our tori from now on by S instead of T unless they are maximal in a larger Lie group under consideration; in the cases of interest to us, they will not be.

Lemma 9.2.8 ([GGK02, Cor. C.27]). *Let S be a torus and M a compact S-manifold. Then the action of S on M is equivariantly formal if and only if*

$$h^\bullet(M) = h^\bullet(M^S).$$

Proof. By Proposition 9.1.2, the action of S on M is equivariantly formal if and only if $h^\bullet(M) = \operatorname{rk}_{H_S^*} H_S^*(M)$, but by Corollary 9.2.6, $h^\bullet(M) = \operatorname{rk}_{H_S^*} H_S^*(M) = h^\bullet(M^S)$. $\square$

In fact, one inequality in this lemma holds regardless of whether the action is equivariantly formal.

Lemma 9.2.9 (Borel [BBF+60, IV 5.5, p. 62]). *Let S be a torus and M a compact S-manifold. Then*

$$h^\bullet(M) \geqslant h^\bullet(M^S). \tag{9.2}$$

Proof [GGK02, Lemma C.24]. Consider the spectral sequence (E_r) of the Borel fibration $M \to M_S \to BS$. Since the E_2 page is $H_S^* \otimes H^*(M)$, we have $\operatorname{rk}_{H_S^*} E_2 = h^\bullet(M)$. Because E_∞ is a subquotient of E_2, we know

$$\operatorname{rk}_{H_S^*} E_2 \;\geqslant\; \operatorname{rk}_{H_S^*} E_\infty;$$

rank at most stays the same as the sequence unwinds. Now $E_\infty = \operatorname{gr}_\bullet H_S^*(M)$ is isomorphic to $H_S^*(M)$ as an H_S^*-module, so that

$$h^\bullet(M) \;=\; \operatorname{rk}_{H_S^*} E_2 \;\geqslant\; \operatorname{rk}_{H_S^*} E_\infty \;=\; \operatorname{rk}_{H_S^*} H_S^*(M).$$

But by Corollary 9.2.5,

$$\operatorname{rk}_{H_S^*} H_S^*(M) = h^\bullet(M^S). \qquad \square$$

Remark 9.2.10. Though we cite Lemma 9.2.8 to Ginzburg et al. [GGK02], the result as stated there is over-optimistic, as can be seen from examples of Franz and Puppe [FP08, Sec. 5]. It becomes true if "torsion-free" in the statement in [GGK02] is everywhere replaced by "free."

Chapter 10

Equivariant formality of isotropy actions

As we first stated in Section 2.2, in the event G is a Lie group and K a closed subgroup, the orbit space G/K of the right K-action is a *homogeneous space*. The left action of the isotropy subgoup K on this space given by $k \cdot gK = (kg)K$ is the ***isotropy action***. Recall from the beginning of Chapter 8 that when G is a compact, connected Lie group and K a closed, connected subgroup, we say (G, K) is a *compact pair* of Lie groups.

Definition 10.0.1. A compact pair (G, K) is said to be ***isotropy-formal*** if the isotropy action of K on G/K is equivariantly formal in the sense of Definition 9.1.1.

To motivate discussion of the isotropy action, note that K is essentially the largest subgroup of G which can act equivariant formality on G/K.

Proposition 10.0.2. *Let (G, K) be a compact connected pair and H a closed subgroup of G. If the natural action of H on G/K, obtained from restricting the defining G-action on G/K, is equviariantly formal, then the maximal torus T_H of H is conjugate in G to a subtorus of K.*

Proof. By Lemma 9.1.3, if H acts equivariant formality on G/K, then so does T_H. By Lemma 9.2.8, this occurs if and only if G/K and the fixed point set $(G/K)^{T_H}$ have equal total Betti number. In particular, the total Betti number of the fixed point set must be positive. But CreffpGGT shows that $(G/K)^{T_H}$ is nonempty if and only if T_H is conjugate in G to a subgroup of K. $\square$

The present chapter, with the exception of Section 10.1, will lay the groundwork for the theory of isotropy-formal actions and summarize its condition as of April 2014. A few epicycles on the Shiga–Takahashi criterion devised by the author are included. The following chapter will recount the author's original contributions.

10.1. Reduction of isotropy-formality to a maximal torus

Let (G, K) be a compact connected pair and S a maximal torus of K. From Lemma 9.1.3, we know (G, K) is isotropy-formal if and only if the restricted action of S on G/K is also equivariantly formal. What may be a surprise—or may seem perfectly natural, but in any event was not known before last April—is that the other instance of K can also be replaced by S: that is, K acts equivariant formality on G/K if and only if K acts equivariant formality on G/K, which is to say, the pair (G, S) is isotropy-formal.

To prove the result, we will need to simultaneously consider left and right actions and homotopy quotients. From our construction of the Milnor EK in Section 3.3, we know that EK admits as well as the natural right action a natural left action.

Definition 10.1.1. As with a left action, given a *right* action of a group K on a space X, there is a diagonal action of K on $X \times EK$ given by $(x, e) \cdot k = (xk, k^{-1}e)$, and a *right* homotopy quotient

$$X_K := \frac{X \times EK}{(xk, e) \sim (x, ke)}.$$

We denote the orbit of (x, e) under K by $[x, e]_K \in X_K$. We temporarily denote by ${}_K X$ the *left* homotopy quotient defined in Section 4.2.3. We still use the left homotopy quotient to define equivariant cohomology:

$$H^*_K(X) := H^*({}_K X).$$

Also as with a left action, given a right action of K on X, there is a map $X_K \twoheadrightarrow X/K$, functorial in K and in X, which is a weak homotopy equivalence if the action is free, and satisfies the statements in Lemma 6.4.1.

Remark 10.1.2. Although the notation is inconsistent with that we used previously, under which X_K was a *left* homotopy quotient, all statements about homotopy quotients are equivalent for the left and the right actions.

The key point of this equivalence is the following. Let $CK = \big(K \times [0,1]\big)/\big(K \times \{0\}\big)$ be the cone on K. The Milnor EK can be viewed as that subspace of the countably infinite product $\prod_{n=1}^{\infty} CK$ populated by lists of pairs (t_n, k_n) such that $\sum_{n=1}^{\infty} t_n = 1$ and only finitely many $t_n \neq 0$. The group K acts diagonally *both* on the left and on the right of EK by $k \cdot (t_n, k_n) = (t_n, kk_n)$ and $(t_n, k_n) \cdot k = (t_n, k_n k)$ respectively. There is a natural self-homeomorphism of EK given by

$$e = (t_n, k_n) \longmapsto (t_n, k_n^{-1}) =: e^{-1},$$

which takes $ke \longmapsto e^{-1}k^{-1}$. This homeomorphism allows us to exchange the right actions we have used heretoforth for left actions as often as we wish, and also descends to a homeomorphism

$$K\backslash EK \xrightarrow{\approx} EK/K = BK.$$

It is only for matters related to the next definition that distinguishing between left and right actions becomes important, and it is for this definition we have temporarily abandoned our previous notation.

Definition 10.1.3. If X admits both a left S-action and a right K-action, then the right homotopy quotient X_K admits a left S-action and the left homotopy quotient ${}_S X$ admits a right K-action, and we can form the ***homotopy biquotient***

$${}_S X_K := \frac{ES \times X \times EK}{(e_1 s, x, ke_2) \sim (e_1, sxk, e_2)} \approx {}_S(X_K) \approx ({}_S X)_K,$$

whose elements are denoted ${}_S[e_1, x, e_2]_K$.

With the new definitions we can prove the result.

Theorem 10.1.4. *Let (G, K) be a compact pair and S a maximal torus of K. The pair (G, K) is isotropy-formal if and only if (G, S) is.*

Proof. By Lemma 9.1.3, it is enough to show that K acts equivariant formality on G/S if and only if it does on G/K.

For the forward direction, assume K acts equivariant formality on G/S. Since K acts freely by right multiplication on G and ${}_K G$, by Observation 6.4.3, the Weyl group $W = N_K(S)/S$ of K acts on $H^*(G/S)$ and $H^*_K(G/S)$, and

$$H^*(G/S) \longleftarrow H^*_K(G/S)$$

is W-equivariant. Because, by assumption, this map is surjective, by Lemma 6.4.4, the restriction $H^*(G/S)^W \longleftarrow H^*_K(G/S)^W$ to subrings of W-invariants is also surjective; but by the naturality statement in Lemma 6.4.5, this surjection is equivalent to the map

$$H^*(G/K) \longleftarrow H^*_K(G/K)$$

induced by the fiber inclusion $G/K \hookrightarrow {}_K G/K$.

For the reverse direction, assume the fiber inclusion $G/K \hookrightarrow {}_K G/K$ is H^*-surjective. Since the right K-actions on G and G_K are free, by Lemma 6.4.1, we may replace the fiber inclusion with $\varkappa\colon G_K \longhookrightarrow {}_K G_K$. Write $\xi_0\colon BS \to BK$ again. Applying the right homotopy-quotient functor $(-)_{S\hookrightarrow K}$ of Corollary 6.4.6 to the fiber inclusion $G \longhookrightarrow G_K$ of the Borel fibration yields the following map in F-Bun/ξ_0:

$$\begin{tikzcd}
G_S \arrow[r, hook, "\lambda"] \arrow[d] & {}_K G_S \arrow[r, two heads] \arrow[d] & BS \arrow[d] \\
G_K \arrow[r, hook, "\varkappa"'] & {}_K G_K \arrow[r, two heads] & BK,
\end{tikzcd}
\qquad
\begin{tikzcd}
{[g,e_2]_S} \arrow[r, mapsto] \arrow[d, mapsto] & {}_K[e_1,g,e_2]_S \arrow[r, mapsto] \arrow[d, mapsto] & Se_2 \arrow[d, mapsto] \\
{[g,e_2]_K} \arrow[r, mapsto] & {}_K[e_1,g,e_2]_K \arrow[r, mapsto] & Ke_2,
\end{tikzcd}$$

where the inclusions in the left square are the fiber inclusions in the Borel fibrations in question.[1]

Applying Theorem 4.4.1 to this diagram, the upper-left map λ^*, which we wish to show is surjective, is equivalent to

$$\varkappa^* \otimes \mathrm{id}_{H_S^*}\colon H^*({}_K G_K) \otimes_{H_K^*} H_S^* \longrightarrow H^*(G_K) \otimes_{H_K^*} H_S^*.$$

But by assumption, $\varkappa^*$ is surjective, and $- \otimes_{H_K^*} H_S^*$ is right exact. □

Remarks 10.1.5. (*a*) The original proof of the "if" implication in Theorem 10.1.4 relied on the enhanced form of Corollary 6.4.6 stated in Theorem B.4.9, which can be used to show the map λ^* is equivalent to $\varkappa^* \otimes \mathrm{id}_{H_S^*}$. We insist upon the naturality statements because without them, Theorem 10.1.4 would no longer follow.

(*b*) One might wonder if the passage to the notationally cumbersome homotopy biquotient in this proof is avoidable, but our proof relies crucially on the notion of being a bundle over $BS \to BK$,

[1] N.B.: The rows are *not* themselves bundles: the maps from the total spaces to the common base space BK of the Borel fibrations $G_S \to {}_K G_S \to BK$ and $G_K \to {}_K G_K \to BK$ are given by ${}_K[e_1,g,e_2]_S \longmapsto e_1 K$ and ${}_K[e_1,g,e_2]_K \longmapsto e_1 K$.

and the necessary maps do not exist from $G/S \to G/K$. Indeed, consider the horn

$$\begin{aligned} G/S \longleftarrow G_S \longrightarrow BS, \\ gS \longleftarrow [g,e]_S \longmapsto Se. \end{aligned}$$

A map $G/S \longrightarrow BS$ completing this horn to a triangle would have to take $gS \longmapsto Se$ for all $g \in G$ and $e \in ES$, an infeasibly tall order. Morally, since the map $G_S \longrightarrow G/S$ quotients ES out of $G_S = G \times_S ES$, while $G_S \longrightarrow BS$ quotients out G, the quotients G/S and BS have "nothing left in common" to construct a nontrivial map from. One can still use Lemma 6.4.1 to get the necessary right H^*_K- and H^*_S-algebra structures on $H^*(G/S)$ and $H^*_K(G/S)$, and then reason using Corollary 6.4.6 on right tensor factors, but this subterfuge seems more circuitous and less honest (and less natural, no pun intended), than simply invoking homotopy biquotients.

(*c*) That said, we shall not need biquotients again, so from now on the reader can consider anything denoted X_K to be the left or the right homotopy quotient as she or he pleases.

10.2. Isotropic torus actions and fixed points

A key component of the proofs of our results in Chapter 11, is the manifestation in the case of an isotropy action of the fixed-point characterization Lemma 9.2.8, which recall:

Lemma 9.2.8 ([GGK02, Cor. C.27]). *Let S be a torus and M a compact S-manifold. Then the action of S on M is equivariantly formal if and only if*

$$h^\bullet(M) = h^\bullet(M^S).$$

Let (G,K) be a compact pair with and S a maximal torus. of K To understand isotropy-formality of (G, S), we should understand the fixed point set $(G/S)^S$. Let $N = N_G(S)$ be the normalizer of S and $Z = Z_G(S)$ its centralizer. Recall from the beginning of Appendix B that $\pi_0 N$ denotes the component group of N.

Now recall Lemma 8.3.8 from Section 8.3.1.

Lemma 8.3.8. *Let S be a torus in a compact, connected Lie group G and $Z = Z_G(S)$ its centralizer in Z. The cohomology of Z decomposes as*

$$H^*(Z) \cong H^*(S) \otimes H^*(Z/S).$$

Consequently, $H^*(Z/S)$ is an exterior algebra on $\operatorname{rk} G - \operatorname{rk} S$ generators and $h^\bullet(Z/S) = 2^{\operatorname{rk} G - \operatorname{rk} S}$.

This result makes available a useful dimension computation.

Corollary 10.2.1 (Goertsches–Noshari, 2014 [GN15, Prop. 3.1])**.** *Let (G, K) be a compact pair with S a maximal torus of K. Then*

$$h^\bullet\big((G/K)^S\big) = \frac{|\pi_0 N|}{|W_K|} \cdot 2^{\operatorname{rk} G - \operatorname{rk} S}$$

Proof. We know from Corollary 2.4.8 that N/S has $|\pi_0 N|/|W_K$ components, each homeomorphic to Z/S and from Lemma 8.3.8 that $h^\bullet(Z/S) = 2^{\operatorname{rk} G - \operatorname{rk} S}$. □

Proposition 10.2.2 (Goertsches–Noshari, 2014 [GN15, Prop. 3.2])**.** *Let (G, K) be a compact pair with S a maximal torus of K. The action of S on G/K is equivariantly formal if and only if*

$$h^\bullet(G/S) \leqslant \frac{|\pi_0 N|}{|W_K|} \cdot 2^{\operatorname{rk} G - \operatorname{rk} S}. \tag{10.1}$$

Proof. By Lemma 9.2.8, the action is equivariantly formal if and only if $h^\bullet(G/K) = h^\bullet\big((G/K)^S\big)$, and we always have the $\geqslant$ direction by Lemma 9.2.9, so we only need the reverse inequality.

From Corollary 10.2.1, the right-hand side is $h^\bullet(G/S) = \dfrac{|\pi_0 N|}{|W_K|} \cdot 2^{\operatorname{rk} G - \operatorname{rk} S}$. □

Corollary 10.2.3. *Let (G,S) be a compact pair with S a torus. Then (G,S) is isotropy-formal if and only if*

$$h^\bullet(G/S) \leqslant |\pi_0 N| \cdot 2^{\operatorname{rk} G - \operatorname{rk} S}. \tag{10.2}$$

10.3. Earlier work on equivariant formal isotropy actions

The groundwork in equivariant cohomology and cohomology of homogeneous spaces being laid, we in this section summarize the state of knowledge on isotropy-formality as of April 2014.

10.3.1. Isotropy-formality of equal-rank and generalized symmetric pairs

It is classical that a generalized flag manifold has equivariantly formal isotropy action [Bri98, Prop. 1].

Proposition 10.3.1. *An equal-rank pair (G,K) is isotropy-formal.*

Proof. Consider the spectral sequence of the Borel fibration $G/K \to (G/K)_K \to BK$. Recall from Theorem 7.6.1 and Theorem 8.3.11 that the rings H_K^* and $H^*(G/K)$ are both concentrated in even degree. By Corollary 4.3.11, the spectral sequence collapses at E_2, so by Proposition 9.1.2, the action is equivariantly formal. □

Recall (Definition 8.3.1) that (G,K) is said to be *cohomology-surjective* if $H^*(G) \longrightarrow H^*(K)$ is surjective; this was the other

Proposition 10.3.2 ([Shi96, Cor. 4.2, p. 180]). *A cohomology-surjective pair (G,K) is isotropy-formal.*

In fact, cohomology-surjectivity admits a characterization in terms of isotropy-formality, due to Oliver Goertsches and Sam Noshari.

Proposition 10.3.3 (Goertsches–Noshari, 2014 [GN15, Prop. 3.3]). *Let (G,K) be a compact pair. Then the following are equivalent:*

1. *(G,K) is isotropy-formal and $N_G(S)/N_K(S)$ is connected.*

2. *(G,K) is a cohomology-surjective.*

Proof. Let S be a maximal torus of K. If (G,K) is cohomology-surjective, then by Theorem 8.3.2, we have $H^*(G/K) \cong \Lambda PG \,/\!/\, \Lambda PK$, so

$$h^\bullet(G/K) = 2^{\operatorname{rk} G - \operatorname{rk} K} \leqslant \frac{\left|\pi_0 N_G(K)\right|}{\left|\pi_0 N_G(S)\right|} 2^{\operatorname{rk} G - \operatorname{rk} K} = h^\bullet\big((G/K)^S\big),$$

by Corollary 10.2.1, which implies by Proposition 10.2.2 that (G,K) is isotropy-formal and by Lemma 9.2.9 that $\left|\pi_0 N_G(K)\right| = 1$.

On the other hand, if $\left|\pi_0 N_G(K)\right| = \left|\pi_0 N_G(S)\right|$, then by Proposition 10.2.2 again, isotropy-formality is just the demand $h^\bullet(G/K) = 2^{\operatorname{rk} G - \operatorname{rk} K}$, which happens if and only if (G,K) is cohomology-surjective, by Proposition 8.3.6. □

Recall from Definition A.4.8 that a *generalized symmetric pair* (G,K) is a compact pair such that K is the identity component of the fixed point set of some finite-order continuous automorphism $\theta \in \operatorname{Aut} G$, and that (G,K) is a *symmetric pair* in the event $\theta^2 = \mathrm{id}$. Goertsches proved in 2011 that such symmetric pairs are isotropy-formal.

Theorem 10.3.4 (Goertsches [Goe12, Sec. 1, Theorem]). *All symmetric pairs (G,K) are isotropy-formal.*

Noshari, in his master's thesis work, generalized the result to generalized symmetric pairs.

Theorem 10.3.5 (Goertsches–Noshari [GN14, Thm. 5.6]). *All generalized symmetric pairs (G,K) are isotropy-formal.*

It is worth summarizing the proofs of these results, as they comprise a substantial portion of the existing literature on isotropy-formality.

The original proof for symmetric pairs [Goe12] breaks symmetric pairs into equal-rank pairs, *split-rank* pairs, and *outer symmetric* pairs. The equal-rank case follows from Proposition 10.3.1; split-rank pairs turn out to be cohomology-surjective, so that Proposition 10.3.2 applies; and the outer symmetric pairs form a small, completely classified set of exceptional cases which are checked individually.

The proof [GN14] for generalized symmetric pairs (G, K) proceeds on related but distinct lines. Let S be a maximal torus of K. The analysis of the outer symmetric pairs (G, K) in the earlier Goertsches paper was abetted by a subdivision of the Cartan subalgebra $\mathfrak{s}$ of K into polyhedral cones called *compartments*, which are analogous to Weyl chambers of W_G but instead are permuted simply transitively by $\pi_0 N_G(K) \cong N_G(K)/Z_G(K)$. There exists a closed, connected group H, the *folded subgroup*, sharing a maximal torus S with K, intermediate between K and G in that $K \leqslant H \leqslant G$, and such that the compartments in $\mathfrak{s}$ are the Weyl chambers for H. The group H is called "folded" because, on the Lie algebra level, its Dynkin diagram Δ_H is a quotient of Δ_G by a graph automorphism.

This *folded pair* (G, H) is again a generalized symmetric pair, and the earlier arguments about compartments are now reflected in the replacement of the isotropy subgroup K with this folded H. Noshari shows that the folded pair (G, H) is in fact cohomology-surjective, so that Proposition 10.3.2 or Proposition 10.3.3 applies and (G, H) is both isotropy-formal and (by Theorem 8.3.2) formal. Noshari has an argument, subsumed by our Theorem 10.1.4, that if (G, H) is formal and K shares a maximal torus with H, then (G, H) is isotropy-formal if and only if (G, K) is, so this shows the initial generalized symmetric pair was isotropy-formal.

Breaking news. Personal communication with Oliver Goertsches informs us that work into this question proceeds apace and the finite-order restriction on the automorphism θ has been bypassed. The state of the art is now thus:

Theorem 10.3.6 (Goertsches–Noshari [GN15])**.** *Let (G, K) be a compact pair such that $K = (G^\theta)_0$ is the identity component of the fixed point set of a continuous group automorphism θ of G. Then (G, K) is isotropy-formal and formal.*

10.3.2. The criterion of Shiga and Takahashi

On a related note, in 1996, Hiroo Shiga took up the question of isotropy-formality [Shi96], and working in the Cartan model, discovered the following sufficient condition.

Theorem 10.3.7 (Shiga)**.** *Let (G, K) be a compact pair and $N = N_G(K)$ the normalizer. Suppose*

1. *(G, K) is formal, and*
2. *the map $H^*(G/K)^N \hookrightarrow H^*(G/K) \to H^*(G)$ induced by $G \twoheadrightarrow G/K$ is injective.*

Then (G, K) is isotropy-formal.

We have discussed the notion of a formal pair at some length in Section 8.4, but the other condition, that $H^*(G/K)^N \longrightarrow H^*(G)$ be injective, is less transparent. Fortunately, Shiga provided more easily interpreted equivalent conditions.

Definition 10.3.8. Let (G, K) be a compact pair with normalizer $N = N_G(K)$. We say the pair (G, K) is ***invariant-surjective*** if the map $H_G^* \longrightarrow H_K^N$ of polynomial invariants is a surjection.

Then Theorem 10.3.7 admits the following equivalent restatement.

Proposition 10.3.9 (Shiga)**.** *If a compact pair (G, K) is formal and invariant-surjective, then it is isotropy-formal.*

Shiga actually includes his statement of this result the condition that (G, K) be formal, but by Theorem 8.3.2, this demand is redundant. In a later-written (but earlier-published) technical report [ST95] coauthored with Hideo Takahashi, Shiga also provides a partial converse to this result.

Theorem 10.3.10 (Shiga–Takahashi). *Let (G, S) be a formal pair with S a torus. Then (G, S) is isotropy-formal if and only if it is invariant-surjective.*

Shiga and Takahashi actually also require S to contain regular elements invoke regular elements to ensure the centralizer $Z = Z_G(S)$ is a maximal torus of G, so that Z/S is a torus and $h^\bullet(Z/S) = 2^{\operatorname{rk} G - \operatorname{rk} S}$, but only this dimensional condition is used in the proof, and by Lemma 8.3.8, it goes through *whether or not* S contains regular elements. Shiga informs us via email that he is aware the regularity condition is redundant.

In fact, we noticed the S in the theorem need not be a torus either.

Corollary 10.3.11. *Let (G, K) be a formal pair. Then (G, K) is isotropy-formal if and only if it is invariant-surjective.*

Proof. Let S be a maximal torus of K. Then (G, K) is formal, isotropy-formal, or invariant-surjective if and only if (G, S) is, by, respectively, Proposition 8.4.10, Theorem 10.1.4, and the following lemma, so the result follows from Theorem 10.3.10. □

Lemma 10.3.12. *The map $H_G^* \longrightarrow (H_K^*)^{N_G(K)}$ is surjective if and only if the map $H_G^* \longrightarrow\!\!\!\!\to (H_S^*)^{N_G(S)}$ is.*

Proof. Retreating to real coefficients everywhere, there is a natural isomorphism $H_S^* \cong \mathbb{R}[\mathfrak{s}]$, and it will be enough to show that under this identification, the image of $(H_K^*)^{N_G(K)}$ in $\mathbb{R}[\mathfrak{s}]$ is $\mathbb{R}[\mathfrak{s}]^{N_G(S)}$.

Given $g \in N_G(K)$, since $gKg^{-1} = K$, it follows gSg^{-1} is a maximal torus of K, so since all maximal tori of K are conjugate, there exists $k \in K$ such that $kgS(kg)^{-1} = S$, or in other words, $kg \in N_G(S)$. Thus $K \cdot N_G(K) = K \cdot N_G(S)$ as subgroups of G.

There is a canonical isomorphism $R[\mathfrak{k}]^K \xrightarrow{\sim} H_K^*$, the ***Chern–Weil isomorphism*** (see, e.g., [GGK02, p. 209]) so we can write

$$(H_K^*)^{N_G(K)} \cong \big(\mathbb{R}[\mathfrak{k}]^K\big)^{N_G(K)} = \mathbb{R}[\mathfrak{k}]^{K \cdot N_G(K)} = \mathbb{R}[\mathfrak{k}]^{K \cdot N_G(S)}.$$

The Chevalley restriction theorem (see, e.g., [GGK02, Prop. C.12, p. 200]) states that restriction of invariant polynomials from $\mathfrak{k}$ to $\mathfrak{s}$ yields an isomorphism $\mathbb{R}[\mathfrak{k}]^K \xrightarrow{\sim} \mathbb{R}[\mathfrak{s}]^{W_K}$. Since $W_K = N_K(S)/S$ injects into $N_G(S)/S$, we then have

$$(H_K^*)^{N_G(K)} \cong \big(\mathbb{R}[\mathfrak{k}]^{W_K}\big)^{N_G(S)} \cong \mathbb{R}[\mathfrak{s}]^{N_G(S)} \cong (H_S^*)^{N_G(S)}. \qquad \square$$

The Shiga–Takahashi criterion, moreover admits of a stronger characterization, at least in the event that the action of the normalizer on $\mathfrak{s} \otimes \mathbb{C}$ is by pseudoreflections.

Definition 10.3.13. Let k be a field, V is a finite-dimensional k–vector space, and Γ a finite subgroup of $\mathrm{GL}(V)$. An element $\gamma \in \Gamma$ fixing a hyperplane (a codimension-1 subspace) in Γ is a ***pseudoreflection***. The image of G in $\mathrm{GL}(V)$ is called a ***pseudoreflection group*** if it is generated by pseudoreflections.

Example 10.3.14. (a) The Weyl group of G acts as a reflection group on $\mathfrak{t}$, the Cartan subalgebra.

(b) Fix an integer $n \geqslant 2$. The natural multiplication representation of the group $\langle e^{2\pi i/n}\rangle$ of n^{th} roots of unity on $\mathbb{C}$ is by pseudoreflections, for the fixed point set 0 is a hyperplane.

(c) The normalizer in $\mathrm{Sp}(3)$ of the torus $S = \big\{\mathrm{diag}(z, z^3 w, w^{-5}) : z, w \in S^1\big\}$ does not act on $\mathfrak{s}$ by pseudoreflections. Indeed $\mathfrak{s}$ is two-dimensional and $\pi_0 N_{\mathrm{Sp}(3)}(S) \cong \{\pm 1\}$ contains only the identity and the antipodal map $v \longmapsto -v$, which is not a pseudoreflection.

Leray's theorem that $H^*(G/T)$ is the regular representation of W_G, Corollary 5.2.4, is an instance of a more general algebraic fact on pseudoreflection groups, proven by Chevalley in 1955. Note that an action of a group Γ on a k–vector space V naturally induces actions on both the symmetric algebra $S[V]$ of V and that on its dual, $S[V^\vee]$. Borrowing a notation from algebraic geometry, we write $k[V] := S[V^\vee]$ for the ring of polynomial functions on V.

Theorem 10.3.15 (Chevalley [Che55]). *Suppose* $\operatorname{char} k = 0$ *and* Γ *is a pseudoreflection group. Then* $S[V] /\!/ S[V]^W$ *is the regular representation of* W.

Now we can state another version of Shiga–Takahashi. We tacitly extend to $\mathbb{C}$ coefficients, noting that vector space dimension of cohomology rings is unaffected.

Theorem 10.3.16. *If* (G, K) *is a compact pair, if* S *is a maximal torus of* K*, and if* $N = \pi_0 N_G(S)$ *acts as a pseudoreflection group on the complexified Lie algebra* $\mathfrak{s} \otimes \mathbb{C}$ *of* S*, then any two of the following statements imply the third.*

- *The pair* (G, K) *is formal.*
- *The pair* (G, K) *is isotropy-formal.*
- *The pair* (G, K) *is invariant-surjective.*

Proof. By Proposition 8.4.9, the pair (G, S) is formal if and only if

$$h^\bullet(G/S) = 2^{\operatorname{rk} G - \operatorname{rk} S} \dim H_S^* /\!/ H_G^*. \tag{10.3}$$

The second condition, invariant-surjectivity, can be stated as

$$H_S^* /\!/ H_G^* = H_S^* /\!/ (H_S^*)^N,$$

Since H_S^* is a polynomial ring, this is the regular representation of N, by the Chevalley reflection Theorem 10.3.15. Since the image of $H_G^* \to H_S^*$ lies in $(H_S^*)^N$, this condition is satisfied if and only if

$$\dim H_S^* /\!\!/ H_G^* = |N|. \tag{10.4}$$

The condition that (G, S) be isotropy-formal can be restated thus:

$$h^\bullet(G/S) = 2^{\operatorname{rk} G - \operatorname{rk} S}|N|, \tag{10.5}$$

But any two elements of $\{(10.3), (10.4), (10.5)\}$ imply the third. $\square$

Remark 10.3.17. The Shiga–Takahashi criterion initially appeared to us to be an inapplicable curiosity, its proof *sui generis* and honestly a bit impenetrable, but as we have seen, the criterion turns out to be quite general and to admit a proof that falls naturally out of the cohomology theory of homogeneous spaces as developed in Chapter 8. This embellished criterion is very powerful, and with proper interpretation our result Theorem 11.1.7 can be seen as a consequence. Indeed, the work of Goertsches and Noshari on isotropy-formality of generalized symmetric pairs can be seen as a consequence as well.

Goertsches, Noshari, and the author are aware of no examples of isotropy-formal pairs that are not also formal; the standard examples of *in*formal pairs all fail to satisfy the dimension condition Proposition 10.2.2. The author suspects isotropy-formal pairs may all be formal and the formality condition entirely redundant, in which case Corollary 10.3.11 implies that isotropy-formality and invariant-surjectivity are equivalent conditions on compact pairs.

Chapter 11

Equivariant formality of isotropic torus actions

This chapter forms the core of our original work on equivariant formality of isotropy actions. We begin with a summary.

11.1. Survey of original work on isotropy-formality

The reduction of an isotropy action to a toral action was sufficiently important to our discussion of the literature that we were forced to state it earlier, but we reproduce it here.

Theorem 10.1.4. *Let (G, K) be a compact pair and S a maximal torus of K. The pair (G, K) is isotropy-formal if and only if (G, S) is.*

This result reduces the study of isotropy-formality to understanding embeddings of tori in Lie groups, an already more feasible-looking endeavor. Further, the question reduces to the case the commutator subgroup K is simply-connected.

Theorem 11.1.1. *Let (G, S) be a compact pair with S a torus. If $\widetilde{G}$ is a finite central covering of G and $\widetilde{S}$ the identity component of the preimage of S in $\widetilde{G}$, then (G, S) is isotropy-formal if and only if $(\widetilde{G}, \widetilde{S})$ is.*

The proof is in Section 11.2. In fact, the question nearly reduces to the case G itself is simply-connected.

Theorem 11.1.2. *Let (G, S) be a compact pair with S a torus, K the commutator subgroup of G, and $S' = (S \cap K)_0$ the identity component of its intersection with S. Then the pair (G, S) is isotropy-formal if and only if both*

1. *(K, S') is isotropy-formal and*
2. *any element of K which normalizes S' normalizes all of S.*

The proof is in Section 11.3. These reductions having been achieved, we completely determine, as a proof of concept, whether (G, S^1) is isotropy-formal, where S^1 is any circle subgroup of G.

Definition 11.1.3. Let (G, S) be a compact pair, with S a torus. Then S is said to be ***reflected*** in G if there exists $g \in G$ normalizing S and such that $gsg^{-1} = s^{-1}$ for all $s \in S$.

Put another way, a torus is reflected if conjugation by some $g \in G$ acts as inversion on S.

Algorithm 11.1.4. Let (G, S) be a compact pair with S a circle. The following steps determine whether (G, S) is isotropy-formal.

1. Check whether S lies in the commutator subgroup K of G, a semisimple group. If not, then (G, S) is isotropy-formal.

2. If S is contained in K, continue. We may assume by Theorem 11.1.1 that K is a product of simple Lie groups K_j as listed in Proposition B.4.5. Find the images S_j of S under the compositions $S \hookrightarrow K \twoheadrightarrow K_j$.

3. For each K_j, determine from Table 11.1.5 whether S_j is reflected in K_j.

4. If each S_j is reflected in K_j, then (G, S) is isotropy-formal.

5. If some S_j is not reflected in K_j, then (G, S) is not isotropy-formal.

Table 11.1.5: Reflected lines in simple Lie algebras

Type of K	The circle S in K is reflected ...
A_n	if the exponent multiset J satisfies $J = -J$ (see Remarks 11.1.6(b)).
B_n	always.
C_n	always.
D_{2n}	always.
D_{2n+1}	if S is contained in a D_{2n} subgroup.
G_2	always.
F_4	always.
E_6	if S is contained in a D_4 subgroup.
E_7	always.
E_8	always.

Remarks 11.1.6. (*a*) To ask a circle S lie in the commutator subgroup K is equivalent to asking the image of $\pi_1 S \longrightarrow \pi_1 G$ be infinite or the image of $H^1(G) \longrightarrow H^1(S)$ be trivial (Proposition 11.4.2 and Remark 11.4.3).

(*b*) The ***exponent multiset*** J is the sequence of integers $a_1, \ldots, a_n \in \mathbb{Z}$, considered without order, but with multiplicity, such that the injection $S^1 \rightarrowtail (S^1)^n$ realizing S as a circular subgroup of a maximal torus of $\mathrm{U}(n)$ (the diagonal subgroup, say) is given by $z \longmapsto (z^{a_1}, \ldots, z^{a_n})$. We write $-J$ for the multiset $\{-a_j\}_{1 \leqslant j \leqslant n}$ whose entries are the opposites of those of J; that is to say, for each $a \in \mathbb{Z}$, the element $-a$ occurs in $-J$ with the same multiplicity that a occurs in J. See Proposition E.1.2.

(c) The demand a circle S in a group of type D_{2n+1} lie within a D_{2n} subgroup means the Lie algebra $\mathfrak{s}$ of S is conjugate into the subspace $\mathfrak{so}(2)^{\oplus 2n} \oplus \{1\}^{\oplus 2}$ of the standard (block diagonal) infinitesimal maximal torus $\mathfrak{so}(2)^{\oplus 2n+1}$. See Proposition E.1.3.

(d) The manifestation of the condition that a circle $S \leqslant E_6$ lying within a given maximal torus T^6 of E_6 also be contained in a D_4 subgroup is somewhat complicated and is the subject of Appendix E.1.3. One succinct statement is Proposition E.1.14.

The succinct statement Algorithm 11.1.4 is the concatenation of Theorem 11.1.7, Proposition 11.4.2, Proposition 11.5.3, and Proposition 11.5.4. The construction of Table 11.1.5 is taken up in Section 11.6. The constituent Theorem 11.1.7 in particular is important enough to be stated in this introduction.

Theorem 11.1.7. *Let G be a compact, connected Lie group and S a circular subgroup of G. There are the following three mutually exclusive cases.*

1. *The inclusion $S \hookrightarrow G$ surjects in cohomology and S is not reflected in G.*
2. *The inclusion $S \hookrightarrow G$ is trivial in cohomology and*
   - 2a. *S is reflected in G.*
   - 2b. *S is not reflected in G.*

Only in the case 2b is (G, S) not isotropy-formal.

Here are two sample consequences.

Proposition 11.1.8 (anonymous referee, 2015)**.** *Let (G, K) be a compact pair, where $K \cong \mathrm{SU}(2) \cong \mathrm{Sp}(1)$ or else $K \cong \mathrm{SO}(3)$. Then (G, K) is isotropy-formal.*

Proof. These K are precisely those of type A_1, whose maximal torus is a circle S. This circle S is reflected in K because the Weyl group $W_{A_1} \cong \{\pm 1\}$. It follows that S is also reflected in G, so by Theorem 11.1.7, the pair (G, S) is isotropy-formal. By Theorem 10.1.4, (G, K) is isotropy-formal as well. □

Example 11.1.9. If S is a circle in the unitary group $\mathrm{U}(n)$, then $\big(\mathrm{U}(n), S\big)$ is or is not isotropy-formal as indicated in Table 11.1.10.

Table 11.1.10: The classification for circles in $\mathrm{U}(n)$

Embedding of S	Is $\big(\mathrm{U}(n), S\big)$ isotropy-formal?
$S \not\leqslant \mathrm{SU}(n)$	Yes
$S \leqslant \mathrm{SU}(n)$ and $J = -J$	Yes
$S \leqslant \mathrm{SU}(n)$ and $J \neq -J$	No

11.2. Lifting to the universal compact cover

Recall the structure theorem Theorem B.4.4 for compact Lie groups, which states, *inter alia*, that a compact, connected Lie group G admits a finite central extension $\pi\colon \widetilde{G} \longrightarrow G$ which is the direct product of a semisimple, simply-connected Lie group K and a torus A: if the fiber of π is F, we may write

$$G \cong \widetilde{G}/F = {}^{A \times K}\big/{}_{F}.$$

This $\widetilde{G}$ is a *universal compact cover*.

In determining which toral isotropy actions are equivariantly formal, we would like to replace G with $\widetilde{G}$ and the torus S in G with the identity component $\widetilde{S} = \pi^{-1}(S)_0$ of its preimage in $\widetilde{G}$, a torus of equal rank. Note that $\pi^{-1}(S) = F\widetilde{S}$, and recall this cohomological lifting lemma proven in Appendix B.

Proposition B.3.4. *Let G be a compact connected Lie group and K a closed, connected subgroup, let $\widetilde{G}$ be the universal compact cover of G (see Theorem B.4.4), and $\widetilde{K}$ the identity component of the preimage of K in $\widetilde{G}$, and let F be the kernel of $p\colon \widetilde{G} \longrightarrow G$. If $|F/(F \cap \widetilde{K})|$ is invertible in k, then*

$$H^*(G/K) \cong H^*(\widetilde{G}/\widetilde{K}).$$

As total Betti number is unchanged under the substitution $(G,S) \longmapsto (\widetilde{G},\widetilde{S})$, we want to see the same is true of normalizer components. Write $\widetilde{N} = N_{\widetilde{G}}(\widetilde{S})$ and $N = N_G(S)$ for the normalizers of the tori.

Proposition 11.2.1. *Under the foregoing assumptions, the projection $\pi\colon \widetilde{G} \longrightarrow G$ induces an isomorphism $\pi_0\widetilde{N} \xrightarrow{\sim} \pi_0 N$.*

Proof. Since π is a homomorphism, it sends $\widetilde{N} \longrightarrow N$, inducing the claimed map $\pi_0\widetilde{N} \longrightarrow \pi_0 N$.

To see this map is injective, suppose $\widetilde{w} \in \widetilde{N}$ is such that $w = \pi(\widetilde{w})$ induces the identity on S. Then for all $s \in S$ we have $wsw^{-1}s^{-1} = 1$, so since π is a homomorphism, $\widetilde{w}\widetilde{s}\widetilde{w}^{-1}\widetilde{s}^{-1} \in F$ for all $\widetilde{s} \in \widetilde{S}$. Since $\widetilde{s} \longmapsto \widetilde{w}\widetilde{s}\widetilde{w}^{-1}\widetilde{s}^{-1}$ is a continuous function $\widetilde{S} \longrightarrow F$ to a discrete space, sending $1 \mapsto 1$, $\widetilde{w}$ must centralize $\widetilde{S}$.

To see the map is surjective, given $w \in N$, let $\widetilde{w}$ be any preimage in $\widetilde{G}$. Since π is a homomorphism, conjugation by $\widetilde{w}$ must take $\widetilde{S}$ into $\pi^{-1}(S)$, and since it fixes 1, it must in fact take $\widetilde{S} \longrightarrow \widetilde{S}$, so that $\widetilde{w} \in \widetilde{N}$. ☐

These facts in hand, we conclude the proof of Theorem 11.1.1.

Theorem 11.1.1. *Let (G,S) be a compact pair with S a torus. If $\widetilde{G}$ is a finite central covering of G and $\widetilde{S}$ the identity component of the preimage of S in $\widetilde{G}$, then (G,S) is isotropy-formal if and only if $(\widetilde{G},\widetilde{S})$ is.*

Proof. We know from Proposition 10.2.2 that (G,S) is isotropy-formal if and only if

$$h^\bullet(G/S) = \big|\pi_0\big(N_G(S)\big)\big|2^{\operatorname{rk} G - \operatorname{rk} S}.$$

But evidently $\operatorname{rk}\widetilde{G} = \operatorname{rk} G$ and $\operatorname{rk}\widetilde{S} = \operatorname{rk} S$; from Proposition B.3.4, we know $h^\bullet(\widetilde{G}/\widetilde{S}) = h^\bullet(G/S)$; and from Proposition 11.2.1, we know $\pi_0\big(N_{\widetilde{G}}(\widetilde{S})\big) \longleftrightarrow \pi_0\big(N_G(S)\big)$. $\square$

In what follows, we can therefore replace G with its universal compact cover $\widetilde{G} = A \times K$. For later, when we specialize to circles, we note the following corollary of Proposition 11.2.1.

Corollary 11.2.2. *Under these hypotheses, the torus S is reflected in G just if $\widetilde{S}$ is reflected in $\widetilde{G}$.*

11.3. Reduction to a semisimple group

In this section, G is the product of a torus A and a simply-connected Lie group K. Let S be a torus in G and

$$\overline{K} = K/(S \cap K),$$
$$\overline{G} = G/S,$$
$$A^{\perp S} = \operatorname{coker}(S \to A).$$

The trivial bundle $K \to G \to A$ induces a bundle $\overline{K} \to \overline{G} \to A^{\perp S}$ we claim is also trivial.

Proposition 11.3.1. *Given the above hypotheses, $\overline{G} \approx \overline{K} \times A^{\perp S}$.*

Proof. The kernel of the composition $S \hookrightarrow G \twoheadrightarrow A$ is $S \cap K$, so there is an isomorphism

$$\overline{S} := S/(S \cap K) \xrightarrow{\sim} \operatorname{im}(S \to A) =: A^{\| S}$$

of tori. Factoring out $S \cap K$, we have

$$\overline{G} = \frac{K \times A}{S} \approx \frac{\overline{K} \times A}{\overline{S}}$$

by the third isomorphism theorem. As A is a torus, the projection $A \twoheadrightarrow A^{\perp S}$ splits, by Proposition A.4.2, so that A may be factored as $A \cong A^{\perp S} \times A^{\| S}$ in the display, and

$$\frac{\overline{K} \times A}{\overline{S}} \approx \frac{\overline{K} \times A^{\| S}}{\overline{S}} \times A^{\perp S} \approx \frac{\overline{K} \times \overline{S}}{\overline{S}} \times A^{\perp S}$$

But $(\overline{K} \times \overline{S})/\overline{S} \approx \overline{K}$ by Lemma 2.1.1. $\square$

Now we can carry through the claimed (near-)reduction to the semisimple case.

Theorem 11.1.2. *Let (G, S) be a compact pair with S a torus, K the commutator subgroup of G, and $S' = (S \cap K)_0$ the identity component of its intersection with S. Then the pair (G, S) is isotropy-formal if and only if both*

1. *(K, S') is isotropy-formal and*
2. *any element of K which normalizes S' normalizes all of S.*

Proof. By Proposition 11.2.1, we may as well assume that $G = A \times K$, so that $N = N_G(S) = A \times N_K(S)$. Note further that because K is normal in G, any group element normalizing S also must normalize S'. Write $N' = N_K(S')$. Thus we have $\pi_0 N \cong \pi_0\big(N_K(S)\big) \leqslant \pi_0 N'$. From Lemma 9.2.9 as applied to the pair (K, S') and Corollary 10.2.1, we have

$$h^\bullet(K/S') \geqslant |\pi_0 N'|\, 2^{\operatorname{rk} K - \operatorname{rk} S'} \geqslant |\pi_0 N|\, 2^{\operatorname{rk} K - \operatorname{rk} S'}. \tag{11.1}$$

Write $A^{\perp S} = \operatorname{coker}(S \to A)$ and $A^{\| S} = \operatorname{im}(S \to A)$, so that $\operatorname{rk} A = \operatorname{rk} A^{\| S} + \operatorname{rk} A^{\perp S}$ and

$\operatorname{rk} S = \operatorname{rk} A^{\parallel S} + \operatorname{rk} S'$. Then we know

$$\operatorname{rk} G - \operatorname{rk} S = (\operatorname{rk} A + \operatorname{rk} K) - (\operatorname{rk} A^{\parallel S} + \operatorname{rk} S') = \operatorname{rk} A^{\perp S} + \operatorname{rk} K - \operatorname{rk} S'.$$

From Proposition 11.3.1, we have $h^\bullet(G/S) = 2^{\operatorname{rk} A^{\perp S}} h^\bullet(K/S')$, so multiplying (11.1) by $2^{\operatorname{rk} A^{\perp S}}$ yields

$$h^\bullet(G/S) \geqslant |\pi_0 N'|\, 2^{\operatorname{rk} G - \operatorname{rk} S} \geqslant |\pi_0 N|\, 2^{\operatorname{rk} G - \operatorname{rk} S}. \tag{11.2}$$

Corollary 10.2.3 states that (G, S) is isotropy-formal if and only if the inequalities (11.2) are in fact equalities, which is equivalent to (11.1) being equalities. But by Corollary 10.2.3 again, this can only happen if (K, S') is isotropy-formal and additionally $\pi_0(N') \cong \pi_0(N)$. □

Remarks 11.3.2. *(a)* There do exist cases where the inequality $|\pi_0 N| \leqslant |\pi_0 N'|$ is strict, making Theorem 11.1.2 something less than a full reduction to the semisimple case. For instance, let $G = A \times K = S^1 \times \mathrm{SU}(2)^2$, pick a circle S^1 in $\mathrm{SU}(2)$, let T be the maximal torus $A \times S^1 \times S^1$ of G, and let

$$S = \big\{(z; w, zw^{-1}) : z, w \in S^1\big\}$$

be a rank-two subtorus of T, so that

$$S' = \big\{(1; w, w^{-1}) : w \in S^1\big\}.$$

Then $|\pi_0 N'| = 2$, the nontrivial element of $\pi_0 N'$ acting on T as $(z; t, u) \longmapsto (z; t^{-1}, u^{-1})$ since A is central. But this map takes $(z; w, zw^{-1})$ to $(z; w^{-1}, z^{-1}w)$, which is not in S unless $z = z^{-1} = \pm 1$, so $\pi_0 N = 1$.

(b) Write T' for a maximal torus of K containing $\operatorname{im}(S \to K)$. The demand each element $w \in N'$

normalize S is equivalent to the demand that for each $(a,t) \in S \leqslant A \times T'$, we have $(a, wtw^{-1}) \in S$, so that $wtw^{-1}t^{-1} \in S' = S \cap K$ for all $t \in \operatorname{im}(S \to T')$. Equivalently, the induced action of N' on $\operatorname{im}(S \to T')/S'$ is trivial. It is not clear, however, this reformulation is more enlightening or applicable than the original.

11.4. Equivariant formality of isotropic circle actions

It is now our goal to demonstrate the statements Algorithm 11.1.4 and Table 11.1.5 regarding equivariant formality of circle actions.

We first make some preliminary remarks on $\pi_0 N$. In general, continuous automorphisms of a torus $S \cong (\mathbb{R}/\mathbb{Z})^r = \mathbb{R}^r/\mathbb{Z}^r$ are induced by linear isometries of $\mathbb{R}^r$ fixing the integer lattice $\mathbb{Z}^r$, which correspond to elements of $\mathrm{GL}(r,\mathbb{Z})$, so that $\operatorname{Aut} S \cong \mathrm{GL}(r,\mathbb{Z})$. When S is a circle, $r = 1$ and $\mathrm{GL}(1,\mathbb{Z}) = \{[\pm 1]\}$, so by Lemma 2.4.7 the component group $\pi_0 N$ must be trivial or be generated by the involution $s \longmapsto s^{-1}$. Thus to determine $|\pi_0 N|$ in this case, it will suffice to determine whether there is any element $g \in G$ such that $g^{-1}sg = s^{-1}$ for all $s \in S$. Recall from Definition 11.1.3 that a circle is said to be *reflected* if there is some such element g. We summarize:

Proposition 11.4.1. *If (G,S) is a compact pair and S a circle, then the cardinality of $\pi_0\big(N_G(S)\big)$ is 2 if S is reflected in G and 1 otherwise.*

A complete analysis of when a circle is reflected is conducted in Section 11.5. To be able to apply Corollary 10.2.3, it remains to understand $h^\bullet(G/S)$, which was a consequence of the theorem Proposition 8.5.1 of Leray and Koszul.

Proposition 8.5.1. *Let G be a compact, connected Lie group and S a circle subgroup. Then the rational cohomology ring $H^*(G/S)$ has one of the following forms.*

1. *If $H^1(G) \longrightarrow H^1(S)$ is surjective, then there is $z_1 \in H^1(G)$ such that*

$$H^*(G/S) \cong H^*(G)/(z_1).$$

In terms of total Betti number, $h^\bullet(G) = \frac{1}{2}h^\bullet(G/S)$.

2. *If $H^1(G) \longrightarrow H^1(S)$ is zero, there are $z_3 \in H^3(G)$ and $s \in H^2(G/S)$ such that*

$$H^*(G/S) \cong \frac{H^*(G)}{(z_3)} \otimes \frac{\mathbb{Q}[s]}{(s^2)}.$$

In terms of total Betti number, $h^\bullet(G) = h^\bullet(G/S)$.

Now we can prove Theorem 11.1.7.

Theorem 11.1.7. *Let G be a compact, connected Lie group and S a circular subgroup of G. There are the following three mutually exclusive cases.*

1. *The inclusion $S \hookrightarrow G$ surjects in cohomology and S is not reflected in G.*

2. *The inclusion $S \hookrightarrow G$ is trivial in cohomology and*

   2a. *S is reflected in G.*

   2b. *S is not reflected in G.*

Only in the case 2b is (G, S) not isotropy-formal.

Proof. Recall (Corollary 10.2.3) that $h^\bullet(N/S) = |\pi_0 N|\, 2^{\operatorname{rk} G - \operatorname{rk} S}$ and that (G, S) is isotropy-formal if and only if $h^\bullet(G/S) = h^\bullet(N/S)$. Proposition 11.4.1 imposes the constraint $|\pi_0 N| \in \{1, 2\}$, and Proposition 8.5.1 the constraint $h^\bullet(G/S) \in \left\{\frac{1}{2}h^\bullet(G), h^\bullet(G)\right\}$. By Lemma 9.2.9, it is impossible that both $h^\bullet(G/S) = \frac{1}{2}h^\bullet(G)$ and $|\pi_0 N| = 2$, so there are only the following three cases.

1. We have $h^\bullet(G/S) = \frac{1}{2}h^\bullet(G)$ and $|\pi_0 N| = 1$. Equivariant formality is achieved.

2. We have $h^\bullet(G/S) = h^\bullet(G)$, and

   2a. $|\pi_0 N| = 2$. Equivariant formality is achieved.

   2b. $|\pi_0 N| = 1$. Equivariant formality is not achieved. □

It is convenient to be able to express the conditions on the map $H^1(G) \longrightarrow H^1(S)$ in terms of the intersection of S with the commutator subgroup K.

Proposition 11.4.2. *Let G be a compact, connected Lie group and S a toral subgroup. The inclusion $S \hookrightarrow G$ is trivial in cohomology if and only if S is contained in the commutator subgroup K.*

Proof. Note that being contained in the commutator subgroup is invariant under taking covers, and recall from Section 11.2 that the rank of $H^1(G) \longrightarrow H^1(S)$ is as well, so we may assume $G = A \times K$ with A a torus and K simply-connected. Note that $S \leqslant K$ just when the composition $S \to G \to A$ is trivial.

If $S \to G \to A$ is trivial, then $H^1(A) \to H^1(G) \to H^1(S)$ is trivial. Since $\pi_1 K = H_1(K;\mathbb{Z})$ is zero, so also are $H_1(K)$ and $H^1(K)$, meaning $H^1(A) \cong H^1(G)$ by the Künneth theorem. Thus in this case $H^1(G) \longrightarrow H^1(S)$ is also trivial.

On the other hand, if $S \to G \to A$ is not trivial, its image is a circle in A, possibly traversed multiple times, so that $S \longrightarrow A$ induces a nonzero map $H_1(S) \longrightarrow H_1(A)$. Dualizing, $H^1(A) \longrightarrow H^1(S)$ is nonzero. □

Remark 11.4.3. The torus S is actually contained in the commutator subgroup K of G if and only if its image in $A = G/K$ is trivial. Since $H^*(A;\mathbb{Q}) \cong H^*(G;\mathbb{Q})$ by the lifting lemma Proposition B.3.4 and the topological Künneth Theorem B.2.2, it follows that this occurs if and only if $H^*(G;\mathbb{Q}) \longrightarrow H^*(S;\mathbb{Q})$ is trivial. Dualizing, that happens if and only if $H_*(S;\mathbb{Q}) \longrightarrow H_*(G;\mathbb{Q})$ is trivial. Because

$H_*(S;\mathbb{Q})$ is an exterior algebra on generators $H_1(S;\mathbb{Q})$, that is equivalent to $H_1(S;\mathbb{Q}) \longrightarrow H_1(G;\mathbb{Q})$ being trivial, or the image of $H_1(S;\mathbb{Z}) \longrightarrow H_1(G;\mathbb{Z})$ being torsion, or equivalently, the kernel having full rank. Since S and G are groups, their fundamental groups are abelian, so this is actually the map $\pi_1 S \longrightarrow \pi_1 G$. Thus one has the statement

$$S \leqslant K \iff \operatorname{im}(\pi_1 S \longrightarrow \pi_1 G) \text{ is finite.}$$

In the event $S \cong S^1$, the statement is that S lies in K if and only if $\pi_1 S \longrightarrow \pi_1 G$ fails to be injective, and conversely, that $\pi_1 S \longrightarrow \pi_1 G$ is injective if and only if S does not lie in K.

Remark 11.4.4. The results on circles can be seen as a consequence of the (slightly enhanced) Shiga–Takahashi criterion Theorem 10.3.10, though only the sufficiency direction follows from the published version.

Let (G, S) be a compact pair, S a circle, and $N = N_G(S)$. The pair is formal by Lemma 8.5.5, so (G, S) is isotropy-formal if and only if $H^*_G \longrightarrow H^N_S$ is surjective by Corollary 10.3.11. Note that $|N| \leqslant 2$ by Proposition 11.4.1. If $|N| = 1$, then $H^N_S = H^*_S$, so Shiga–Takahashi says that (G, S) is isotropy formal if and only if $H^*_G \longrightarrow H^*_S$ is surjective, which happens if and only $H^*(G) \longrightarrow H^*(S)$ is surjective by Proposition 8.3.5, the cohomology-surjective case.

If $|N| = 2$, on the other hand so that S is reflected in G, then H^N_S is the fixed point of $H^*_S = \mathbb{Q}[s]$ under the involution $s \longmapsto -s$, which is the subring $\mathbb{Q}[s^2]$, and the image of $H^*_G \longrightarrow H^N_S$ contains this subring by Lemma 7.8.3.

The reflected circles still must be determined in some way. Shiga himself [Shi96, pp. 80–82] carries through this program in the special case $(\mathrm{SU}(n), S^1)$, showing both necessity and sufficiency for his conditions in this case and reassuringly arriving at the same result as we did.

11.5. Reflectibility of circles: reduction

In the next two sections, we continue the classification implied by Algorithm 11.1.4 by determining when circles are reflected in a compact Lie groups G. This section reduces reflectibility in stages to the case G is a semisimple group, then a simply-connected group, and finally a simple group. First note that elements of $N_G(S)$ reflecting S, if they exist, can be represented by elements of the Weyl group $W_G = N_G(T)/T$ of G.

Lemma 11.5.1 ([Bou68, Ex. IX.2.4, p. 391][DW98, Lem. 9.7, p. 20])**.** *Let G be a compact, connected Lie group, T a maximal torus, and S a subtorus. Given an automorphism of S induced by conjugation by a normalizing element $n \in N_G(S)$, there exists an element $w \in N_G(T)$ inducing the same automorphism.*

Proof. Conjugation by n stabilizes the centralizer $Z := Z_G(S)$ of S, for given $z \in Z$ and $s \in S$, since $nsn^{-1} \in S$ by normality, we have

$$(nzn^{-1})s(nzn^{-1})^{-1} = nz(n^{-1}sn)z^{-1}n^{-1} = n(n^{-1}sn)n^{-1} = s.$$

T is a maximal torus of Z, so also must nTn^{-1} be, by Theorem B.4.9. All maximal tori of Z are conjugate, so there exists $z \in Z$ such that $znTn^{-1}z^{-1} = T$, or $zn \in N_G(T)$. Since $z \in Z$ centralizes S and $nSn^{-1} = S$, conjugation by $w = zn$ induces the same automorphism of S as n does. □

Write $\mathrm{Fix}_{N_G(T)}(S)$ for the set of elements of $N_G(T)$ fixing S pointwise. This association $N \rightsquigarrow N_G(T)$ is not a function, but if $n \rightsquigarrow w_1$ and $n \rightsquigarrow w_2$, then conjugation by $w_1^{-1}w_2$ fixes S pointwise. so $N \rightsquigarrow N_G(T)$ descends to a well-defined homomorphism

$$N \longrightarrow \frac{N_G(T)}{Z_G(S) \cap N_G(T)} \cong \frac{W_G}{\mathrm{Fix}_{W_G}(S)}$$

with kernel Z; thus, to determine $|\pi_0 N|$, we need only survey Weyl group elements. We state this as a proposition.

Proposition 11.5.2. *Let G be a compact, connected Lie group and S a torus in G. Write $N = N_G(S)$ for its normalizer and $Z = Z_G(S)$ for its centralizer. Then the conjugation action of G induces an injection $\pi_0 N \cong N/Z \rightarrowtail W_G/\mathrm{Fix}_{W_G}(S)$. In particular, S is reflected in G if and only if conjugation by some element of the Weyl group W_G of G induces $s \longmapsto s^{-1}$ on S.*

Note that for the purposes of the previous definition, S need not be a circle. Now, however assume S is a circle in G. From Proposition 11.2.1 and Corollary 11.2.2, we may assume G is the product $A \times K$ of a torus A and a simply-connected Lie group K.

Proposition 11.5.3. *Let G be a compact, connected Lie group and S a toral subgroup. Then S is reflected in G if and only if it is reflected in the commutator subgroup K.*

Proof. Assuming $G = A \times K$, from Theorem 11.1.7 and Proposition 11.4.2, we know that S is not reflected unless it is contained in K. Further, since the conjugation action of A is trivial, if S is reflected, it is also reflected in K. $\square$

Reflectibility of a circle in a simply-connected group in turn depends only on simple factors, for the fact Lemma B.4.10 that Weyl group of a product is the product of the Weyl groups of the factors then translates into the expected statement about reflectibility in products.

Proposition 11.5.4. *Let K be a product of Lie groups K_j. If S is a toral subgroup of K and S_j, for each j, its image under the projection $K \longrightarrow\!\!\!\!\rightarrow K_j$, then S is reflected in K just if each S_j is reflected in K_j.*

Proof. Write T_j for maximal tori of K_j containing S_j. If S is reflected in K, then there is some element $(w_j) \in W = \prod W_{K_j}$ such that (w_j) reflects S. Since the projections $\prod T_j \longrightarrow T_j$ are W-equivariant, it follows each S_j is reflected by w_j in K_j.

Now suppose each S_j is reflected in K_j. Then each S_j is reflected by some $w_j \in W_j$, so the list $w = (w_j)$ reflects the torus $\prod S_j$ in K; and S is a subtorus of $\prod S_j$. $\square$

So, to determine reflectibility of a given circle in a compact, connected Lie group, we need only lift it to the universal cover, verify that it intersects any toral component trivially, and check whether or not its projection to each simple factor is reflected.

We will make repeated use of the following.

Observation 11.5.5. Let G be a compact, connected Lie group and H a closed, connected subgroup. If a torus S of H is reflected in H, it is also in G.

11.6. Reflected circles in simple groups

In this section we let G be a simple Lie group and classify reflected circles S in G. If we fix a maximal torus T of G, then by Theorem B.4.9, some conjugate gSg^{-1} is contained in T, and conjugation $x \longmapsto gxg^{-1}$ takes $N_G(S)$ to $N_G(gSg^{-1})$, so that S is reflected in G if and only if its conjugate gSg^{-1} is. So from now on, we can just assume S is in our chosen maximal torus T.

Proposition 11.6.1. *Let G be a simple Lie group and S a circular subgroup. Then S is reflected in G if and only if it is reflected by the longest word w_0 in the Weyl group W of G.*

Proof. The circle S determines a line $\mathfrak{s}$ in the Lie algebra $\mathfrak{t}$ of T. Arbitrarily orienting $\mathfrak{s}$, we can write it as the union $\mathfrak{s}^+ \cup -\mathfrak{s}^+$ of a "positive" and "negative" ray. By Proposition B.4.18, the nonzero points of $\mathfrak{s}^+$ lie in a unique closed Weyl chamber C, and by Definition B.4.17 there exists a system $\Delta = \{\alpha_1, \ldots, \alpha_{\dim T}\}$ of simple roots such that C is the "positive" closed Weyl chamber in $\mathfrak{t}$, where all $\alpha_j \geqslant 0$. It then follows that $-\mathfrak{s}^+$ lies in the negative closed Weyl chamber $-C$ where all $\alpha_j \leqslant 0$. The Weyl group acts simply transitively on Weyl chambers by Proposition B.4.18, so only the longest word $w_0 \in W$ sends $\Delta \longrightarrow -\Delta$ and hence $C \longrightarrow -C$, and there will exist an element $w \in W$ taking $\mathfrak{s}^+ \longrightarrow -\mathfrak{s}^+$ if and only if w_0 does so. □

Chi-Kwong Fok (personal communication) pointed out this fact as well a representation-theoretic consequence.

Proposition 11.6.2. *Let G be a simple Lie group and S a circular subgroup. Then S is reflected in G if and only if the irreducible representation of G determined by S is self-dual.*

Proof. Identifying $\mathfrak{t}$ with $\mathfrak{t}^\vee$ through the Killing form, (the image of) $\mathfrak{s}^+$ meets the weight lattice of G and determines a minimal dominant weight λ; and S is reflected if and only if some Weyl group element takes λ to $-\lambda$. Again, if any Weyl element does so, then w_0 will, so S is reflected if and only if $\lambda = -w_0\lambda$. The representation dual to the irreducible representation with highest weight λ is that with highest weight $-w_0\lambda$. □

The construction of Table 11.1.5 now proceeds case by case through the Killing–Cartan classification Proposition B.4.5.

Proposition 11.6.3. *Let T be a maximal torus in a simple Lie group G whose type is one of*

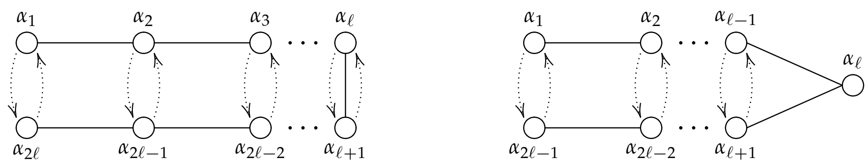
$$B_n, \quad C_n, \quad D_{2n}, \quad G_2, \quad F_4, \quad E_7, \quad E_8.$$

Then S is reflected in G.

First proof. If G is a simple Lie group of one of these types, its Weyl group W is a Coxeter group of corresponding type. For Coxeter groups of these types, the longest word w_0 acts as $-\mathrm{id}$ on the vector space (here $\mathfrak{t}$) carrying the defining representation of W [Kan01, Lem. 27-2, p. 283][Hum92, p. 82], so $\mathfrak{t}$ is reflected. □

Second proof. It is known that central involutions of a Weyl group act as $X \longmapsto -X$ on the Lie algebra $\mathfrak{t}$ of a maximal torus [DW01, Thm. 1.8] and that the center of W is isomorphic to $\mathbb{Z}/2$ for the Weyl groups of types B_n, C_n, D_{2n}, G_2, F_4, E_7, and E_8 [DW01, Rmk. 1.9]. □

Figure 11.6.5: The graph involutions of A_n



In the remaining cases, the longest word $w_0 \in W$ does not act as $-\operatorname{id}$ on $\mathfrak{t}$, so more work is required.

Proposition 11.6.4. *Let G be a simple Lie group with trivial center* (viz., *of type A_n, D_{2n+1}, or E_6) and let S be a circle in G. Then S is reflected in G if and only if there is some $w \in W$ such that $w \cdot \mathfrak{s}$ lies in the fixed point subalgebra $\mathfrak{t}^\theta$ of the Cartan subalgebra under an automorphism $\theta \in \operatorname{Aut}(\mathfrak{t})$ induced by a nontrivial diagram automorphism of the Dynkin diagram of G.*

Proof. From Proposition 11.6.1 we know S is reflected if and only if $\mathfrak{s}$ is reflected by w_0. Note that this occurs if and only if $\mathfrak{s}$ is fixed pointwise by the nontrivial automorphism $-w_0 \in \operatorname{Aut}(\mathfrak{t})$. This automorphism $-w_0$ stabilizes but does not fix the positive closed Weyl chamber C, and so cannot be induced from W, which acts simply transitively on Weyl chambers. Likewise, any extension of $-w_0$ to an automorphism of $\mathfrak{g}$ is an outer automorphism in that it is not in the image of $\operatorname{Ad} G$. But it is known [FH91, Prop. D.40, p. 498] that all outer automorphisms of $\mathfrak{g}$ are induced by a graph automorphism of the Dynkin diagram Γ of G in that

$$\operatorname{Out}\mathfrak{g} := \frac{\operatorname{Aut}\mathfrak{g}}{\operatorname{Ad} G} \cong \operatorname{Aut}\Gamma. \qquad \square$$

It thus remains only to understand the fixed point subalgebras of diagram automorphisms for Lie algebras of type A_n, D_{2n+1}, and E_6.

Proposition 11.6.6. *In a Lie algebra of type A_n, a point $v \in \mathfrak{t}^\vee < \mathbb{R}^{n+1}$ of the dual Cartan algebra is fixed by an automorphism of the Dynkin diagram if and only if some reversing the coordinates of v yields $-v$.*

Proof. The Dynkin diagram of A_n and its only automorphism θ are depicted in Figure 11.6.5; the map θ acts on simple roots of A_n by exchanging $\alpha_j \longleftrightarrow \alpha_{n-j}$, as in the figure. The fixed point subspace of $\mathfrak{t}^\vee \cong \mathbb{R}^n$ is spanned by the vectors $\alpha_j + \alpha_{n-j}$ so it contains those vectors $v' = [a_1 \ \cdots \ a_n] \in \mathbb{R}^n$ for which $a_j = a_{n-j}$. These α_j are usually identified with $e_j - e_{j+1} \in \mathbb{R}^{n+1}$, where $(e_j)_{1 \leqslant j \leqslant n+1}$ is the standard basis on $\mathbb{R}^{n+1}$.[1] This embedding $v' \longmapsto v \in \mathbb{R}^{n+1}$ takes

$$[a_1 \ a_2 \ \cdots \ a_{n-1} \ a_n] \longmapsto \big[a_1 \ \ (a_2 - a_1) \ \ \cdots \ \ (a_n - a_{n-1}) \ \ -a_n\big],$$

so the symmetry requirement $a_j = a_{n-j}$ translates into the antisymmetry condition $v_j = -v_{n+1-j}$ on the coordinates of v. $\square$

Corollary 11.6.7. *Let $K = \mathrm{SU}(n)$ and let S be a circular subgroup. Then S is reflected if and only if the exponent multiset*[2] *J of the inclusion $S \longhookrightarrow T$ satisfies $J = -J$.*

For example, $[-1\ 0\ 1] \in \mathbb{R}^3$ meets this condition and $[2\ 1\ -3]$ does not.

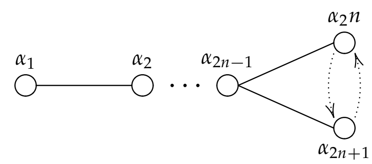
Proof. Using the W-equivariant isomorphism $\mathfrak{t}^\vee \xrightarrow{\sim} \mathfrak{t}$ discussed in Corollary B.4.16, we may identify $\mathfrak{t}^\theta$ with $(\mathfrak{t}^\vee)^\theta < \mathbb{R}^{n+1}$. The Weyl group $W_{A_n} = S_{n+1}$ acts on $\mathbb{R}^{n+1}$ by permutation of coordinates, so given $v \in \mathfrak{t}$, there exists a $w \in S_{n+1}$ such that $w \cdot v \in \mathfrak{t}^\theta$ if and only if some permutation of the entries of $-v$ yields v. The result then follows immediately from Proposition 11.6.6 and Proposition 11.6.4. $\square$

Remarks 11.6.8. (a) The root subsystems of $A_{2\ell}$ and $A_{2\ell-1}$ fixed by θ are respectively of types B_n and C_n. The former corresponds on the group level to the inclusion $\mathrm{SO}(2\ell+1) \longhookrightarrow \mathrm{SU}(2\ell+1)$ and the latter to the $\mathrm{Sp}(n)$ embedded in $\mathrm{SU}(2n)$ via the block map on coordinates induced by

[1] Thus $\mathfrak{t}^\vee$ is usually considered as the subspace of elements whose coordinates sum to zero in the Cartan algebra $\mathbb{R}^{n+1}$ for $\mathrm{U}(n+1)$.

[2] A multiset is like a sequence, except elements are *not* ordered, but *are* counted with multiplicity. Other collections often considered as multisets are roots of a polynomial and eigenvalues of an operator.

Figure 11.6.9: The graph involution of D_{2n+1}



the ring injection $H^* \rightarrowtail \mathbb{C}^{2\times 2}$. These subgroups are fixed points of involutive automorphisms of $\mathrm{SU}(n)$ yielding the classical symmetric spaces $\mathrm{SU}(n)/\mathrm{SO}(n)$ and $\mathrm{SU}(2n)/\mathrm{Sp}(n)$.

(b) The self-duality criterion Proposition 11.6.2 rears its head here as follows. The representation τ of S on $\mathbb{C}^n$ given by restricting the defining representation of $\mathrm{SU}(n)$ to S is a direct sum $\bigoplus_{j=1}^n \rho^{\otimes a_j}$ of tensor powers of the defining representation ρ of S^1 on $\mathbb{C}$. The dual representation $\tau^\vee = \bigoplus_{j=1}^n \rho^{\otimes(-a_j)}$, will be isomorphic to τ just if $J = -J$.

Proposition 11.6.10. *In a Lie algebra of type D_{2n+1}, a point $v \in \mathfrak{t}^\vee < \mathbb{R}^{2n+1}$ of the dual Cartan algebra is fixed by an automorphism of the Dynkin diagram if and only if the last coordinate of v is zero.*

Proof. The Dynkin diagram of D_{2n+1} and its lone graph automorphism θ are shown in Figure 11.6.9. This θ fixes all simple roots except α_{2n} and α_{2n+1}, which it exchanges. The fixed point subspace of $(\mathfrak{t}^\vee)^\theta$ is spanned by the vectors $\{\alpha_j\}_{j<2n}$ and by $\alpha_{2n} + \alpha_{2n+1}$. The roots α_j for $j \leqslant 2n$ are usually identified with $e_j - e_{j+1} \in \mathbb{R}^{2n+1}$, and a_{2n+1} with $e_n + e_{n+1}$, where $(e_j)_{1\leqslant j\leqslant n+1}$ is the standard basis on $\mathbb{R}^{2n+1}$. The image of the composite embedding $(\mathfrak{t}^\vee)^\theta \hookrightarrow \mathfrak{t}^\vee \to \mathbb{R}^{2n+1}$ is $\mathbb{R}^{2n} \times \{0\}$ since $\alpha_{2n} + \alpha_{2n+1} = 2e_{2n}$. $\square$

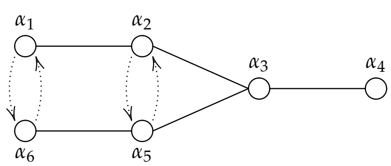
Corollary 11.6.11. *Let S be a circle in* $\mathrm{Spin}(4n+2)$. *Then S is reflected in* $\mathrm{Spin}(4n+2)$ *if and only if it is conjugate into a* $\mathrm{Spin}(4n)$ *subgroup.*

Proof. Using the W-equivariant isomorphism $\mathfrak{t}^\vee \xrightarrow{\sim} \mathfrak{t}$ discussed in Corollary B.4.16, we may identify $(\mathfrak{t}^\vee)^\theta$ with $\mathfrak{t}^\theta = \mathbb{R}^{2n} \times \{0\} < \mathbb{R}^{2n+1}$. The Weyl group $W_{D_{2n+1}} = \{\pm 1\}^{2n} \rtimes S_{2n+1}$ acts on

$\mathbb{R}^{2n+1}$ by permutating its coordinates and inverting an even number of them, so given $v \in \mathfrak{t}$, there exists a $w \in W_{D_{2n+1}}$ such that $w \cdot v \in \mathfrak{t}^{\theta}$ if and only if one coordinate is zero. The result then follows immediately from Proposition 11.6.10 and Proposition 11.6.4. $\square$

Remark 11.6.12. The sublattice of a D_{2n+1} lattice fixed by θ is of type B_{2n} and corresponds to a $\mathrm{SO}(4n)$ subgroup of $\mathrm{SO}(4n+2)$. This subgroup is the fixed point set of an involutive automorphism of $\mathrm{SO}(4n+2)$ yielding the classical symmetric space $\mathrm{SO}(4n+2)/\mathrm{SO}(4n) = V_2(\mathbb{R}^{4n+2})$.

Figure 11.6.13: The graph involution of E_6



Proposition 11.6.14. *In a Lie algebra of type E_6, a point $v \in \mathfrak{t}^\vee$ of the dual Cartan algebra is fixed by an automorphism of the Dynkin diagram if and only if it lies in a certain F_4 sublattice.*

Proof. The Dynkin diagram of E_6 and its only automorphism θ are depicted in Figure 11.6.13; the map θ acts on simple roots of E_6 as indicated in the picture, and $(\mathfrak{t}^\vee)^\theta$ is spanned by $\Delta = \{\alpha_1+\alpha_6, \alpha_2+\alpha_5, \alpha_3, \alpha_4\}$. By assumption, we have $\alpha_i \cdot \alpha_j = -2|\alpha_i||\alpha_j|$ for adjacent α_i, α_j and $\alpha_i \cdot \alpha_j = 0$ otherwise, so Δ is a simple root system of type F_4 with $a_1 + \alpha_6$ and $\alpha_2 + \alpha_5$ long and α_3 and α_4 short. $\square$

Proposition 11.6.15. *A circular subgroup S of E_6 or its universal cover $\widetilde{E}_6$ is reflected just if it is conjugate into a* $\mathrm{Spin}(8)$ *subgroup.*

Proof. Using the W-equivariant isomorphism $\mathfrak{t}^\vee \xrightarrow{\sim} \mathfrak{t}$ discussed in Corollary B.4.16, we may identify $(\mathfrak{t}^\vee)^\theta$ with $\mathfrak{t}^\theta$. It follows immediately from Proposition 11.6.14 and Proposition 11.6.4 that the tangent spaces $\mathfrak{s} < \mathfrak{t}$ to reflected circles S are precisely those such that there exists $w \in W_{E_6}$ such that $w \cdot \mathfrak{s} < \mathfrak{t}^\theta$. Now because $(\mathfrak{t}^\vee)^\theta$ is spanned by an F_4 sublattice of the E_6 root lattice, its dual $\mathfrak{t}^\theta$ is tangent to the maximal torus T^4 of an F_4 subgroup. In the classic series of inclusions $\mathrm{Spin}(8) < F_4 < E_6$, the first two share a maximal torus T^4, so $\mathfrak{t}^\theta$ is actually tangent to the maximal torus of a $\mathrm{Spin}(8)$. □

Remark 11.6.16. We had a much more intricate proof of the classification in Table 11.1.5 before it was pointed out to us by Chi-Kwong Fok (personal communication) and Jay Taylor [Tay] that the longest word of the Weyl group yields a simpler (no pun intended) classification. This original proof is reproduced for comparison in Appendix E.

Included in particular are detailed original proofs that there are precisely *forty-five* such T^4 in a given maximal torus T^6 of E_6. containing all and only reflected circles. A reader who finds our current justification that $(\mathfrak{t}^\vee)^\theta$ corresponds to an F_4 subgroup of E_6 sketchy will find there a much more detailed explanation why this must be the case and why in fact $\mathfrak{t}^\theta$ must be are tangent to maximal tori of $\mathrm{Spin}(8)$ subgroups. We will also show there that each of the associated $\mathfrak{t}^4$ are spanned by four mutually orthogonal coroots in the E_6 coroot lattice. As a consequence, we accidentally demonstrate the well-known fact that W_{F_4} injects into W_{E_6} with index forty-five. Finally, we will provide an explicit description of the 135 mutually orthogonal bases of roots of $(\mathfrak{t}^6)^\vee$ which span the forty-five reflected subspaces $(\mathfrak{t}^4)^\vee$.

Appendix A

Algebraic background

In this appendix we gather a ragtag assortment of algebraic preliminaries. Notationally, in all that follows we denote containment of an algebraic substructure by "$\leqslant$," containment of an ideal by "$\trianglelefteq$," isomorphism by "$\cong$," and bijection by "$\leftrightarrow$." The restriction of a map $f\colon A \longrightarrow B$ to a subset $U \subseteq A$ is written $f \restriction U$.

A.1. Commutative algebra

We will only need a very little commutative algebra, but it will include the following. Let A be an ungraded commutative ring and B a unital A-algebra. We say an element $a \in A$ ***annihilates*** B if $a1 = 0$ in B. Given any $a \in A$ we may ***localize***, or ***invert*** a in A, to form a new A-algebra

$$A_a := A[a^{-1}] \cong A[x]/(ax - 1).$$

The class of x in A_a, called a^{-1}, serves as an inverse to a in $A[a^{-1}]$; if $A[a^{-1}] = 0$ is the zero ring, this still holds for uninteresting reasons.

We define localization of an A-module M by

$$M_a := M[a^{-1}] := A_a \underset{A}{\otimes} M.$$

Localization is a ***exact functor*** in that it takes an exact sequence of A-modules to an exact sequence of $A[a^{-1}]$-modules.

The localization $M[a^{-1}]$ turns out to be trivial if and only if each element $m \in M$ is annihilated by some power a^n of M ([AM69, Ex. 3.1, p. 43]); the idea is that then $m = x^n a^n m = x^n 0 = 0$ in $M[a^{-1}]$. We say that M is ***a-torsion*** in this instance. For a unital A-algebra B to be a-torsion, it is necessary and sufficient that a power of a annihilate the unity $1 \in B$, and then $B[a^{-1}] = 0$. We have the following useful lemma.

Lemma A.1.1. *Let A be a ring, a an element of A, and*

$$M \to N \to P$$

an exact sequence of A-module homomorphisms. Then if M and P are a-torsion, so also is N.

Proof. Since localization is an exact functor, the localized sequence

$$\underbrace{M[a^{-1}]}_{0} \longrightarrow N[a^{-1}] \longrightarrow \underbrace{P[a^{-1}]}_{0},$$

is also exact, meaning $N[a^{-1}]$ is zero. □

A.2. Chain complexes

Our cohomology theories will always take coefficients in an ungraded, commutative ring k with unity; usually, k will be $\mathbb{Q}$ or $\mathbb{R}$. The category of k-modules and k-module homomorphisms is denoted k-Mod. A ***differential group*** is a pair (A, d), where $A \in k$-Mod is a k-module and $d \in \operatorname{End}_k A$, the ***differential***, is a nilsquare endomorphism, so that the composition $d^2 := d \circ d = 0$ is the constant map to the zero element of A. A morphism $f\colon (A, d) \to (B, \delta)$ in the category of differential groups is a ***chain map***, a group homomorphism $f\colon A \to B$ such that $fd = \delta f$.

A $\mathbb{Z}$-***graded*** k-***module*** is an $A \in k$-Mod admitting a direct sum decomposition $A = \bigoplus_{n\in\mathbb{Z}} A_n$. An element $a \in A$ is ***homogeneous*** if there exists some integer $|a| = \deg a$, the ***degree*** of A, such that $a \in A_{\deg a}$. We blur the distinction between $0 \in A_n$ and $0 \in A$, and leave the degree of the latter indeterminate. A k-module homomorphism $f\colon A \longrightarrow B$ between *graded* k-modules is said to be a ***graded*** k-***module homomorphism*** of ***degree*** $n = \deg f$ if

$$\deg f(a) = n + \deg a = \deg f + \deg a$$

for all homogeneous $a \in A$. We let gr-k-Mod be the category of graded k-modules and graded k-module homomorphisms.

A ***chain complex*** (A, d) is a differential group such that $A \in$ gr-k-Mod and additionally d is of *degree* 1. We write $d \upharpoonright A_n =: d_n$. A map $f\colon (A, d) \to (B, \delta)$ of chain complexes is a chain map of differential groups that is additionally a graded map of *degree* 0, so that $fA_n \leqslant B_n$. We let k-Ch denote the category of chain complexes and chain maps of k-modules,

The ***cohomology*** $H(A, d)$ of a differential k-module (A, d) is the quotient $(\ker d)/(\operatorname{im} d)$, which makes sense because $d^2 = 0$. We also write this as $H_d(A)$. The differential group is ***exact*** if $H_d(A) = 0$. A chain map $f\colon (A, d) \to (B, \delta)$ induces a homomorphism $f^*\colon H(A, d) \to H(B, \delta)$ of k-modules. If this map is an isomorphism, then one says f is a ***quasi-isomorphism***.

If A is a chain complex, then $H(A, d)$ is graded by

$$H^n(A, d) := H^*(A, d)_n := \ker d_n / \operatorname{im} d_{n-1}.$$

Then a (graded) chain map induces a map of graded modules, so cohomology is a functor $k\text{-}\mathsf{Ch} \longrightarrow \mathsf{gr}\text{-}k\text{-}\mathsf{Mod}$. A chain complex (A, d) is said to be ***acyclic*** if $H^*(A, d) = H^0(A, d) = k$, meaning $H^n(A, d) = 0$ for $n \neq 0$.

We will say a map $A \longrightarrow B$ of differential groups ***surjects in cohomology*** or is ***H^*-surjective*** if it induces a surjection $H^*(A) \twoheadrightarrow H^*(B)$. In the opposite extreme case, that the map $H^*(A) \longrightarrow H^*(B)$ is zero in dimensions $\geqslant 1$ and is the isomorphism $H^0(A) \longrightarrow H^0(B)$ in dimension 0, we call this map ***trivial***, and say the map $X \longrightarrow Y$ is ***trivial in cohomology***. If $A \longrightarrow B$ is the map $f^*\colon H^*(Y) \longrightarrow H^*(X)$ in cohomology induced by a continuous map $f\colon X \longrightarrow Y$, then we likewise say f is surjective in cohomology or trivial in cohomology if f^* is.

A.2.1. Polynomials and numbers associated to a graded module

A graded k-module A is said to be of ***finite type*** if each graded component A_n has finite k-rank. Given a graded k-module A of finite type, we define the ***Poincaré polynomial*** of A to be the formal rational function

$$p(A) := \sum_{n \in \mathbb{Z}} (\operatorname{rk}_k A_n) t^n.$$

The sum $p(X)|_{t=1} = \sum \operatorname{rk}_k A_n$ is the ***total rank***. If we evaluate at $t = -1$ instead, we get the ***Euler characteristic*** $\chi(A) := p(X)|_{t=-1} = \sum (-1)^n \operatorname{rk}_k A_n$; this last is only well defined if the total Betti number is finite.

Given a chain complex, Euler characteristic is preserved under cohomology: one has the following corollary of the rank–nullity theorem of introductory linear algebra as applied to the differential d.

Proposition A.2.1. *Let (A, d) be a chain complex over k of finite total Betti number. Then*

$$\chi(A) = \chi\big(H^*(A, d)\big).$$

A.3. Commutative and differential graded algebras

A cohomology ring is a commutative graded algebra, and it is defined as the cohomology of a chain complex which is itself a graded algebra. We set out some commonplaces of these objects.

A.3.1. Commutative graded algebras

A cohomology ring A will be a ***graded commutative k-algebra***. This means A is a graded k-module, and additionally the product is such that

$$A_m \cdot A_n \leqslant A_{m+n};$$

and for all homogeneous elements $a, b \in A$, one has

$$ba = (-1)^{|a||b|}ab.$$

Mostly, these rings will actually be ***$\mathbb{N}$-graded***, so that $A_n = 0$ for $n < 0$, and the absolute cohomology rings $H^*(X)$ (as opposed to relative cohomology rings $H^*(X, Y)$) will be unital, so that the map $x \mapsto x \cdot 1$ embeds $k \rightarrowtail A_0 \hookrightarrow A$ and the k-algebra structure can be seen as the restriction of the ring multiplication $A \times A \longrightarrow A$. We will call these ***$k$-CGAs*** for short, and the category of graded commutative k-algebras and degree-preserving k-algebra homomorphisms will be written k-CGA.

The cohomology theories of interest to us will also be algebras over CGAs E^*, or ***E^*-algebras***. For us an E^*-algebra structure on a CGA A is a k-bilinear map $E^* \times A \longrightarrow A$ that "adds degrees" in that

$$E^*_m \cdot A_n \leqslant A_{m+n}.$$

For absolute (as opposed to relative) cohomology theories, as we shall deal in almost exclusively, these maps will arise from the multiplication of A via precomposition with k-algebra homomorphisms $E^* \longrightarrow A$, just as the action of k often is. The associated morphisms in the category E^*-CGA are those that preserve the multiplication; for absolute cohomologies, such a map $f\colon A \longrightarrow B$ is one these are precisely those where the composition $E^* \to A \to B$ equals the structure map $E^* \to B$.

The product in k-CGA is the ring product $A \times B$, graded by $(A \times B)_n = A_n \times B_n$. The coproduct is the ***graded tensor product***: this is $A \otimes_k B$ as a group, with the grading

$$(A \underset{k}{\otimes} B)_n = \bigoplus_{\ell+m=n} A_\ell \underset{k}{\otimes} B_m$$

and the commutation rule $(1 \otimes b)(a \otimes 1) = (-1)^{|a||b|} a \otimes b$ for $a \in A_{|a|}$ and $b \in B_{|b|}$. As often as feasible, we suppress ring subscripts on tensor signs, and in elements, we omit the tensor signs themselves, letting $a \otimes b =: ab$, so that for example we recover the reassuring expression $ba = (-1)^{|a||b|} ab$.

Given a graded unital k-algebra A with a preferred basis (a_j) of $A_0 \neq 0$, the map $A_0 \xrightarrow{\sim} k\{a_j\} \xrightarrow{+} k$ given by $\sum \gamma_j a_j \longmapsto \sum \gamma_j$ induces a natural ring homomorphism $A \twoheadrightarrow A_0 \longrightarrow k$ called the ***augmentation***. Its kernel $\widetilde{A}$ is called the ***augmentation ideal***; the notation is in analogy with reduced cohomology.[1] If $A_0 \cong k$, we say A is ***connected***; the terminology is because the singular cohomology of a connected space satisfies this condition. In this case, the augmentation ideal is $\bigoplus_{n \geqslant 1} A_n$.

Given homomorphism $f\colon A \longrightarrow B$ of graded k-algebras, write

$$B /\!\!/ A := B/\big(f(\widetilde{A})\big).$$

[1] Industry standard seems to be $\overline{A}$, but I prefer to reserve this notation for quotients.

This is the right conception of cokernel for maps between cohomology rings: one wants the 0-graded component to stay the same and the rest of the image of f to vanish. If A is a graded subalgebra of B, then one wants to think of

$$0 \to A \to B \to A /\!\!/ B \to 0$$

as a "short exact sequence" of rings, but of course this doesn't make sense: the sequence $A \to B \to C$ of k-modules is exact at B if $\operatorname{im}(A \to B) = \ker(B \to C)$, but the image of a ring map is a ring, while the image is an ideal, a different type of object. The compromise solution is the following definition.

Definition A.3.1. A sequence $A \to B \to C$ of ring maps is said to be ***coexact*** at B if

$$\ker(B \to C) = \big(\operatorname{im}(\widetilde{A} \to \widetilde{B})\big).$$

Example A.3.2. Let A be a graded k-subalgebra of a graded k-algebra B. Then $0 \to A \to B \to A /\!\!/ B \to 0$ is a short coexact sequence, by design. If A and C are k-algebras, free as k-modules (in the applications we care most about, $k = \mathbb{Q}$), then taking $B = A \otimes C$, we see the sequence

$$0 \to A \longrightarrow A \otimes C \longrightarrow C \to 0$$

is short coexact.

Remark A.3.3. The reasoning for the somewhat intimidating term *coexact* is thus [MS68, p. 762]: the sequence $A \to B \to C$ of k-modules is exact if

$$\operatorname{im}(A \to B) = \ker(B \to C).$$

Recall that the ***coimage*** of a map $f\colon A \longrightarrow B$ is the quotient $A/\ker f$, the object that the first isomorphism theorem tells us is naturally isomorphic to the image of f (which is probably why people usually don't bother defining coimage). Let $A \to B \to C$ be a sequence of graded k-algebras. Dualizing the definition of exactness gives the condition

$$\operatorname{coker}(A \to B) = \operatorname{coim}(B \to C), \tag{A.1}$$

which makes sense because both sides are rings (and which would be equivalent to the original exactness condition if these were modules instead). Now

$$\operatorname{coker}(A \to B) = B/\big(\operatorname{im}(\widetilde{A} \to \widetilde{B})\big),$$
$$\operatorname{coim}(B \to C) = B/\ker(B \to C),$$

so that

$$\operatorname{coker}(A \to B) = \operatorname{coim}(B \to C) \qquad \Longleftrightarrow \qquad \ker(B \to C) = \big(\operatorname{im}(\widetilde{A} \to \widetilde{B})\big),$$

the left-hand equation being the dualized exactness condition (A.1) and the right-hand the condition we dubbed coexactness in Definition A.3.1.

A.3.2. Bigraded algebras

Some k-algebras A we will encounter will have a ***bigrading***:

$$A = A^{\bullet,\bullet} = \bigoplus_{p,q \in \mathbb{Z}} A^{p,q}$$

in such a way that the ***bidegrees*** (p,q) add under multiplication:

$$A^{i,j} \cdot A^{p,q} \leqslant A^{i+p,j+q}.$$

We conventionally visualize such a ring as a grid in the xy-plane, with the p^{th} ***column***

$$A^{p,\bullet} = \bigoplus_q A^{p,q}$$

residing in the strip $p \leqslant x \leqslant p+1$ and the q^{th} ***row***

$$A^{\bullet,q} = \bigoplus_p A^{p,q}$$

residing in the strip $q \leqslant y \leqslant q+1$. For us, such gradings will always reside in the first quadrant: $(p,q) \in \mathbb{N} \times \mathbb{N}$.

A linear map $f\colon A \to B$ of bigraded algebras is said to have bidegree $\operatorname{bideg}(f) = (p,q)$ if $f(A^{i,j}) \leqslant B^{i+p,j+q}$. A bigraded ring will be said to ***commutative*** if the associated singly-graded k-algebra $A^\bullet = \bigoplus_n A^n$, graded by $A^n := \bigoplus_{p+q=n} A^{p,q}$ and called the ***total complex***, is a CGA. A ***differential bigraded algebra*** (A,d) is a bigraded algebra such that d is an antideriviation on the associated singly-graded algebra $A^\bullet$ of degree 1. We make no further demands initially as to how d interacts with the bigrading, but note that since $dA^n \leqslant A^{n+1}$, one has for each bidegree (i,j) that $dA^{i,j} \leqslant \bigoplus_\ell A^{i+\ell,j+1-\ell}$, and composing with projections to $A^{i+\ell,j+1-\ell}$, one obtains ***component*** maps $d^\ell\colon A^{i,j} \longrightarrow A^{i+\ell,j+1-\ell}$ of bidegree $(\ell, 1-\ell)$ such that

$$d = \sum_{\ell \in \mathbb{Z}} d^\ell.$$

Free graded algebras

Suppose that $\operatorname{char} k \neq 2$. As with modules, there are free objects in the category of k-CGAs, which have the following description. Given a free graded k-module V if we separate it into even- and odd-degree factors V_{even} and V_{odd}, then the ***free graded commutative k-algebra*** on V is the graded tensor product

$$SV_{\text{even}} \otimes_k \Lambda V_{\text{odd}}$$

of the symmetric algebra SV_{even} on the even-degree generators and the exterior algebra ΛV_{odd} on the odd-degree generators. Given k-bases $\vec{t} = (t_1, \ldots, t_m)$ of V_{even} and $\vec{z} = (z_1, \ldots, z_n)$ of V_{odd}, we also write these as

$$S[\vec{t}] := SV_{\text{even}};$$

$$\Lambda[\vec{z}] := \Lambda V_{\text{odd}}.$$

Write

$$\Delta[z_m] := k\{1, z_m\},$$

for the unique rank–two unital k-algebra with elements of degrees zero and m, which is the cohomology of an m-sphere. This is $\Lambda[z_m]$ for m odd and $S[z_m]/(z_m^2)$ for m even.

In the event $\operatorname{char} k = 2$, the graded commutativity relation $xy = (-1)^{|x|\,|y|}yx$, or equivalently $xy \pm yx = 0$, forces genuine commutativity $xy = yx$ for all elements since $1 = -1$ in k. Thus a free k-CGA is a symmetric algebra SV in characteristic 2, independent of the grading on V. Algebras which merely *resemble* ΛV still play an important role in characteristic two.

Definition A.3.4. Let k be a commutative ring. A k-algebra A (*not* assumed graded commutative), free as a k-module, is said to have a ***simple system of generators*** $v_1, \ldots, v_n, \ldots$ if a k-basis for A

is given by the monomials

$$v_{j_1} \cdots v_{j_\ell}, \qquad j_1 < \cdots < j_\ell,$$

where each generator occurs at most once. If A has a simple system of generators, we write

$$A = \Delta[v_1, \ldots, v_n, \ldots]$$

despite the fact that this description does not specify A up to algebra isomorphism.

Example A.3.5. The exterior algebra $\Lambda[z_1, \ldots, z_n]$ admits $z_1, \ldots, z_n$ as a simple system of generators.

This is of course the motivating case. The multiplication need not be anticommutative, as one can see from the following example.

Example A.3.6. Borel [Bor54, Théorème 16.4] found that the mod 2 homology ring of Spin(10) is given by

$$H_*(\mathrm{Spin}(10); \mathbb{F}_2) = \Delta[v_3, v_5, v_6, v_7, v_9, v_{15}],$$

where all $v_j^2 = 0$ and all pairs of v_j commute except for (v_6, v_9), which instead satisfy

$$v_6 v_9 = v_9 v_6 + v_{15}.$$

For a last example consider polynomial rings.

Example A.3.7. The polynomial ring $k[x]$ admits $x, x^2, x^4, x^8, \ldots$ as a simple system of generators, as consequence of the binary representation for natural numbers.

Poincaré polynomials for free algebras

In most cases we care about, the Poincaré polynomial will applied to a nonnegatively-graded k-CGA of finite type. The Poincaré polynomial is a homomorphism $\mathsf{gr}\text{-}k\text{-}\mathsf{Mod} \longrightarrow k[t]$ in the sense that

$$p(A \times B) = p(A) + p(B), \qquad p(A \otimes B) = p(A) \cdot p(B).$$

Usually the CGA in question will be the cohomology ring $H^*(X;k)$ of a space, and we will write

$$p(X) := p\big(H^*(X;k)\big) = \sum_{n \in \mathbb{N}} \operatorname{rk}_k H^n(X;k) t^n.$$

The individual ranks $h^k(X) := \operatorname{rk}_k H^k(X;k)$ are called the ***Betti numbers*** of X; the associated total rank $p(X)|_{t=1} = \sum h^n(X)$ is called the ***total Betti number*** of the space and denoted $h^\bullet(X)$. The Euler characteristic $p(X)|_{t=-1} = \sum (-1)^n h^n(X)$ of $H^*(X;k)$ is called the Euler characteristic *of the space*, and written $\chi(X)$. If we write $h^{\text{even}}(X) = \sum h^{2n}(X)$ and $h^{\text{odd}}(X) = \sum h^{2n+1}(X)$, then

$$\begin{aligned} h^\bullet(X) + \chi(X) &= 2 \cdot h^{\text{even}}(X); \\ h^\bullet(X) - \chi(X) &= 2 \cdot h^{\text{odd}}(X) \end{aligned}$$

Free CGAs behave pleasantly under Poincaré polynomial because $p(-)$ is multiplicative. If $\deg x = n$ is odd, then $p\big(\Lambda[x]\big) = 1 + t^n$. Thus given an exterior algebra ΛV on an oddly-graded free k-module V of finite type, with Poincaré polynomial $p(V) = \sum t^{n_j}$ (it is okay if certain n_j occur more than once), the tensor rule yields

$$p(\Lambda V) = \prod (1 + t^{n_j}).$$

Likewise, if $\deg x = n$ is even, then $S[x] = k[x]$ is spanned by $1, x, x^2, \ldots$, so

$$p\big(S[x]\big) = \sum_{j\in\mathbb{N}} t^{jn} = \frac{1}{1-t^n}.$$

Given an exterior algebra SV on an evenly-graded free k-module V of finite type, $p(V) = \sum t^{n_j}$, then, the tensor rule yields

$$p(SV) = \prod \frac{1}{1-t^{n_j}}. \tag{A.2}$$

Proposition A.3.8. *Let k be a field, V be a positively-graded k–vector space, SV the symmetric algebra, and W a graded vector subspace of SV such that the subalgebra A it generates is a free CGA and SV is a free A-module. Then*

$$p(SV /\!\!/ A) = \frac{p(SV)}{p(SW)}.$$

Proof. Let (q_α) be a homogeneous A-basis for SV. Then $(q_\alpha \otimes 1)$ forms a graded basis for $SV /\!\!/ A = SV \otimes_{\tilde{A}} k$, so on the level of graded k-modules, one has $SV \cong A \otimes_k k\{q_\alpha \otimes 1\} \cong A \otimes (SV /\!\!/ A)$. Taking Poincaré polynomials and dividing through by $p(SV /\!\!/ A)$ gives the result. □

This sort of quotient will become relevant to us in Section 8.4, where it will be found that an important subring of the cohomology ring $H^*(G/K;\mathbb{Q})$, of a compact homogeneous space, namely the image of the characteristic map $\chi^*\colon H^*(BK;\mathbb{Q}) \longrightarrow H^*(G/K;\mathbb{Q})$, is frequently of this form.

Indecomposables

The ***indecomposable*** elements of A are, informally, those of positive degree that cannot be written as sums of products of lower-degree elements; the idea is to find an analogy for irreducible polynomials for rings with more complex ideal structure. Recalling that The most convenient definition turns out to be this: the ***module of indecomposables*** is the k-module

$$Q(A) := \widetilde{A}/\widetilde{A}\widetilde{A} \cong \widetilde{A} \underset{A}{\otimes} k$$

where $\widetilde{A}$ is the augmentation ideal and the denominator denoted $\widetilde{A}\widetilde{A}$ is understood to be the module spanned by products ab for $a, b \in \widetilde{A}$ of positive-degree elements. Under this definition we see Q is functorial, since a graded homomorphism $A \longrightarrow B$ takes $\widetilde{A} \longrightarrow \widetilde{B}$ and hence $\widetilde{A}\widetilde{A} \longrightarrow \widetilde{B}\widetilde{B}$.

If A is a free k-module, then so is $Q(A)$, so the k-module surjection $\widetilde{A} \longrightarrow\!\!\!\!\rightarrow Q(A)$ splits, and we can consider $Q(A)$ (in a badly noncanonical way) as a k-submodule of algebra generators for A. Because it satisfies a product rule, an (anti)derivation d on A, like a ring homomorphism, is uniquely determined by its values on such a lifted $Q(A)$, so a linear map on $Q(A)$ determines at most one antiderivation of A. .

A.3.3. Differential graded algebras

A chain complex (A, d) concentrated in nonnegative degree such that A is also a graded ring satisfying the product rule

$$d(ab) = da \cdot b + (-1)^{|a|} a \cdot db$$

for homogeneous elements a, b is a ***differential graded algebra*** (or k-*DGA*). A differential d on a graded ring satisfying this condition is called an ***antiderivation***. A ***derivation***, on the other hand, satisfies

$$d(ab) = da \cdot b + a \cdot db.$$

An (anti)derivation on a unital k-algebra satisfies $d1 = 0$ and hence $d(k \cdot 1) = 0$. A morphism of DGAs is a k-algebra map that is simultaneously a chain map. If A was a k-CGA, then we say (A, d) is a ***commutative differential graded algebra*** (henceforth k-*CDGA*).

The kernel $Z = \ker d$ of an (anti)derivation d is a subalgebra, because d is additive and because if $da = db = 0$, then $d(ab) = (da)b \pm a(db) = 0$. The image $B = \operatorname{im} d$ is an ideal of $Z = \ker d$, because if $b = da \in B$ and $c \in Z$, then $b \in Z$ and $d(ac) = (da)c + a(dc) = bc$.

The product in the category of DGAs is the graded ring direct product $A \times B$, equipped with the differential $d(a, b) := (da, db)$. The coproduct is the same tensor product $A \underset{k}{\otimes} B$ as for CGAs, equipped with the unique antiderivation given by

$$d(a \otimes b) = da \otimes b + (-1)^{|a|} a \otimes db$$

on pure tensors. If we omit the tensor signs, this gives back, formally, the same product rule.

A.3.4. The algebraic Künneth theorem

It is trivial that a product of DGAs remains a product on taking cohomology. In an ideal world, the same would remain true of coproducts, and this ideal world is achieved in the event one of the DGAs lacks torsion.

Theorem A.3.9. *Let k be a principal ideal domain and suppose A and C are graded differential k-modules, each free as a graded k-module. Then*

$$H^n(A \underset{k}{\otimes} C) \cong \bigoplus_{0 \leqslant j \leqslant n} \big(H^j(A) \otimes H^{n-j}(C)\big) \oplus \bigoplus_{0 \leqslant j \leqslant n} \operatorname{Tor}_1^k\big(H^{j+1}(A), H^{n-j}(C)\big).$$

Proof. Write $Z_n = \ker(d_n \colon A_n \longrightarrow A_{n+1})$ and $B_n = \operatorname{im}(d_{n-1} \colon A_{n-1} \longrightarrow A_n)$. Then one has a short exact sequence

$$0 \to Z \longrightarrow A \longrightarrow B_{\bullet+1} \to 0$$

of complexes where the differentials on Z and $B_{\bullet+1}$ are 0. Since we have assumed C is flat, on tensoring these complexes with C, we obtain a short exact sequence

$$0 \to Z \otimes C \longrightarrow A \otimes C \longrightarrow B_{\bullet+1} \otimes C \to 0$$

of complexes, where the differentials on $Z_\bullet \otimes C$ and $B_{\bullet+1} \otimes C$ are both $\mathrm{id}_A \otimes d_C$ and the differential on $A_\bullet \otimes C$ is the expected $d_A \otimes \mathrm{id}_C \pm \mathrm{id}_A \otimes d_C$. Write $i_\bullet \colon B_\bullet \longrightarrow Z_\bullet$ for the inclusion; then it is not hard to see the the connecting map in the long exact sequence in cohomology is the map $(i \otimes \mathrm{id}_C)^* \colon B_\bullet \otimes H^*(C) \longrightarrow Z_\bullet \otimes H^*(C)$ induced by $i \otimes \mathrm{id}_C$. Thus we get a short exact sequence

$$0 \to \operatorname{coker}(i \otimes \mathrm{id}_C)^* \longrightarrow H^*(A \otimes C) \longrightarrow \ker(i_{\bullet+1} \otimes \mathrm{id}_C)^* \to 0.$$

Because $0 \to B_{\bullet+1} \longrightarrow Z_{\bullet+1} \longrightarrow H^{\bullet+1}(A) \to 0$ is exact, the first term is $H^*(A) \otimes_k H^*(B)$ and the last is $\operatorname{Tor}_1^k\big(H^{*+1}(A), H^*(C)\big)$. Resorting summands to gather equal total degrees yields the statement of the theorem. □

In particular, one has the following.

Corollary A.3.10. *Let A and C be k-DGAs free as k-modules and such that $H^*(C)$ is flat over k. Then*

$$H^*(A \underset{k}{\otimes} C) \cong H^*(A) \underset{k}{\otimes} H^*(C)$$

as k-algebras.

Proof. The hypotheses precisely ensure the Tor_1^k term vanishes. □

Note that it more than suffices k be a field.

A.3.5. Poincaré duality algebras

The real cohomology ring of a compact manifold exhibits an important phenomenon which we generalize to an arbitary CGA.

Definition A.3.11. Let A be a k-CGA, free as a k-module. Suppose there exists a maximum $n \in \mathbb{N}$ such that $A_n \neq 0$, that $A_n \cong k$, and that for all $j \in [0, n]$ the natural pairing

$$A_j \times A_{n-j} \longrightarrow A_n$$

obtained by restricting the multiplication of A is nondegenerate. Then we call A a ***Poincaré duality algebra*** (or ***PDA***) and a nonzero element of A_n a ***fundamental class*** for A, which we write as $[A]$. If we fix a homogeneous basis (v_j) of A, we can define a linear map $a \longmapsto a^*$ on A by setting $v_j^* := v_{n-j}$ whenever $v_j v_{n-j} = [A]$ and extending linearly. Such a linear map is called a ***duality map***.

Theorem A.3.12 (Poincaré; [BT82, I.(5.4), p. 44])**.** *If M is a compact manifold, the real singular cohomology ring $H^*(M; \mathbb{R})$ is a PDA.*

Example A.3.13. Let V be a finitely generated, oddly-graded free k-module. Then the exterior algebra ΛV is a Poincaré duality algebra with fundamental class given by the product of a basis of V.

Poincaré duality is a severe restriction on the structure of a ring, with powerful consequences, and it is inherited by tensor-factors.

Proposition A.3.14. *Let A and B be k-CGAs, free as k-modules, and suppose B is a PDA. Then $A \otimes B$ exhibits Poincaré duality just if A does.*

Sketch of proof. If A and B are PDAs with duals given by $a \longmapsto a^*$ and $b \longmapsto b^*$, then $a \otimes b \longmapsto a^* \otimes b^*$ is easily seen to be a duality on $A \otimes B$ up to sign. If, on the other hand, $b \longmapsto b^*$ is a duality on A and $a \otimes b \longmapsto \overline{a \otimes b}$ is a duality on $A \otimes B$, then for any homogeneous $a \in A$ one has $\overline{a \otimes 1} = a^* \otimes [B]$ for some $a^* \in A$, and $a \longmapsto a^*$ is a duality on A. □

A.4. Splittings and formality

An epimorphism $A \longrightarrow\!\!\!\!\!\rightarrow B$ is said to ***split*** if there exists a monomorphism $B \rightarrowtail A$, called a ***section***, such that the composition $B \to A \to B$ is the identity on B. This section is virtually never canonical, but it is frequently useful to be able to lift the structure of B back into A, in however haphazard a manner.

Surjective homomorphisms onto free objects always split in categories whose objects carry a group structure (we always assume the axiom of choice), and we use this simple fact repeatedly.

Proposition A.4.1. *Let $\pi\colon A \twoheadrightarrow F$ be a surjection in* gr-k-Mod *and suppose F is free. Then π splits.*

Proof. Let S be a k-basis for F and for each $s \in S$ pick a preimage $a_s \in \pi^{-1}\{s\}$. This assignment extends to the needed section. □

Restricting to the case everything lies in one graded component, one obtains the result in k-Mod. Specializing to the category S^1-Mod of modules over $S^1 \cong \mathbb{R}/\mathbb{Z}$ (which are projective limits of tori, hence called *pro-tori*), one obtains the following useful statement.

Proposition A.4.2. *Any exact sequence $0 \to A \longrightarrow B \longrightarrow C \to 0$ of tori splits: we can write $B \cong A \oplus \sigma(C)$ as an internal direct sum of topological groups for some suitable section $\sigma\colon C \rightarrowtail B$ of the projection to C.*

Alternate proof. Any short exact sequence of free abelian groups splits, and the functors

$$A \longmapsto A \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z},$$
$$\pi_1(T,1) \longleftarrow T$$

furnish an equivalence of categories between finitely generated free abelian groups and tori. □

We will also need to apply this principle to CGAs.

Proposition A.4.3. *Let F be a free k-CGA and $\pi\colon A \longrightarrow\!\!\!\!\rightarrow F$ a surjective k-CGA homomorphism. Then there exists a section $i\colon F \rightarrowtail A$ of π.*

Proof. Suppose F is free on the graded k-module V. Since V is free as a graded module, there exists a section $i\colon V \rightarrowtail A$ of π over V by Proposition A.4.1. As π is a ring homomorphism, the subalgebra A' generated in A by iV projects back onto F under π. Were A' not itself a free k-CGA, there would be some relation between homogeneous elements of A' other than those ensured by the CGA axioms, and it would not be possible for $\pi|_{A'}$ to be surjective, so there is no such relation. Thus $\pi|_{A'}$ is a CGA isomorphism; now extend i to be its inverse. □

When we deal with principal bundles, the following simple proposition will be useful.

Proposition A.4.4. *Let $0 \to A \longrightarrow B \longrightarrow F \to 0$ be a coexact sequence of k-CGA maps with F free and B of finite type. Suppose further that for each degree n we have $\mathrm{rk}_k B_n = \mathrm{rk}_k(A \underset{k}{\otimes} F)_n$. Then $B \cong A \otimes F$.*

Proof. The projection $B \longrightarrow F$ splits by Proposition A.4.3, and together with $\widetilde{B}$, the lift of $\widetilde{F}$

generates A as an algebra, so there is a commutative diagram

$$\begin{tikzcd}
 & A \otimes F \arrow[dd] \arrow[dr, shift right] & \\
A \arrow[ur] \arrow[dr] & & F, \arrow[ul, shift right] \arrow[dl, dotted, shift left] \\
 & B \arrow[ur, shift left] &
\end{tikzcd}$$

of ring maps with the vertical map surjective. If this vertical map failed to also be injective, the rank assumption would fail, so it is an isomorphism. □

Here is an example application of such a splitting.

Example A.4.5. We will show in Section 7.2 that the cohomology ring $H^*(G;\mathbb{R})$ of a compact, connected Lie group is an exterior algebra. If we compute this cohomology using the de Rham complex $\Omega(G)$, an $\mathbb{R}$-CDGA, and write $Z^*(G) = \ker d$ for the ring of closed forms, then $Z^*(G)$ is an $\mathbb{R}$-CGA, so the natural projection $Z \longrightarrow\!\!\!\!\!\rightarrow H^*(G;\mathbb{R})$ admits a section, and we can embed

$$H^*(G) \rightarrowtail Z^*(G) \hookrightarrow \Omega(G)$$

as a subalgebra.

One can of course always find representative forms, but the ability to select them to form a subring on the nose, rather than up to homotopy, is rather special, and this circumstance is of sufficient utility that we here extract a formal statement of this behavior for later use (no pun intended).

Definition A.4.6. A differential graded k-algebra (A, d) is said to be ***formal*** if there exists a zig-zag of k-DGA quasi-isomorphisms

$$\begin{tikzcd}[column sep=small] & (B_1,d_1) \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & (A,d) \arrow[dl] \\ \big(H^*(A),0\big) & & (B_2,d_2) & & (B_n,d_n) & \end{tikzcd}$$

connecting $\big(H^*(A),0\big)$ and (A,d). A path-connected topological space X is said to be ***formal*** if there exists a formal $\mathbb{Q}$-DGA with cohomology $H^*(X;\mathbb{Q})$.

Example A.4.7. We will find in Section 7.6 that for a compact, connected Lie group G, the cohomology of its classifying space BG (see Appendix B.1.3 for the definition and Chapter 3 for the construction) is a symmetric algebra, hence a free CGA. There exists a $\mathbb{Q}$-CDGA computing rational singular cohomology, called A_{PL} [FHT01, Ch. 10], and it follows there exists a $\mathbb{Q}$-CDGA map $H^*(BG) \rightarrowtail A_{\mathrm{PL}}(BG)$.

Now seems as good a place as any to define generalized symmetric spaces, which we will invoke occasionally later.

Definition A.4.8. Let G be a connected Lie group and $\theta \in \operatorname{Aut} G$ a smooth automorphism of finite order. Then the fixed point set G^θ is a closed subgroup of G. Let K be a subgroup of G^θ containing its identity component $(G^\theta)_0$. Then G/K is called a ***generalized symmetric space***. In the event θ is an involution, G/K is a ***symmetric space***. If in addition G and K are compact and connected, we call (G,K) a ***(generalized) symmetric pair***.

The generalized symmetric spaces and especially the symmetric spaces *per se* form a completely classified system of examples which have been intensively studied since the early 1900s.

Example A.4.9. Élie Cartan demonstrated that symmetric spaces G/K are formal, in fact showing that the collection of *harmonic* forms on a symmetric space forms a subring of the differential forms $\Omega(G/K)$ consisting of one element from each class in $H^*(G/K)$. We will produce a version of this proof in Proposition 8.4.11.

Svjetlana Terzić [Ter01] and independently Zofia Stępień [Ste02] have also shown that compact generalized symmetric spaces G/K with isotropy group K connected are formal. It is *not*, however, the case that wedge products of harmonic forms on such spaces are again harmonic (that such should happen is called *geometric formality*); see Terzić's later joint article with Dieter Kotschick [KT03].

A.5. Filtrations and spectral sequences

This section comprises a few words, without proofs, about the spectral sequence associated to a filtered differential group.

A.5.1. Filtered differential groups

Consider the total order $\mathbb{Z}$ as a category in the opposite of the standard way, so that there exists a (unique) arrow $n \to m$ just when $n \geqslant m$, and write $\mathsf{Sub}(A)$ for the category of k-submodules of A and inclusions therebetween.

A ***filtered group*** is a pair $(A, F_\bullet)$, where A is a k-module and $F_\bullet\colon \mathbb{Z} \to \mathsf{Sub}(A)$ is a functor with $F_0 = A$. In other words, this is an infinite descending sequence

$$\cdots = F_{-1} = F_0 = A \geqslant F_1 \geqslant F_2 \geqslant \cdots .$$

We also write $F_n = F_n A$. One can recompile this information into a $\mathbb{Z}$-graded group

$$\bigoplus F_\bullet A := \bigoplus_{r\in\mathbb{Z}} F_r A$$

and the inclusions $F_{n+1}A \hookrightarrow F_n A$ into an injective endomorphism $i \in \operatorname{End}_k\big(\bigoplus F_\bullet A\big)$ of degree -1 which stabilizes to an isomorphism in nonpositive degrees. Viewing i as a direct system and taking the direct limit recovers the original filtered group, so we may regard $(A, F_\bullet)$ and $(\bigoplus F_\bullet A, i)$ as equivalent objects and denote both, slightly abusively, by (A, i). Say a filtration is ***complete*** if $F_p A = 0$ for all sufficiently large p. The group

$$\operatorname{gr}_\bullet A := \operatorname{coker} i_\bullet = \bigoplus_{r \geqslant 0} F_r A / F_{r+1} A$$

is the ***associated graded*** group of (A, i). A map $f\colon A \longrightarrow B$ is said to ***preserve filtrations*** (A, i) and (B, ι) if $f(F_p A) \leqslant F_p B$ (or equivalently $f \circ i = \iota \circ f$). We write such a map as $f\colon (A, i) \longrightarrow (B, \iota)$. Such a map induces an associated graded map $\operatorname{gr}_\bullet f\colon \operatorname{gr}_\bullet A \longrightarrow \operatorname{gr}_\bullet B$. We have the following recurring result on such maps.

Proposition A.5.1. *Let $f\colon (A, i) \longrightarrow (B, \iota)$ be a filtration-preserving chain map of filtered groups. Suppose that both filtrations are complete. Then if $\operatorname{gr}_\bullet f$ is an isomorphism, so also must be f itself.*

Proof. Fix a filtration degree p sufficiently large that $F_{p+1}A = 0 = F_{p+1}B$. We have a map

$$\begin{tikzcd}
0 \arrow[r] & F_{p+1}A \arrow[r] \arrow[d, "f"] & F_p A \arrow[r] \arrow[d, "f"] & \operatorname{gr}_p A \arrow[r] \arrow[d, "\operatorname{gr}_\bullet f", "\wr"'] & 0 \\
0 \arrow[r] & F_{p+1}B \arrow[r] & F_p B \arrow[r] & \operatorname{gr}_p B \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences; by the five lemma, it follows $F_p f\colon F_p A \longrightarrow F_p B$ is an isomorphism. This begins a decreasing induction on p, which terminates in $f\colon A \xrightarrow{\sim} B$ when $p = 0$. $\square$

A ***filtered differential group*** is a triple (A, d, i) such that (A, d) is a differential group, (A, i) a filtered group, and d preserves the filtration, inducing restricted differentials $F_n d \in \operatorname{End}_k F_n A$. (It is also true that then i is a chain map $(\bigoplus F_\bullet A, d) \to (\bigoplus F_\bullet A, d)$.) A homomorphism of filtered differential groups is a chain map commuting with the filtration.

Given a filtered differential group $(A, \tilde{d}, \tilde{\imath})$, the differential $\tilde{d}$ descends to a differential $\bar{d}$ on $\operatorname{gr}_\bullet A$, inducing a short exact sequence of differential groups

$$0 \to A \xrightarrow{\tilde{\imath}} A \xrightarrow{\tilde{\jmath}} \operatorname{gr}_\bullet A \to 0.$$

This induces a triangular exact sequence

$$\begin{tikzcd} H(A) \arrow[rr, "i"] & & H(A) \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

of cohomology groups, where $E = H_{\bar{d}}(\operatorname{gr}_\bullet A)$. Such a triangle is traditionally called an ***exact couple***. Here i and j are induced from $\tilde{\imath}$ and $\tilde{\jmath}$ respectively, and k arises from the snake lemma and takes $[a + \tilde{\imath}A] \mapsto [\tilde{\imath}^{-1}da]$. If we set $d = jk$, then $d^2 = j(kj)k = 0$, so d is a differential on E. The original filtration $\tilde{\imath}$ descends to a filtration $F_n H^*(A) = i^n H^*(A) = H^*_{\tilde{d}}(\operatorname{im} \tilde{\imath}^n)$.

Note that a map of filtered differential groups induces a map of short exact sequences and a map of exact couples in cohomology (a triangular prism), and so in particular a map of differential groups on the E components. We have $\deg i = \deg j = 0$ and $\deg k = 1$.

A.5.2. The spectral sequence of an exact couple

There is a functor

$$\begin{tikzcd} A_0 \arrow[rr, "i"] & & A_0 \arrow[dl, "j"] \\ & E_0, \arrow[ul, "k"] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} A_1 \arrow[rr, "i_1"] & & A_1 \arrow[dl, "j_1"] \\ & E_1, \arrow[ul, "k_1"] & \end{tikzcd}$$

taking an exact couple to the ***derived couple*** whose objects are $A_1 = iA_0$ and $E_1 = H(E_0, d_0)$, and whose maps are given by $i_1 = (i \restriction iA_0)$ and $j_1 \colon ia \longmapsto [ja]$ and $k_1 \colon [e] \longmapsto ke$.

This process can be iterated, and the sequence (E_r, d_r) of differential groups so derived is called the ***spectral sequence of the exact couple***. Each E_r is traditionally called a ***page***. Note that (A_0, i) is a filtered group with $F_r A_0 = A_r = i^r A_0$. Write A_∞ for the intersection $\bigcap_{r \geqslant 0} A_r = \bigcap_{r \geqslant 0} i^r A_0$. If for each $a \in A_0$, the sequence $(i^r a)$ eventually stabilizes, then $i_\infty = i \restriction A_\infty$ is injective, and fits into a short exact sequence

$$\begin{tikzcd} A_\infty \arrow[rr, "i_\infty"] & & A_\infty \arrow[dl, "j_\infty"] \\ & E_\infty; \arrow[ul, "0"] & \end{tikzcd}$$

that is, $E_\infty = \operatorname{coker} i_\infty$, and we may think as this E_∞ as the limiting behavior of the pages (E_r) and call it the ***limiting page***. If $E_r \cong E_\infty$ for some finite r, we say the sequence ***collapses*** at E_r.

To see how E_∞ arises, recall that $E_{r+1} = (\ker d_r)/(\operatorname{im} d_r)$, where $\ker d_r$ and $\operatorname{im} d_r$ are in turn subgroups of $E_r = (\ker d_{r-1})/(\operatorname{im} d_{r-1})$. Thus, under the quotient map $\pi \colon \ker d_{r-1} \twoheadrightarrow E_r$, the groups $\ker d_r$ and $\operatorname{im} d_r$ lift to subgroups $\pi^{-1} \operatorname{im} d_r \leqslant \pi^{-1} \ker d_r$ of $\ker d_{r-1}$, each containing $\operatorname{im} d_{r-1}$. By the third isomorphism theorem, we still have $(\pi^{-1} \ker d_r)/(\pi^{-1} \operatorname{im} d_r) \cong (\ker d_r)/(\operatorname{im} d_r) = E_{r+1}$. Iteratively pulling back all the way to E_0, we get a sequence of subgroups

$$B_0 \leqslant B_1 \leqslant B_2 \leqslant B_3 \leqslant \cdots \leqslant Z_3 \leqslant Z_2 \leqslant Z_1 \leqslant Z_0$$

of E_0 such that $E_{r+1} = Z_r/B_r$. Now we can unambiguously declare $Z_\infty := \bigcap Z_r$ and $B_\infty := \bigcup B_r$, and take $E_\infty = Z_\infty / B_\infty$.

A ***homomorphism of spectral sequences*** is a sequence $\big(\psi_r \colon (\widetilde{E}_r, \tilde{d}_r) \longrightarrow (E_r, d_r)\big)_{r \geqslant n}$ of chain maps of differential groups such that each ψ_{r+1} for $r \geqslant n$ is induced by ψ_r, which is to say $\psi_{r+1} = H^*(\psi_r)$. From the remark at the end of Appendix A.5.1, it follows that a map of filtered differential groups induces a map of exact couples and iteratively a map of spectral sequences.

A.5.3. The spectral sequence of a filtered differential graded algebra

Now let us consider a ***filtered chain complex*** $(A, \tilde{\imath}, d)$. This is a filtered differential group such that $A = \bigoplus_{n \in \mathbb{N}} A_n$ is a nonnegatively-graded group, $dA_n \leqslant A_{n+1}$, and $\tilde{\imath}$ is a graded homomorphism of degree 0, so that $iA_n \leqslant A_n$. Thus each A_n inherits a filtration, as well. In the resulting exact couple $H(A) \to H(A) \to H(E)$, one has $i\colon H^n(A) \longrightarrow H^n(A)$ and $j\colon H^n(A) \longrightarrow H^n(E)$ of degree zero, but $k\colon H^n(E) \longrightarrow H^{n+1}(E)$ of degree 1.

Theorem A.5.2 (Koszul)**.** *Let (A, d, i) be a filtered chain complex. Then in the spectral sequence associated to the exact couple $0 \to A \xrightarrow{i} A \longrightarrow \operatorname{gr} A \to 0$, one has*

- $(E_0, d_0) = (\operatorname{gr}_\bullet A, \operatorname{gr}_\bullet d)$,
- $E_1 \cong H^*(\operatorname{gr}_\bullet A)$,
- $E_\infty \cong \operatorname{gr}_\bullet H^*(A)$.

We call this the ***filtration spectral sequence*** *of the filtered chain complex (A, d, i). All pages inherit a bigrading $E_r^{p,q}$ induced from the grading $E_0^{p,q} = \operatorname{gr}_p A^q$ of $E_0 = \operatorname{gr}_\bullet A$, and the differential d_r is of bidegree $(r, 1-r)$. This is* ***first-quadrant spectral sequence*** *in that $E_r^{p,q} = 0$ if $p < 0$ or $q < 0$.*

Given a bigraded differential algebra A, a natural decreasing filtration, the ***horizontal filtration***, is given by

$$F_p A := \bigoplus_{i \geqslant p} A^{i,\bullet}.$$

If in the decomposition $d = \sum_{\ell \in \mathbb{Z}} d^\ell$ of Appendix A.3.2 one has $d^\ell = 0$ for $\ell < 0$, then d is filtration-preserving, and we call (A,d,i) a ***filtered differential bigraded algebra***. In this case, the theorem applied to (A,d,i) yields a spectral sequence (E_r,d_r) with $E_0 \cong A$ again and the differentials d_r induced from the components d^r.

Corollary A.5.3. *Let (A,d,i) be a filtered, nonnegatively-bigraded DGA. Then in the spectral sequence associated to the horizontal filtration one has*

- $(E_0,d_0) \cong (A,d^0)$,
- $E_1 \cong \bigoplus_{p\in\mathbb{N}} H^*(A^{p,\bullet},d^0), \quad d_1 = H^*_{d^0}(d^1)$,
- $E_2 \cong H^*_{d^1}H^*_{d^0}(A)$,
- $E_\infty \cong \operatorname{gr}_\bullet H^*(A)$.

In one key situation in which we apply this spectral sequence, we are able to say even more about E_2.

Corollary A.5.4. *If (A,d,i) is a filtered, nonnegatively-bigraded DGA such that $A = A^{\bullet,\bullet}$ is a tensor product $A^{\bullet,0}\otimes A^{0,\bullet}$, and i is the horizontal filtration, then d^0 is induced by a map $A^{0,\bullet} \longrightarrow A^{0,\bullet}$ and d^1 by a map $A^{\bullet,0} \longrightarrow A^{\bullet,0}$, and we have*

- $E_0 \cong A, \qquad d_0 = \mathrm{id}\otimes d^0$,
- $E_1 \cong A^{\bullet,0}\otimes H^*_{d^0}(A^{0,\bullet}), \qquad d_1 = d^1\otimes \mathrm{id}$,
- $E_2 \cong H^*_{d^1}(A^{\bullet,0})\otimes H^*_{d^0}(A^{0,\bullet})$,
- $E_\infty \cong \operatorname{gr}_\bullet H^*(A)$.

In Section 4.3, after we have discussed fiber bundles, we will introduce the Serre spectral sequence, which will have also have a tensor structure on E_2, for more topological reasons.

Remark A.5.5. The algebraic Künneth Theorem A.3.9 of this chapter and the universal coefficient Theorem B.2.1 of the next both generalize to filtration spectral sequences if we do not assume that the modules in question are not free over the base ring k or that k is not a principal ideal domain.

A.5.4. Fundamental results on spectral sequences

A common way to understand the cohomology ring of a filtered DGA is to engage in wishful thinking: one finds another spectral sequence one would *like* to approximate that of the DGA in question, contrives a map between the idealized sequence and the actual sequence, and shows it yields an isomorphism on a late enough page. The theoretical justification behind this chicanery has at most two steps.

Theorem A.5.6 (Zeeman–Moore comparison theorem, [McC01, Thm. 3.26, p. 82])**.** *Let* $(\psi_r)\colon ('E_r, 'd_r) \longrightarrow (E_r, d_r)$ *be a map of spectral sequences such that* $E_2 \cong E_2^{\bullet,0} \otimes E_2^{0,\bullet}$ *and* $'E_2 \cong {}'E_2^{\bullet,0} \otimes {}'E_2^{0,\bullet}$ *decompose as tensor products. Then any two of the following three conditions imply the third:*

- $\psi_2^{\bullet,0}\colon E_2^{\bullet,0} \longrightarrow {}'E_2^{\bullet,0}$ *is an isomorphism,*
- $\psi_2^{0,\bullet}\colon E_2^{0,\bullet} \longrightarrow {}'E_2^{0,\bullet}$ *is an isomorphism,*
- $\psi_\infty\colon\ E_\infty \longrightarrow {}'E_\infty$ *is an isomorphism.*

Given an isomorphism of E_2 pages or E_∞ pages then shows that the inducing map of DGAs was a quasi-isomorphism.

Proposition A.5.7. *Let* $f\colon A \longrightarrow B$ *be a map of filtered DGAs and* $(\psi_r)\colon ('E_r, 'd_r) \longrightarrow (E_r, d_r)$ *the associated map of filtration spectral sequences. Suppose that both filtrations are complete in each degree: for each* $n \in \mathbb{N}$ *there exists* $p(n) \in \mathbb{N}$ *such that when* $p \geqslant p(n)$ *one has* $F_pA^n = 0 = F_pB^n$*. If* ψ_r *is an isomorphism for any* $r \geqslant 0$*, then* $f^*\colon H^*(A) \longrightarrow H^*(B)$ *is an isomorphism.*

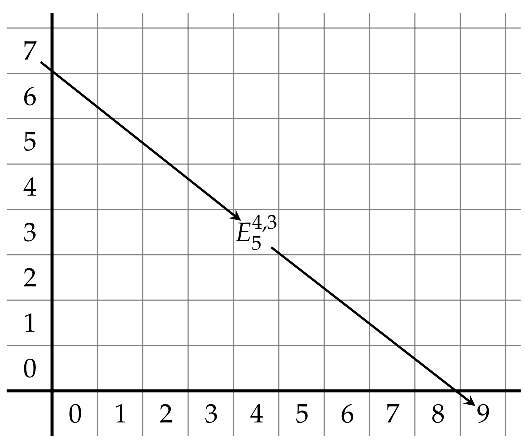
Proof. If any ψ_r is an isomorphism, then since ${}'d_r\psi_r = \psi_r d_r$, it follows that all later ψ_r and ψ_∞ are isomorphisms. By Corollary A.5.3, ψ_∞ is the isomorphism $\mathrm{gr}_\bullet f^*\colon \mathrm{gr}_\bullet H^*(A) \longrightarrow \mathrm{gr}_\bullet H^*(B)$. For any given total degree n, we can apply Proposition A.5.1 to the map $\psi_\infty^n\colon \mathrm{gr}_\bullet H^n(A) \longrightarrow \mathrm{gr}_\bullet H^n(B)$ to conclude $H^n(f)$ is an isomorphism. □

A much simpler fact that comes up frequently is the following:

Proposition A.5.8. *Let (E_r, d_r) be a first-quadrant spectral sequence. If $p < r$ and $q < r-1$, then $E_r^{p,q} = E_\infty^{p,q}$.*

Proof. Because the bidegree of d_r is $(r, 1-r)$, the domain $E_r^{p-r,q+r-1}$ of the component of d_r with codomain $E_r^{p,q}$ lies in the second quadrant, and the codomain $E_r^{p+r,q+1-r}$ of the component of d_r with domain $E_r^{p,q}$ lies in the fourth quadrant. See Figure A.5.9. Since these quadrants are inhabited only by zero groups, the differentials in and out of $E_r^{p,q}$ are zero, so $E_r^{p,q} = E_{r+1}^{p,q}$. All later differentials out of this square must also be zero for the same reason. □

Figure A.5.9: The differentials to and from $E_5^{4,3}$ leave the first quadrant



This genre of reasoning, that something must stabilize at a certain page—or vanish before a certain page, lest it survive to E_∞—goes by the trade name of "*lacunary considerations*." One uses such considerations as frequently as possible because they are usually far simpler than actually computing differentials. Occasionally this kind of spatial reasoning enables one to understand what happens in a spectral sequence without having done any algebra at all.

Proposition A.5.10 ([McC01, Example 1.K, p. 25])**.** *Let (A, d) be a filtered differential bigraded k-algebra, free as a k-module. If E_∞ is a free k-CGA, then $E_\infty \cong H^*(A, d)$ as a k-CGA.*

This is analogous to the statement of Proposition A.4.3, except here the "quotient" object is actually the associated graded algebra, and there is not a priori an algebra map $H^*(A) \longrightarrow \operatorname{gr}_\bullet H^*(A) = E_\infty$.

Proof. Consider the indecomposables $Q(E_\infty)$ as a bigraded k-submodule of algebra generators of E_∞. Because $E_\infty = \operatorname{gr}_\bullet H^*(A)$ and $H^*(A)$ are isomorphic on the level of bigraded k-modules, we can map $Q(E_\infty)$ to $H^*(A)$ in such a way as to preserve the bidegree. This yields a bigraded module injection $Q(E_\infty) \rightarrowtail H^*(A)$. Since E_∞ is the free CGA on $Q(E_\infty)$, this injection extends uniquely to the filtration-preserving CGA map $f\colon E_\infty \longrightarrow H^*(A)$ in the diagram below (the bottom row will be explained).

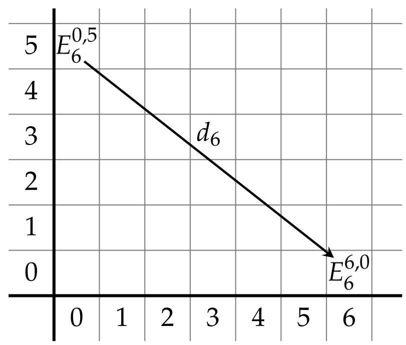
$$\begin{tikzcd}
Q(E_\infty) \arrow[d, hook] \arrow[dr, tail] & \\
E_\infty \arrow[r, "f"'] \arrow[d, rightsquigarrow, "\wr"] & H^*(A) \arrow[d, rightsquigarrow] \\
E_\infty \arrow[r, "\operatorname{gr}_\bullet f"'] & E_\infty .
\end{tikzcd}$$

Since f lifts generators of $\operatorname{gr}_\bullet E_\infty$ to $H^*(A)$ in a filtration-preserving manner and $\operatorname{gr}_\bullet E_\infty = E_\infty$, the map $\operatorname{gr}_\bullet f\colon E_\infty \longrightarrow E_\infty$ is the identity on $Q(E_\infty)$ by the identity map. Since $\operatorname{gr}_\bullet f$ is a ring map, it is then the identity map on E_∞ (and a fortiori an isomorphism) so an application of Proposition A.5.1 shows the original $f\colon E_\infty \longrightarrow H^*(A)$ is a CGA isomorphism. $\square$

A.5.5. The transgression

Early on in the history of bigraded spectral sequences of the form discussed above, it was noticed that the ***edge maps*** $d_r\colon E_r^{0,r-1} \longrightarrow E_r^{r,0}$ from the left column to the bottom row have a special importance.

Figure A.5.11: An edge map



Definition A.5.12 (Koszul, 1950 [Kos50, Sec. 18])**.** Let (E_r, d_r) be the filtration spectral sequence of a filtered DGA (A, d). If $z \in E_2^{0,r-1}$ is in the kernel of each d_p for $p < r$, so that $d_r z \in E_r^{0,r}$ is defined (that is, if z survives long enough to be in the domain of an edge homomorphism), then z is said to ***transgress***. The dotted map τ in the diagram

$$\begin{tikzcd}
 & E_r^{0,r-1} \arrow[r, hook] \arrow[d, "d_r"'] & E_2^{0,r-1} \arrow[dll, dotted, "\tau"'] \\
E_2^{r,0} \arrow[r, two heads] & E_r^{r,0},
\end{tikzcd}$$

viewed as a k-linear map from a submodule of $E_2^{0,r-1}$ to a quotient module of $E_2^{r,0}$, is the ***transgression***.

Proposition A.5.13. *Let (E_r, d_r) be the filtration spectral sequence of a filtered DGA (A, d). An element $z \in E_2^{0,r-1}$ transgresses to $\tau z \in E_2^{r,0}$ if and only if there exists $y \in A$ which represents z in $\operatorname{gr}_0 A = E_2^{0,r-1} = H^{r-1}(\operatorname{gr}_0 A, d^0)$ and such that $dy \in A$ represents τz in $E_2^{r,0} = H^0(\operatorname{gr}_r A, d^0)$.*

Historical remarks A.5.14. According to the concluding notes in Greub *et al.* [GHV76], instances of transgressions were first identified by Shiing-Shen Chern [Che46] and Guy Hirsch [Hir48] before Koszul observed the pattern and coined the term "transgression" in his thesis work.

The filtration spectral sequence is first described in Koszul's *Comptes Rendus* note [Kos47a], and is extracted from Leray's earlier work as described in a 1946 *Comptes Rendus* notice [Ler46a]. Koszul was the first other person to work through and understand Leray's post-war topological output, and was the chief instigator of the simplifications that made spectral sequences accessible to the rest of the mathematical community [Mil00].

Appendix B

Topological background

In this appendix we define some relevant topological categories and state some well-known results in algebraic topology and Lie theory. We will take homotopy groups and singular homology and cohomology groups as known concepts, and cite basic results in algebraic topology without proof, but will restate that the ***0th homotopy set*** $\pi_0 X$ of a space X is its set of path-components, which inherits a group structure if X is a group. We denote homotopy equivalences by "$\simeq$," homeomorphisms by "$\approx$," and Lie group isomorphisms by "$\cong$." If a group G acts on a space X via $\phi\colon G \times X \longrightarrow X$, we write $\phi\colon G \curvearrowright X$. The interior of a manifold M with boundary is $\mathring{M}$ and its boundary is ∂M. The complement of a set $A \subseteq B$ is $B \backslash A$.

B.1. Topological structures of interest

Let Top be the category whose objects are pairs (X, A) of topological spaces, A closed in X, with morphisms $(X, A) \longrightarrow (Y, B)$ those continuous maps $f\colon X \longrightarrow Y$ such that $f(A) \subseteq B$. The category whose objects are individual topological spaces and morphisms continuous maps is included as a full category through the inclusion $X \longmapsto (X, \varnothing)$, where $\varnothing$ is the empty space.

B.1.1. Cell complexes

A ***CW complex*** is a topological space X equipped with a decomposition into a union of disks of increasing dimension. Less elliptically, such an X must admit a filtration (X^n) into ***n-skeleta*** meeting the following conditions:

- The 0-skeleton X^0 is a discrete space.
- Given the n-skeleton X^n, index a collection of distinct $(n+1)$-disks as $(D^{n+1}_\alpha)_{\alpha\in\mathrm{A}}$. From each boundary S^n_α, let a continuous map $\varphi_\alpha\colon S^n_\alpha \longrightarrow X^n$, the ***attaching map***, be given. These maps assemble into a map $\varphi\colon \coprod_{\alpha\in\mathrm{A}} S^n_\alpha \longrightarrow X^n$, and X^{n+1} is defined to be the quotient space
$$X^n \amalg \coprod_{\alpha\in\mathrm{A}} D^{n+1}_\alpha \Big/ s \sim \varphi(s)$$
of the disjoint union: we've identified the boundaries of the D^{n+1}_α with their images in X^n.
- We let $X = \bigcup_{n\in\mathbb{N}} X^n$ be the colimit, with the direct limit topology. This amounts to saying $U \subseteq X$ is open just if each $U \cap X^n$ is open in X^n.

A map $f\colon X \longrightarrow Y$ between two CW complexes is said to be ***cellular*** if it respects the skeleta: $f\colon X^n \longrightarrow Y^n$ for all n.

Write CW for the subcategory of Top whose objects are ***CW pairs*** consisting of a CW complex X and closed subcomplex A, and whose maps $(X,A) \longrightarrow (Y,B)$ are required to be cellular, meaning both $X \longrightarrow Y$ and the restriction $A \longrightarrow B$ are cellular. The category CW is a homotopy-theoretic skeleton of Top in the sense that given any $(X,A) \in$ Top there exists $(\widetilde{X},\widetilde{A}) \in$ CW and a weak homotopy equivalence $(\widetilde{X},\widetilde{A}) \longrightarrow (X,A)$ in Top. This map (or (X,A) itself) is called a *CW approximation* [Hat02, Example 4.15, p. 353]. Moreover, any map of pairs is the same up to homotopy as a map between CW complexes: given a map $(X,A) \longrightarrow (Y,B)$ of pairs there exists a map between CW approximations making the following square commute up to homotopy:

$$\begin{tikzcd} (\widetilde{X}, \widetilde{A}) \arrow[r] \arrow[d, dotted] & (X, A) \arrow[d] \\ (\widetilde{Y}, \widetilde{B}) \arrow[r] & (Y, B). \end{tikzcd}$$

Although CW is unstable under the formation of mapping spaces, with judicious use of CW approximations, we may basically assume every space that follows is a CW complex.

B.1.2. Fiber bundles

A ***fiber space*** with is a continuous surjection $E \longrightarrow B$ such that for each $b \in B$, we have $h^{-1}\{b\} \approx F$ for some fixed space F, the ***fiber***. Each $h^{-1}\{b\}$ is also called a fiber, E is the ***total space***, and B the ***base***. We abbreviate this assemblage as $F \to E \to B$. Given two fiber spaces $p\colon E \to B$ and $p'\colon E' \to B'$, a map $h\colon E \longrightarrow E'$ of total spaces is ***fiber-preserving*** if it sends fibers into fibers. Equivalently, there is a map $\bar{h}$ of bases making the following diagram commute:

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ B \arrow[r, dotted, "\bar{h}"] & B'. \end{tikzcd}$$

Then $hp^{-1}\{b\} \subseteq (p')^{-1}\{\bar{h}(b)\}$ for all $b \in B$. Fiber spaces with fiber F and fiber-preserving maps form a category whose ***isomorphisms*** are fiber-preserving homeomorphisms.

A fiber space $p\colon E \to B$ with fiber F is a ***fiber bundle***, or an ***F-bundle*** (or *locally trivial*), if

- the base B admits an open cover of sets U such that there are fiber space isomorphisms $\phi_U\colon p^{-1}(U) \xrightarrow{\approx} U \times F$, called *(local)* ***trivializations***, and
- these trivializations are compatible in the sense that given two overlapping trivializing

opens U and V, the ***transition functions*** $g_{U,V}$ defined by the composite homeomorphism

$$\phi_{U,V}\colon (U \cap V) \times F \xrightarrow{\phi_V^{-1}} p^{-1}(U \cap V) \xrightarrow{\phi_U} (U \cap V) \times F$$

$$(x, f) \longmapsto \big(x, g_{U,V}(x)(f)\big),$$

are continuous maps $U \cap V \longrightarrow \operatorname{Homeo} F$. Morally, different coordinatizations of the same trivial subbundle should differ continuously.

Given a fiber space $F \to E \xrightarrow{p} B$ and an subset $U \subseteq B$, the ***restriction*** $E|_U$ is the F-bundle $(p \restriction U)\colon p^{-1}(U) \to U$. This generalizes to the following construction. Given a continuous map $h\colon X \longrightarrow B$ (for restrictions, an inclusion), we can construct a ***pullback*** space $h^*E \to X$ with fiber F fitting into the commutative square

$$\begin{tikzcd} h^*E \arrow[r, "\tilde{h} = \mathrm{pr}_2"] \arrow[d, "h^*p := \mathrm{pr}_1"'] & E \arrow[d, "p"] \\ X \arrow[r, "h"] & B, \end{tikzcd}$$

where the new total space is

$$f^*E = X \underset{B}{\times} E := \big\{(x, e) \in X \times E : h(x) = p(e)\big\} \subsetneq X \times E$$

and the new maps the restrictions of the factor projections from $X \times E$. This total space is called the ***fiber product***, and (with the maps), it is the pullback of the diagram $X \to B \leftarrow E$ in Top.[1] If $E \to B$ was an F-bundle, so also is $h^*E \to X$: given a local trivialization

$$\phi = (p, \rho)\colon p^{-1}U \xrightarrow{\approx} U \times F,$$

[1] This notation $X \underset{B}{\times} E$, now universal, is due to Paul Baum [Smi67, p. 68].

a trivialization of the pullback $(h^*E)|_{h^{-1}(U)}$ is given by

$$\mathrm{id}_X \times \rho\colon (x, e) \longmapsto \big(x, \rho(e)\big),$$

and such sets $h^{-1}(U)$ cover X. The resulting bundle is a ***pullback bundle***.

If F, E, B are all smooth manifolds and the fiber inclusion, projection, and transition functions are all C^∞, we say $F \to E \to B$ is a ***smooth bundle***. One can similarly define holomorphic and algebraic bundles, but smooth and merely continuous bundles are all we shall work with.

B.1.3. Principal bundles

Suppose we are given a fiber bundle $F \to E \to B$ admitting trivializations $(\phi_U)_{U \in \mathscr{U}}$, such that each transition function $g_{U,V}$ takes values in some subgroup G of the group $\operatorname{Homeo} F$ of self-homeomorphisms of the fiber. As G is a topological group, its multiplication is continuous, and left multiplication ℓ_g by any element of $g \in G$ is a self-homeomorphism of G. In this way the transition function values $g_{U,V}(x) \in G$ can be viewed as elements of $\operatorname{Homeo} G$, and we can form a G-bundle $G \to P \to B$ by starting with the disjoint union $\coprod_{U \in \mathscr{U}} U \times G$ and gluing the pieces by the relations

$$(x, g) \sim \big(x, g_{U,V}(x) \cdot g\big)$$

for all nonempty intersections $U \cap V$ of sets in $\mathscr{U}$ and all $x \in U \cap V$ and $g \in G$.

The disjoint union we started with admits a global right G-action $(u, g) \cdot g' = (u, gg')$, which descends to a right G-action on P since the transition functions act on the *left* of the fibers G. This right action is simply transitive on each fiber. We call a G-bundle admitting a right G-action acting simply transitively on each fiber a ***principal G-bundle***; this motivating bundle $G \to P \to B$ is one such.

We can recover the original $F \to E \to B$ from $G \to P \to B$ and the map $\psi \colon G \longrightarrow \operatorname{Homeo} F$ by a pushout construction:

$$E \approx P \underset{G}{\times} F := \frac{P \times F}{\big([x,g],f\big) \sim \big([x,1]\psi(g)f\big)} \approx \frac{\coprod_{U \in \mathscr{U}} U \times G \times F}{\big(x, g_{U,V}(x)g, f\big) \sim (x,g,f) \sim \big(x,1,\psi(g)f\big)} \tag{B.1}$$

Verbally, this can be seen as extracting the G-valued transition functions from a principal G-bundle and applying them to F instead of G. For this reason, the bundles $E \to B$ and $P \to B$ are said to be ***associated***. Because this correspondence is reversible, principal G-bundles essentially carry all information about fiber bundles.

Further, in Chapter 3, we construct a *universal principal G-bundle $EG \to BG$* that every principal G-bundle is a pullback of. Given such a bundle, a space F, and a homomorphism $\psi \colon G \longrightarrow \operatorname{Homeo} F$, it follows the the associated F-bundle $EG \times_G F \to BG$ is universal for F-bundles with transition functions in $\psi(G)$; for example, $E\mathrm{GL}(n,\mathbb{R}) \underset{\mathrm{GL}(n,\mathbb{R})}{\times} \mathbb{R}^n \to B\mathrm{GL}(n,\mathbb{R})$ is a universal vector bundle.

B.2. Algebraic topology grab bag

This section is just a collection of useful algebro-topological results we will need later, presented without much in the way of motivation.

The algebraic Künneth Theorem A.3.9 has at least two major topological repercussions.

Theorem B.2.1 (Universal coefficients [Hat02, Thms. 3.2, 3.A.3, pp. 195, 264])**.** *Let X be a topological space and k a principal ideal domain. For each $n \in \mathbb{N}$ one has the following short exact sequences of abelian groups:*

$$0 \to H_n(X;\mathbb{Z}) \underset{\mathbb{Z}}{\otimes} k \longrightarrow H_n(X;k) \longrightarrow \operatorname{Tor}_1^k\big(H_{n-1}(X;\mathbb{Z}),k\big) \to 0,$$

$$0 \to \operatorname{Ext}_k^1\big(H_{n-1}(X;\mathbb{Z}),k\big) \longrightarrow H^n(X;k) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}\big(H_n(X;\mathbb{Z}),k\big) \to 0.$$

Proof. The homology sequence follows from Theorem A.3.9 by taking $C = C_0 = k$ and $A = C_\bullet(X)$ the singular chain complex, taking into account the differentials go in the opposite direction expected. The cohomology sequence arises from taking $C = k$ and $A = \operatorname{Hom}_{\mathbb{Z}}\big(C_\bullet(X), \mathbb{Z}\big)$ the singular cochain complex, noting $A \otimes_{\mathbb{Z}} k \cong \operatorname{Hom}_{\mathbb{Z}}\big(C_\bullet(X), k\big)$. □

Theorem B.2.2 (Topological Künneth [Hat02, Thm. 3B.6, 3.21][Mas91, Thm. 11.2, p. 346])**.** *Let X and Y be topological spaces and k an abelian group. Suppose $H^*(X)$ is of finite type. Then for each $n \in \mathbb{N}$ one has the following split short exact sequences of abelian groups:*

$$0 \to \bigoplus_{0\leqslant j\leqslant n} \big(H_j(X) \otimes H_{n-j}(Y)\big) \longrightarrow H_n(X \times Y; k) \longrightarrow \bigoplus_{0\leqslant j\leqslant n} \operatorname{Tor}_1^k\big(H_j(X;k), H_{n-j-1}(Y;k)\big) \to 0;$$

$$0 \to \bigoplus_{0\leqslant j\leqslant n} \big(H^j(X;Z) \otimes H^n(Y;k)\big) \longrightarrow H^n(X \times Y; k) \longrightarrow \bigoplus_{0\leqslant j\leqslant n+1} \operatorname{Tor}_1^{\mathbb{Z}}\big(H^j(X;\mathbb{Z}), H^{n+1-j}(Y;k)\big) \to 0.$$

When one of the rings $H^*(X;k)$ or $H^*(Y;k)$ is free as a k-module, the Ext and Tor terms disappear and these isomorphisms assume a product form

$$H^*(X \times Y) \cong H^*X \otimes H^*Y.$$

One also obtains the following relation between integral homology and cohomology.

Proposition B.2.3. *Let X be a topological space. The torsion subgroups and torsion-free quotients of the singular homology and cohomology groups $H_*(X;\mathbb{Z})$ and $H^*(X;\mathbb{Z})$ satisfy*

$$H^n(X;\mathbb{Z}) \cong H_n(X;\mathbb{Z})_{\text{free}} \oplus H_{n-1}(X;\mathbb{Z})_{\text{tors}}$$

We will use fiber bundles frequently, and need a criterion for determining when the fundamental groups of their base spaces are trivial.

Theorem B.2.4 ([Hat02, Thm. 4.3])**.** *Let $F \to E \to B$ be a fiber bundle. Then there is associated a long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_2(F) \longrightarrow \pi_2(E) \longrightarrow \pi_2(B) \longrightarrow \pi_1(F) \longrightarrow \pi_1(E) \longrightarrow \pi_1(B) \longrightarrow \pi_0(F) \longrightarrow \pi_0(E) \to 0.$$

There are important but subtle relations between the homology and homotopy groups.

Proposition B.2.5. *The first singular homology group of a space X is the abelianization of its fundamental group: $H_1(X;\mathbb{Z}) \cong \pi_1(X)^{\mathrm{ab}}$.*

Theorem B.2.6. *Let X be a simply-connected topological space and let $n > 1$ be the least natural number such that $\pi_n X$ is nontrivial. Then the same n is also minimal such that $H_n X$ is nontrivial, and the natural* ***Hurewicz map***

$$\begin{aligned} \pi_n X &\longrightarrow H_n X, \\ [\sigma \colon S^n \longrightarrow X] &\longmapsto \sigma_*[S^n], \end{aligned}$$

taking the homotopy class of a map from a sphere to the pushforward of the fundamental class, is an isomorphism.

The homotopy groups completely determine homotopy type in the following sense.

Theorem B.2.7 (Whitehead [Hat02, Thm. 4.5, p. 346])**.** *Let $f \colon X \longrightarrow Y$ be a map of CW complexes such that $\pi_n f \colon \pi_n X \xrightarrow{\sim} \pi_n Y$ is an isomorphism for all $n \geqslant 0$ (a* ***weak homotopy equivalence****). Then f is a homotopy equivalence.*

Theorem B.2.8 (Whitehead [Hat02, Thm. 4.21, p. 356])**.** *] Let $f \colon X \longrightarrow Y$ be a weak homotopy equivalence of topological spaces. Then $H^n f \colon H^n Y \xrightarrow{\sim} H^n X$ is an isomorphism for all n.*

We will also need the Lefschetz fixed point theorem. Note that if X is of finite type, the natural maps $H^n(X;\mathbb{Z}) \twoheadrightarrow H^n(X;\mathbb{Z})_{\text{free}} \rightarrowtail H^n(X;\mathbb{Q})$ carry a $\mathbb{Z}$-basis of the free $\mathbb{Z}$-module $H^n(X;\mathbb{Z})_{\text{free}}$ to a $\mathbb{Q}$-basis of $H^n(X;\mathbb{Q})$.

Definition B.2.9. Let $f\colon X \to X$ be a continuous self-map of a topological space X of finite type. Then associated endomorphisms $H^n(f) \in \operatorname{Aut}_{\mathbb{Q}} H^n(X;\mathbb{Q})$ are defined for each $n \geqslant 0$. The ***Lefschetz number***

$$\chi(f) := \sum_{n\geqslant 0}(-1)^n \operatorname{tr} H^n(f)$$

is the alternating sum of these traces, where each trace is taken with respect to a basis of $H^n(X;\mathbb{Q})$ inherited from $H^n(X;\mathbb{Z})_{\text{free}}$.

Since the trace of the identity map of a vector space is just the dimension of that space and $H^n(\mathrm{id}_X) = \mathrm{id}_{H^n(X;\mathbb{Q})}$ one immediately has the following.

Proposition B.2.10. *Let $f\colon X \longrightarrow X$ be a continuous self-map of a topological space of finite type. Then the Lefschetz number of the identity map id_X is the Euler characteristic of X:*

$$\chi(X) = \chi(\mathrm{id}_X).$$

The more interesting fact about the Lefschetz number is the Lefschetz fixed point theorem.

Theorem B.2.11 (Lefschetz, [Hat02, Thm. 2C.3, p. 179])**.** *Let X be a topological space which is a deformation retract of a simplicial complex and $f\colon X \longrightarrow X$ a continuous map without fixed points. Then the Lefschetz number $\chi(f)$ is 0.*

B.3. Covers and transfer isomorphisms

In this section, we leverage a standard result on the cohomology of covers to a statement we use later about the cohomology of homogeneous spaces.

Proposition B.3.1 ([Hat02, Prop. 3G.1]). *Let F be a finite group acting by homeomorphisms on a space X, so that $p\colon X \longrightarrow X/F$ is a finite covering. Suppose $|F|$ is invertible in k. Then the map*

$$p^*\colon H^*(X/F;k) \longrightarrow H^*(X;k)$$

is an injection with image the invariant subring $H^*(X;k)^F$.

Proof. Since simplices Δ^n are simply-connected, each singular simplex $\sigma\colon \Delta^n \longrightarrow X/F$ lifts to a singular simplex $\widetilde{\sigma}\colon \Delta^n \longrightarrow X$. The map $\tau\colon \sigma \longmapsto \sum_{f\in F} f\circ\widetilde{\sigma}$ summing over all such lifts then induces a ***transfer map*** $\tau\colon C_n(X/F) \longrightarrow C_n(X)$ of singular chain groups. For each lift $f\widetilde{\sigma}$ we have $p(f\widetilde{\sigma}) = \sigma$, so $p\circ\tau = |F|\cdot\mathrm{id}$ on $C_n(X/F)$. Dualizing yields a cochain map $\tau^*\colon C^n(X;k) \longrightarrow C^n(X/F;k)$ such that $\tau^*\circ p^* = |F|\cdot\mathrm{id}$ on $C^n(X;k)$, so the same holds in $H^*(X;k)$.

If we demand $|F|$ be a unit in k, then $\tau^*\circ p^*$ is an isomorphism, so p^* is injective. Since $p\circ f = p$ for all $f\in F$, it follows $\mathrm{im}\, p^*$ is contained in the invariant subring $H^*(X;k)^F$. On the other hand, since $\tau\circ p$ sends $\widetilde{\sigma} \longmapsto \sum_{f\in F} f\circ\widetilde{\sigma}$, it follows $p^*\tau^*\alpha = \sum_{f\in F} f^*\alpha$ for all $\alpha\in H^*(X;k)$. In particular, if $\alpha\in H^*(X;k)^F$ is F-invariant, then $p^*\tau^*\alpha = |F|\alpha$, so p^* surjects onto $H(X;k)^F$. □

Corollary B.3.2. *In the situation of Proposition B.3.1, suppose the action of F on X is the restriction of a continuous action of a path-connected group Γ on X. Then*

$$H^*(X/F;k) \cong H^*(X;k)$$

Proof. Let $f\in F$. Since Γ is path-connected, the left translation $\gamma \longmapsto f\gamma$ on Γ is homotopic to the identity. It follows f acts trivially on H^*X. Thus $(H^*X)^F \cong H^*X$ for any coefficient ring. □

Proposition B.3.3. *Let Γ be a path-connected group, H_0 a connected subgroup, and F a finite central subgroup of Γ. Write $F_0 = F \cap H_0$ and suppose $|F/F_0|$ is invertible in k. Then*

$$H^*(\Gamma/FH_0) \cong H^*(\Gamma/H_0).$$

Proof. The space Γ/FH_0 is the quotient of Γ/H_0 by the left action of F/F_0 given by $fF_0 \cdot \gamma H_0 = \gamma f H_0$, which is well defined because F is central in Γ. But F/F_0 is a subgroup of the path-connected group Γ/F_0, so the result follows from Corollary B.3.2. □

Proposition B.3.4. *Let G be a compact connected Lie group and K a closed, connected subgroup, let $\widetilde{G}$ be the universal compact cover of G (see Theorem B.4.4), and $\widetilde{K}$ the identity component of the preimage of K in $\widetilde{G}$, and let F be the kernel of $p\colon \widetilde{G} \longrightarrow G$. If $|F/(F \cap \widetilde{K})|$ is invertible in k, then*

$$H^*(G/K) \cong H^*(\widetilde{G}/\widetilde{K}).$$

Proof. By Theorem B.4.4, F is central, so taking $\widetilde{G} = \Gamma$ and $H_0 = \widetilde{K}$ in the statement of Proposition B.3.3 we have $\Gamma/H_0 = \widetilde{G}/\widetilde{K}$ and $\Gamma/FH_0 \approx G/p(\widetilde{K}) = G/K$ and the result follows.. □

The two preceding lifting results are too simple not to have been known, yet the author knows no reference.

Proposition B.3.5. *Let $F \to X \to B$ be a finite-sheeted covering. If either of the Euler characteristics $\chi(X), \chi(B)$ is finite, then so is the other, and $\chi(X) = \chi(B) \cdot |W|$.*

Proof sketch. Taking a CW approximation, we may assume X and B to be CW complexes and $X \to B$ cellular. Each cell of B is covered by $|F|$ cells in X, so the result follows from cellular homology. □

B.4. The structure of compact Lie groups

In this section, we record—without much explanation, interstitial verbiage, or real comprehensibility—the background we require on compact Lie groups. Dwyer and Wilkinson [DW98] develop this material in an atypical algebro-topological manner concordant with the approach adopted here. Bröcker and tom Dieck [BtD85] is another standard reference.

Proposition B.4.1. *There exists a smooth map* $\exp\colon \mathfrak{g} \longrightarrow G$, *the **exponential**, which is surjective if* G *is compact and connected, which restricts to a homomorphism on the preimage of any connected abelian subgroup (in particular, on any line), and whose inverse in a neighborhood of* $1 \in G$ *serves as a smooth chart.*

Proposition B.4.2 ([Wik14])**.** *The fundamental group of a topological group is abelian.*

Theorem B.4.3 ([War71, Thm. 3.58, p. 120][GGK02, Prop. B.18, p. 179])**.** *Let* G *be a Lie group and* K *a closed subgroup. Then* G/K *is a manifold and* $K \to G \to G/K$ *a principal* K*-bundle.*

One of the main structure theorems for compact Lie groups is the following.

Theorem B.4.4 ([HM06, Thm. 2.19, p. 207])**.** *Every compact, connected Lie group* G *admits a finite central extension*

$$0 \to F \longrightarrow \widetilde{G} \longrightarrow G \to 0$$

such that $\widetilde{G}$ *is the direct product of a compact, simply-connected Lie group* K *and a torus* A*. Thus*

$$G \;\cong\; {}^{A \times K}\!\big/_{F}\,.$$

We call $\widetilde{G}$ the ***universal compact cover*** of G; it is uniquely determined up to isomorphism.[2]

Proposition B.4.5 (Élie Cartan–Wilhelm Killing)**.** *Every simply-connected Lie group K is a direct product of finitely many **simple groups**, groups whose maximal proper normal subgroups are finite. A simply-connected simple group is one of the following:*

$$\mathrm{SU}(n),\quad \mathrm{Sp}(n),\quad \mathrm{Spin}(n),\quad G_2,\quad F_4,\quad \widetilde{E}_6,\quad \widetilde{E}_7,\quad E_8,$$

with the exception of $\mathrm{Spin}(1) = \mathrm{O}(1)$ *and* $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$; *the three infinite families comprise the simply-connected **classical groups** and the last five the **exceptional groups**.*

We will not explain the exceptional groups, but the groups $\mathrm{Spin}(n)$ are double covers of $\mathrm{SO}(n)$ for $n \geqslant 3$ (when $\pi_1\mathrm{SO}(n) \cong \mathbb{Z}/2$) and $\mathrm{Spin}(2) = \mathrm{SO}(2) \cong S^1$.

A compact group finitely covered by a simply-connected group is called ***semisimple***.

Proposition B.4.6. *Let G be a compact, semisimple Lie group. Then $H^1(G;\mathbb{Q}) = 0$.*

Proof. It is an artifact of our definitions that G admits a simply-connected finite cover $\widetilde{G}$. By the universal coefficient theorem Theorem B.2.1, we have $H^1(\widetilde{G};\mathbb{Q}) \cong H_1(\widetilde{G};\mathbb{Q}) \cong H_1(\widetilde{G};\mathbb{Z}) \otimes \mathbb{Q}$, and by Proposition B.4.2 and Proposition B.2.5 we know $H_1(\widetilde{G};\mathbb{Z}) \cong \pi_1\widetilde{G}$, which we have assumed to be a finite group. □

A classification-type result in the opposite direction is that all compact Lie groups can be seen as closed subgroups of $\mathrm{GL}(n,\mathbb{C})$.

Theorem B.4.7 (Fritz Peter–Hermann Weyl [BtD85, Thm. III.4.1, p. 136])**.** *Every compact Lie group G admits a faithful representation.*

[2] This is arguably a misnomer; this object cannot be initial in that we can always cover the toral factor A with another torus, and in particular the fiber F is not uniquely determined by this characterization.

This is a corollary of the Peter–Weyl theorem, and implies in particular that every compact Lie group embeds as a closed subgroup of $\mathrm{U}(n)$ for n sufficiently large.

B.4.1. The maximal torus

A *real* ***torus*** is a Lie group smoothly isomorphic to the direct product of finitely many copies of the complex circle group $S^1 \cong \mathrm{U}(1)$; for us tori are always considered as Lie groups. A one-dimensional torus is a ***circle***. Much of the structure of the structure of compact, connected Lie groups arises due to tori they contain. The ***centralizer*** $Z_G(K)$ of a subgroup K of a group G is the set of $g \in G$ such that $gkg^{-1} = k$ for each $k \in K$.

Lemma B.4.8. *Any torus T contains a* ***topological generator****, an element generating a dense subgroup.*

Sketch of proof. Any element of $\mathbb{R}^\ell$ none of whose coordinates is a rational multiple of any other will project to such an element in $(\mathbb{R}/\mathbb{Z})^\ell \cong T$. □

Theorem B.4.9. *Let G be a compact, connected Lie group. Every torus S of G is contained in a torus T which is properly contained in no other torus; such a T is called a* ***maximal torus*** *of G. Every element lies in some maximal torus, each maximal torus is its own centralizer in G, and all maximal tori are conjugate in G.*

Given a group G, and a subgroup K of G, we write $N_G(K)$ for the ***normalizer*** of K in G, the set of elements $g \in G$ such that $gKg^{-1} = K$. The ***Weyl group*** W_G of G is defined to be the quotient group $N_G(T)/T$, the collection of nontrivial symmetries of T induced by conjugation. It is always a finite reflection group.

Lemma B.4.10. *Let $G = \prod G_j$ be a product of finitely many connected Lie groups G_j. Then the Weyl group of G is the direct product $\prod W(G_j)$.*

Proof. This is an immediate consequence of the fact the functor $(G, K) \longmapsto N_G(K)$ preserves products. □

Proposition B.4.11. *Let G be a connected, compact Lie group. Then the center $Z(G)$ is the intersection of all maximal tori in G.*

Proof [DW98, Prop. 7.1]. Any element of G lies in some conjugate gTg^{-1} of any given maximal torus T of G, so if $z \in G$, there exists $t \in T$ with $z = gtg^{-1}$, or equivalently $t = g^{-1}zg$. If $z \in Z(G)$ is central, then $g^{-1}zg = z$, so $z \in T$. So a central element lies in every maximal torus.

On the other hand, if $x \in G$ fails to lie in some maximal torus T, it has some conjugate in T, say $gxg^{-1} \neq g$, meaning $gx \neq xg$. □

B.4.2. The root and weight lattices

This section contains a terse recounting of some standard results in Lie theory we use in Section 11.6 and Appendix E.1.3 to complete the classification of equivariantly formal isotropic circle actions.

Proposition B.4.12 ([BtD85, Prop. V.(5.13), p. 214])**.** *On the Lie algebra $\mathfrak{g}$ of G there exists a symmetric bilinear form $B(-,-)$, the **Killing form**, which is invariant under adjoint action of G. This form is negative definite if G is compact.*

Sketch of proof. Each element $v \in \mathfrak{g}$ defines a linear endomorphism $\operatorname{ad} v \in \operatorname{End} \mathfrak{g}$ by $(\operatorname{ad} v)w := [v, w]$. Once a basis of $\mathfrak{g}$ is selected, a trace is well defined, and one sets $B(v, w) := \operatorname{tr}(\operatorname{ad} v \circ \operatorname{ad} w)$. □

Let G be a compact Lie group and T a maximal torus. The adjoint action of T on $\mathfrak{g}$ yields, on tensoring with $\mathbb{C}$, a representation $T \longrightarrow \operatorname{Aut}_{\mathbb{C}}(\mathfrak{g} \otimes \mathbb{C})$. Because T is abelian and $\mathbb{C}$ algebraically closed, this representation decomposes as a direct sum of joint eigenspaces $\mathfrak{g}_\vartheta$ given by

$$\mathfrak{g}_\vartheta = \{v \in \mathfrak{g} \ : \ t \cdot v = \vartheta(t)v \ \text{ for each } t \in T \ \}$$

for some continuous homomorphisms

$$\vartheta \colon T \longrightarrow S^1.$$

On the Lie algebra level, these homomorphisms correspond to ***roots***

$$\alpha \colon \mathfrak{t} \longrightarrow \mathbb{R}.$$

such that $\vartheta(\exp v) = e^{2\pi i\alpha(v)}$ for all $v \in \mathfrak{t}$. We think of these roots α as lying in the dual vector space $\mathfrak{t}^\vee := \mathrm{Hom}_\mathbb{R}(\mathfrak{t}, \mathbb{R})$. They form a finite subset $\Phi \subsetneq \mathfrak{t}^\vee$, the ***root system*** of G, spanning the ***root lattice*** $Q(G) := \mathbb{Z}\Phi < \mathfrak{t}^\vee$. A generic half-space H^+ of $\mathfrak{t}^\vee$ contains precisely half these roots, and determines a notion of positivity according to which the roots $\Phi^+ := \Phi \cap H^+$ are the ***positive roots***. An $\mathbb{N}$-basis for Φ^+ comprises a set Δ of ***simple roots*** for G, and a set of simple roots forms an $\mathbb{R}$-basis for $\mathfrak{t}^\vee$.

The negative of the Killing form defines an inner product on $\mathfrak{g}$ which we will simply denote by $v \cdot w$. This inner product defines on $\mathfrak{t}$ the ***Killing isomorphism***

$$\begin{aligned} \varkappa \colon \mathfrak{t} &\xrightarrow{\sim} \mathfrak{t}^\vee, \\ v &\longmapsto (w \mapsto v \cdot w). \end{aligned}$$

of $\mathfrak{t}$ with its dual [BtD85, p. 194], which is W-equivariant with respect to the standard action described in Proposition B.4.15. In particular, associated to each root $\alpha \in \mathfrak{t}^\vee$ is a ***coroot*** $\alpha^\vee = \frac{2}{\alpha\cdot\alpha}\varkappa^{-1}(\alpha) \in \mathfrak{t}$. The $\mathbb{Z}$-span of these is the ***coroot lattice*** $Q^\vee(G) < \mathfrak{t}$.

The roots are completely determinative of simply-connected Lie groups.

Proposition B.4.13. *A connected, compact Lie group G is characterized uniquely by its maximal torus, its center, and its roots.*

The coroot lattice is contained in the ***integer lattice*** $\Lambda(T) := \ker(\exp\colon \mathfrak{t} \to T) \cong \pi_1 T$ as the kernel [BtD85, p. 223] of a canonical surjection $\Lambda(T) \longrightarrow\!\!\!\!\rightarrow \pi_1 G$. We will need roots and coroots just for the following three results.

Proposition B.4.14. *If a Lie group G is simply-connected, then*

$$Q^\vee(G) = \Lambda(T).$$

Proposition B.4.15. *The faithful representation of the Weyl group W_G on $\mathfrak{t}$ induced by the adjoint representation and the dual representation on $\mathfrak{t}^\vee$ are generated by the* ***root reflections***

$$\begin{aligned} s_\alpha\colon v &\longmapsto v - 2\alpha(v)\alpha^\vee && \text{for } v \in \mathfrak{t}, \\ s_\alpha\colon \beta &\longmapsto \beta - 2\frac{\alpha\cdot\beta}{\alpha\cdot\alpha}\alpha && \text{for } \beta \in \mathfrak{t}^\vee \end{aligned}$$

in simple roots $\alpha \in \Delta$.

Corollary B.4.16. *If the roots $\alpha \in \Phi(G)$ all satisfy $\alpha\cdot\alpha = 2$, as when $G \in \{A_n, E_6, D_4\}$, then Killing isomorphism $\varkappa\colon \mathfrak{t} \xrightarrow{\sim} \mathfrak{t}^\vee$ takes $Q^\vee(G)$ isomorphically onto $Q(G)$.*

The roots and the Weyl group determine a canonical decompositions of $\mathfrak{t}$ and $\mathfrak{t}^\vee$.

Definition B.4.17. Let $\Delta = \{\alpha_1, \ldots, \alpha_{\operatorname{rk} G}\}$ be a collection of simple roots for a Lie group G, and let $(\varepsilon_j) \in \{\pm\}^{\operatorname{rk} G}$ be a sequence of signs. Any intersection of the half-planes $\alpha_j^{-1}(\varepsilon_j \mathbb{R}^+)$ is a ***Weyl***

chamber and its closure $\bigcap_j (\varepsilon_j \alpha_j)^{-1}[0,\infty)$ is a ***closed Weyl chamber***. The chamber corresponding to all positive signs, $\bigcap_j \alpha_j^{-1}(\mathbb{R}^+)$, is the ***positive*** or *fundamental* Weyl chamber, and likewise for the closed Weyl chambers. Since given any root α, its opposite $-\alpha$ is also a root, for any Weyl chamber $\mathring{C}$, there is a system of simple roots Δ such that $\mathring{C}$ is the positive Weyl chamber.

Proposition B.4.18 ([Ada69, Thm. 5.13, Cor. 5.16])**.** *Let G be a compact, connected Lie group and W its Weyl group. The Weyl group W acts simply transitively on the set of Weyl chambers, and given any closed Weyl chamber C, each orbit of the adjoint action of W on $\mathfrak{t}\backslash\{0\}$ meets C in precisely one point.*

Appendix C

Classical results on the cheap

Every story has a beginning, however humble. In this appendix, which originated this work, we apply the Atiyah–Bott/Berline–Vergne localization theorem in a calculated manner to achieve simple proofs of classical results. There was no natural place to include these results earlier, as it would not have made sense to do so before developing (1) the cohomology ring $H^*(BT)$, which led into the cohomology of homogeneous spaces (Chapter 7), or (2) the Borel localization theorem, which led naturally into the discussion of equivariant formality of torus actions (Chapter 11); still, the author felt it should be included.

C.1. Equivariant extensions of characteristic classes

All characteristic classes (Section 7.7) admit equivariant extensions.

Lemma C.1.1. *Let G be a topological group, M a G-space, $E \to M$ a vector bundle, and $c(E) \in H^*(M)$ a characteristic class. There exists an equivariant extension $c^G(E) \in H^*_G(M)$.*

Proof. The mixing construction of Section 2.1 yields a vector bundle $E_G \to M_G$. Fix a point $e_0 \in EG$; then $i\colon x \mapsto [e_0, x]$ is the fiber inclusion of the Borel fibration $M \to M_G \to BG$, and the original bundle $E \to M$ is the pullback of the induced bundle $E_G \to M_G$ under i:

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & E_G \arrow[r, two heads] \arrow[d] & BG \arrow[d, equal] \\ M \arrow[r, hook, "i"'] & M_G \arrow[r, two heads] & BG. \end{tikzcd}$$

The class

$$c^G(E) := c(E_G) \in H_G^*(M)$$

pulls back to $c(E)$ by naturality:

$$i^* c(E_G) = c(i^* E_G) = c(E). \qquad \square$$

In particular, this holds for the Euler class of an oriented vector bundle: given an oriented vector bundle $\nu\colon E \to M$ over a G-manifold M, with fiber of dimension n, the Euler class $e(\nu) \in H^n(M)$ as defined in Section 7.7 has an equivariant extension $e^G(\nu) \in H_G^n(M)$, called the ***equivariant Euler class***.

C.2. The Berline–Vergne/Atiyah–Bott localization theorem

In this section we provide a brief statement of a simple version of the Atiyah–Bott formulation of the Berline–Vergne/Atiyah–Bott localization theorem, which suffices to recover some classical resutls.

Recall from (9.1) that if T acts trivially on a manifold N, then $H_T^*(N) \cong H_T^* \otimes H^*(N)$ as an H_T^*-algebra. As H_T^* is a polynomial ring $\mathbb{Z}[u_1, \ldots, u_{\dim T}]$ by (Section 7.4), one can view elements of $H_T^*(N)$ as polynomials with coefficients in $H(N)$:

$$H_T^*(N) \cong H^*(N)[u_1, \ldots, u_{\dim T}].$$

The classical evaluation map (or *Kronecker pairing*)

$$H^n(N) \times H_n(N) \longrightarrow \mathbb{Z},$$

induced by evaluating a singular cochain on a chain, then extends to an evaluation map

$$H_T^*(N) \times H_*(N) \longrightarrow \mathbb{Z}[u_1, \ldots, u_{\dim T}]$$

given by extending additively the maps

$$\begin{aligned}\big(H^n(N)[u_1, \ldots, u_{\dim T}]\big) \times H_n(N) &\longrightarrow \mathbb{Z}[u_1, \ldots, u_{\dim T}], \\ (a \cdot u^I, x) &\longmapsto a(x) \cdot u^I.\end{aligned}$$

Rather than the traditional angle-bracket pairing, we denote evaluation simply by juxtaposition. If we extend the coefficient ring H_T^* to its localization $\widehat{H}_T^* \cong \mathbb{Q}(u_1, \ldots, u_{\dim T})$, then one similarly has a pairing

$$\widehat{H}_T^*(N) \times H_*(N; \mathbb{Q}) \longrightarrow \mathbb{Q}(u_1, \ldots, u_{\dim T}),$$

which obeys the following powerful and influential theorem, the "hammer" spoken of in the introduction.

Theorem C.2.1 (Berline–Vergne/Atiyah–Bott localization [BV82][AB84])**.** *Let M be a compact, oriented manifold admitting an action by a torus T. Suppose $a \in H^{\dim M}(M; \mathbb{Q})$ is a cohomology class admitting a T-equivariant extension $a^T \in H_T^{\dim M}(M; \mathbb{Q})$. For each connected component N of the fixed point set M^T, write i_N for the inclusion $N \hookrightarrow M$, write ν_N for the normal bundle to N in M, with orientation induced from M, write $e^T(\nu_N)$ for the T-equivariant Euler class of ν_N, and denote homological* *fundamental classes with brackets. Then*

$$a[M] = \sum_{N \subseteq M^T} \frac{i_N^* a^T}{e^T(\nu_N)}[N].$$

Remarks C.2.2. *(a)* The left-hand side of the display lies in H_T^*, whilst the terms of the right-hand side *a priori* lie only in the field of fractions $\widehat{H}_T^*$; it is part of the theorem that the sum of these rational functions is a polynomial.

(b) Evaluation against $[N]$ annihilates all of $H_T^*(N)$ except $H^{\dim N}(N) \otimes H_T^*$, so the theorem says something rather subtle about the nature of the "lower-degree" coefficients of an equivariant extension.

(c) This is a weak version of the Atiyah–Bott formulation of the theorem: the full statement instead uses cohomological *pushforwards*, on elements $a \in H_G(M;\mathbb{Q})$ on both sides of the equation. The Berline–Vergne formulation uses the Cartan model, replaces evaluation with integration, and evaluates the resulting equivariant forms at a vector $X \in \mathfrak{t}$. Given how exclusively this dissertation has relied on the Borel model, it made sense to opt for the less analytic formulation here.

C.3. The hammer applied

As promised, in this section we reobtain some classical theorems by application of Berline–Vergne/Atiyah–Bott localization.

Proposition C.3.1 (Hopf–Samelson, [HS40, pp. 240–251]). *Let G be a compact Lie group and H a subgroup of lesser rank. Then the Pontrjagin numbers and Euler characteristic of G/H vanish.*

Proof. Let T be a maximal torus in G and consider the equivariant cohomology $H_T^*(G/H)$ under the left action of T. By Lemma C.1.1, the characteristic classes of the tangent bundle $E = T(G/H)$ have equivariant extensions $c^T(E)$.

For a characteristic class $c \in H^{\mathrm{top}}(G/H;\mathbb{R})$, with corresponding characteristic number $c[G/H]$, by Theorem C.2.1,

$$c[G/H] = \sum_{N \subseteq (G/H)^T} \frac{i_N^* c^T}{e^T(\nu_N)}[N]$$

which by Corollary 2.4.5 is an empty sum. □

The localization formula also allows us a simple way to calculate Euler characteristics in the event a space admits a torus action with isolated fixed points.

Proposition C.3.2. *Let T be be a torus and M a compact T-manifold with isolated fixed points. Then the Euler characteristic $\chi(M)$ is $|M^T|$, the number of fixed points.*

Proof [Mei06, Example 9.5]. Write ν_p for the normal bundle to a singleton $\{p\} \subseteq M$, and $i_p \colon \{p\} \hookrightarrow M$ for the inclusion. By Theorem C.2.1,

$$\chi(M) = e(TM)[M] = \sum_{p \in M^T} \frac{i_p^* e^T(TM)}{e^T(\nu_p)}.$$

But $i_p^* TM \cong T_p M = \nu_p$ (since $\{0\} = T_p\{p\} \leqslant T_p M$), so by naturality,

$$i^* e^T(TM) = e^T\big(i^*(TM)\big) = e^T(\nu_p),$$

and we arrive at the less intimidating formula

$$\chi(M) = \sum_{p \in M^T} 1.$$ □

This formula provides a slightly different explanation for the observation that $\chi(G/H) = 0$ when H is not of full rank in G, a result usually obtained by use of the Serre spectral sequence, since by Corollary 2.4.5, the action of a maximal torus T of G on G/H has no fixed point. One can also obtain this result from an application of

Proposition C.3.2 also allows us to cheaply reclaim a familiar result on homogeneous spaces, using the fixed point set of the left action of T on G/H calculated in Lemma 2.4.4.

Proposition C.3.3 (Leray)**.** *Let G be a compact Lie group and H a subgroup containing a maximal torus T. Then*

$$\chi(G/H) = |W_G| \,/\, |W_H|,$$

where $W_G = N_G(T)/T$ is the Weyl group of G. In particular, $\chi(G/T) = |W_G|$.

Proof. Use Proposition C.3.2, counting fixed points with Lemma 2.4.4. □

Historical remarks C.3.4. It was this writer's advisor Loring W. Tu who suggested to him Proposition C.3.1 might be true and might admit of an equivariant proof. As far as we are aware, this proof is new; however, given that our initial literature search failed to uncover that this result was over seventy years old, the reader could be forgiven for imagining the equivariant proof might also not be original. The equivariant proof of Proposition C.3.2 was also independently obtained, despite this proof being so standard as to be found in introductory expository accounts [Mei06].

In fact, the author seems to recall something stronger than Proposition C.3.1 holds, although he cannot recall the reference: the tangent bundle of such a homogeneous space is stably parallelizable. One can find the analogous observations for Chern classes of stable almost complex structures of homogeneous spaces G/S, for S a torus, not necessarily maximal, in the last paper of the Borel–Hirzebruch trilogy on characteristic classes of homogeneous spaces [BH58; BH59; BH60].

According to Dieudonné's history [Die09], the result $\chi(G/T) = |W_G|$ was first proven by Weil in 1935 [Wei35, p. 520] and rediscovered by again by Hopf and Samelson [p. 251][HS40] (they write $W = \mathfrak{S}$ and $\sigma = |W|$). In fact, they prove the full Proposition C.3.3 at the end of this paper.

Appendix D

Borel's proof of Chevalley's theorem

We will require a very little standard material on sheaves and sheaf cohomology to proceed ([War71, Ch. 5], [ET14, Sec. 2]).

D.1. Sheaf cohomology

Taking as known the concepts of sheaf, constant sheaf, fine sheaf, acyclic sheaf, resolution, and sheaf cohomology, let k be a principal ideal domain and $\underline{k}$ the constant sheaf in the rest of this subsection.

Definition D.1.1. Let $\mathscr{C}^\bullet$ be any acyclic resolution of the constant sheaf $\underline{k}$ on a paracompact topological space X. Let $\mathscr{A}$ be any other sheaf on X. Then the ***sheaf cohomology*** $H^*(X;\mathscr{A})$ is the cohomology of the complex

$$0 \to (\mathscr{C}^0 \otimes \mathscr{A})(X) \longrightarrow (\mathscr{C}^1 \otimes \mathscr{A})(X) \longrightarrow (\mathscr{C}^2 \otimes \mathscr{A})(X) \longrightarrow \cdots$$

of k-modules of global sections. In particular, the sheaf cohomology $H^*(X;\underline{k})$ is $H^*\big(\mathscr{C}^\bullet(X)\big)$.

Proposition D.1.2. *Let X be a topological space homotopy equivalent to a finite CW complex. Then the* *singular cohomology and sheaf cohomology rings*

$$H^*(X;k) \cong H^*(X;\underline{k})$$

are isomorphic.

Definition D.1.1 holds if the resolution $\mathscr{F}_\bullet$ is a fine sheaf of $\mathbb{R}$-DGAs on a paracompact space X, and it is possible to find these.

Proposition D.1.3 (Cartan)**.** *Let X be a paracompact space. Then there exists a fine sheaf of $\mathbb{R}$-CDGAs resolving the constant sheaf $\underline{\mathbb{R}}$.*

Proposition D.1.4. *Let $\mathscr{F}$ be a sheaf of k-DGAs and $\mathscr{G}$ a fine sheaf of k-DGAs on a paracompact topological space X. Then the sheaf tensor product $\mathscr{F} \otimes \mathscr{G}$ is again fine.*

This has the following somewhat surprising corollary.

Corollary D.1.5. *Let $\mathscr{F}$ be a sheaf of torsion-free k-DGAs and $\mathscr{G}$ a fine sheaf of k-DGAs on a paracompact topological space X. Then the canonical sheaf map $\mathscr{G} \longrightarrow \mathscr{F} \otimes \mathscr{G}$ induces isomorphisms between the cohomology rings of the DGAs $\mathscr{G}(X)$ and $(\mathscr{F} \otimes \mathscr{G})(X)$ with the expected differentials.*

We will need to apply this result only in a rather specific case.

Example D.1.6. Let $\pi\colon E \longrightarrow B$ be a smooth fiber bundle with compact total space. Let Ω_E be the sheaf de Rham complex on E, assigning to each open $U \subseteq E$ the de Rham complex $\Omega(U)$; this is a fine sheaf. Let Ω_B be the sheaf de Rham complex on the base space B, and $\pi^*\Omega_B$ its pullback to E: given $U \subseteq B$, one assigns

$$(\pi^*\Omega_B)(U) = \Omega_B\big(\pi(U)\big).^{1}$$

[1] This simplification of the usual definition suffices because π is an open map: since $\pi(U)$ is itself open, the direct limit $\varinjlim_{V \supseteq \pi(U)} \Omega_B(V)$ is just $\Omega_B\big(\pi(U)\big)$ again.

Now we can formally construct the tensor product $\mathscr{C} := \pi^*\Omega_B \otimes \Omega_E$, which will be another fine sheaf of $\mathbb{R}$-DGAs on E by Corollary D.1.5. It follows the de Rham cohomology $H^*(E;\mathbb{R})$ is the cohomology of the complex

$$A = \mathscr{C}(E) = \big(\pi^*\Omega_B \otimes_{\mathbb{R}} \Omega_E\big)(E).$$

This is disorienting, because it looks much as if we are claiming the cohomology of the complex $A' = \pi^*\Omega(B) \otimes \Omega(E)$ is $H^*(E)$, whereas by the Künneth theorem Corollary A.3.10 and the fact that π^* is injective on forms (so that $\pi^*\Omega(B) \cong \Omega(B)$), we should have

$$H^*(A') = H^*(B) \otimes H^*(E).$$

Our escape is that the sheaf $\mathscr{C}$ is radically different from the presheaf $U \longmapsto \Omega\big(\pi(U)\big) \otimes \Omega(U)$ on E it is associated to, so that A and A' are far from being quasi-isomorphic.

Since a sheaf is determined completely by its stalks, it suffices to recall that at some (hence any) $x \in E$, the stalk is

$$\big(\pi^*\Omega_B \otimes \Omega_E\big)|_x := \Omega_B|_{\pi(x)} \otimes \Omega_E|_x.$$

So a form $\tau \otimes \omega \in \mathscr{C}(U) = \Omega_B\big(\pi(U)\big) \otimes \Omega_E(U)$ becomes zero in the stalk if and only if $\tau = 0$ on some sufficiently small neighborhood of $\pi(x)$ or else $\omega = 0$ on a neighborhood of x. Since global sections of $\mathscr{C}$ are pieced together from stalks, a tensor $\tau \otimes \omega$ of forms represents zero if and only if τ and ω are never simultaneously nonzero.

Analogously, one has the following.

Proposition D.1.7. *Let $\pi \colon E \longrightarrow B$ be a smooth fiber bundle with compact total space, $\mathscr{B}$ be a fine sheaf* *of $\mathbb{R}$-DGAs on B, and $\mathscr{E}$ a fine sheaf of $\mathbb{R}$-DGAs on E. Then*

$$(\pi^*\mathscr{B}\otimes\mathscr{E})(E) \;\cong\; {(\pi^*\mathscr{B})(B)\otimes\mathscr{E}(E)}\Big/\Big(\big\{\tau\otimes\omega : \operatorname{supp} b\cap\pi(\operatorname{supp} e)=\varnothing\big\}\Big).$$

We will need only two more results on sheaf cohomology.

Proposition D.1.8. *Let $\mathscr{F}$ be a sheaf of k-DGAs and $\mathscr{G}$ an acyclic sheaf of k-DGAs on topological space X. Then the natural sheaf map $\mathscr{F}\longrightarrow\mathscr{F}\otimes\mathscr{G}$ induces isomorphisms in sheaf cohomology.*

Definition D.1.9. Let X be a topological space and $\mathscr{F}$ a sheaf of k-modules on X. Suppose $\mathscr{C}^\bullet$ is a resolution of $\underline{k}$, so that

$$0\to\underline{k}\longrightarrow\mathscr{C}^0\longrightarrow\mathscr{C}^1\longrightarrow\mathscr{C}^2\longrightarrow\cdots$$

is an exact sequence of sheaves. Then the ***cohomology sheaf*** $\mathscr{H}^p(\mathscr{F})$ is the sheaf given by the p^{th} cohomology of the complex of sheaves

$$0\to\mathscr{C}^0\otimes\mathscr{F}\longrightarrow\mathscr{C}^1\otimes\mathscr{F}\longrightarrow\mathscr{C}^2\otimes\mathscr{F}\longrightarrow\cdots;$$

proceeding stalkwise, one has

$$\mathscr{H}^p(\mathscr{F})_x = {\ker(\mathscr{C}^p|_x\otimes\mathscr{F}|_x\longrightarrow\mathscr{C}^{p+1}|_x\otimes\mathscr{F}|_x)}\Big/{\operatorname{im}(\mathscr{C}^{p-1}|_x\otimes\mathscr{F}|_x\longrightarrow\mathscr{C}^p|_x\otimes\mathscr{F}|_x)}.$$

Historical remarks D.1.10. Cartan's original proof of the existence of a fine resolution of $\mathbb{R}$-DGAs was never published, part of a veritable rash of non-publication that afflicted the work of this school and made this exposition a bit difficult more difficult than otherwise it might have been.

D.2. The Leray spectral sequence

We paraphrase Borel's 1951 ETH exposition of the Leray spectral sequence [Bor51, Exposé VII-3].

Let $f\colon X \longrightarrow Y$ be a map of spaces, with Y paracompact, and let $\mathscr{F}$ be a fine sheaf of k-DGAs on X and $\mathscr{G}$ a fine sheaf of k-DGAs on Y. Then the pushforward $f^*\mathscr{F}$ is again a sheaf on Y, and the sheaf $\mathscr{G} \otimes f_*\mathscr{F}$ is again fine, so that the sheaf cohomology $H^*(X; f^*\mathscr{G} \otimes \mathscr{F})$ is just $H^*(X;k)$. Now $\mathscr{F}$ and $f^*\mathscr{G}$ are k-DGAs, so $f^*\mathscr{G} \otimes \mathscr{F}$ admits a natural bigrading

$$(\mathscr{G} \otimes f_*\mathscr{F})^\ell = \bigoplus_{p+q=\ell} \mathscr{G}^p \otimes f_*\mathscr{F}^q,$$

and one can filter it by p:

$$F_p(\mathscr{G} \otimes f_*\mathscr{F})\colon U \longmapsto \bigoplus_{i \geqslant p} (\mathscr{G}^i \otimes f_*\mathscr{F}^\bullet)(U).$$

We now restrict to global sections and consider the filtration spectral sequence of the horizontal filtration

$$F_p(\mathscr{G} \otimes f_*\mathscr{F})(Y) := \bigoplus_{i \geqslant p} (\mathscr{G}^i \otimes f_*\mathscr{F})(Y).$$

as described in Corollary A.5.4.

To make this sequence usable requires a bit of unpacking. The global sections of $\mathscr{G} \otimes f_*\mathscr{F}$ and of $f^*\mathscr{G} \otimes \mathscr{F}$ are both given by

$$(\mathscr{G} \otimes f_*\mathscr{F})(Y) \cong (f^*\mathscr{G} \otimes \mathscr{F})(X) \ \cong \ {\mathscr{G}(Y) \otimes \mathscr{F}(X)}\Big/\Big(\{\tau \otimes \omega : f^{-1}(\operatorname{supp} \tau) \otimes \operatorname{supp} \omega = \varnothing\}\Big) \tag{D.1}$$

as described in Proposition D.1.7. Since $f^*\mathscr{G} \otimes \mathscr{F}$ is a fine sheaf on X, we have $H^*(X; f^*\mathscr{G} \otimes \mathscr{F}) \cong H^*(X;k)$ by Proposition D.1.2 and hence $E_\infty^{p,q} \cong \operatorname{gr}_p H^{p+q}(X;k)$, where associated graded modules are taken with respect to the filtration

$$F_p(f^*\mathscr{G} \otimes \mathscr{F})(X) := \bigoplus_{i \geqslant p} (f^*\mathscr{G}^i \otimes \mathscr{F})(Y).$$

on $(f^*\mathscr{G} \otimes \mathscr{F})(X)$ lifting the filtration on $(\mathscr{G} \otimes f_*\mathscr{F})(Y)$.

Because $\mathscr{G}$ is already graded, the associated graded construction changes nothing on the algebra level, so

$$E_0 = (\mathscr{G} \otimes f_*\mathscr{F})(Y)$$

again. The differential d_0 is induced by the differential $d_{\mathscr{F}}$ of $\mathscr{F}$ and is zero on $\mathscr{F}$, so we have

$$E_1^{p,q} = \big(\mathscr{G}^p \otimes \mathscr{H}^q(f_*\mathscr{F})\big)(Y).$$

These are global sections of a product sheaf, as in Proposition D.1.7; support of an element of $\mathscr{H}^q(f_*\mathscr{F})(Y)$ is determined by which stalks it vanishes on. The differential d_1 is zero on $\mathscr{H}^q(f_*\mathscr{F})$ and extends the differential $d_{\mathscr{G}}$ of $\mathscr{G}$. Recall from Definition D.1.1 that since $\mathscr{G}$ is acyclic, sheaf cohomology on Y with coefficients in any sheaf $\mathscr{A}$ is given by

$$H^*(Y;\mathscr{A}) = H^*\big((\mathscr{G}^\bullet \otimes \mathscr{A})(Y); d_{\mathscr{G}} \otimes \mathrm{id}_{\mathscr{A}}\big);$$

in particular one finds $E_2^{p,q}$ is the sheaf cohomology

$$E_2^{p,q}(Y) \cong H^p\big(Y;\mathscr{H}^q(f_*\mathscr{F})\big).$$

To understand this last better, note that the pushforward $f_*\mathscr{F}$ is the sheaf whose stalk at $y \in Y$ is the direct limit of $\mathscr{F}(U)$ over neighborhoods U of $f^{-1}\{y\}$, so

$$(f_*\mathscr{F})|_y = H^*\big(f^{-1}\{y\};k\big).$$

Now, cohomology with coefficients in $f_*\mathscr{F}$ and in $\mathscr{G}\otimes f_*\mathscr{F}$ are the same by Corollary D.1.5, and we can write by abuse of notation

$$\mathscr{E}_2^{p,q}(Y) = H^p\Big(Y;H^q\big(f^{-1}\{y\}\big)\Big).$$

Essentially, this is the cohomology of Y with coefficients varying over the cohomology of the fibers. This spectral sequence (E_r, d_r) is the ***Leray spectral sequence*** of the map $f\colon X \longrightarrow Y$.

In the event $f\colon X \longrightarrow Y$ was a bundle, the fibers are homeomorphic, so the cohomology groups $H^*(F)$ of individual fibers are isomorphic, related to one another by isomorphisms $\gamma_*\colon H^*(E|_{\gamma(0)}) \longrightarrow H^*(E|_{\gamma(1)})$ induced by lifting paths $\gamma\colon [0,1] \longrightarrow Y$ in the base to homeomorphisms between fibers. Thus the Leray spectral sequence of a bundle projection $\pi\colon E \longrightarrow B$ is the Serre spectral sequence of the bundle $F \to E \to B$, at least after the E_2 page. Just as with the Serre spectral sequence, if the coefficients are trivial—and critically for us, if $F = G$ is a group and the bundle is principal, by Proposition 4.3.6—the coefficient sheaf in E_2 can be seen as constant, and then if $H^*(G)$ is a free k-module, we have $E_2 = H^*(B)\otimes H^*(G)$.

We summarize this discussion:

Theorem D.2.1 (Leray). *Let $f\colon X \longrightarrow Y$ be a map of spaces, with Y paracompact. Let $\mathscr{F}$ be a fine sheaf of k-DGAs on X and $\mathscr{G}$ a fine sheaf of k-DGAs on Y. Equip the sheaf $\mathscr{G}\otimes f_*\mathscr{F}$ with the horizontal filtration induced by the grading of $\mathscr{G}$. Then associated to the map f is a spectral sequence of k-DGAs satisfying*

- $E_0^{p,q} \cong (\mathscr{G}^p \otimes f_*\mathscr{F}^q)(Y)$,

- $E_2^{p,q} \cong H^p\big(Y; \mathscr{H}^q(f_*\mathscr{F})\big)$,
- $E_\infty^{p,q} \cong \operatorname{gr}_p H^{p+q}(X;k)$.

In the event $f\colon X \longrightarrow\!\!\!\!\!\rightarrow Y$ is the projection of a fiber bundle, this sequence agrees with the Serre spectral sequence after E_2; if further $G \to X \xrightarrow{f} Y$ is a principal G-bundle for G a topological group, then

$$E_2 \cong H^*(Y) \otimes H^*(G).$$

Remark D.2.2. It is because the Serre spectral sequence is a very special instance of the Leray spectral sequence that it is often called the *Leray–Serre spectral sequence*. The Leray spectral sequence of a fibration a priori contains strictly more information than the Serre spectral sequence; it is precisely because the entire Leray spectral sequence contains such a surfeit of data that Serre's paring it down to a spectral sequence of singular cohomology groups in the event of a fibration revolutionized homotopy theory.

D.3. Borel's proof

In this section, we provide a proof of Chevalley's theorem close to Borel's original. Most of it is in the setup; once the relevant DGAs are defined, the quasi-isomorphisms are nearly immediate.

Let $k = \mathbb{R}$, let G be a compact, connected Lie group, and let $G \to E \xrightarrow{\pi} B$ be a smooth principal G-bundle. Write $P = PG$ for the space of primitives of $H^*(G) = H^*(G;\mathbb{R})$, so that $H^*(G) \cong \Lambda P$. Fix a transgression

$$\tau\colon P \xrightarrow{\sim} QH^*(BG) \rightarrowtail H^*(BG).$$

As $\pi\colon E \to B$ is a principal G-bundle, there is a classifying map $\chi\colon B \longrightarrow BG$. Let $[x_j]$ be a basis of P and $[b_j] = \chi^*\tau(x_j) \in H^*(B)$ for each j.

Let $\mathscr{B}$ be an fine sheaf of $\mathbb{R}$-CDGAs resolving the constant sheaf $\underline{\mathbb{R}}$ on B, as guaranteed by Proposition D.1.3 and likewise $\mathscr{E}$ be a fine sheaf of $\mathbb{R}$-CDGAs resolving the constant sheaf $\underline{\mathbb{R}}$ on E, so that by Definition D.1.1 and Proposition D.1.2,

$$H^*(B) \cong H^*\big(\mathscr{B}(B)\big); \qquad H^*(E) \cong H^*\big(\mathscr{E}(E)\big).$$

We can pull $\mathscr{B}$ back to a sheaf $\pi^*\mathscr{B}$ on E and then the tensor product $\pi^*\mathscr{B} \otimes \mathscr{E}$ is another fine sheaf on E. If we set $C = (\pi^*\mathscr{B} \otimes \mathscr{E})(E)$ with the expected differential, then by Corollary D.1.5,

$$H^*(C) = H^*\big((\pi^*\mathscr{B} \otimes \mathscr{E})(E)\big) \cong H^*(E)$$

as well. Recall from Proposition D.1.7 that C can be seen as the quotient of $(B) \otimes \mathscr{E}(E)$ by the ideal $\mathfrak{n}$ spanned by elements of empty support; accepting

By Theorem 7.6.2, the classes $[x_j] \in PG$ are universally transgressive, which in particular means in this instance they transgress in the filtration spectral sequence (E_r, d_r) of C as filtered by

$$C^p := \bigoplus_{i \geqslant p} (\pi^*\mathscr{B}^i \otimes \mathscr{E})(E).$$

By Theorem D.2.1, this is a version of the Leray spectral sequence of $\pi\colon E \longrightarrow B$, which from $E_2 \cong H^*(B) \otimes H^*(G)$ on, is isomorphic to the Serre spectral sequence of this bundle. Thus, as discussed in Proposition 4.3.15, the transgression of the primitive classes $[x_j] \in PG$ means there exist elements $c_j \in C$ such that $d_C c_j = \pi^* b_j \otimes 1 \pmod{\mathfrak{n}}$,

These transgressive cochains allow us to compile a simpler model C' of $H^*(E)$ as in the previously cited version Theorem 8.1.3 of Theorem D.3.1. As ΛP is a free CGA, we can lift it to a subalgebra $\Lambda\{x_j\}$ of $(\pi^*\mathscr{B} \otimes \mathscr{E})(E)$ generated by global sections x_j of $\pi^*\mathscr{B} \otimes \mathscr{E}$. Let $C' =$

$\mathscr{B}(B) \otimes \Lambda\{x_j\}$, with differential the unique antiderivation $d_{C'}$ satisfying

$$d_{C'}(b \otimes 1) = d_{\mathscr{B}} b \otimes 1, \qquad d_{C'}(1 \otimes x_j) = b_j \otimes 1$$

and filtration

$$(C')^p := \bigoplus_{i \geqslant p} \mathscr{B}^i(B) \otimes H^*(G).$$

Then the map

$$\begin{aligned} \lambda\colon C' &\longrightarrow C : \\ b \otimes 1 &\longmapsto \pi^* b \otimes 1, \\ 1 \otimes [x_j] &\longmapsto c_j \end{aligned}$$

is a filtration-preserving DGA homomorphism, which we hope to show is a quasi-isomorphism.

Theorem D.3.1 (Chevalley)**.** *This map λ is a quasi-isomorphism completing a commutative diagram*

$$\begin{tikzcd} & H^*(C') \arrow[dr] \arrow[dd, "\lambda^*"] & \\ H^*(G) \arrow[ur] \arrow[dr] & & H^*(B). \\ & H^*(E) \arrow[ur] & \end{tikzcd}$$

Proof (Borel). Apply the filtration spectral sequence of (Corollary A.5.4) to both DGAs and the map λ^*. The spectral sequence (E_r, d_r) of C is the Leray spectral sequence of $\pi\colon E \longrightarrow B$, as discussed in Appendix D.2, by the identification (D.1). Write $({}'E_r, {}'d_r)$ for the spectral sequence of

C'. The 0^{th} page is the associated graded algebra of the filtration:

$$'E_0^{p,\bullet} = \mathscr{B}^p(B) \otimes H^*(G).$$

Since $\deg x_j \geqslant 1$, we have $\deg b_j \geqslant 2$, so $d_{C'}$ increases the filtration degree of each element of $H^*(G)$ by at least 2, and the filtration degree of elements of $\mathscr{B}(B)$ by 1. Thus no image of $d_{C'}$ survives the associated graded procedure, so $'d_0 = 0$ and

$$'E_1 = {}'E_0 = \mathscr{B}(B) \otimes H^*(G).$$

The differential $'d_1$ still sends generators of $H^*(G)$ at least two filtration degrees forward, but acts as $d_{\mathscr{B}}$ on $\mathscr{B}(B)$, so that $'d_1 = \delta_{\mathscr{B}} \otimes \mathrm{id}_{H^*(G)}$ and

$$'E_2 \cong H^*(B) \otimes H^*(G).$$

Thus $'E_2 \cong E_2$; it just remains to see the map $\lambda_2 \colon {}'E_2 \longrightarrow E_2$ itself is such an isomorphism in a manner making the diagram commute. But $1 \otimes [x_j] \in C'$ and $1 \otimes x_j \pmod{\mathfrak{n}} \in C$ both become $1 \otimes [x_j]$ in $H^*(B) \otimes H^*(G)$, and $b \otimes 1 \in C'$ and $b \otimes 1 \pmod{\mathfrak{n}} \in C$ both become $[b] \otimes 1$ in $H^*(B) \otimes H^*(G)$. □

Historical remarks D.3.2. The proof presented above is in terms of a historically late formulation of Leray's technology; there were several such accounts, of gradually improving comprehensibility. The entirety of the account that follows is derived from the work of Borel expositing Leray's work, both in 1951 and 1997 [Bor51; Bor98].

Leray's original motivation for the topological edifice he erected seems to have been the de Rham complex. This is an $\mathbb{R}$-CGA of forms supported on various subsets, yielding a complex which Poincaré already had shown to be trivial on Euclidean subsets, but which collate together nonetheless to contain global information about a manifold, as conjectured by Élie Cartan and proven by Georges de Rham in his thesis. Recall that the support of forms ω, τ on a manifold M satisfies these axioms:

$$\operatorname{supp}(\tau + \omega) \subseteq \operatorname{supp} \tau \cup \operatorname{supp} \omega; \qquad \operatorname{supp}(\tau \wedge \omega) \subseteq \operatorname{supp} \tau \cap \operatorname{supp} \omega; \qquad \operatorname{supp}(\alpha \cdot \omega) \subseteq \operatorname{supp} \omega;$$
$$\operatorname{supp} d\omega \subseteq \operatorname{supp}(\omega); \qquad \operatorname{supp} 0 = \varnothing; \qquad \operatorname{supp} 1 = M.$$

Leray's idea, beautiful in its audacity, is to equip a topological space X with a complex (*complexe concrete*) K of "forms on a space," equipped with a *support function*

$$\begin{aligned} K &\longrightarrow \{\text{closed subsets of } X\}, \\ k &\longmapsto |k|, \end{aligned}$$

satisfying the same axioms as differential forms despite the absence of any native notion of smoothness, and despite elements of k not being functions in a real sense. As a purely algebraic object, a complex is a CGA over a commutative coefficient ring (which we will write as A to allow $k \in K$); only the support function imparts any topological content.

With this setup, and some further definitions, Leray is able to reprove a good amount of existing algebraic topology as of 1945, proving that the cohomology of certain types of complexes recovers Hopf's and Samelson's theorems on Lie groups, the Lefschetz fixed-point theorem, the Brouwer fixed-point theorem, invariance of domain, Poincaré duality, and Alexander duality. Building up *couvertures* (defined below) associated to nerves of a cofinal sequence of closed covers of a topological space, Leray can show this cohomology is isomorphic to Čech cohomology on compact Hausdorff spaces X.

Here are some of those further definitions. If $F \subseteq X$ is a closed subset, the ***intersection*** $F.K$ is defined to be the quotient

$$F.K := K/\{k \in K : |k| \cap F = \varnothing\}$$

with support function $k \longmapsto |k| \cap F$, and there is a natural restriction homomorphism $K \longrightarrow F.K$. If $F = \{x\}$ is a singleton, one writes xK; these are the germs of forms if $K = \Omega(M)$ is the de Rham complex. The system of such restrictions $F \longmapsto F.K$ becomes a ***sheaf*** (*faisceau*) under Leray's later (1946) definition, which should be contrasted with the modern definition depending on an open cover; Leray's definition arises because he is at this point interested in cohomology with compact supports on a locally compact space. The tensor product $K \otimes K'$ of two complexes is assigned supports

$$|z| := \{x \in X : \text{the image of } z \text{ is nonzero under } K \otimes K' \longrightarrow xK \otimes xK'\},$$

where the maps determining the support of z are the tensor products of the restriction maps discussed previously. The ***intersection*** $K \bigcirc K'$ is then given by

$$K \bigcirc K' := {}^{K \otimes K'}\!/\{z \in K \otimes K' : |z| = \varnothing\}.$$

Our Proposition D.1.7 can be seen as a later sheaf-theoretic redaction of this definition.

An A-complexe is *fine* (*fin*) if every finite cover (U_j) of X admits a partition of unity, which is a set of A-endomorphisms $\varphi_j \colon K \longrightarrow K$ such that

$$\operatorname{supp} \varphi_j(k) \subseteq \overline{U_j} \cap \operatorname{supp} k, \qquad \sum \varphi_j(k) = k$$

for all $k \in K$. An A-complex is a ***couverture*** if its stalks are acyclic: i.e., if $H^*(xK) = H^0(xK) \cong A$ for every $x \in X$. Leray's original cohomology on a normal space is that of the union of all couvertures.

The closest thing to a fine *couverture* in modern language seems to be the space of global sections of a fine resolution of the constant sheaf. The intersection $K \bigcirc \mathscr{F}$ of a sheaf and a complex is as expected, and one can take coefficients in an A-module M merely by forming the tensor product $K \otimes M$ or $K \bigcirc \mathscr{F} \otimes M$, where the elements of $A \otimes M$ are viewed as having global support. A complex K (for example, a *couverture*) can be pushed forward under proper maps $f\colon X \longrightarrow Y$ and pulled back under any map $g\colon Z \longrightarrow Y$: as algebraic structures $fK \cong K \cong g^{-1}K$, but the supports of k in fK and $g^{-1}K$ are respectively $f\big(|k|\big)$ and $g^{-1}\big(|k|\big)$.

By the time of Borel's 1951 lectures on Leray's work [Bor51], a sheaf (Borel credits this definition to Lazard) has become the *espace étalé* associated to a presheaf satisfying the gluing axioms, so essentially the modern, open-set definition. The statement of the Leray spectral sequence in these lecture notes is as follows.

Theorem D.3.3 (Leray). *Let $f\colon X \longrightarrow Y$ be a continuous map, K and L fine A-*couvertures*, M an A-algebra, and $\underline{F}$ the sheaf associated to $f(K \otimes M)$. Then there exists a spectral sequence in which*

$$E_0 = \mathrm{gr}\left(f^{-1}(L) \bigcirc K \otimes M\right), \qquad E_1 = L \bigcirc H(\underline{F}), \qquad E_2 = H\big(L \bigcirc H(\underline{F})\big),$$

(d_0 is the derivation induced by that of K, and d_1 the derivation induced by that of L) and which terminates in the associated graded algebra of $H^*(X, M)$, compatibly filtered.

Appendix E

The original classification of reflected circles in simple groups

The construction of Table 11.1.5 proceeds case by case through the Killing–Cartan classification Proposition B.4.5. We do most of this theoretically, and then computationally verify a key detail in the E_6 case.

E.1. Theoretical reflections

E.1.1. The case all circles are reflected

It is known that if one exists, a central involution of the Weyl group acts as $X \longmapsto -X$ on the Lie algebra $\mathfrak{t}$ of a maximal torus [DW01, Theorem 1.8] and further, that if there does exist a central involution, it is represented by the longest word $w_0 \in W$. We could therefore simply note that the center of W is $\mathbb{Z}/2$ for the Weyl groups of types B_n, C_n, D_{2n}, G_2, F_4, E_7, and E_8 [Kan01, Lem. 27-2, p. 283], but we can also argue directly.

Proposition E.1.1. *Let K be a finite quotient of any of the simply-connected classical simple Lie groups* $\mathrm{Spin}(2n+1)$, $\mathrm{Sp}(n)$, *and* $\mathrm{Spin}(4n)$, *or the simply-connected exceptional Lie groups of types G_2, F_4, E_7, and E_8, and S a maximal torus of K. Then S is reflected in K.*

Proof. It will be conventient to write $J := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{O}(2)$. Here are the cases.

- $\mathrm{Spin}(2n+1)$. In $\mathrm{SO}(2n+1)$, conjugations by the orthogonal block-diagonal matrices $g_1 = \mathrm{diag}\big(J,\ldots,J,[1]\big)$ and $g_{-1} = \mathrm{diag}\big(J,\ldots,J,[-1]\big)$ reflect the block-diagonal maximal torus $T = \mathrm{SO}(2)^n \times \big\{[1]\big\}$, and one of $g_{\pm 1}$ is in $\mathrm{SO}(2n+1)$. Since $\mathrm{Spin}(2n+1)$ is a double cover of $\mathrm{SO}(2n+1)$, the maximal torus $\widetilde{T}$ of $\mathrm{Spin}(2n+1)$ is reflected by Corollary 11.2.2.

- $\mathrm{Sp}(n)$. Choose as maximal torus the subgroup $T = \mathrm{U}(1)^n < \mathrm{Sp}(1)^n$ of diagonal elements with coordinates in the $\{1,i\}$-plane of $\mathbb{H}$. Then conjugation by the diagonal matrix $\mathrm{diag}(j,\cdots,j) = j \cdot I \in \mathrm{Sp}(n)$ reflects T.

- $\mathrm{Spin}(4n)$. Conjugation by the block-diagonal matrix $\mathrm{diag}(J,\ldots,J) \in \mathrm{SO}(4n)$ reflects each element of the block-diagonal maximal torus $T = \mathrm{SO}(2)^{2n}$. Since $\mathrm{Spin}(4n)$ is a double cover of $\mathrm{SO}(4n)$, the maximal torus $\widetilde{T}$ of $\mathrm{Spin}(4n)$ is reflected by Corollary 11.2.2.

- G_2. The roots are the vertices of a nested pair of regular hexagons in the plane $\mathfrak{t}^* \cong \mathfrak{t}$. The Weyl group is the dihedral group D_6 of order 12, which contains rotation by π.

- F_4. Recall [MT00, Theorem 7.4(1), p. 357] that F_4 acts transitively on the octonionic plane $\mathbb{O}\mathrm{P}^2$ with point-stabilizer $\mathrm{Spin}(9)$. A maximal torus T^4 of $\mathrm{Spin}(9)$ is reflected in $\mathrm{Spin}(9)$, so by Observation 11.5.5, it is also reflected in F_4. But T^4 is also maximal in F_4.

- E_7. Recall [Wol67, p. 285] that E_7 admits a local product subgroup $H \cong \mathrm{SO}(12)\cdot\mathrm{SU}(2)$. Since this subgroup is of rank 7, it contains a maximal torus of E_7. By Observation 11.5.5, it suffices to know this maximal torus is reflected in H. Because H is finitely covered by the direct product $\mathrm{SO}(12)\times\mathrm{SU}(2)$, it is enough, by Corollary 11.2.2, to see that a maximal torus of $\mathrm{SO}(12)\times\mathrm{SU}(2)$ is reflected. Finally, by Proposition 11.5.4, it is enough to know the maximal tori of $\mathrm{SO}(12)$ and $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$ are reflected; but we proved so above.

- E_8. Recall [Wol67, p. 285] that E_8 has a subgroup $\mathrm{SO}(16)$ of rank 8. Let T^8 be a maximal torus of E_8 lying in this SO(16). We showed above that T^8 is reflected in SO(16), so by Observation 11.5.5, it also is reflected in E_8. $\square$

E.1.2. The cases of $\mathrm{SU}(n)$ and $\mathrm{Spin}(4n+2)$

In the remaining cases, there is no Weyl element reflecting the entire maximal torus, and we have to distinguish between circles.

Proposition E.1.2. *Let $K = \mathrm{SU}(n)$ and let S be a circular subgroup. Then S is reflected if and only if the exponent multiset J of the inclusion $S \hookrightarrow T$ satisfies $J = -J$.*

Proof. Let T be the diagonal maximal torus $\mathrm{U}(1)^n \cap \mathrm{SU}(n)$ of $\mathrm{SU}(n)$. Then S is conjugate into T, and we may replace S with this conjugate, which is the image of a map $z \longmapsto \mathrm{diag}(z^{a_1}, \ldots, z^{a_n})$, where the $a_j \in \mathbb{Z}$ have greatest common divisor 1 and $\sum a_j = 0$. Write $J = \{a_1, \ldots, a_n\}$ for the multiset of exponents characterizing this embedding of S.[1]

The Weyl group of $\mathrm{SU}(n)$ is the symmetric group S_n, acting on T through permutation of coordinates, so S is reflected if and only if there exists $\sigma \in S_n$ such that $\mathrm{diag}\big(z^{a_{\sigma(1)}}, \ldots, z^{a_{\sigma(n)}}\big) = \mathrm{diag}(z^{-a_1}, \ldots, z^{-a_n})$ for all $z \in S^1$. This is the case if and only if J and $-J$ are equal as multisets. $\square$

Proposition E.1.3. *Let S be a circle in* $\mathrm{Spin}(4n+2)$. *Then S is reflected in* $\mathrm{Spin}(4n+2)$ *if and only if it is conjugate into a* $\mathrm{Spin}(4n)$ *subgroup.*

Proof. By Proposition 11.2.1, it is equivalent to work in $\mathrm{SO}(4n+2)$, where the equivalent statement is this: a circle S in $\mathrm{SO}(4n+2)$, is reflected if and only if it is conjugate into the block-diagonal subgroup $\mathrm{diag}\big(\mathrm{SO}(4n), 1, 1\big)$ of $\mathrm{SO}(4n+2)$.

For sufficiency, if $S < T$ is contained in some $\mathrm{SO}(4n) \times \{1\}^2$ subgroup, then conjugation by a block-diagonal matrix with $2n$ blocks J and one 2×2 block I induces $s \longmapsto s^{-1}$ on S.

[1] That is, as with the collection of roots of a polynomial, elements are *not* ordered, but *are* counted with multiplicity. Though the sequence $(a_1, \ldots, a_n)$ is not uniquely determined by S, the pair of multisets $\{J, -J\}$ is; equivalently, J is well-defined up to sign.

For necessity, if $S < T$ is not contained in any $\mathrm{SO}(4n)$ subgroup, there are no 0 exponents a_j in the exponent multiset $J(\phi) = \{a_1, \ldots, a_n\}$ of the embedding

$$\phi\colon S^1 \xrightarrow{\sim} S \hookrightarrow \mathrm{SO}(2)^{2n+1} \xrightarrow{\sim} (S^1)^{2n+1},$$

$$z \longmapsto (z^{a_1}, \ldots, z^{a_n}),$$

so all exponents a_j are either negative or positive. If there are $n_+(\phi)$ positive exponents and $n_-(\phi)$ negative exponents, then since $n_+(\phi) + n_-(\phi) = 2n + 1$ is odd, the numbers $n_+(\phi), n_-(\phi)$ are of *differing parity*. By Proposition 11.5.2, any reflection of S must be induced by the Weyl group $W = W_{D_{2n+1}} = H_{2n} \rtimes S_{2n+1}$, where $H_{2n} < \{\pm 1\}^{2n+1}$ is the hyperplane defined by $\prod_{j=1}^{2n+1} \varepsilon_j = 1$ and S_{2n+1} acts on T and on the normal factor H_{2n} by permuting entries. Thus an even number of exponent signs in $J(w \circ \phi)$ differ from the corresponding signs in $J(\phi)$, so any $w \in W$ preserves the parities of n_+ and n_-:

$$n_+(w \circ \phi) \equiv n_+(\phi) \pmod 2;$$

$$n_-(w \circ \phi) \equiv n_-(\phi) \pmod 2.$$

If w were to reflect S, however, we would need $J(w \circ \phi) = -J(\phi)$, so that $n_+(w \circ \phi) \equiv n_-(\phi)$ and $n_-(w \circ \phi) \equiv n_+(\phi) \pmod 2$, which is impossible since $n_+(\phi)$ and $n_-(\phi)$ are of different parity. Thus S cannot be reflected in $\mathrm{SO}(4n + 2)$. $\square$

E.1.3. E_6

We come to the final and most interesting case. In this section we allow ourselves to freely quote non-introductory material on reflection groups from the literature. Recall from Table 11.1.5 the desired conclusion:

Proposition E.1.4. *A circular subgroup S of E_6 or its universal cover $\widetilde{E}_6$ is reflected just if it is conjugate into a* $\mathrm{Spin}(8)$ *subgroup.*

To motivate this conclusion, consider the famous chain of inclusions

$$\mathrm{SU}(2) < \mathrm{SU}(3) < G_2 < \mathrm{Spin}(8) < \mathrm{Spin}(9) < F_4 < E_6 < E_7 < E_8,$$

or more specifically the subchain $\mathrm{Spin}(8) < F_4 < E_6$. The three-fold universal cover $\widetilde{E}_6 \twoheadrightarrow E_6$ restricts to covers of F_4 and $\mathrm{Spin}(8)$. Since these groups are simply-connected, they are isomorphic to the identity components of their preimages, so we get a lifted chain $\mathrm{Spin}(8) < F_4 < \widetilde{E}_6$. Because a maximal torus T^4 of $\mathrm{Spin}(8)$ is reflected in $\mathrm{Spin}(8)$, it is also reflected in E_6 and $\widetilde{E}_6$, by Observation 11.5.5 and Corollary 11.2.2:

Proposition E.1.5. *Any circular subgroup S of E_6 or $\widetilde{E}_6$ contained in a* $\mathrm{Spin}(8)$ *subgroup is reflected.*

To prove Proposition E.1.4, we must show this sufficient condition is also necessary. Fix a maximal torus T^6 of $\widetilde{E}_6$. We create a collection of 4-spaces in the dual $(\mathfrak{t}^6)^\vee$ to the Lie algebra of T^6 which contains all reflected subspaces of $(\mathfrak{t}^6)^\vee$, transport those spaces to $\mathfrak{t}^6$ through the Killing isomorphism, then show these subspaces are the Lie algebras of maximal tori of $\mathrm{Spin}(8)$ subgroups of $\widetilde{E}_6$.

Definition E.1.6. Let XIV be the collection of subspaces $\mathbb{R}\Xi$ of $(\mathfrak{t}^6)^\vee$ spanned by quadruples Ξ of mutually orthogonal roots of $\widetilde{E}_6$. Write $\varkappa$ for the W-equivariant isomorphism $\mathfrak{t}^6 \xrightarrow{\sim} (\mathfrak{t}^6)^\vee$ between the adjoint and coadjoint representations of $\widetilde{E}_6$ induced through the Killing form, as described in Proposition B.4.12.

Proposition E.1.7. *The union $\bigcup_{w \in W_{E_6}} \{v \in \mathfrak{t}^6 : w \cdot v = -v\}$ of all (-1)-eigenspaces of the adjoint action of W_{E_6}) is $\bigcup$ XIV.*

Proof. Any collection $\{\alpha_j\} \subset (\mathfrak{t}^6)^\vee$ of jointly orthogonal roots is reflected by the product $\prod w_{\alpha_j} \in W$ of commuting root reflections corresponding to those roots. Because $\varkappa$ is W-equivariant, each $\varkappa^{-1}(\mathbb{R}\Xi) \in \varkappa^{-1}(\text{XIV})$ is contained in the (-1)-eigenspace of some Weyl element.

For the other direction, if $v \in \mathfrak{t}$ is a nonzero vector reflected by an element $w \in W$ of order $2^k \cdot m$, where m is odd, then $w^m \cdot v = -v$ as well. So all reflected lines are reflected by elements of even-power order.

Each involution $w \in W$ can be expressed as a product of orthogonal root reflections [Car72, Lem. 5, p. 5], and it will become clear later in this section (or alternately from the results of Appendix E.2) that the largest such collection is of cardinality 4, so all (-1)-eigenspaces of involutions in W are contained in $\bigcup \text{XIV}$.

We now need only show the (-1)-eigenspace of each element of orders $4, 8, \ldots$ in W is also in $\bigcup \text{XIV}$. Given a pair $w, w' \in W$ whose (-1)-eigenspaces satisfy $V_{-1}(w) \leqslant V_{-1}(w')$, then for all $g \in W$ we have $V_{-1}(gwg^{-1}) = g \cdot V_{-1}(w) \leqslant g \cdot V_{-1}(w') = V_{-1}(gw'g^{-1})$ as well, so we only need to check for one element in each conjugacy class of even-power order elements whether its (-1)-eigenspace is contained in $\bigcup \text{XIV}$.

One may verify this using the GAP 4 program comprising Appendix E.2, which returns a positive result on a mid-2011 iMac with 4 gigabytes of RAM in approximately 1.5 seconds. □

It remains to show elements of XIV are tangent to Spin(8) subgroups. Because it will turn out (Proposition E.1.13) that $|\text{XIV}| = 45$, it is impractical to demonstrate individually for each $\mathbb{R}\Xi$ that it arises from a Spin(8), so we will show (Proposition E.1.8) that W acts transitively on XIV, and then it will suffice to prove a single $\varkappa^{-1}(\mathbb{R}\Xi)$ is tangent to the maximal torus of a Spin(8) subgroup.

But to do any of this, we need to be more explicit about what these spaces and roots are. Though the space $(\mathfrak{t}^6)^\vee$ is six-dimensional, it standard to identify it not with $\mathbb{R}^6$ but rather with a six-dimensional subspace of $\mathbb{R}^8 = \mathbb{R}^5 \times \mathbb{R}^3$, the larger space conceived of as a Cartan algebra for E_8. A standard system of simple roots for E_6, already invoked in the program in Appendix E.2, is [CCN+85, p. 26][Bou68, Planche V, p. 260]

$$\begin{aligned}
\Delta \;:=\; \Big\{ & \tfrac{1}{2}(1,\ 1,\ 1,\ 1,\ 1;\ 1,\ 1,\ 1),\\
& -(1,\ 1,\ 0,\ 0,\ 0;\ 0,\ 0,\ 0),\\
& (1,-1,\ 0,\ 0,\ 0;\ 0,\ 0,\ 0),\\
& (0,\ 1,-1,\ 0,\ 0;\ 0,\ 0,\ 0),\\
& (0,\ 0,\ 1,-1,\ 0;\ 0,\ 0,\ 0),\\
& (0,\ 0,\ 0,\ 1,-1;\ 0,\ 0,\ 0)\Big\}.
\end{aligned}$$

These simple roots Δ span the six-dimensional subspace

$$(\mathfrak{t}^6)^\vee \;=\; \big(\mathbb{R}^5 \times \{0\}^3\big) \;\oplus\; \mathbb{R}\cdot(1,1,1,1,1;1,1,1)$$

and generate the system Φ of 72 roots obtained by permuting the first five coordinates of

$$\begin{aligned}
\gamma_{12} &= && (1,\ 1,\ 0,\ 0,\ 0;\ 0,\ 0,\ 0),\\
\delta_{12} &= && (1,-1,\ 0,\ 0,\ 0;\ 0,\ 0,\ 0),\\
\varepsilon_1 &= \tfrac{1}{2} && (1,-1,-1,-1,-1;\ 1,\ 1,\ 1),\\
\zeta &= \tfrac{1}{2} && (1,\ 1,\ 1,\ 1,\ 1;\ 1,\ 1,\ 1),\\
\eta_{12} &= \tfrac{1}{2} && (-1,-1,\ 1,\ 1,\ 1;\ 1,\ 1,\ 1),
\end{aligned} \tag{E.1}$$

and multiplying the resulting elements by ± 1. Among these we declare the 36 positive roots to be[2]

$$\Phi^+ := \{\gamma_{ab}, \delta_{ab}, \eta_{ab}\}_{1 \leqslant a < b \leqslant 5} \cup \{\varepsilon_a\}_{1 \leqslant a \leqslant 5} \cup \{\zeta\}.$$

Now we can show the elements of Ξ are conjugate.

Proposition E.1.8. *The Weyl group W_{E_6} acts transitively on* XIV.

This follows immediately from the following stronger statement. (Cf. Carter [Car72, Lemma 11.(i), p. 14], where transitivity is proved for triples.)

Lemma E.1.9. *The Weyl group W_{E_6} acts simply transitively on the set of ordered quadruples of mutually orthogonal roots in $(\mathfrak{t}^6)^\vee$.*

Proof. Given any ordered quadruple $(\alpha, \beta, \gamma, \delta)$ in Φ, we find a unique element of W_{E_6} sending it to $(\zeta, -\varepsilon_1, \delta_{23}, \delta_{45})$.

Since W acts transitively and isometrically on Φ, there exists a $w \in W$ such that $w\alpha = \zeta$, and $w \cdot \{\beta, \gamma, \delta\}$ is contained in the set $\Phi(A_5) = \{\pm\varepsilon_a\}_{1 \leqslant a \leqslant 5} \cup \{\pm\delta_{ab}\}_{1 \leqslant a < b \leqslant 5}$ of roots orthogonal to ζ.

Amongst these, a spanning sequence is $\Delta(A_5) = (-\varepsilon_1, \delta_{12}, \delta_{23}, \delta_{34}, \delta_{45})$. Each entry has inner product -1 with adjacent entries and is orthogonal to the others, so $\Phi(A_5)$ is a root system of type A_5, and root reflections in $\Phi(A_5)$ generate a Weyl subgroup $W(A_5) < W_{E_6}$ fixing ζ. As $W(A_5)$ acts transitively on $\Phi(A_5)$, there is a $w' \in W(A_5)$ such that $w'w\beta = -\varepsilon_1$. Since w' is an isometry, the roots $w'w \cdot \{\gamma, \delta\}$ are contained in the set $\Phi(A_3) \subsetneq \Phi(A_5) = \{\pm\delta_{ab}\}_{1 \leqslant a < b \leqslant 5}$ of roots orthogonal to $\{\zeta, -\varepsilon_1\}$.

Amongst these, a spanning sequence is $\Delta(A_3) = (\delta_{23}, \delta_{34}, \delta_{45})$, so $\Phi(A_3)$ is a root system of type A_3, and root reflections in $\Phi(A_3)$ generate a Weyl subgroup $W(A_3) < W(A_5)$ fixing $(\zeta, -\varepsilon_1)$. As $W(A_3)$ acts transitively on $\Phi(A_3)$, there is a $w'' \in W(A_3)$ such that $w''w'w\gamma = \delta_{23}$. Since w''

[2] Here, to be clear, in δ_{ab} the 1 appears in the a^{th} coordinate and the -1 in the b^{th}, but we will let $\eta_{ab} = \eta_{ba}$ and $\gamma_{ab} = \gamma_{ba}$ for notational convenience. For example, $\delta_{24} = -\delta_{42} = (0, 1, 0, -1, 0; 0; 0, 0)$ and $\eta_{43} = \eta_{34} = \frac{1}{2}(1, 1, -1, -1, 1; 1, 1, 1)$ and $\varepsilon_5 = \frac{1}{2}(-1, -1, -1, -1, 1; 1, 1, 1)$.

is an isometry, the root $w''w'w\delta$ is contained in the set $\Phi(A_1) = \{\pm\delta_{45}\}$ of roots orthogonal to $\{\zeta, -\varepsilon_1, \delta_{23}\}$, a system of type A_1.

The root reflection in $\Phi(A_1)$ generates a Weyl subgroup $W(A_1) < W(A_3)$ fixing $(\zeta, -\varepsilon_1, \delta_{23})$. As $W(A_1)$ acts transitively on $\Phi(A_1)$, there is exists a $w''' \in W(A_1)$ such that $w'''w''w'w\delta = \delta_{45}$.

Thus the composition $w'''w''w'w$ does what we want to the quadruple $(\alpha, \beta, \gamma, \delta)$. That it is the unique Weyl group element doing so follows from a sequence of applications of the orbit–stabilizer theorem. Since W_{E_6} has order 51,840 and transitively permutes 72 roots, it follows Stab(ζ) has order 720. Since Stab(ζ) transitively permutes the 30 roots $\Phi(A_5)$, it follows that the ordered pair stabilizer Stab$(\zeta, -\varepsilon_1)$ has order 24. Since this stabilizer transitively permutes the 12 roots $\Phi(A_3)$, it follows the triplet-stabilizer Stab$(\zeta, -\varepsilon_1, \delta_{23})$ has order 2. And this stabilizer acts simply transitively on $\Phi(A_1) = \{\pm\delta_0\}$, so Stab$(\zeta, -\varepsilon_1, \delta_{23}, \delta_{45})$ is trivial and W_{E_6} acts simply transitively on ordered quadruples of mutually orthogonal roots. □

Now we show such a $\Phi \cap \mathbb{R}\Xi$ is a D_4 root system.

Lemma E.1.10. *The D_4 root subsystems of Φ are precisely $\{\Phi \cap \mathbb{R}\Xi : \mathbb{R}\Xi \in \mathrm{XIV}\}$.*

Proof. A root system of type D_4 is spanned by four mutually orthogonal roots, so a D_4 subsystem of Φ must be $\Phi \cap \mathbb{R}\Xi$ for some $\mathbb{R}\Xi \in \mathrm{XIV}$.

On the other hand, let $\Xi = \{\gamma_{12}, \delta_{12}, \gamma_{34}, \delta_{34}\}$; then $\mathbb{R}\Xi \in \mathrm{XIV}$ is generic by Proposition E.1.8. The quadruple $\Delta = (\gamma_{12}, \gamma_{34}, \delta_{34}, -\gamma_{23})$ in $\Phi \cap \mathbb{R}\Xi$ spans $\mathbb{R}\Xi$, its first three entries are mutually orthogonal, and $\gamma_{23} \cdot \gamma_{12} = \gamma_{23} \cdot \gamma_{34} = \gamma_{23} \cdot \delta_{34} = -1$, so $\Phi \cap \mathbb{R}\Xi$ is a root system of type D_4. □

Now we can finally show XIV does have something to do with Spin(8).

Proposition E.1.11. *Given a maximal torus T^6 of $\widetilde{E}_6$, the duals in $\mathfrak{t}^6$ to the spaces XIV in $(\mathfrak{t}^6)^\vee$ are precisely the Lie algebras $\mathfrak{t}^6$ of those maximal tori T^4 of* Spin(8) *subgroups such that T^4 is contained in T^6.*

Proof. Recalling the discussion of integer and coroot lattices in Appendix B.4.2, consider a Spin(8) in $\widetilde{E}_6$ whose maximal torus T^4 lies in T^6. The inclusion $T^4 \hookrightarrow T^6$ induces inclusions $\mathfrak{t}^4 \hookrightarrow \mathfrak{t}^6$ and $\Lambda(T^4) \hookrightarrow \Lambda(T^6)$.[3] Since Spin(8) and $\widetilde{E}_6$ are both simply-connected, their coroot lattices are their integer lattices by Proposition B.4.14, so this last inclusion can also be written $Q^\vee(\mathrm{Spin}(8)) \hookrightarrow Q^\vee(\widetilde{E}_6)$. Selecting all elements of length $\sqrt{2}$ in these lattices respectively yields root systems $\Phi^\vee(\mathrm{Spin}(8)) \subsetneq \Phi^\vee(\widetilde{E}_6)$ of types D_4 and E_6 within $\mathfrak{t}^4$ and $\mathfrak{t}^6$. Because all roots of both groups are long, by Corollary B.4.16 the Killing isomorphism $\varkappa$ takes these systems respectively to a root system $\Phi(D_4)$ of type D_4 and the root system Φ of $\widetilde{E}_6$ described above. By Lemma E.1.10, the span of $\Phi(D_4)$ is $\Phi \cap \mathbb{R}\Xi$ for some $\Xi \in \mathrm{XIV}$. That is, $\varkappa(\mathfrak{t}^4) \in \mathrm{XIV}$. □

The transitivity of the action on orthogonal quadruples makes it relatively simple to enumerate XIV.

Lemma E.1.12. *A vector space $\mathbb{R}\Xi \in \mathrm{XIV}$ is spanned by precisely three distinct sets of four mutually orthogonal roots.*

Proof. Let $\Xi = \{\gamma_{12}, \delta_{12}, \gamma_{34}, \delta_{34}\}$ again. The positive roots $\Phi^+ \cap \mathbb{R}\Xi$ are $\{\gamma_{ab}, \delta_{ab} \colon 1 \leqslant a < b \leqslant 4\}$. Among these, the three roots orthogonal to δ_{ab} are $\{\gamma_{ab}, \gamma_{cd}, \delta_{cd}\}$, where $\big|\{a,b,c,d\}\big| = 4$ and $c < d$, and likewise the three roots orthogonal to γ_{ab} are $\{\delta_{ab}, \gamma_{cd}, \delta_{cd}\}$, so the spanning quadruples are determined by the partitions of $\{1,2,3,4\}$ into pairs of pairs $\big\{\{a,b\},\{c,d\}\big\}$, of which there are three. □

Now we can conclude the enumeration.

Proposition E.1.13. *We have $|\mathrm{XIV}| = 45$.*

Proof. The Weyl group W_{E_6} acts simply transitively on the set of ordered quadruples $(\alpha, \beta, \gamma, \delta)$ of mutually orthogonal roots, by Proposition E.1.8, and $\big|\{\pm 1\}^4 \rtimes S_4\big|$ distinct such quadruples

[3] To tie this back to circles, the exponential of each line $\mathfrak{s}$ in $\mathfrak{t}^6$ containing a point of $\Lambda(T^6)$ is a circle in T^6, so the set of circles in T^6 can be identified with the projectivization $P\Lambda(T^6)$.

correspond to any given *set* $\{\alpha, \beta, \gamma, \delta\}$ of four mutually orthogonal *positive* roots, so there are $|W_{E_6}|/|\{\pm 1\}^4 \rtimes S_4| = 51{,}840/384 = 135$ such sets. By Lemma E.1.12, each such 4-space admits three distinct unordered bases of positive roots, so there are 135/3 = 45 distinct such spaces. $\square$

To summarize, we have proven the following.

Proposition E.1.14. *If a circular subgroup S lies within a maximal torus T^6 of E_6, then S is reflected just if it is contained one of forty-five Weyl-conjugate maximal tori lying in T^6 which are tangent spaces of* Spin(8) *subgroups of E_6*

Remarks E.1.15. *(a)* Proposition E.1.13 also shows that the stabilizer of any element of XIV in W_{E_6} is a group of order $|W_{E_6}|/45 = 1{,}152 = |W(F_4)|$. If we write $N = N_{E_6}(T^4)$ and $Z = Z_{E_6}(T^4)$, then by the proof of Observation 11.5.5 we have the chain of injections

$$W(F_4) \rightarrowtail \frac{N}{Z} \rightarrowtail \frac{W_{E_6}}{\mathrm{Fix}_{W_{E_6}}(T^4)},$$

and we have in essence just shown the map $W(F_4) \rightarrowtail N/Z$ is actually an isomorphism and $\mathrm{Fix}_{W_{E_6}}(T^4)$ is trivial, so that $W(F_4)$ injects into W_{E_6} as the set of elements normalizing T^4. The author is advised this result can be understood from Carter's book [Car85, Sec. 13.3].

(b) We originally found the 135 maximal mutually orthogonal sets of positive roots through brute force, then checked with Python we had accounted for them all. Explicitly, in terms of (E.1), they

are

(60)	$\{\varepsilon_a, \eta_{ab}, \gamma_{ac}, \delta_{de}\}$,	where $\|\{a,b,c,d,e\}\| = 5$ and $d < e$,
(30)	$\{\eta_{ab}, \eta_{cd}, \gamma_{ac}, \gamma_{bd}\}$,	where $\|\{a,b,c,d\}\| = 4$,
(15)	$\{\eta_{ab}, \eta_{cd}, \delta_{ab}, \delta_{cd}\}$,	where $\|\{a,b,c,d\}\| = 4$ and $a < b$ and $c < d$,
(15)	$\{\gamma_{ab}, \gamma_{cd}, \delta_{ab}, \delta_{cd}\}$,	where $\|\{a,b,c,d\}\| = 4$ and $a < b$ and $c < d$,
(15)	$\{\zeta, \varepsilon_a, \delta_{bc}, \delta_{de}\}$,	where $\|\{a,b,c,d,e\}\| = 5$ and $b < c$ and $d < e$.

E.2. The GAP verification of Proposition E.1.7

```
# Supply a standard system of simple roots for E_6.
E6_simple_roots := [
 [1,-1,-1,-1,-1,1,1,1]/2,
-[1,1,0,0,0,0,0,0],
 [1,-1,0,0,0,0,0,0],
 [0,1,-1,0,0,0,0,0],
 [0,0,1,-1,0,0,0,0],
 [0,0,0,1,-1,0,0,0]
];;

# Simple reflections generate the Weyl group of E_6. Check the Weyl group is the right size.
simple_refl := List(E6_simple_roots,ReflectionMat);; W := Group(simple_refl); Size(W);

# Find its conjugacy classes. How big are they?
conj_raw := ConjugacyClasses(W);; List(conj_raw,x->Size(x));

# There are no elements of order 16.
Filtered(conj_raw,c->Order(Representative(c))=16);

# We find all conjugacy classes of elements of orders 2, 4, and 8.
conj_cl248 := [];;
for expon in [1..3] do
        conj_cl248[expon]:= Filtered(conj_raw,c->Order(Representative(c))=2^expon);;
od;

# How big are these conjugacy classes of elements of orders 2, 4, and 8?
List(conj_cl248,x->List(x,y->Size(y)));

# There are four conjugacy classes of involutions.
```

```
# We know these are all products of four or fewer orthogonal root reflections.

# This function finds the (-1)-eigenspace of a matrix.
reflected_space := function(matrix)
        local e;
        for e in Eigenspaces(Rationals,matrix) do
                if matrix*Representative(e) = -Representative(e) then
                        return e;
                fi;
        od;
        return [];
end;;

# This function finds a reflected space of some element in a conjugacy class.
refl_sp_of_rep := class->reflected_space(Representative(class));;

# This function orders a set by its function values.
ord := fcn->(set->Concatenation(List(Set(set,fcn),x->Filtered(set,v->fcn(v)=x))));

# We reorder the conjugacy classes of involutions so that the fourth one is the
# one with the highest-dimensional reflected spaces.
conj_cl248[1] :=
ord(x->Dimension(refl_sp_of_rep(x)))(conj_cl248[1]);;

# The largest (-1)-eigenspaces of involutions have dimension 4.
# We know these contain all (-1)-eigenspaces of other involutions.
# We generate all these (45) four-dimensional (-1)-eigenspaces.
invol_refl_spaces := List(conj_cl248[1][4],x->reflected_space(x));;

# We generate one reflecting space for each conjugacy class of elements of
# of orders 2, 4, and 8.
rep248_refl_spaces := Concatenation(List(conj_cl248,y->List(y,x->refl_sp_of_rep(x))));;

# This function checks if a vector space is contained in such an eigenspace.
subset_invol_refl_space := function(vector_space)
        local V, answer;
        answer := "No.";
        for V in invol_refl_spaces do
                if IsSubset(V,vector_space) then
                                answer := "Yes.";
                fi;
        od;
        return answer;
end;;
```

```
# We run through representatives of conjugacy classes of elements of orders 4 and 8
# and check whether their (-1)-eigenspaces are contained in a four-dimensional one.
for class in [5..Size(rep248_refl_spaces)] do;
        Print("Class ");
        Print(class);
        Print(": ");
        Print(subset_invol_refl_space(rep248_refl_spaces[class]));
        Print("\n");
od;

# For every class, a "Yes" shows up, demonstrating the four-dimensional (-1)-eigenspaces
# of involutions in fact contain all (-1)-eigenspaces.
```

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