GROUPS QUASI-ISOMETRIC TO $H \times \mathbb{R}^n$

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Abstract

We describe a conjectural characterization of all groups quasi-isometric to $H \times \mathbb{R}^n$, where H is any non-elementary hyperbolic group, and we provide an outline of the steps required to establish such a characterization. We carry out several steps of this plan. We consider those lines $\widehat{L}$ in the asymptotic cone $\mathrm{Cone}_\omega(H)$ which, in a precise sense, "arise from lines L in H". We give a complete description of such lines, showing (in particular) that they are extremely rare in $\mathrm{Cone}_\omega(H)$. Given a top-dimensional quasi-flat in $H \times \mathbb{R}^n$, we show the induced bi-Lipschitz embedded flat in $\mathrm{Cone}_\omega(H \times \mathbb{R}^n)$ must lie uniformly close to some $\widehat{L} \times \mathbb{R}^n$, where $\widehat{L}$ is one of these rare lines. As a result, we conclude that quasi-actions on $H \times \mathbb{R}^n$ must project to quasi-actions on H and therefore to homeomorphic actions on ∂H. Finally, we show that such an action on ∂H is a convergence action which is uniform if it is discrete, and we discuss the work that remains to complete the conjectured characterization.

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Chapter 1

Introduction

Geometric group theory explores the geometric properties of infinite groups, following seminal papers of Mikhael Gromov (e.g., [Gro81, Gro84, Gro93]). A standard model space for a given group Γ is the *Cayley graph*, $\mathrm{Cay}(\Gamma, S)$. This is a graph with vertex set Γ and edges determined by a chosen finite generating set S for Γ so that the word metric on Γ with respect to S is precisely the distance function in the graph. The natural action of Γ on itself by left multiplication extends to an action of Γ on $\mathrm{Cay}(\Gamma, S)$ which is properly discontinuous, cocompact, and isometric. Different generating sets for Γ produce non-isometric Cayley graphs. Nevertheless, any two Cayley graphs for Γ are *quasi-isometric*. With the motivating example of Cayley graphs in mind, features preserved by quasi-isometries are often called *(coarse) geometric features* of groups, and actions which are properly discontinuous, cocompact, and isometric are referred to as *geometric* actions. To an extent, we should only ascribe geometric characteristics to a group in so far as they are independent of the generating set (that is, up to quasi-isometry), and we will often study the geometry of infinite groups via their geometric actions on particular model spaces. (For more detailed definitions, see Section 2.1.)

A central goal of geometric group theory is to classify groups up to quasi-isometry, and so problems such as *Characterize those groups which act geometrically on a given space X* are fundamental. For examples, it is well known that any group which is quasi-isometric to $\mathbb{Z}^n$ must be virtually $\mathbb{Z}^n$ (due to Gromov and Bridson–Gersten); that any group which is quasi-isometric to a tree is virtually free (Stallings); and that any group which is quasi-isometric to $\mathbb{H}^2$ must be virtually cocompact Fuchsian [Gab92, CJ94, Tuk88]. (By a *Fuchsian group*, we mean a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$.) Eleanor Rieffel [Rie01] provides the following characterization of groups quasi-isometric to $\mathbb{H}^2 \times \mathbb{R}$:

Theorem ([Rie01])**.** For any group Γ quasi-isometric to the hyperbolic plane cross the real line, there is a short exact sequence

$$0 \to A \to \Gamma \to G \to 0,$$

where A is virtually $\mathbb{Z}$ and G is a finite extension of a cocompact Fuchsian group.

As Rieffel notes in the introduction, this result contributed to a program attempting to classify all finitely generated groups quasi-isometric to each of William Thurston's eight 3-dimensional geometries. It was also part of a series of papers (including [KKL98, KL01, KL97a]), exploring the extent to which "product decompositions" are geometric features (that is, preserved by quasi-isometries). A major result in this category is:

Theorem ([KKL98])**.** Suppose M, N are closed, nonpositively curved Riemannian manifolds, and consider the de Rham decomposition of their universal covers $\widetilde{M} = \mathbb{E}^m \times \prod_{i=1}^{k} M_i$, $\tilde{N} = \mathbb{E}^n \times \prod_{i=1}^{\ell} N_i$. Then, for every $K \geqslant 1, C \geqslant 0$, there is a constant D so that for each (K, C)-quasi-isometry $\phi\colon \widetilde{M} \to \tilde{N}$ we have: $k = \ell$, $m = n$, and, after reindexing the factors N_j, there are quasi-isometries $\phi_i\colon M_i \to N_i$ such that every i, the following diagram commutes up to error at most D:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\ \phi\ } & \tilde{N} \\ \downarrow & & \downarrow \\ M_i & \xrightarrow[\ \phi_i\ ]{} & N_i \end{array}$$

We will develop some of the tools necessary to generalize this characterization to the setting of groups quasi-isometric to $H \times \mathbb{R}^n$, where H is any non-elementary word hyperbolic group. We will make use of three major tools: quasi-actions, asymptotic cones, and convergence groups. A *quasi-action* of a group Γ on a space X is a generalization of an isometric action in which the group elements act like quasi-isometries. There are a variety of reasons to consider this sort of generalization, the primary reason being that any group quasi-isometric to X will quasi-act on X, even if it cannot isometrically act on X. A fairly complete treatment of the basic facts about quasi-actions may be found in Appendix A. (The facts presented there are not new, but a treatment including all details does not seem to appear in existing literature, and some subtleties may be taken for granted in other sources.) The terminology and facts necessary for our present purposes are included in Section 2.2.

The *asymptotic cone* of a space is a metric space which is a topological invariant under quasi-isometries. That is, a quasi-isometry between two spaces induces a homeomorphism between the asymptotic cones. Section 2.3 provides the necessary background on asymptotic cones. If H is a non-elementary hyperbolic group, then its asymptotic cone $\mathrm{Cone}_\omega(H)$ is a homogeneous $\mathbb{R}$-tree with uncountable branching at every point. In Section 3.2.1, we provide a detailed description of those lines $\mathrm{Cone}_\omega(H)$ which (in a sense made precise in that section) "arise from lines in H". There is a natural base point $\widehat{e} \in \mathrm{Cone}_\omega(H)$ which splits any such line into rays, which we call *midribs*. We call the connected components of $\mathrm{Cone}_\omega(H)\backslash\{\widehat{e}\}$ *leaves*, and we show:

Theorem 3.2.2. Let H be a non-elementary hyperbolic group with asymptotic cone $\mathrm{Cone}_\omega(H)$.

1. For each $a \in \partial H$, there is a unique corresponding midrib.
2. Each leaf contains at most one midrib.
3. If $f\colon \mathbb{R} \to H$ is any quasi-isometric embedding with induced map $\widehat{f}$ on the asymptotic cones, then the image of $\widehat{f}$ is precisely the union of the two midribs corresponding to the boundary points of H which are the endpoints of f.

In Section 3.2.2, we will make use of this description to prove a rigidity result about the *quasi-flats* (quasi-isometric embeddings of copies of $\mathbb{R}^{n+1}$) inside of $H \times \mathbb{R}^n$.

Theorem 3.2.10. Let $f\colon \mathbb{R}^{n+1} \to H \times \mathbb{R}^n$ be a quasi-isometric embedding and let $\widehat{f}$ be the induced map on the asymptotic cones. Then

1. the image of $\widehat{f}$ is of the form $\widehat{L} \times \mathbb{R}^n$, where $\widehat{L}$ is a line in $\mathrm{Cone}_\omega(H)$,

2. $\widehat{L}$ is the union of two midribs corresponding to some distinct boundary points $a, b \in \partial H$, and

3. for any line L in H with endpoints a, b, $\widehat{L}$ is the line in the asymptotic cone arising from L, and the image of f is uniformly close $L \times \mathbb{R}^n$.

As a corollary, we may project quasi-actions:

Theorem 3.2.8. Let $f \colon H \times \mathbb{R}^n \to H \times \mathbb{R}^n$ be a quasi-isometry and write

$$\pi \colon H \times \mathbb{R}^n \to H$$

for the natural projection. Then there is some constant D such that, for each $x \in H$, there exists $y \in H$ such that $f(\{x\} \times \mathbb{R}^n)$ and $\{y\} \times \mathbb{R}^n$ are at Hausdorff distance at most D. The assignment $x \mapsto y$ is a quasi-isometry (which is equivalent to $\pi \circ f$) whose quasi-isometry constants depend only on those of f.

In particular, if Γ quasi-acts on $H \times \mathbb{R}^n$ via φ, then $\psi_g = \pi \circ \varphi_g$ gives a quasi-action of Γ on H, and ψ is cobounded if φ is.

A *convergence action* is an action with a particular dynamical property. Gehring and Martin [GM87] defined this property and showed that Möbius groups acting on spheres have this property. In particular, the actions of Fuchsian groups on $\mathbb{H}^2$ extend to actions by homeomorphisms on the boundary S^1, and these actions on the boundary are convergence actions. [Gab92] and, independently, [CJ94, Tuk88] showed that a subgroup of $\mathrm{Homeo}(S^1)$ is conjugate to the restriction of a Fuchsian group if and only if it has the convergence property. [Fre94] and [Tuk94] independently showed that all non-elementary hyperbolic groups acting on their boundary have the convergence property, and [Bow98] showed that a slightly stronger property, the *uniform convergence property*, characterizes non-elementary hyperbolic groups. A slightly more complete introduction to convergence groups is given in Section 2.4. In Section 3.2.3, we show that if Γ is quasi-isometric to $H \times \mathbb{R}^n$ where H is a non-elementary hyperbolic group, then there is some homomorphism $\Xi \colon \Gamma \to F \leqslant \mathrm{Homeo}(\partial H)$, where F acts like a convergence group. In the case that this convergence group is also discrete, we show that F is a uniform convergence group.

In Chapter 4, we will discuss the work that remains to complete a characterization of all groups quasi-isometric to $H \times \mathbb{R}^n$, where H is any non-elementary group. In addition, we will discuss some areas of potential future research that could build off of the methods discussed throughout. In particular, we consider a potential characterization of groups quasi-isometric to $G \times \mathbb{R}^n$, where G is a *relatively hyperbolic group* (see Section 4.2 for definitions and some background).

Chapter 2

Background

2.1 Hyperbolic Spaces and Groups

In this section, we give a brief introduction to the terminology, notation, and basic facts of hyperbolic groups and spaces. We describe the maps (*quasi-isometries*) and the group actions (*geometric actions*) which preserve the coarse geometric features in which we will be interested. We describe the *boundary* of a hyperbolic space, a topological invariant of primary importance. And we describe the extension of quasi-isometries to the boundary. A more complete exposition can be found in any introductory notes or text on geometric group theory (e.g., [BH99]).

A space (X, d) is called *δ-hyperbolic* if, for any geodesic triangle with sides p, q, r, each side is contained in the δ-neighborhood of the other two sides. We say X is *hyperbolic* if it is δ-hyperbolic for some δ.

Given a group Γ with a finite generating set S, the *Cayley graph of* Γ *with respect to* S is the graph $\mathrm{Cay}(\Gamma, S)$ with vertices given by Γ, where two vertices $g, h \in \Gamma$ are connected by an edge if there exists an $s \in S$ such that $h = gs$. (Here it is usually assumed that S includes all inverses of elements in S.) Taking edges lengths to be 1, a Cayley graph has a natural metric, and the action of Γ on itself by left multiplication extends to an action of Γ on $\mathrm{Cay}(\Gamma, S)$ by isometries.

We say a group Γ is *hyperbolic* if some Cayley graph of Γ is hyperbolic. In this case, any Cayley graph is hyperbolic (although possibly with different constants δ).

Definition 2.1.1. Let X and Y be metric spaces. A (K, C)-*quasi-isometric embedding* is a map $f : X \to Y$ such that

$$\frac{1}{K} d(x, x') - C \leqslant d(f(x), f(x')) \leqslant K d(x, x') + C$$

for all $x, x' \in X$. We say f is a (K, C, δ)-*quasi-isometry* if, in addition, for each

$y \in Y$ there exists $x \in X$ such that $d(f(x), y) \leqslant \delta$. We say that f is a *quasi-isometry (or quasi-isometric embedding)* if it is a (K, C, δ)-quasi-isometry (or (K, C)-quasi-isometric embedding) for some constants K, C, δ. We note that increasing the constants preserves the property of being a quasi-isometry.

The fundamental example of a quasi-isometry comes from a map $\text{Cay}(\Gamma, S) \to \text{Cay}(\Gamma, S')$ extending the identity map on the vertex set. That is, any two Cayley graphs of Γ with different (finite) generating sets are quasi-isometric. A question of central interest in geometric group theory is to classify finitely generated groups up to quasi-isometry. The following lemma justifies defining the hyperbolicity of a group by using an arbitrary Cayley graph.

Lemma 2.1.2. If X is a hyperbolic space and $f\colon X \to Y$ is a quasi-isometry, then Y is a hyperbolic space.

An isometric embedding $\gamma\colon I \to X$, where I is an interval (or ray, or line), is called a *geodesic segment (or ray, or line)*. A quasi-isometric embedding of an interval (or ray, or line) into X is called a *quasi-geodesic segment (or ray, or line)*.

Let (X, d) be a hyperbolic space with a fixed base point $x \in X$. For points $y, z \in X$, we will write $[y, z]$ for any choice of geodesic from y to z. The *Gromov inner product of y and z based at x* is

$$(y \cdot z)_x = \frac{1}{2}(d(x, y) + d(x, z) - d(y, z)).$$

Up to a bounded error (depending only on δ), the Gromov inner product $(y \cdot z)_x$ measures the distance from x to a geodesic between $[y, z]$.

We briefly note an alternate definition of hyperbolicity. Given a triangle Δ, there is a unique corresponding metric tripod T and a collapsing map $\chi\colon \Delta \to T$ so that the restriction of χ to each side of Δ is an isometry. X is said to be *δ-hyperbolic* if, for every geodesic triangle Δ with corresponding collapsing map $\chi\colon \Delta \to T$ and for any $t \in T$, the diameter of $\chi^{-1}(t)$ is at most δ. This δ and the δ in the previous definition may not be equal.

The Gromov inner product $(y \cdot z)_x$ is precisely the distance from $\chi(x)$ to o in the tripod corresponding to $\Delta = \Delta(x, y, z)$. So, if $(y \cdot z)_x \geqslant C$, then the geodesics $[x, y]$ and $[x, z]$ travel δ-close to each other for at least time C. (See Figure 2.1.)

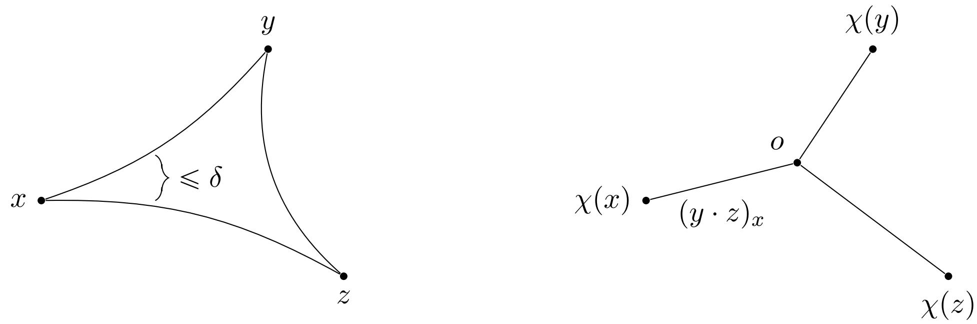

Figure 2.1: Hyperbolicity and the Gromov inner product.

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Given two (quasi-)geodesic rays γ_1, γ_2, we say they are *equivalent* if they are finite Hausdorff distance apart. A sequence $(y_n)_n$ in X is said to *converge at infinity* if $(y_i \cdot y_j)_x \to \infty$ as $i, j \to \infty$. Two sequences $(y_n)_n$ and $(z_n)_n$ are said to be *equivalent* if $(y_i \cdot z_j)_x \to \infty$ as $i, j \to \infty$. There are three standard ways to view the *boundary of* X:

1. ∂X, the set of equivalence classes of geodesic rays,
2. $\partial_q X$, the set of equivalence classes of quasi-geodesic rays, or
3. $\partial_s X$, the set of equivalence classes of sequences converging at infinity.

Each of these sets is given a topology which makes precise the intuition that "nearby rays or sequences are those which stay close for a long time". For example, if we fix a constant C, a basis of open sets for the ∂X is given by sets of the form

$$B_r([\gamma]) = \{[\gamma'] \mid d(\gamma(t), \gamma'(t)) \leqslant C, 0 \leqslant t \leqslant r\}.$$

In a hyperbolic space, quasi-geodesics are always close to actual geodesics:

Theorem 2.1.3 (Stability of quasi-geodesics)**.** If X is a δ-hyperbolic space and $c \colon I \to X$ is a (K, C)-quasi-isometric embedding, then there exists some R depending only on δ, K, C such that the Hausdorff distance between $c(I)$ and a geodesic connecting its endpoints is at most R.

With the topologies appropriately defined, the three notions of boundary are homeomorphic. For example, by the stability of quasi-geodesics, we can choose geodesic representatives of classes in $\partial_q X$. So we may refer to ∂X unambiguously as *the boundary of* X, whether we think of its elements as equivalence classes of geodesic rays, quasi-geodesic rays, or sequences converging at infinity.

Proposition 2.1.4. Let X, Y be hyperbolic spaces. A quasi-isometry $f\colon X \to Y$ induces a homeomorphism $\partial f\colon \partial X \to \partial Y$ on the boundaries via

$$\partial f([(\gamma)]) = [(f \circ \gamma)].$$

The map ∂f is sometimes referred to as the *boundary value* of f.

Definition 2.1.5. An action of Γ on a metric space X is called *properly discontinuous* if, for any compact set $K \subset X$,

$$|\{g \in \Gamma \mid g \cdot K \cap K \neq \varnothing\}| < \infty.$$

The action is called *cocompact* if there exists a compact set $K \subset X$ such that

$$X = \bigcup_{g \in \Gamma} g \cdot K.$$

An action which is properly discontinuous, cocompact, and by isometries is called a *geometric action*.

Lemma 2.1.6 (Schwarz–Milnor)**.** If Γ acts geometrically on a proper metric space X and $x \in X$ is any chosen base point, then the orbit map $g \mapsto g \cdot x$ is a quasi-isometry between Γ and X.

Because of this fundamental result, one might hope that we could characterize those groups quasi-isometric to X by characterizing the groups which act geometrically on X. We note, however, that the converse of the Schwarz–Milnor lemma is false, so these two questions are not precisely the same.

Definition 2.1.7. Two quasi-isometries $f, f' \colon X \to Y$ are *equivalent* if

$$\sup_{x \in X} d(f(x), f'(x)) \leqslant C < \infty.$$

We write $f \sim f'$, or $f \sim_C f'$ if we wish to keep track of the constant C. It is straightforward to check that this is an equivalence relation. Let $f \colon X \to Y$ be a quasi-isometry. A *quasi-inverse of* f is a quasi-isometry $g \colon Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$.

It is easy to see that quasi-inverses are not, in general, unique, so the collection of quasi-isometries on a given space X does not form a group. However, the collection of equivalence classes of quasi-isometries on X does form a group, called the *quasi-isometry group*, denoted $\mathcal{QI}(X)$. This group is, in general, extremely large and complicated.

2.2 Quasi-actions

As mentioned in the previous section, the Schwarz–Milnor Lemma does not quite say that the collection of groups quasi-isometric to a given space X is the same as the collection of groups which act geometrically on X. [MSW03] gives an example of two groups which are quasi-isometric to one another but do not act geometrically on any common space X. A *quasi-action by quasi-isometries* relaxes the notion of an action by isometries. In particular, we will see that, in a "quasi" version of the Schwarz–Milnor Lemma, the converse will hold. In this section, we present some basic definitions, notation, and facts about quasi-actions. For a complete collection of detailed proofs, see Appendix A.

Definition 2.2.1. Let Γ be a group and X a metric space. A *quasi-action by quasi-isometries* φ is an assignment to each $g \in \Gamma$ of a (K, C, δ)-quasi-isometry φ_g, satisfying the following:

1. $\varphi_e \sim \mathrm{id}_X$,

2. there is a constant α such that

$$\sup_{x \in X} d(\varphi_g \varphi_h(x), \varphi_{gh}(x)) \leqslant \alpha, \text{ and}$$

3. the constants K, C, δ, α are uniform over g, h.

We remark that any quasi-action of Γ on X induces a natural homomorphism $\Gamma \to \mathcal{QI}(X)$. However—unlike most other group actions by some type of morphism on some type of space—the converse is false. See Example A.1.2 for an example of a $\mathbb{Z}$ subgroup of $\mathcal{QI}(\mathbb{R})$ which cannot quasi-act on $\mathbb{R}$.

Definition 2.2.2 (Cobounded, Proper)**.** Let φ be a quasi-action of Γ on X. We say that φ is *cobounded* if there exists a constant C such that, for each $x, y \in X$, there exists $g \in \Gamma$ such that $d(\varphi_g(x), y) < C$. We will write

$$\Gamma^{x,y}_{\varphi,R} = \{g \in \Gamma \mid \varphi_g(N(x,R)) \cap N(y,R) \neq \varnothing\}| \leqslant M.$$

We say that φ is *proper* if, for each R, there exists M such that, for all $x, y \in X$, $\left|\Gamma^{x,y}_{\varphi,R}\right| \leqslant M$. If φ is a cobounded and proper quasi-action by quasi-isometries, we will call φ *geometric*.

Any action by isometries is, in particular, a quasi-action by quasi-isometries. In this case, proper is the same as properly discontinuous, and cobounded is the same as cocompact.

Definition 2.2.3 (Equivalence of Quasi-Actions)**.** We say two quasi-actions φ and ψ of Γ on X are *equivalent* if there exists a constant $C > 0$ such that

$$\sup_{x \in X} d(\varphi_g(x), \psi_g(x)) < C.$$

We will write $\varphi \sim \psi$, or $\varphi \sim_C \psi$ if we wish to keep track of the constant.

We observe that the constant C in this definition is uniform over $g \in \Gamma$. It is not enough that $\varphi_g \sim \psi_g$ for each $g \in \Gamma$.

Definition 2.2.4. Suppose Γ quasi-acts on X via φ and on Y via ψ. A quasi-isometry $f\colon X \to Y$ is called a *quasi-conjugacy between φ and ψ* if there is a constant C such that $d(f(\varphi_g(x)), \psi_g(f(x))) \leqslant C$ for all $g \in \Gamma$ and $x \in X$. That is, the following diagram commutes up to a uniformly bounded error of at most C:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \varphi_g}\downarrow & \circlearrowright & \downarrow{\scriptstyle \psi_g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

If φ is a quasi-action of Γ on X and $f\colon X \to Y$ is a quasi-isometry with quasi-inverse f', then

$$\psi_g(y) = f \circ \varphi_g \circ f'(y)$$

defines a quasi-action of Γ on Y which is quasi-conjugate to φ (and f, f' are both quasi-conjugacies).

Equivalence of quasi-actions is an equivalence relation on the set of quasi-actions of Γ on a fixed space X. Quasi-conjugacy gives an equivalence relation among all quasi-actions of Γ on any space. We also have the notion of equivalence of quasi-isometries. The interactions between these three equivalence relations and the quasi-action properties are recorded in the following theorem:

Theorem 2.2.5. Suppose φ is a quasi-action of Γ on X, ψ is a quasi-action of Γ on Y, and $f\colon X \to Y$ is a quasi-conjugacy between φ and ψ.

1. Any quasi-action φ' is equivalent to φ if and only if φ' is quasi-conjugate to ψ via f.

2. Any quasi-isometry equivalent to f and any coarse inverse for f is also a quasi-conjugacy between φ and ψ.

3. φ is cobounded (proper) if and only if ψ is cobounded (proper).

4. If φ' is equivalent to φ, then φ is cobounded (proper) if and only if φ' is cobounded (proper).

Lemma 2.2.6 (Quasi-Schwarz–Milnor, [Nek97])**.** A group Γ is quasi-isometric to a proper metric space X if and only if there is a geometric quasi-action of Γ on X.

Proof. Supposing Γ quasi-acts on X via φ, the same orbit map as in the Schwarz–Milnor Lemma gives a quasi-isometry. That is, for a fixed $x \in X$, $g \mapsto g \cdot x$ is a quasi-isometry.

Conversely, if $f \colon \Gamma \to X$ is a quasi-isometry with quasi-inverse f' and φ is the natural action of Γ on itself by left multiplication, then $\psi_g = f \circ \varphi_g \circ f'$ gives a quasi-conjugate quasi-action of Γ on X. It is a geometric quasi-action since the action of Γ on itself by left multiplication is a geometric action. □

2.3 Asymptotic Cones

We briefly recall some basic definitions and notations. For a more comprehensive introduction to asymptotic cones, see, for example, [BH99].

Definition 2.3.1. A *filter on* $\mathbb{N}$ is a collection ω of subsets of $\mathbb{N}$ such that

1. if $A \subset B$ and $A \in \omega$, then $B \in \omega$,
2. if $A, B \in \omega$, then $A \cap B \in \omega$, and
3. $\varnothing \notin \omega$.

The set of filters on $\mathbb{N}$ is partially ordered by inclusion. An *ultrafilter on* $\mathbb{N}$ is a filter which is maximal with respect to this partial order. An ultrafilter which contains all cofinite subsets of $\mathbb{N}$ is called a *non-principal ultrafilter.* We note that it is common to think of an ultrafilter as a measure on $\mathbb{N}$, so we may write $A \in \omega$ or $\omega(A) = 1$ interchangeably.

Given a non-principal ultrafilter ω and a bounded sequence $(a_n)_n$ of real numbers, there exists a unique $a \in \mathbb{R}$ such that

$$\omega\{n \colon |a_n - a| < \epsilon\} = 1$$

for every $\epsilon > 0$. This value a is called the *ultralimit of* $(a_n)_n$ *with respect to* ω, and is denoted $\omega\text{-}\lim a_n$.

Let (X, d) be a metric space with a chosen base point $x_0 \in X$ and a sequence of scaling factors $D_n \to \infty$. We write $(X_n, d_n) = \left(X, \frac{1}{D_n} d\right)$ for the sequence of scaled metric spaces. Let X_∞ denote the collection of sequences $(x_n)_n$, $x_n \in X_n$, such that $d_n(x_n, x_0)$ is bounded independent of n. The relation $(x_n)_n \sim (y_n)_n$ if $\omega\text{-}\lim d_n(x_n, y_n) = 0$ is an equivalence relation on X_∞. We write X_ω for the set of equivalence classes, and we give X_ω a metric d_ω as follows:

$$d_\omega((x_n)_n, (y_n)_n) = \omega\text{-}\lim d_n(x_n, y_n).$$

The space (X_ω, d_ω) is called an *asymptotic cone of* X, and is denoted $\mathrm{Cone}_\omega(X)$.

It is common to also choose a sequence of distinct base points in this construction. Throughout, we will use a fixed base point x_0 and assume the scaling constants are simply $D_n = n$. We will make use of the following well-known facts about asymptotic cones (see [KL95]):

1. $\mathrm{Cone}_\omega(\mathbb{E}^n)$ is isometric to $\mathbb{E}^n$.

2. If X is geodesic, then $\mathrm{Cone}_\omega(X)$ is geodesic.

3. If X is δ-hyperbolic, then $\mathrm{Cone}_\omega(X)$ is 0-hyperbolic.

4. $\mathrm{Cone}_\omega(X \times Y) = \mathrm{Cone}_\omega(X) \times \mathrm{Cone}_\omega(Y)$.

5. If $f\colon X \to Y$ is a quasi-isometric embedding, then f induces a bi-Lipschitz embedding $\widehat{f}\colon \mathrm{Cone}_\omega(X) \to \mathrm{Cone}_\omega(Y)$, via

   $$\widehat{f}([(x_n)_n]) = [(f(x_n))_n].$$

   Moreover, if f is a quasi-isometry, then $\widehat{f}$ is a bi-Lipschitz homeomorphism.

In Section 3.2.1, we will make use of the fact that we have a characterization of asymptotic cones of non-elementary hyperbolic groups (namely, that they are

$\mathbb{R}$-trees). With an eye toward Chapter 4, we observe a few other scenarios in which there are known characterizations of the asymptotic cones of a class of groups. For example:

Theorem 2.3.2 ([Gro81])**.** A finitely generated group is virtually nilpotent if and only if its asymptotic cones are locally compact.

The asymptotic cone was used in [KL97b] and [KL01] to show that quasi-isometries of certain products of symmetric spaces and Euclidean buildings (in the former paper) and symmetric spaces with simply connected nilpotent Lie groups (in the latter) project to quasi-isometries of the factors.

Definition 2.3.3. Let X be a complete geodesic metric space and $\mathscr{P}$ a collection of closed geodesic subsets. We call the elements of $\mathscr{P}$ *pieces*, and we say that X *is tree-graded with respect to* $\mathscr{P}$ if the following two conditions are satisfied:

1. Every two different pieces have at most one common point.
2. Every simple geodesic triangle in X is contained in one piece.

Theorem 2.3.4 ([DS05, Theorem 8.5])**.** A finitely generated group G is relatively hyperbolic with respect to finitely generated subgroups $H_1, \ldots, H_n$ if and only if every asymptotic cone $\mathrm{Cone}_\omega(G)$ is tree-graded with respect to the ω-limits of cosets of the subgroups H_i.

Again, the characterization of the asymptotic cones seems to be the first step towards a rigidity result. [DS05] gives a partial rigidity result:

Theorem 2.3.5 ([DS05, Corollary 5.22])**.** Let G be a finitely generated group that is hyperbolic relative to unconstricted subgroups $H_1, \ldots, H_m$. Let G' be a group that is quasi-isometric to G. Then G' is hyperbolic relative to subgroups $H'_1, \ldots, H'_n$, each of which is quasi-isometric to one of the $H_1, \ldots, H_m$.

2.4 Convergence Groups

In addition to being a topological invariant, the boundary of a hyperbolic group is studied for another reason. Hyperbolic groups act on their boundaries with a particular dynamical property which turns out to characterize hyperbolic groups.

Definition 2.4.1. We say that $\Gamma \leqslant \mathrm{Homeo}(M)$ is a *convergence group* if, given any infinite sequence $(g_i)_i$ of distinct elements of Γ, there is a subsequence $(g_{n_i})_i$ such that one of the following occurs:

1. there exists $g \in G$ so that $g_{n_i} \to g$ uniformly on M, or
2. there exist points $a, b \in M$ so that $g_n(z) \to a$ uniformly on compact sets away from b, and $g_n^{-1}(z) \to b$ uniformly on compact sets away from a. Following [Bow99], in this case we will say that $(g_i)_i$ is a *collapsing sequence*.

We say Γ is a *discrete convergence group* if the first case does not occur.

Convergence groups were introduced by Gehring and Martin [GM87]. A major result by Gabai and, independently, Casson–Jungreis and Tukia established that subgroups of $\mathrm{Homeo}(S^1)$ which are the restrictions of Fuchsian groups (i.e., discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$) are characterized by the convergence property:

Theorem ([Gab92, CJ94, Tuk88])**.** *$G \leqslant \mathrm{Homeo}(S^1)$ is a discrete convergence group if and only if G is conjugate in $\mathrm{Homeo}(S^1)$ to the restriction of a Fuchsian group.*

Bowditch [Bow99] has given the following reformulation of a discrete convergence group. (In fact, Bowditch attributes this equivalence to Gehring and Martin in the case that M is topologically a sphere. However, the only reference given is to a set of hand-written notes. In any case, a fully general proof is given in [Bow99].) We will write $\Theta(M)$ for the space of ordered, distinct triples of M. Then the action of Γ on M naturally extends to an action on $\Theta(M)$.

Definition 2.4.2. We say that $\Gamma \leqslant \mathrm{Homeo}(M)$ is a *discrete convergence group* if the natural action on $\Theta(M)$ is properly discontinuous. If the action on $\Theta(M)$ is also cocompact, we say that Γ is a *uniform convergence group*.

Freden [Fre94] and Tukia [Tuk94] independently showed that all hyperbolic groups act like discrete convergence groups on their boundaries. Bowditch [Bow98] showed that acting like a uniform convergence group characterizes hyperbolic groups:

Theorem (Bowditch)**.** Suppose that M is a perfect metrisable compactum. Suppose that a group Γ acts by homeomorphisms on M as a uniform convergence group. Then Γ is hyperbolic. Moreover, there is a Γ-equivariant homeomorphism of M onto $\partial\Gamma$.

In [Bow98], Bowditch gives the following alternate characterization of uniformity:

Definition 2.4.3. Let $\Gamma \leqslant \text{Homeo}(M)$. A point $x \in M$ is called a *conical limit point* if there are distinct points $b, c \in M$ and a sequence (g_i) in Γ such that $g_i(x) \to b$ and $g_i(y) \to c$ for all $y \in M \backslash \{x\}$.

Theorem (Bowditch)**.** If $\Gamma \leqslant \text{Homeo}(M)$ is a convergence group such that every point of M is a conical limit point, then Γ is a uniform convergence group.

We remark that the terminology regarding convergence groups has changed somewhat in the literature. [GM87] and [Tuk94] make a distinction between discrete and not necessarily discrete convergence groups. Since [Gab92] is primarily concerned with Fuchsian groups (which are, in particular, discrete), any convergence action is already a discrete convergence action, and so Gabai simply uses the term "convergence group". Similarly, Bowditch only uses the term "convergence group", but if $g_i \to g$ uniformly on M for some infinite collection of distinct $g_i \in \Gamma$, then clearly Γ cannot act properly discontinuously on $\Theta(M)$. The convergence actions in [Bow98, Bow99] are implicitly discrete.

Chapter 3

Groups QI to $H \times \mathbb{R}^n$

3.1 Summary of Rieffel's Proof

Rieffle's main result in [Rie01] is the following theorem:

Theorem (Rieffel)**.** For any group Γ quasi-isometric to the hyperbolic plane cross the real line, there is an exact sequence

$$0 \to A \to \Gamma \to G \to 0$$

where A is virtually infinite cyclic and G is a finite extension of a cocompact Fuchsian group.

Here, we will give a brief outline of the ideas of the proof, since we will follow the structure as closely as possible in the following sections. Given that Γ is quasi-isometric to $\mathbb{H}^2 \times \mathbb{R}$, there is a quasi-action φ of Γ on $\mathbb{H}^2 \times \mathbb{R}$. Let $\pi \colon \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2$ be the projection onto a horizontal slice $\mathbb{H}^2 \times \{0\}$. Rieffel shows that φ projects to a quasi-action by composition with π, essentially by showing that a horizontal (hyperbolic) plane has too much area to be mapped into a vertical (Euclidean) plane. The projected quasi-action ψ is clearly cobounded, since φ is. (However, it will no longer be proper.)

Given the quasi-action ψ of Γ on $\mathbb{H}^2$, taking boundary values gives an action by homeomorphisms on $\partial\mathbb{H}^2 \cong S^1$. That is, there is a map $\Xi \colon \Gamma \to F \leqslant \mathrm{Homeo}(S^1)$. Applying previous results [Gab92, CJ94, Tuk88], Rieffel concludes that the orientation-preserving subgroup $F^+ \leqslant F$ is quasisymmetrically conjugate to a Möbius group. That is, there is a homomorphism $\Phi \colon \Gamma \to G$, where G is either a Möbius group or a $\mathbb{Z}/2\mathbb{Z}$ extension of a Möbius group.

The action of G on $\partial\mathbb{H}^2$ is a convergence action. Rieffel goes on to show that this must be a discrete convergence action, that $\ker \Phi$ contains an element of infinite order, and that $\ker \Phi$ must be quasi-isometric to $\mathbb{R}$. These last two facts taken together imply that $\ker \Phi$ is virtually infinite cyclic, establishing the main theorem. We note that these last results rely on G acting like a discrete convergence action, and the proof of discreteness makes use of machinery that is somewhat specific to the setting of $\mathbb{H}^2$.

Essentially, the argument for discreteness goes as follows: fix a left-invariant metric ρ on $\mathrm{PSL}(2, \mathbb{R})$. For a fixed ϵ, consider the asymptotic growth rate as a function of n of the number of elements in G which are simultaneously ϵ-close to the identity in $\mathrm{PSL}(2, \mathbb{R})$ and also have word length at most n in G. Rieffel calls this the *semilocal growth* of G, and shows that it must be linear. On the other hand, assuming the action is non-discrete, Rieffel uses a modified version of the Tits alternative to find elements $a, b \in G$ which are close to the identity in $\mathrm{PSL}(2, \mathbb{R})$ and generate a free group. By a lemma of Zassenhaus, the iterated commutators of a, b remain close to the identity in $\mathrm{PSL}(2, \mathbb{R})$. In this way, Rieffel demonstrates exponential semilocal growth of any non-discrete group. Therefore G must be discrete.

There are many details throughout Rieffel's proof that won't directly generalize to the setting of $H \times \mathbb{R}^n$. But the basic structure will be somewhat preserved:

1. Build a quasi-action of Γ on $H \times \mathbb{R}^n$. (For details, see Section 2.2.)

2. Project this quasi-action to a quasi-action on H. (For details, see Sections 3.2.1 and 3.2.2.)

3. Consider the induced action $\Phi \colon \Gamma \to \mathrm{Homeo}(\partial H)$. Examine the convergence properties of this action. (For details, see Section 3.2.3 and Chapter 4 for comments on work that remains to be done.)

4. Calculate $\ker \Phi$. (See Chapter 4 for comments on work that remains to be done.)

3.2 $H \times \mathbb{R}^n$

3.2.1 Lines in the Asymptotic Cone

Throughout this section, we suppose that H is a non-elementary hyperbolic group with Cayley graph X. Then $\mathrm{Cone}_\omega(X)$ is an $\mathbb{R}$-tree (since it is geodesic and 0-hyperbolic). In fact, $\mathrm{Cone}_\omega(X)$ is a homogeneous $\mathbb{R}$-tree with uncountable branching at every point ([KL95]). We will explore some features of lines in the asymptotic cone $\mathrm{Cone}_\omega(X)$ which arise in a natural way from lines in X.

In this setting, it is natural to choose as our base point the group identity $e \in X$. Taking the constant sequence, we have a natural base point $\widehat{e} = [(e)_n] \in \mathrm{Cone}_\omega(X)$. We will use the term *leaf* to mean any connected component of $\mathrm{Cone}_\omega(X) \setminus \{\widehat{e}\}$.

Definition 3.2.1. Let $L \colon \mathbb{R} \to X$ be an infinite geodesic line. We define $\widehat{L} \colon \mathbb{R} \to \mathrm{Cone}_\omega(X)$ via

$$\widehat{L}(t) = [(L(tn))_n].$$

Then $\widehat{L}$ is a geodesic line in $\mathrm{Cone}_\omega(X)$, and $\widehat{L}(0) = \widehat{e}$. We will say that a line in the asymptotic cone *arises from a line in* X if it can be expressed as $\widehat{L}$ for some line L in X. Such a line $\widehat{L}$ is divided into two rays by removing $\widehat{e}$. We will refer to such rays as *midribs.*[1]

In this section, we will establish the following theorem, which describes all lines in $\mathrm{Cone}_\omega(H)$ which arise from lines in H:

Theorem 3.2.2. Let H be a non-elementary hyperbolic group with asymptotic cone $\mathrm{Cone}_\omega(H)$.

1. For each $a \in \partial H$, there is a unique corresponding midrib.
2. Each leaf contains at most one midrib.
3. If $f \colon \mathbb{R} \to H$ is any quasi-isometric embedding with induced map $\widehat{f}$ on the asymptotic cones, then the image of $\widehat{f}$ is precisely the union of the two midribs corresponding to the boundary points of H which are the endpoints of f.

[1]The term *midrib* in biology refers to the central vein of a leaf, from which other veins emanate.

By definition, midribs each lie in a single leaf of $\mathrm{Cone}_\omega(H)$. We begin by showing that each point in $\mathrm{Cone}_\omega(H)$, then each leaf, and therefore each midrib may be labeled uniquely by a point of ∂H.

Lemma 3.2.3. Let $\widehat{y} \in \mathrm{Cone}_\omega(X)$, and suppose we choose two representative sequences $[(y_n)_n] = [(y'_n)_n] = \widehat{y}$ so that $y_n \to a$ and $y'_n \to b$, where $a, b \in \partial X$. Then $a \neq b$ implies that $\widehat{y} = \widehat{e}$.

Proof. Since $(y_n)_n \sim (y'_n)_n$, we have

$$\omega\text{-}\lim \frac{d(y_n, y'_n)}{n} = 0.$$

Write z_n for the point in $[y_n, y'_n]$ which is closest to e. For large enough n, there is some constant C such that $d(e, z_n) \leqslant C$. Now:

$$\begin{aligned}\omega\text{-}\lim \frac{d(e, y_n)}{n} &\leqslant \omega\text{-}\lim \frac{d(e, z_n) + d(z_n, y_n)}{n} \\ &\leqslant \omega\text{-}\lim \frac{C + d(y_n, y'_n)}{n} \\ &= 0,\end{aligned}$$

so $\widehat{e} = \widehat{y}$. □

That is, except at the base point $\widehat{e}$, for any point $\widehat{y}$ in the asymptotic cone, there is a unique point $a \in \partial X$ such that any representative sequence $(y_n)_n$ for $\widehat{y}$ converges (ω-almost everywhere) to a. We will say that a is the *boundary point (of X) corresponding to $\widehat{y}$*. (However, note that this is not a bijective correspondence.) We will write $\partial\widehat{y} = a$.

Lemma 3.2.4. Suppose $\widehat{y} = [(y_n)_n]$ and $\widehat{y}' = [(y'_n)_n]$ are not $\widehat{e}$. If $\partial\widehat{y} = a$ and $\partial\widehat{y}' = b$ with $a \neq b$, then the geodesic connecting $\widehat{y}$ to $\widehat{y}'$ passes through $\widehat{e}$. In particular, $\widehat{y}$ and $\widehat{y}'$ are in distinct leaves.

Proof. Let $\gamma_n \colon [0,1] \to X$ be a geodesic connecting y_n to y'_n. Since $y_n \to a$ and $y'_n \to b$, we have $(y_n \cdot y'_n)_e \leqslant C$ for some constant C and large enough n. Thus, there is a sequence of points $w_n \in [y_n, y'_n]$ with $d(e, w_n) \leqslant C$. Let $t_n \in [0,1]$ such that $\gamma_n(t_n) = w_n$. Then t_n is a bounded sequence of real numbers, so there is an ultralimit $t = \omega\text{-}\lim t_n$. Since w_n is a bounded sequence, $\widehat{e} = [(\gamma_n(t))_n]$ is on the geodesic connecting $\widehat{y}$ to $\widehat{y}'$. □

It follows that, for any leaf $\widehat{B}$ of $\mathrm{Cone}_\omega(X)$, there is a unique point $a \in \partial X$ such that $\partial\widehat{y} = a$ for all $\widehat{y} \in \widehat{B}$. We will say that a is the *boundary point of* $\widehat{B}$, and we'll write $\partial\widehat{B} = a$. Since a midrib is entirely contained in a single branch, we may also speak of the boundary point of a midrib as the boundary point of the branch in which it is contained. The correspondence between midribs and points in ∂X is bijective:

Proposition 3.2.5. For each $a \in \partial X$, there is a unique midrib with associated boundary value a.

Proof. Let $\gamma_1, \gamma_2 \colon [0,\infty) \to X$ be geodesic rays with $\gamma_1(\infty) = \gamma_2(\infty) = a$. Write $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ for the corresponding midribs. Now, the rays γ_1, γ_2 K-fellow travel for some K (see, e.g., [BH99, Lemma 3.3]). That is, there exists a constant K such that $d(\gamma_1(t), \gamma_2(t)) \leqslant K$ for all t. By definition, $\widehat{\gamma}_1(t) = [(\gamma_1(tn))_n]$, and similarly for γ_2. Now

$$
\begin{aligned}
d(\widehat{\gamma}_1(t), \widehat{\gamma}_2(t)) &= \omega\text{-}\lim \frac{d(\gamma_1(tn), \gamma_2(tn)))}{n} \\
&\leqslant \omega\text{-}\lim \frac{K}{n} \\
&= 0,
\end{aligned}
$$

so $\widehat{\gamma}_1 = \widehat{\gamma}_2$. □

Example 3.2.6. In the previous lemmas, we establish that each leaf contains at most one midrib. We now present an example showing that there are (many) leaves with no midrib. Let $\Gamma = F_2 = \langle a, b \rangle$, the free group of rank 2. Consider the following sequences:

$$y_n = a^{\lfloor \ln n \rfloor} b^n$$
$$z_n = a^{\lfloor \ln n \rfloor} b^{-n}$$

Write $\widehat{y} = [(y_n)]$ and $\widehat{z} = [(z_n)]$. Then $y_n \to a^\infty$ and $z_n \to a^\infty$ as $n \to \infty$, hence $\partial\widehat{y} = \partial\widehat{z} = a^\infty$. Moreover, $d(y_n, z_n) = 2n$, so $\widehat{y} \neq \widehat{z}$. However, $d(e, [y_n, z_n]) = \lfloor \ln n \rfloor$, which grows sublinearly, therefore the geodesic connecting $\widehat{y}$ and $\widehat{z}$ must pass through $\widehat{e}$. That is, $\widehat{y}$ and $\widehat{z}$ are in different leaves. Since there is exactly one midrib corresponding to the boundary point a^∞, at least one of the leaves containing $\widehat{y}$ or $\widehat{z}$ does not contain a midrib. (In fact, neither $\widehat{y}$ nor $\widehat{z}$ is in the leaf containing the midrib corresponding to a^∞. This midrib is in the leaf containing $[(a^n)_n]$.) It is fairly straightforward to adjust this example to construct many leaves without midribs.

We complete the proof of Theorem 3.2.1 by showing that the induced image of quasi-geodesic line is precisely the union of two midribs.

Proposition 3.2.7. Suppose $f\colon \mathbb{R} \to H$ is a quasi-isometric embedding, and let $\widehat{f}\colon \mathbb{R} \to \mathrm{Cone}_\omega(H)$ be the induced map on the asymptotic cone. Then the image of $\widehat{f}$ is the union of two midribs. Conversely, any union of two midribs is the image of the induced map of a quasi-geodesic line.

Proof. If f is any quasi-isometric embedding, its image lies uniformly close to a geodesic line L. Therefore, the image of the induced map on the asymptotic cones is the line $\widehat{L}$ which arises from L, and this is the union of two midribs.

Conversely, any two midribs correspond to two distinct boundary points $a, b \in \partial H$. The geodesic line $\widehat{L}$ which is the union of the midribs arises from the geodesic line L connecting a and b in H. □

3.2.2 Rigidity of Quasi-flats in the Asymptotic Cone

In this section, we establish the following:

Theorem 3.2.8. Let $f\colon H \times \mathbb{R}^n \to H \times \mathbb{R}^n$ be a quasi-isometry and write

$$\pi\colon H \times \mathbb{R}^n \to H$$

for the natural projection. Then there is some constant D such that, for each $x \in H$, there exists $y \in H$ such that $f(\{x\} \times \mathbb{R}^n)$ and $\{y\} \times \mathbb{R}^n$ are at Hausdorff distance at most D. The assignment $x \mapsto y$ is a quasi-isometry (which is equivalent to $\pi \circ f$) whose quasi-isometry constants depend only on those of f.

In particular, if Γ quasi-acts on $H \times \mathbb{R}^n$ via φ, then $\psi_g = \pi \circ \varphi_g$ gives a quasi-action of Γ on H, and ψ is cobounded if φ is.

This result is not entirely new. It is a special case of Theorem B in [KKL98] that any quasi-isometry of $H \times \mathbb{R}^n$ is a product of quasi-isometries on the factors up to a bounded error. We note, however, that the statement of that theorem is not *quantitative* (i.e., it is not claimed that the quasi-isometry constants of the factor maps depend only on those of the original map). This is not guaranteed of an arbitrary quasi-isometry, so it is important that our quasi-isometries come from a particularly nice quasi-action. For example, consider the $\mathbb{Z}$-action by an infinite order rotation on $\mathbb{R}^2$. Each power of the generator g projects to a scaling of the x-axis, but the scaling constants depend on the angle of rotation of the power. Since the rotation is infinite order, this angle can approach $\pi/2$ arbitrarily closely, so the scaling constants of the projected maps can be arbitrarily close to 0. Such behavior is not permitted in a quasi-action.

We also note that the proof in the following sections is substantially different. The proof in [KKL98] involves a global topological property of the asymptotic cone $\mathrm{Cone}_\omega(H)$ (which they call *nontranslatability*), whereas the following proof involves the scarcity of lines in $\mathrm{Cone}_\omega(H)$ arising from lines in H established in the previous section. In particular, we hope that some of the methods used in the following proof may apply in some situations in which asymptotic cones may not be topologically nontranslatable.

Definition 3.2.9. By an *n-quasi-flat in Y*, we will mean either a quasi-isometric embedding $f\colon \mathbb{R}^n \to Y$ or the image of such an embedding (hopefully the distinction will be clear from context).

Throughout this section, Y will be $H \times \mathbb{R}^n$, and any quasi-flat will be assumed to be an $(n+1)$-quasi-flat.

Theorem 3.2.10. Let $f\colon \mathbb{R}^{n+1} \to H \times \mathbb{R}^n$ be a quasi-isometric embedding and let $\widehat{f}$ be the induced map on the asymptotic cones. Then

1. the image of $\widehat{f}$ is of the form $\widehat{L} \times \mathbb{R}^n$, where $\widehat{L}$ is a line in $\mathrm{Cone}_\omega(H)$,
2. $\widehat{L}$ is the union of two midribs corresponding to some distinct boundary points $a, b \in \partial H$, and
3. for any line L in H with endpoints a, b, $\widehat{L}$ is the line in the asymptotic cone arising from L, and the image of f is uniformly close $L \times \mathbb{R}^n$.

Proof. The first part of the theorem is established in the case $n = 1$ by [KL95, Lemma 2.14]. To generalize, we need only note that the reduced homology $\widetilde{H}_1(S^{n+1}; \mathbb{Z})$ is trivial for all $n \geqslant 1$, so that the Alexander Duality theorem applies. The remainder of the proof is unchanged. (The reader may refer to, e.g., [Mas91] for the definition of the Alexander–Spaniel cohomology and the Alexander duality theorem.)

Now the image of $\widehat{f}$ is of the form $\widehat{L} \times \mathbb{R}^n$ for some line $\widehat{L}$ in T. We next show that $\widehat{L}$ is the union of two midribs. We will suppose without loss of generality that $e = f(e')$ is in the image of f.

The base point $\widehat{e}$ splits $\widehat{L}$ into two rays. Consider a point $\widehat{y} = [(y_n)_n] = [(f(x_n))_n]$ on one of these rays. Passing to an ω-full measure index set, we will suppose that $y_n \to a \in \partial H$. We will write γ for the geodesic ray emanating from e and converging to a, and we will write γ_n for the geodesic segment from e to y_n. Let z_n be the point on γ which is closest to y_n. Let $t > 0$, and choose n large enough so that $(y_n \cdot z_n)_e > t$. (Since $y_n \to a$ and $z_n \to a$, $(y_n \cdot z_n)_e \to \infty$, so we can always find such an n.)

By definition of δ-hyperbolicity, $d(\gamma(t), \gamma_n(t)) \leqslant \delta$. Moreover, by stability of quasi-geodesics in a hyperbolic space, γ_n is K-Hausdorff close to the image of the geodesic segment $\gamma'_n = [e', x_n]$, where K depends only on δ and the quasi-isometry constants of f. Thus every point $\gamma(t)$ lies $\delta + K$ close to the image of f. (See Figure 3.1.)

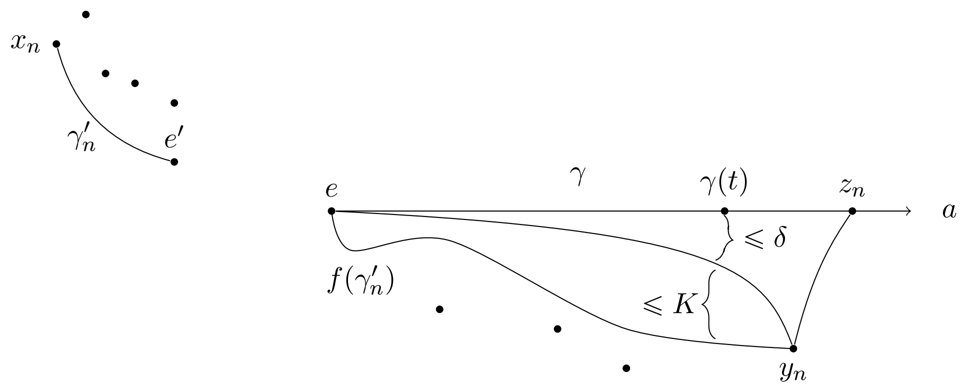

Figure 3.1: A quasi-flat is uniformly close to an "obvious" flat.

Let w_n be a point in the image of f within $\delta + K$ of $\gamma(n)$. Now $[(\gamma(n))_n] = [(w_n)_n] = \widehat{w}_1$ must be a point of $\widehat{L}$. Since γ is a geodesic ray converging to a, the point $\widehat{w}_1$ lies on the midrib corresponding to a. Considering the sequences $\widehat{w}_k = [(w_{kn})_n] = [(\gamma(kn))_n]$ for each $k \in \mathbb{N}$, we find points arbitrarily far along the midrib corresponding to a which are also in $\widehat{L}$. It follows that $\widehat{L}$ contains the midrib corresponding to a.

The same argument applied to the other ray of $\widehat{L}$ implies that it contains another midrib. Since distinct midribs have distinct associated boundary points, this other midrib has some boundary point b which is distinct from a. Since $\widehat{L}$ is a line, it must precisely be the union of the midribs corresponding to a and b. Let L be an infinite geodesic line with endpoints a and b. Then $\widehat{L}$ is the line arising from L.

Finally, we must show that the image of f is uniformly close to $L \times \mathbb{R}^n$. The following argument is adapted from [SW02, Lemma 2.6]. The image of f coarsely contains $L \times \mathbb{R}^n$. That is, there is some D so that every point of $L \times \mathbb{R}^n$ is within D of some point in the image of f.

Let $f^{-1}\colon f(\mathbb{R}^{n+1}) \to \mathbb{R}^{n+1}$ be a coarse inverse for f. Then $f(\mathbb{R}^{n+1})$ cannot contain any points whose distance from $L \times \mathbb{R}^n$ is more than $KC + D$, since f^{-1} cannot identify points which are more than KC away from each other. $\square$

We can now complete the main theorem of the previous this section:

Theorem 3.2.8. Let $f \colon H \times \mathbb{R}^n \to H \times \mathbb{R}^n$ be a quasi-isometry and write

$$\pi \colon H \times \mathbb{R}^n \to H$$

for the natural projection. Then there is some constant D such that, for each $x \in H$, there exists $y \in H$ such that $f(\{x\} \times \mathbb{R}^n)$ and $\{y\} \times \mathbb{R}^n$ are at Hausdorff distance at most D. The assignment $x \mapsto y$ is a quasi-isometry (which is equivalent to $\pi \circ f$) whose quasi-isometry constants depend only on those of f.

In particular, if Γ quasi-acts on $H \times \mathbb{R}^n$ via φ, then $\psi_g = \pi \circ \varphi_g$ gives a quasi-action of Γ on H, and ψ is cobounded if φ is.

Proof. We apply an argument which can be seen in [FM00, Section 7.2] and [SW02, Section 2.2]. Let $h \in H$. Then h is the quasi-center of some ideal triangle, whose infinite geodesic sides we call L_1, L_2, L_3. Any quasi-isometry $\varphi_g \colon H \times \mathbb{R}^n \to H \times \mathbb{R}^n$ restricts to the quasi-isometric embedding of $(n+1)$-dimensional flats $L_i \times \mathbb{R}^n$. By the previous results, the images of these restrictions are uniformly close to $(n+1)$-flats $L'_i \times \mathbb{R}^n$, where the lines L'_i form an ideal triangle. Let h' be some quasi-center of this ideal triangle. The assignment of $\psi_g(h) = h'$ is a quasi-isometry of H which is equivalent to $\pi \circ \varphi_g$. The resulting quasi-action ψ on H is clearly cobounded. □

3.2.3 The Action on the Boundary

Let Γ quasi-act on $X = H \times \mathbb{R}^n$ via φ. Let $\psi_g = \pi \circ \varphi_g$ be the projected quasi-action of Γ on H. Any quasi-isometry of a hyperbolic group induces a homeomorphism of the boundary, so the quasi-action ψ induces a homomorphism $\Xi \colon \Gamma \to \mathrm{Homeo}(\partial H)$. Write F for the image of Ξ. Two equivalent quasi-isometries induce the same homeomorphism on the boundary, so any quasi-action equivalent to ψ will induce the same homomorphism Ξ. By [Tuk94, Corollary 3G], F acts like a (not necessarily discrete) convergence group on ∂H.

Proposition 3.2.11. If F is a discrete convergence group, then every point $b \in \partial H$ is a conical limit point.

Proof. Let $b \in \partial H$, and let $L\colon \mathbb{R} \to H$ be a geodesic line with $L(\infty) = b$. Since ψ is cobounded, there is some constant C such that, for each $i \in \mathbb{N}$ there is some $g_i \in \Gamma$ so that

$$d(\psi_{g_i}(e), L(i)) \leqslant C.$$

Since this C is uniform, we may without loss of generality assume that $\psi_{g_i}(e) = L(i)$ for each i.

Write $\Xi(\psi_{g_i}) = f_i$. We claim that, possibly after passing to a subsequence, the elements f_i are distinct. Suppose not. Since H is non-elementary, there are at least 3 boundary points. Since any hyperbolic group is a visibility space, there is a geodesic line L' connecting two boundary points which are not b. The quasi-isometries ψ_{g_i} all have the same quasi-isometry constants, so there is a uniform constant K such that $\psi_{g_i}(L')$ lies K-Hausdorff close to a geodesic line. For any g_i, g_j such that $\Xi(\psi_{g_i}) = \Xi(\psi_{g_j})$, ψ_{g_i} and ψ_{g_j} must move L' K-Hausdorff close to the same geodesic line. However, any infinitely many such ψ_{g_i} must move e outside any K-ball, a contradiction. So, the collection $\{f_i\}$ must contain infinitely many distinct elements of F.

By assumption, F acts like a discrete convergence group, so there is some possibly further subsequence (g_{n_i}) and points $a, b' \in \partial H$ so that

$$\psi_{g_{n_i}}(z) \to b'$$
$$\psi_{g_{n_i}}^{-1}(z) \to a$$

uniformly on compact sets of $H \cup \partial H$ away from a and b', respectively. Since $\psi_{g_{n_i}}(e) \to b$, we must have that $b = b'$. Taking the sequence $(f_{n_i}^{-1})$, we have a collection of homeomorphisms showing that the given point $b \in \partial H$ is a conical limit point. $\square$

Applying [Bow98, Theorem 8.1]:

Corollary 3.2.12. If F is a discrete convergence group, then it is a uniform convergence group. Thus F is hyperbolic, and ∂F is homeomorphic to ∂H.

We also observe that, in the case in which H is virtually free, then F is conjugate to a group which acts cocompactly on the space of triples, even without the assumption that F is a discrete convergence group:

Proposition 3.2.13. Suppose H is virtually free. Then F is conjugate to some uniformly quasi-conformal group $F' \leqslant \mathrm{Homeo}(\partial H)$ which acts cocompactly on the space of triples.

Proof. H is quasi-isometric to a bounded valence tree T with boundary a Cantor set. By [MSW03, Theorem 1], the cobounded quasi-action ψ is quasi-conjugated to an isometric action ψ' of Γ on some tree T' which is quasi-isometric to T. More precisely, there is a map $h\colon T \to T'$ and a constant C so that

$$d(h(\psi_g(x)), \psi'_g(h(x)) \leqslant C$$

for all $g \in \Gamma$ and $x \in T$. Since ψ is a cobounded quasi-action, ψ' is a cocompact isometric action. The map h induces a homeomorphism $\partial h\colon \partial T \to \partial T'$. Since the maps $h \circ \psi_g \circ h^{-1}$ and ψ'_g are equivalent as quasi-isometries, they induce the same homeomorphism on $\partial T'$. Now, taking the boundary values of ψ', we have a map

$$\Xi'\colon \Gamma \to F' \leqslant \mathrm{Homeo}(\partial T') \cong \mathrm{Homeo}(\partial T),$$

and the group F' is a conjugate (by ∂h) of F.

Since the quasi-isometry constants of ψ'_g and h are uniformly bounded, we have that F' is a uniformly quasi-conformal group. Since ψ' is a cocompact isometric action on a hyperbolic space, the induced action on the space of distinct triples is cocompact. □

Chapter 4

Future Work

4.1 Characterizing Groups Quasi-isometric to $H \times \mathbb{R}^n$

We have so far established the following:

Theorem 4.1.1. Suppose H is a non-elementary hyperbolic group, and Γ is a group which is quasi-isometric with $H \times \mathbb{R}^n$. Then there is a homomorphism $\Xi \colon \Gamma \to Homeo(\partial H)$ whose image F acts like a convergence group on ∂H. In the case that F is also a discrete convergence group, then F is hyperbolic and $\partial F \cong \partial H$.

It may be the case that F is always a discrete convergence group. In the case of $\mathbb{H}^2 \times \mathbb{R}$, [Rie01, Theorem 5.20] shows that the analogous group (called G in that paper) must be a discrete convergence group. The proof relies heavily on facts about $\mathrm{PSL}(2, \mathbb{R})$ that have no generalization as far as the author is aware to the context of non-elementary hyperbolic groups. Briefly, the proof goes as follows.

Given that G is a Möbius group, we have two notions of the *size* of an element of G. On the one hand, we have the word metric d with respect to a fixed finite generating set. On the other hand, the elements of G are represented as elements of $\mathrm{PSL}(2, \mathbb{R})$, and we can fix some left-invariant metric ρ on $\mathrm{PSL}(2, \mathbb{R})$. Rieffel writes

$$G^n_\epsilon = \{g \in G \mid d(g, e) \leqslant n, \rho(g, e) \leqslant \epsilon\}.$$

The *semilocal growth of G in* $\mathrm{PSL}(2, \mathbb{R})$ is the growth rate of $|G^n_\epsilon|$. Rieffel shows that the semilocal growth of the given G is at most linear, while any non-discrete convergence group must have super-linear semilocal growth. As part of the latter claim, Rieffel shows that any Möbius group acting as a non-discrete convergence group must contain a pair of infinite order elliptic elements which generate a free group. Rieffel provides exponentially many elements of this free group that remain within ϵ of the identity to prove super-linear semilocal growth.

We might hope to provide some formal notion of "close to the identity map on ∂H" which is analogous to "close to the identity matrix in $\mathrm{PSL}(2,\mathbb{R})$". Given such a notion, we could define semilocal growth as Rieffel does. So long as the "closeness to the identity map on ∂H" also limits the distance a point can be moved as a quasi-isometry acts on $H \times \mathbb{R}^n$, it will follow essentially unchanged from [Rie01] that the semilocal growth of F will be linear. It will then remain to show that any non-discrete convergence group would have super-linear semilocal growth.

Supposing that any F is a discrete convergence group, the rest of the characterization lies in calculating the kernel of Ξ. In the case $n = 1$, the proof of [Rie01] goes through unchanged and shows that the kernel is virtually $\mathbb{Z}$. In the general case, we may consider the quasi-action of $\ker \Xi$ on the fibers $\mathbb{R}^n$. This quasi-action is clearly proper, since φ is proper. It is not clear whether or not the quasi-action is cobounded. Establishing coboundedness would establish that $\ker \Xi$ is virtually $\mathbb{Z}^n$, completing the characterization of groups quasi-isometric to $H \times \mathbb{R}^n$.

4.2 Relatively Hyperbolic Groups

As stated in Section 3.2.2, the fact that quasi-actions on the product space $H \times \mathbb{R}^n$ project to quasi-actions on H is not entirely new. That quasi-isometries project in this setting is a special case of the following theorem:

Theorem ([KKL98, Theorem B])**.** Suppose $M = Z \times \prod_{i=1}^{k} M_i$ and $N = W \times \prod_{i=1}^{\ell} N_i$ are geodesic metric spaces such that the asymptotic cones of Z and W are homeomorphic to $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and the components M_i, N_j are of coarse type I and II. Then for every $L \geqslant 1, A \geqslant 0$ there is a constant D so that for each (L, A)-quasi-isometry $\phi\colon M \to N$, we have $k = \ell$, $n = m$, and after reindexing the factors N_j, there are quasi-isometries $\phi_i\colon M_i \to N_i$ such that for every i the following diagram commutes up to error at most D:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow & & \downarrow \\ M_i & \xrightarrow[\phi_i]{} & N_i \end{array}$$

Definition 4.2.1. Let X be a geodesic metric space. For points $p, q \in X$, write $p \sim q$ if there is a continuous path from p to q whose image is contained in the image of any continuous path from p to q. X is said to be of *type I* if each equivalence class is a geodesically complete tree which branches everywhere. A space is said to be of *coarse type I* if every asymptotic cone has type I.

The reader may refer to [KKL98] for the precise definitions of (coarse) type II. For our present purposes, it suffices to point out that a non-elementary hyperbolic group is of coarse type I, and that this implies that any two homeomorphisms $f, g\colon \mathrm{Cone}_\omega(H) \to \mathrm{Cone}_\omega(H)$ which are finite distance apart must coincide, a property referred to as *nonstranslatability*.

There are interesting classes of groups for which one might hope to extend the argument given in the previous chapters. As an example, we consider the case of relatively hyperbolic groups.

Definition 4.2.2. Let G be a finitely generated group with subgroups $H_1, \ldots, H_r$. The *coned-off Cayley graph*, $\mathrm{Cay}(G; H_1, \ldots, H_r)$, is constructed from $\mathrm{Cay}(G, S)$ as follows: add a vertex for each left coset gH_i, and add an edge from this vertex to each element of gH_i. We say that G *is hyperbolic relative to* $H_1, \ldots, H_r$ if the following two conditions are satisfied:

1. The coned-off Cayley graph is hyperbolic.
2. For every integer k, each edge of the coned-off Cayley graph belongs to only finitely many simple cycles of length k.

Relatively hyperbolic groups need not be of coarse type I. The simplest example is that $\mathbb{Z}^n$ is hyperbolic relative to $\mathbb{Z}^n$, but any asymptotic cone of $\mathbb{Z}^n$ is $\mathbb{R}^n$, and any translation of $\mathbb{R}^n$ is a homeomorphism which is finite distance from the identity map without coinciding with the identity. So $\mathbb{Z}^n$ is not nontranslatable, and therefore does not have coarse type I. This example is fairly trivial. Consider the group $G = \mathbb{Z}^2 * \mathbb{Z}^2$. The asymptotic cone of G with respect to a chosen fixed base point consists of uncountably many planes, each with infinitely many planes wedged to every single point in an $\mathbb{R}$-tree like fashion. Each point in $\mathrm{Cone}_\omega(G)$ is its own equivalence class, so G is not of coarse type I. So the results of [KKL98] may not apply to this class of groups.

The methods of Sections 3.2.1 and 3.2.2 do not directly apply in this setting without additional work. However, the first steps of that work already appear in the literature in the form of a description of the asymptotic cones of relatively hyperbolic groups and a characterization of relatively hyperbolic groups by their boundary actions:

Theorem ([DS05, Theorem 8.5])**.** A finitely generated group G is asymptotically tree-graded with respect to subgroups $\{H_1, \ldots, H_m\}$ if and only if G is (strongly) hyperbolic relative to $\{H_1, \ldots, H_m\}$ and each H_i is finitely generated.

[DS05] also includes proofs of some rigidity results, showing that quasi-isometries preserve the property of being asymptotically tree-graded, so that there is some hope for results analogous to those obtained in Section 3.2.2:

Question 4.2.3. Suppose G is hyperbolic relative to some family of subgroups $\{H_1, \ldots, H_m\}$. Does every quasi-action of a group Γ on $G \times \mathbb{R}^n$ project to a quasi-action on G?

An appropriate notion of the boundary of a relatively hyperbolic group is defined in [Bow12]. [Yam04] provides a topological characterization of relatively hyperbolic groups according to the dynamics of their actions on compact spaces, and [Ger09] provides (among other results) a slightly simplified characterization:

Theorem ([Ger09, Yam04])**.** Suppose G acts like an expansive, discrete convergence group on a non-empty, perfect, metrisable compactum M. Then G is hyperbolic relative to the set of its maximal parabolic subgroups, and M is equivariantly homeomorphic to the boundary of G.

So, one might hope to answer a question like the following:

Question 4.2.4. Suppose Γ is quasi-isometric to $G \times \mathbb{R}^n$, where G is a relatively hyperbolic group. Then, is there necessarily a short exact sequence

$$0 \to A \to \Gamma \to G' \to 0,$$

where A is virtually $\mathbb{Z}^n$ and G' is relatively hyperbolic with boundary homeomorphic to ∂G?

Appendices

Appendix A

Quasi-actions

A.1 Definitions, Properties, and Examples

A central idea in geometric group theory is to explore the geometry of groups through the spaces on which they act. With the goal of classifying groups up to quasi-isometry, one often seeks to characterize all groups which act geometrically on a given space X. The Schwarz–Milnor Lemma says that all such groups are quasi-isometric to X, however the converse is false. That is, there exist groups quasi-isometric to spaces on which they cannot act geometrically. However, any group quasi-isometric to X must *quasi-act* on X. A *quasi-action by quasi-isometries* generalizes the usual group action by isometries so that the converse of a "quasi" version of the Schwarz–Milnor Lemma holds.

Many basic definitions and facts about quasi-actions can be seen in some form in [KL01, MSW03, Nek97]. Many small details have elementary proofs, and so they often go unproven (or unstated entirely), and some terminology or conventions have changed somewhat since these papers. We provide here an introduction to the theory of quasi-actions in the hopes of saving some future mathematician some time. All proofs given are essentially elementary, and I make no claim that all of these results are original.

We will begin by defining quasi-actions. There are some aspects of this definition which are, perhaps, unexpected. We will discuss and provide some examples illustrating pathological behavior. We will then define the quasi-action properties in which we will be interested—*cobounded* and *proper*. We will define and discuss three separate equivalence relations—*equivalence of quasi-isometries*, *equivalence of quasi-actions*, and *quasi-conjugacy of quasi-actions*—and discuss their various interactions with each other and with the quasi-action properties. Finally, we show the converse of a "quasi" version of the Schwarz–Milnor Lemma for quasi-actions:

Theorem. A group Γ quasi-acts on a space X coboundedly and properly if and only if Γ is finitely generated and quasi-isometric to X.

Definition A.1.1 (Quasi-action)**.** Let Γ be a group and (X, d) a metric space. A *quasi-action by quasi-isometries of* Γ *on* X associates to each $g \in \Gamma$ a quasi-isometry $\varphi_g \colon X \to X$ satisfying:

1. (φ_e is almost the identity) $\sup_{x \in X} d(\varphi_e(x), x) < \infty$
2. (φ is almost an action) there exists a constant $\alpha > 0$ such that, for all $g, h \in \Gamma$,
$$\sup_{x \in X} d(\varphi_g \varphi_h(x), \varphi_{gh}(x)) < \alpha, \text{ and}$$
3. the quasi-isometry constants K, C, δ and the constant α do not depend on $g, h \in \Gamma$.

We observe that the uniformity of the constants K, C, δ, α may seem unusual. In most cases, a group action of Γ on X by isometries (or homeomorphisms, or linear transformations, etc.) is equivalent to a homomorphism $\Gamma \to \mathrm{Isom}(X)$ (or $\Gamma \to \mathrm{Homeo}(X)$, or $\Gamma \to \mathrm{GL}(X)$, etc.). This is not the case with quasi-actions: any quasi-action φ induces a homomorphism $\Gamma \to \mathcal{QI}(X)$, but the converse is false. The following is a striking counterexample:

Example A.1.2. Let $\Gamma = \mathbb{Z}$ and $X = \mathbb{R}$. The map $f_n \colon \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = nx$ is easily seen to be a $(n, 0)$-quasi-isometry of X. Moreover, f_n is not a (K, C)-quasi-isometry for any $K < n$. It follows that the equivalence classes $[f_n]$ and $[f_m]$ are distinct when $n \neq m$. Consider the assignment $\varphi_n = f_{2^n}$. Taking equivalence classes, this gives a homomorphism $\Gamma \to \mathcal{QI}(X)$ whose image is isomorphic to $\mathbb{Z} = \langle [f_2] \rangle$. However, for any fixed K, there is some n so that f_{2^n} cannot be equivalent to any (K, C)-quasi-isometry. Since the maps φ_n cannot be replaced with equivalent quasi-isometries with uniform constants, we have an infinite cyclic subgroup of $\mathcal{QI}(X)$ which cannot quasi-act on X.

Definition A.1.3 (Cobounded, Proper)**.** Let φ be a quasi-action of Γ on X. We say that φ is *cobounded* if there exists a constant C such that, for each $x, y \in X$, there exists $g \in \Gamma$ such that $d(\varphi_g(x), y) < R$. We say that φ is *proper* if, for each R, there exists M such that, for all $x, y \in X$,

$$|\{g \in \Gamma \mid \varphi_g(N(x,R)) \cap N(y,R) \neq \varnothing\}| \leqslant M.$$

For brevity of notation, we will write

$$\Gamma^{x,y}_{\varphi,R} = \{g \in \Gamma \mid \varphi_g(N(x,R)) \cap N(y,R) \neq \varnothing\}.$$

If φ is a cobounded and proper quasi-action by quasi-isometries, we will call φ *geometric*.

We note that any action by isometries is also a quasi-action by quasi-isometries. In this case, it is cobounded (proper) if and only if it is cocompact (properly discontinuous).

A.2 Equivalence and Quasi-conjugacy of Quasi-actions

There are a variety of ways in which we can adjust quasi-actions, or build new quasi-actions out of old ones, while maintaining the properties of being cobounded or proper.

Definition A.2.1 (Equivalence of Quasi-Actions)**.** We say two quasi-actions φ and ψ of Γ on X are *equivalent* if there exists a constant $C > 0$ such that

$$\sup_{x \in X} d(\varphi_g(x), \psi_g(x)) < C.$$

We will write $\varphi \sim \psi$, or $\varphi \sim_C \psi$ if we wish to keep track of the constant.

Example A.2.2. We remark, again, on the uniformity of the constant C in this definition. Let $\Gamma = \mathbb{Z}$ and $X = \mathbb{R}$. Define $\varphi_n(x) = x$ and $\psi_n(x) = x + n$. Then φ and

ψ are both quasi-actions, and both induce the trivial homomorphism $\Gamma \to \mathcal{QI}(X)$. Moreover, for each n, φ_n and ψ_n are equivalent as quasi-isometries. However, there is no uniform constant C (independent of n) so that $\sup_{x \in X} d(x, x+n) < C$, therefore φ and ψ are not equivalent as quasi-actions. We also note that ψ is a cobounded and proper quasi-action, while φ is not. This example shows that we cannot hope to identify these quasi-action properties by examining the induced map $\Gamma \to \mathcal{QI}(X)$.

For the remainder of the chapter, we will not specify our constants each time they show up. All quasi-isometries will have constants K, C, δ. The constant α will refer to the uniform "almost action" constant from the definition of quasi-action. We will overuse the constant C, which is one of the quasi-isometry constants, the coboundedness constant, the constant in the definition of equivalence of quasi-actions, and the constant in the definition of quasi-conjugacy (below). We note that we do not lose any generality by equating these seemingly different constants—in all our definitions, constants may be increased, and in all of the following claims, there are uniform bounds on the given constants, so that we can take the maximum of each constant.

Proposition A.2.3 (Equivalence preserves cobounded, proper)**.** Suppose φ and ψ are equivalent quasi-actions of Γ on X. Then φ is cobounded if and only if ψ is, and φ is proper if and only if ψ is.

Proof. Let $\varphi \sim_C \psi$, and suppose φ is cobounded. Let $x, y \in X$ be arbitrary, and choose $g \in \Gamma$ such that $d(\varphi_g(x), y) < C$. Then

$$\begin{aligned} d(\psi_g(x), y) &\leqslant d(\psi_g(x), \varphi_g(x)) + d(\varphi_g(x), y) \\ &\leqslant C + C. \end{aligned}$$

C does not depend on x, y, so ψ is cobounded.

Now, suppose φ is proper, and let $R > 0$. We must show that there is some M such that $|\Gamma^{x,y}_{\psi,R}| \leqslant M$ for all $x, y \in X$. We will show that

$$\Gamma^{x,y}_{\psi,R} \subset \Gamma^{x,y}_{\varphi,R'}$$

where $R' = R + C$ (which does not depend on x, y). The set $\Gamma^{x,y}_{\varphi,R'}$ has size bounded independently of x, y since φ is proper.

Let $g \in \Gamma$ such that $\psi_g(N(x,R)) \cap N(y,R) \neq \varnothing$. Then there is some $z \in N(x,R)$ such that $\psi_g(z) \in N(y,R)$. Since $d(\varphi_g(z), \psi_g(z)) \leqslant C$, we have $\varphi_g(z) \in N(y, R+C)$, hence $g \in \Gamma^{x,y}_{\varphi,R+C}$. Therefore, ψ is proper. □

The most useful tool for building new quasi-actions out of quasi-isometries or existing quasi-actions is the notion of *quasi-conjugacy*. Quasi-conjugacies also allow us to push quasi-actions from one space to another without affecting the properties of the quasi-action.

Definition A.2.4 (Quasi-conjugacy)**.** Suppose Γ quasi-acts on X via φ and on Y via ψ. A *quasi-conjugacy between* φ *to* ψ is a quasi-isometry $f \colon X \to Y$ such that there exists a constant $C > 0$ satisfying

$$d(f(\varphi_g(x)), \psi_g(f(x))) < C$$

for all $g \in \Gamma$ and $x \in X$. In this case, we say that φ and ψ are *quasi-conjugate*.

The phrase "quasi-conjugacy between φ and ψ", rather than "from φ to ψ", is justified by the following lemma, which says that coarse inverses to quasi-conjugacies are also quasi-conjugacies.

A quasi-conjugacy consists of three pieces of information: the quasi-actions φ and ψ, and the quasi-isometry f. The following sequence of lemmas up to Corollary A.2.7 explore the relationship between quasi-conjugacy, equivalence of quasi-actions, and equivalence of quasi-isometries. They show that we may freely replace φ or ψ with equivalent quasi-actions, and we may replace f with an equivalent quasi-isometry or coarse inverse. Lemma A.2.8 gives the last detail needed to show that quasi-conjugacy is an equivalence relation.

Propositions A.2.9 and A.2.10 and Theorem A.2.11 are the punchlines of this section. The propositions establish that quasi-conjugacy preserves the quasi-action properties, and that we can build quasi-conjugate pairs of quasi-conjugacies in a natural way. The final theorem is the converse of a Schwarz–Milnor lemma for quasi-actions.

Lemma A.2.5. Suppose Γ quasi-acts on X via φ and on Y via ψ. Let $f_1\colon X \to Y$ be a quasi-conjugacy between φ and ψ. Let $f_1'\colon X \to Y$ be a quasi-isometry which is equivalent to f_1, and let $f_2\colon Y \to X$ be any coarse inverse of f_1. Then f_1' and f_2 are also quasi-conjugacies between φ and ψ.

Proof. We first show that f_1' is a quasi-conjugacy. Since f_1 and f_1' are equivalent as quasi-isometries, they are at some bounded distance C in the sup norm. Now:

$$\begin{aligned} d(f_1'(\varphi_g(x)), \psi_g(f_1'(x))) &\leqslant d(f_1'(\varphi_g(x)), f_1(\varphi_g(x)))+ \\ &\quad\; d(f_1(\varphi_g(x)), \psi_g(f_1(x)))+ \\ &\quad\; d(\psi_g(f_1(x)), \psi_g(f_1'(x))) \\ &\leqslant C + C + (KC + C), \end{aligned}$$

and this bound does not depend on x or g. Therefore, f_1' is a quasi-conjugacy.

We now show that f_2 is a quasi-conjugacy. We have:

$$\begin{aligned} d(f_1(f_2(\psi_g(y))), f_1(\varphi_g(f_2(y)))) &\leqslant d(f_1(f_2(\psi_g(y))), \psi_g(y))+ \\ &\quad\; d(\psi_g(y), \psi_g(f_1(f_2(y))))+ \\ &\quad\; d(\psi_g(f_1(f_2(y))), f_1(\varphi_g(f_2(y)))) \\ &\leqslant C + (KC + C) + C. \end{aligned}$$

Now, applying the quasi-isometry inequalities for f_1, we have:

$$\begin{aligned} & \frac{1}{K} d(f_2(\psi_g(y)), \varphi_g(f_2(y))) - C \leqslant KC + 3C \\ \Rightarrow \quad & d(f_2(\psi_g(y)), \varphi_g(f_2(y))) \leqslant K(KC + 4C), \end{aligned}$$

and this bound does not depend on y or g. Therefore, f_2 is a quasi-conjugacy. ☐

In this lemma, we had two fixed quasi-actions and a given quasi-conjugacy.

The lemma says we may freely replace the quasi-conjugacy with any equivalent quasi-isometry or coarse inverse, giving us many quasi-conjugacies between the same two quasi-actions. We may ask what happens when we start with different given information. The following lemma establishes that, given a quasi-action on X and a quasi-isometry $f\colon X \to Y$, all quasi-actions on Y for which f is a quasi-conjugacy are equivalent as quasi-actions.

Lemma A.2.6. Suppose Γ quasi-acts on X via φ and on Y via ψ and ψ', and let $f\colon X \to Y$ be a quasi-conjugacy between φ and ψ. Then f is a quasi-conjugacy between φ and ψ' if and only if $\psi \sim \psi'$.

Proof. We first suppose that f is a quasi-conjugacy between ψ and ψ'. Let $y \in Y$ and $g \in \Gamma$ be arbitrary. Since f is, in particular, a quasi-isometry, there exists $x \in X$ such that $d(f(x), y) < \delta$. Then:

$$\begin{aligned} d(\psi_g(y), \psi'_g(y)) \leqslant\ & d(\psi_g(y), \psi_g(f(x)))+ \\ & d(\psi_g(f(x)), f(\varphi_g(x)))+ \\ & d(f(\varphi_g(x)), \psi'_g(f(x)))+ \\ & d(\psi'_g(f(x)), \psi'_g(y)) \\ \leqslant\ & (K\delta + C) + C + C + (K\delta + C), \end{aligned}$$

hence ψ and ψ' are equivalent.

Conversely, suppose ψ and ψ' are equivalent. Let $x \in X$ and $g \in \Gamma$ be arbitrary. Then:

$$\begin{aligned} d(f(\varphi_g(x)), \psi'_g(f(x))) &\leqslant d(f(\varphi_g(x)), \psi_g(f(x))) + d(\psi_g(f(x)), \psi'_g(f(x))) \\ &\leqslant C + C, \end{aligned}$$

hence f is a quasi-conjugacy between φ and ψ'. $\square$

The following corollary is immediate:

Corollary A.2.7. Suppose φ, φ' are quasi-actions of Γ on X, and ψ, ψ' are quasi-actions of Γ on Y. Let $f\colon X \to Y$ be a quasi-conjugacy between φ and ψ and also between φ' and ψ'. Then $\varphi \sim \varphi'$ if and only if $\psi \sim \psi'$.

A quasi-conjugacy involves three separate pieces: a quasi-action on X, a quasi-action on Y, and a quasi-isometry $X \to Y$. The previous lemmas establish that we may freely replace each of these pieces by an equivalent piece while maintaining the quasi-conjugacy. The following lemma completes the claim that quasi-conjugacy is an equivalence relation among all quasi-actions of Γ on any metric space.

Lemma A.2.8. Let Γ quasi-act on X via φ, on Y via ψ, and on Z via ξ. Let $f_1\colon X \to Y$ be a quasi-conjugacy between φ and ψ, and let $f_2\colon Y \to Z$ be a quasi-conjugacy between ψ and ξ. Then $f_2 \circ f_1$ is a quasi-conjugacy between φ and ξ.

Proof. We have:

$$\begin{aligned} d(f_2(f_1(\varphi_g(x))), \xi_g(f_2(f_1(x))) &\leqslant d(f_2(f_1(\varphi_g(x))), f_2(\psi_g(f_1(x))))+ \\ &\quad d(f_2(\psi_g(f_1(x))), \xi_g(f_2(f_1(x)))) \\ &\leqslant (Kd(f_1(\varphi_g(x)), \psi_g(f_1(x))) + C) + C \\ &\leqslant (KC + C) + C. \end{aligned}$$

□

Having explored every possible interaction between quasi-conjugacy, equivalence of quasi-actions, and equivalence of quasi-isometries, it remains to show that quasi-conjugacies preserve the quasi-action properties, and that we can build quasi-conjugate pairs of quasi-actions fairly naturally.

Proposition A.2.9 (Quasi-conjugacies preserve cobounded, proper)**.** Suppose Γ quasi-acts on X via φ and on Y via ψ, and let $f\colon X \to Y$ be a quasi-conjugacy between them. Then φ is cobounded (proper) if and only if ψ is cobounded (proper).

Proof. Suppose φ is cobounded. Let $y, y' \in Y$ be arbitrary. Since f is a quasi-isometry, there exist $x, x' \in X$ such that $d(f(x), y) \leqslant \delta$ and $d(f(x'), y') \leqslant \delta$. By coboundedness of φ, there exists $g \in \Gamma$ so that $d(\varphi_g(x), x') < C$. Now:

$$\begin{aligned} d(\psi_g(y), y') &\leqslant d(\psi_g(y), \psi_g(f(x)))+ \\ &\quad d(\psi_g(f(x)), f(\varphi_g(x)))+ \\ &\quad d(f(\varphi_g(x)), f(x'))+ \\ &\quad d(f(x'), y') \\ &\leqslant (K\delta + C) + C + (KC + C) + \delta, \end{aligned}$$

therefore ψ is cobounded.

Now, suppose φ is proper. As in Proposition A.2.3, we seek to show that

$$\Gamma^{x,y}_{\psi,R} \subset \Gamma^{x,y}_{\varphi,R'}$$

which is a finite set. Here, $R' = K(\delta + R + (K\delta + C) + 2C)$ does not depend on x, y. (The reader may wish to refer to Figure A.1.)

Let $R > 0$ and $y, y' \in Y$ be arbitrary. Let $g \in \Gamma$ such that

$$\psi_g(N(y, R)) \cap N(y', R) \neq \varnothing.$$

Then there is some $z \in N(y, R)$ such that $\psi_g(z) \in N(y', R)$. Since f is a quasi-isometry, there exist $x, x', w \in X$ such that

$$\begin{aligned} d(f(x), y) &\leqslant \delta \\ d(f(x'), y') &\leqslant \delta \\ d(f(w), z) &\leqslant \delta \end{aligned}$$

It follows that $d(\psi_g(z), \psi_g(f(w))) \leqslant K\delta + C$. Now, we have $d(f(x'), f(\varphi_g(w))) \leqslant \delta + R + (K\delta + C) + C$ (see Figure A.1).

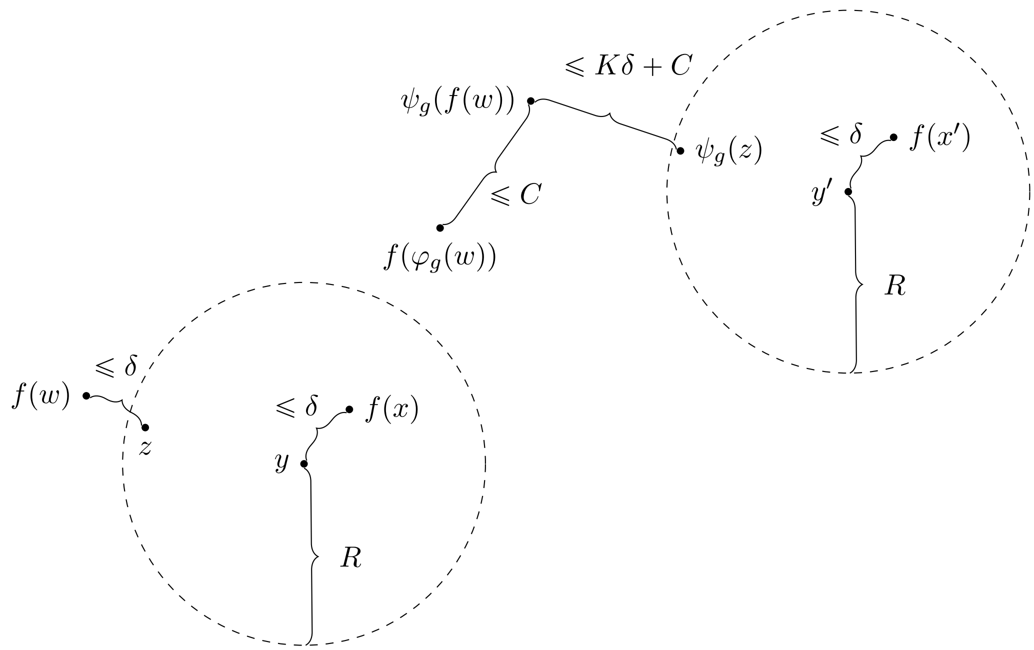

Figure A.1: Quasi-conjugacy preserves properness.

Applying the quasi-isometry inequality of f, we have:

$$\begin{aligned} \frac{1}{K} d(x', \varphi_g(w)) - C &\leqslant \delta + R + (K\delta + C) + C \\ \Rightarrow \qquad d(x', \varphi_g(w)) &\leqslant K(\delta + R + (K\delta + C) + 2C). \end{aligned}$$

We observe that $d(f(x), f(w)) \leqslant R + 2\delta$. Applying the quasi-isometry inequality:

$$d(x, w) \leqslant K(R + 2\delta + C) \leqslant \underbrace{K(\delta + R + (K\delta + C) + 2C)}_{=R'}.$$

That is, $\varphi_g(w) \in \varphi_g(N(x, R')) \cap N(x', R') \neq \varnothing$, completing the claim and the proposition. ☐

Proposition A.2.10 (Building quasi-conjugacies)**.** Suppose φ is a quasi-action of Γ on X. Let $f_1 \colon X \to Y$ be a quasi-isometry with coarse inverse $f_2 \colon Y \to X$. Then the assignment

$$\psi_g(y) := f_1(\varphi_g(f_2(y)))$$

gives a quasi-action ψ of Γ on Y, and f_1, f_2 are quasi-conjugacies between φ and ψ.

Proof. We first note that the quasi-isometry constants of the ψ_g are bounded independently of g since this is true for the φ_g. First, we check that ψ_e is finite distance in the sup norm from the identity map:

$$\begin{aligned}
d(f_1(\varphi_e(f_2(y))), y) &\leqslant d(f_1(\varphi_e(f_2(y))), f_1(f_2(y))) + d(f_1(f_2(y)), y) \\
&\leqslant (Kd(\varphi_e(f_2(y)), f_2(y)) + C) + C \\
&\leqslant (KC + C) + C,
\end{aligned}$$

which is bounded independent of $y \in Y$. Now, we check that ψ is almost an action:

$$\begin{aligned}
d(\psi_g(\psi_h(y)), \psi_{gh}(y)) &= d(f_1(\varphi_g(f_2(\psi_h(y)))), f_1(\varphi_{gh}(f_2(y)))) \\
&\leqslant d(f_1(\varphi_g(f_2(\psi_h(y)))), f_1(\varphi_g(\varphi_h(f_2(y)))))+ \\
&\quad d(f_1(\varphi_g(\varphi_h(f_2(y)))), f_1(\varphi_{gh}(f_2(y)))) \\
&\leqslant [Kd(\varphi_g(f_2(\psi_h(y))), \varphi_g(\varphi_h(f_2(y)))) + C] + \\
&\quad [Kd(\varphi_g(\varphi_h(f_2(y))), \varphi_{gh}(f_2(y))) + C] \\
&\leqslant [K\,[Kd(f_2(\psi_h(y)), \varphi_h(f_2(y))) + C] + C] + \\
&\quad [K\alpha + C] \\
&= [K\,[Kd(f_2(f_1(\varphi_h(f_2(y)))), \varphi_h(f_2(y))) + C] + C] + \\
&\quad [K\alpha + C] \\
&\leqslant [K[KC + C] + C]+ \\
&\quad [K\alpha + C].
\end{aligned}$$

This bound is independent of $g, h \in \Gamma$ and $y \in Y$, so ψ_{gh} and $\psi_g \circ \psi_h$ are uniformly bounded finite distance in the sup norm. Thus, ψ is a quasi-action. Finally:

$$\begin{aligned} d(f_1(\varphi_g(x)), \psi_g(f_1(x))) &= d(f_1(\varphi_g(x)), f_1(\varphi_g(f_2(f_1(x))))) \\ &\leqslant Kd(\varphi_g(x), \varphi_g(f_2(f_1(x)))) + C \\ &\leqslant K[Kd(x, f_2(f_1(x))) + C] + C \\ &\leqslant K[KC + C] + C, \end{aligned}$$

therefore f_1 is a quasi-conjugacy between φ and ψ. □

The converse of a "quasi" version of Schwarz–Milnor is now immediate:

Theorem A.2.11. Suppose Γ is finitely generated and quasi-isometric to X. Then there exists a cobounded and proper quasi-action of Γ on X.

Proof. Γ acts on itself by left multiplication, and this action is properly discontinuous, cocompact, and by isometries. In particular, it is a cobounded and proper quasi-action by quasi-isometries. Using the given quasi-isometry $\Gamma \to X$, we construct a quasi-action of Γ on X which is quasi-conjugate to the action of Γ on itself by left multiplication. Since the quasi-action is quasi-conjugate, it is also cobounded and proper. □

Bibliography

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[Bow98] Brian H. Bowditch. A topological characterisation of hyperbolic groups. *J. Amer. Math. Soc.*, 11(3):643–667, 1998.

[Bow99] B. H. Bowditch. Convergence groups and configuration spaces. In *Geometric group theory down under (Canberra, 1996)*, pages 23–54. de Gruyter, Berlin, 1999.

[Bow12] B. H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra Comput.*, 22(3):1250016, 66, 2012.

[CJ94] Andrew Casson and Douglas Jungreis. Convergence groups and Seifert fibered 3-manifolds. *Invent. Math.*, 118(3):441–456, 1994.

[DS05] Cornelia Druţu and Mark Sapir. Tree-graded spaces and asymptotic cones of groups. *Topology*, 44(5):959–1058, 2005. With an appendix by Denis Osin and Sapir.

[FM00] Benson Farb and Lee Mosher. On the asymptotic geometry of abelian-by-cyclic groups. *Acta Math.*, 184(2):145–202, 2000.

[Fre94] Eric M. Freden. *Negatively curved groups have the convergence property.* ProQuest LLC, Ann Arbor, MI, 1994. Thesis (Ph.D.)–Brigham Young University.

[Gab92] David Gabai. Convergence groups are Fuchsian groups. *Ann. of Math. (2)*, 136(3):447–510, 1992.

[Ger09] Victor Gerasimov. Expansive convergence groups are relatively hyperbolic. *Geom. Funct. Anal.*, 19(1):137–169, 2009.

[GM87] F. W. Gehring and G. J. Martin. Discrete quasiconformal groups. I. *Proc. London Math. Soc. (3)*, 55(2):331–358, 1987.

[Gro81] Mikhael Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53):53–73, 1981.

[Gro84] Mikhael Gromov. Infinite groups as geometric objects. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 385–392, Warsaw, 1984. PWN.

[Gro93] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[KKL98] Michael Kapovich, Bruce Kleiner, and Bernhard Leeb. Quasi-isometries and the de Rham decomposition. *Topology*, 37(6):1193–1211, 1998.

[KL95] M. Kapovich and B. Leeb. On asymptotic cones and quasi-isometry classes of fundamental groups of 3-manifolds. *Geom. Funct. Anal.*, 5(3):582–603, 1995.

[KL97a] Michael Kapovich and Bernhard Leeb. Quasi-isometries preserve the geometric decomposition of Haken manifolds. *Invent. Math.*, 128(2):393–416, 1997.

[KL97b] Bruce Kleiner and Bernhard Leeb. Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *Inst. Hautes Études Sci. Publ. Math.*, (86):115–197 (1998), 1997.

[KL01] Bruce Kleiner and Bernhard Leeb. Groups quasi-isometric to symmetric spaces. *Comm. Anal. Geom.*, 9(2):239–260, 2001.

[Mas91] William S. Massey. *A basic course in algebraic topology*, volume 127 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[MSW03] Lee Mosher, Michah Sageev, and Kevin Whyte. Quasi-actions on trees. I. Bounded valence. *Ann. of Math. (2)*, 158(1):115–164, 2003.

[Nek97] V. Nekrashevych. Quasi-isometry groups. *Mat. Stud.*, 8(2):227–232, 238, 1997.

[Rie01] Eleanor G. Rieffel. Groups quasi-isometric to $\mathbf{H}^2 \times \mathbf{R}$. *J. London Math. Soc. (2)*, 64(1):44–60, 2001.

[SW02] Estelle Souche and Bert Wiest. An elementary approach to quasi-isometries of tree $\times$ $\mathbb{R}^n$. In *Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part II (Haifa, 2000)*, volume 95, pages 87–102, 2002.

[Tuk88] Pekka Tukia. Homeomorphic conjugates of Fuchsian groups. *J. Reine Angew. Math.*, 391:1–54, 1988.

[Tuk94] Pekka Tukia. Convergence groups and Gromov's metric hyperbolic spaces. *New Zealand J. Math.*, 23(2):157–187, 1994.

[Yam04] Asli Yaman. A topological characterisation of relatively hyperbolic groups. *J. Reine Angew. Math.*, 566:41–89, 2004.