FIVE INNOVATIONS IN PLANETARY THEORY

INTRODUCED BY KEPLER IN *ASTRONOMIA NOVA*

1. Lines of apsides -- i.e. lines between aphelia and perihelia -- always pass through the true Sun, and all motions should be referenced to the true Sun rather than the mean Sun.

2. Introduce a bi-section of the eccentricity for the Earth-Sun orbit, in the manner of the other orbits, implying that the Earth-Sun motion involves a true variation in velocity, not just an apparent one; the key consequence of bi-section of the eccentricity is that, with the equant or any velocity rule refining it, the arc length velocities of the orbiting body at its maximum and minimum distances from the central body -- i.e. at aphelion and perihelion when the Sun is the central body -- are in the inverse ratio of these distances.

3. Replace the equant with the area rule -- or its equivalent, *the component of the arc length velocity normal to a radius drawn from the central body varies inversely with the distance from the central body* -- for purposes of determining the (unperturbed) location of the orbiting body versus time from aphelion.

4. Construe the trajectory of the orbiting body (or, at least, its primary motion at any time) as defining an ellipse with the central body at a focus.

5. Construe the orbital motion (at least at any time) as occurring in a plane that passes through the true Sun and is inclined at a measurable angle to the plane defined by the Earth-Sun orbit.

Qualifying remarks in parentheses in the above statements represent allowances Kepler expressly made in subsequent works; they are not inconsistent with the conclusions of *Astronomia Nova*, for what they amount to, in effect, is to allow for the possibility of the values of the elements of the Keplerian orbits changing over time. All five of these innovations are still part of orbital astronomy today.

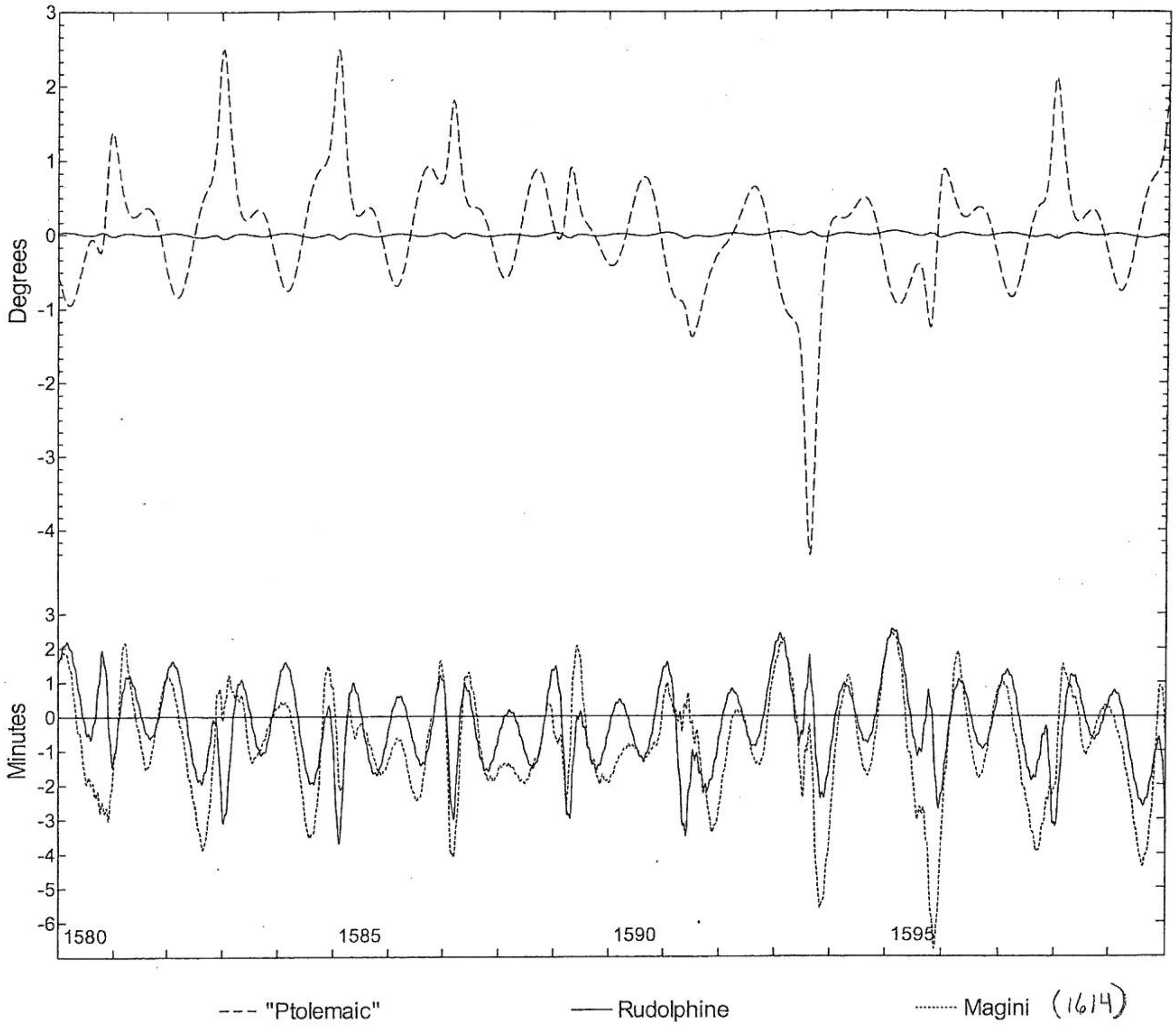


FIG. 5. Errors in geocentric longitudes of Mars. (*Above*) Ptolemaic calculation compared with Kepler's *Rudolphine tables*; (*below, at much enlarged scale*) the *Rudolphine* errors compared with those from Magini's "Keplerian" method in his *Supplementum ephemeridum*.

Tempus	Locus ☉	Solis a Terra distantia	Martis a Sole distantia	Martis eccentricus in ecliptica	Locus computatus	Locus observatus	Differentia	Latitudo
1582. 23 Nove. H. 16. 0	11°. 41′ ♐	98345	158852	0°. 42′. 11″ ♋	26. 40. 0 ♋	26. 38. 30 ♋	1′. 30″ +	Bor. 2. 49
26 Dece. H. 8. 30	15. 4 ♑	98226	162104	16. 7. 18 ♋	17. 44. 19 ♋	17. 40. 30 ♋	3. 49 +	4. 7
30 Dece. H. 8. 10	19. 9 ♑	98252	162443	17. 56. 32 ♋	16. 6. 20 ♋	16. 0. 30 ♋	5. 50 +	4. 8
1583. 26 Janua. H. 6. 15	16. 33 ♒	98624	164421	0. 6. 24 ♌	8. 17. 57 ♋	8. 20. 30 ♋	2. 33 —	2. 52
1584. 21 Dece. H. 14. 0	10. 16 ♑	98207	164907	3. 51. 45 ♌	1. 14. 34 ♍	1. 13. 30 ♍	1. 4 +	3. 31
1585. 24 Janua. H. 9. 0	14. 53 ♒	98595	166210	18. 47. 8 ♌	24. 3. 58 ♌	24. 7. 30 ♌	3. 32 —	4. 31
4 Febr. H. 6. 40	26. 10 ♒	98830	166400	23. 33. 41 ♌	19. 43. 52 ♌	19. 47. 0 ♌	3. 8 —	4. 28
12 Mart. H. 10. 30	2. 16 ♈	99858	166170	9. 23. 14 ♍	11. 43. 31 ♌	11. 46. 0 ♌	2. 29 —	3. 22
1587. 25 Janua. H. 17. 0	16. 1 ♒	98611	166232	8. 13. 40 ♍	4. 41. 50 ♎	4. 42. 0 ♎	0. 10 —	3. 26
4 Mart. H. 13. 24	24. 0 ♓	99595	164737	24. 56. 50 ♍	26. 24. 41 ♍	26. 25. 40 ♍	0. 59 —	3. 38
10 Mart. H. 11. 30	29. 52 ♓	99780	164382	27. 35. 54 ♍	24. 5. 15 ♍	24. 5. 15 ♍	0. 0	3. 29
21 April. H. 9. 30	10. 48 ♉	101010	161027	16. 44. 51 ♎	15. 49. 50 ♍	15. 48. 20 ♍	1. 30 +	1. 48
1589. 8 Mart. H. 16. 24	28. 36 ♓	99736	161000	16. 55. 14 ♎	12. 14. 7 ♏	12. 16. 50 ♏	2. 43 —	2. 4
13 April. H. 11. 15	3. 38 ♉	100810	157141	4. 1. 50 ♏	4. 45. 0 ♏	4. 43. 20 ♏	1. 40 +	1. 10
15 April. H. 12. 5	5. 36 ♉	100866	156900	5. 1. 41 ♏	3. 58. 57 ♏	3. 58. 20 ♏	0. 37 +	1. 4
6 Maji. H. 11. 20	25. 49 ♉	101366	154326	15. 30. 36 ♏	27. 8. 17 ♎	27. 7. 20 ♎	0. 57 +	0. 7
1591. 13 Maji. H. 14. 0	2. 10 ♊	101467	147891	12. 7. 38 ♐	2. 15. 36 ♑	2. 20. 0 ♑	4. 24 —	Aust. 2. 25
6 Junii H. 12. 20	24. 59 ♊	101769	144981	25. 38. 48 ♐	27. 11. 45 ♐	27. 15. 0 ♐	3. 15 —	3. 55
10 Junii H. 11. 50	28. 47 ♊	101789	144526	27. 56. 49 ♐	25. 57. 57 ♐	26. 2. 36 ♐	4. 39 —	4. 8
28 Junii H. 10. 24	15. 51 ♋	101770	142608	8. 29. 32 ♑	21. 4. 21 ♐	21. 10. 0 ♐	5. 39 —	4. 45
1593. 21 Julii H. 14. 0	8. 26 ♌	101498	138376	20. 1. 38 ♒	17. 43. 14 ♓	17. 45. 45 ♓	2. 31 —	5. 46
22 Aug. H. 12. 20	9. 11 ♍	100761	138463	10. 15. 25 ♓	13. 9. 39 ♓	13. 10. 15 ♓	0. 36 —	6. 7
29 Aug. H. 10. 20	11. 54 ♍	100562	138682	14. 37. 15 ♓	11. 11. 41 ♓	11. 14. 0 ♓	2. 19 —	5. 52
3 Octo. H. 8. 0	20. 15 ♎	99500	140697	6. 19. 39 ♈	7. 49. 54 ♓	7. 50. 10 ♓	0. 16 —	3. 17
1595. 17 Sept. H. 16. 45	4. 18 ♎	99990	143222	22. 49. 19 ♈	26. 5. 45 ♉	26. 7. 12 ♉	1. 27 —	1. 42
27 Octo. H. 12. 20	13. 59 ♏	98851	147890	15. 35. 38 ♉	18. 50. 46 ♉	18. 51. 15 ♉	0. 29 —	0. 6
3 Nove. H. 12. 0	21. 2 ♏	98694	148773	19. 26. 33 ♉	16. 18. 33 ♉	16. 18. 30 ♉	0. 3 +	Bor. 0. 17
18 Dece. H. 8. 0	6. 43 ♑	98200	154539	13. 2. 29 ♊	11. 39. 1 ♉	11. 40. 0 ♉	0. 59 —	1. 40

ASTRONOMIA NOVA OVERVIEW

GOALS

1. Introduce reforms to orbital theory for Mars to remove all excessive discrepancies vs. observation
2. Determine true motion of Mars at least with respect to the Sun, if not the fixed stars, for 1580-1604
3. Establish Copernican theory for Mars-Earth-Sun

PROBLEMS

1. Working from a limited set of observations, 1580-1604, restricted to geocentric longitudes & latitudes
2. Must use theory to reach any conclusions re orbital motion, but without begging questions re the motion
3. Observations were of limited precision (though with estimated bounds of the limits), and corrections for parallax and atmospheric refraction were suspect
4. Supplemental theories, e.g. of Earth-Sun orbit, can introduce extraneous sources of discrepancy

SUPPLEMENTAL THEORIES EMPLOYED

1. Tycho's Earth-Sun theory, along the way refined to incorporate bisection of eccentricity
2. Vicarious hypothesis for heliocentric longitudes of Mars about the true Sun
3. (In places) Tycho's theory of heliocentric longitudes for Mars about the mean Sun
4. Assumption that Mars orbits the Sun, not the Earth (while not foreclosing a Ptolemaic representation)

Questions of Evidence

To what extent did the evidence presented in *Astronomia Nova* on the Mars and Earth-Sun orbits exclude alternatives to:

1. **The line of apsides passes through the true Sun?**

2. **The orbit lies on a plane passing through the true Sun at a constant angle of inclination with respect to the ecliptic?**

3. **The bisection of the eccentricity of the Earth-Sun orbit?**

4. **The diametral distance rule?**

5. **The area rule?**

6. **Mars's trajectory is an ellipse?**

To what extent did the evidence presented in *Astronomia Nova* support the Copernican system:

1. **Over the Tychonic system?**

2. **Over the Ptolemaic system?**

Why Kepler?

- Access to Tycho's observations

- Tycho's redefinition of the goal of astronomy: agreement within observational accuracy at all times around the zodiac

- Kepler's demonstration of limitations in relying on acronychal observations

- Idea, inherited from Copernicus, of employing theory-mediated triangulations to determine sun-planet and earth-sun distances vs. time

- Tycho's exceptionally accurate theory of the Sun

- The (perhaps coincidental) choice of Mars

CHRONOLOGY

December 27, 1571
Born in Weil der Stadt, Germany

1584–1588
Attends seminary school at Adelberg and Maulbronn

1589–1594
Attends the University of Tübingen, where he receives a B.A. (by examination, 1588) and M.A. (1591); nearly completes three additional years of study in theology

April 1594
Arrives in Graz, Styria, to assume the position of mathematics teacher and district mathematician

1596–March 1597
Printing of the *Mysterium* cosmographicum in Tübingen

April 27, 1597
Marries Barbara Müller

September 28, 1598
Counter-Reformation begins in Styria; Protestant teachers and preachers expelled from Graz; Kepler is allowed to return after about a month

January-June 1600
Visits Tycho Brahe at Benatky Castle

September 30, 1600
Leaves Styria with his family and all their possessions when all remaining Protestants are banished

October 24, 1601
Tycho Brahe dies; two days later, Kepler is named his successor as imperial mathematician to Rudolf II in Prague

1604
Publishes the *Astronomia pars optica*

Around Easter, 1605
Discovers the elliptical form of Mars's orbit

1609
Astronomia nova finally published

March 1610
Galileo publishes the *Sidereus nuncius* containing his telescopic discoveries; Kepler responds, publishing the *Dissertatio cum nunceo sidereo* in Prague in May

Summer 1611
Kepler publishes *Dioptrice,* containing explanation of the telescope

July 3, 1611
Barbara Kepler dies

January 20, 1612
Holy Roman Emperor Rudolf II dies; Archduke Matthias succeeds him

May 1612
Begins work as mathematician to the Estates of Upper Austria in Linz

October 30, 1613
Marries Susanna Reuttinger

July 1615
Publishes *Nova stereometria doliorum vinariorum* in Linz

Fall 1617
Publishes the first volume of *Epitome astronomiae Copernicanae* in Linz

Fall 1617–early 1618
Returns to Württemberg with his mother, but her court case is delayed

May 15, 1618
Discovers his third law of planetary motion

May 23, 1618
"Defenestration of Prague"; Thirty Years War begins

1619
Harmonice mundi libri V published in Linz

March 20, 1619
Holy Roman Emperor Matthias dies; Archduke Ferdinand II suceeds him 5 months later

Early 1620
Second volume of *Epitome astronomiae Copernicanae* published in Linz

August 7, 1620–August 1621
Katharina Kepler arrested for witchcraft; Johannes Kepler returns to Württemberg to assist in her defense

Fall 1621
Final volume of the *Epitome astronomiae Copernicanae* published in Frankfurt

October 1625
Counter-Reformation begins in Upper Austria

November 1626
Kepler and his family leave Linz

December 1626-September 1627
Rudolfine Tables printed in Ulm

July 1628
Kepler arrives in Sagan to become personal mathematician to General Wallenstein; Counter-Reformation in Sagan begins four months later

November 15, 1630
Dies while visiting a meeting of imperial electors in Regensburg

This detail of the frontispiece of the Rudolfine Tables *shows Kepler as the architect of the achievement. He works by candlelight on a model of the temple above him. The banner lists his important publications. A few coins dropped by the Hapsburg eagle have reached his desk.*

Kepler's Principal Publications

Mysterium Cosmographicum (1596)

On More Certain Foundations of Astrology (1602)

Astronomiae Pars Optica Traditur (1604)

De Stella Nova (1606) – on the nova of 1604

Astronomia Nova (1609)

Dissertatio cum Nuncio siderio (1610)

Narratio de Jovis satellitibus (1611)

Dioptrice (1611) – theory of the telescope

Stereometria dolorioum vinariorum (1615)

Harmonice Mundi (1618)

Epitome Astronomiae Copernicanae (1618-1621)

De cometis libelli tres (1619) – on the comet of 1618

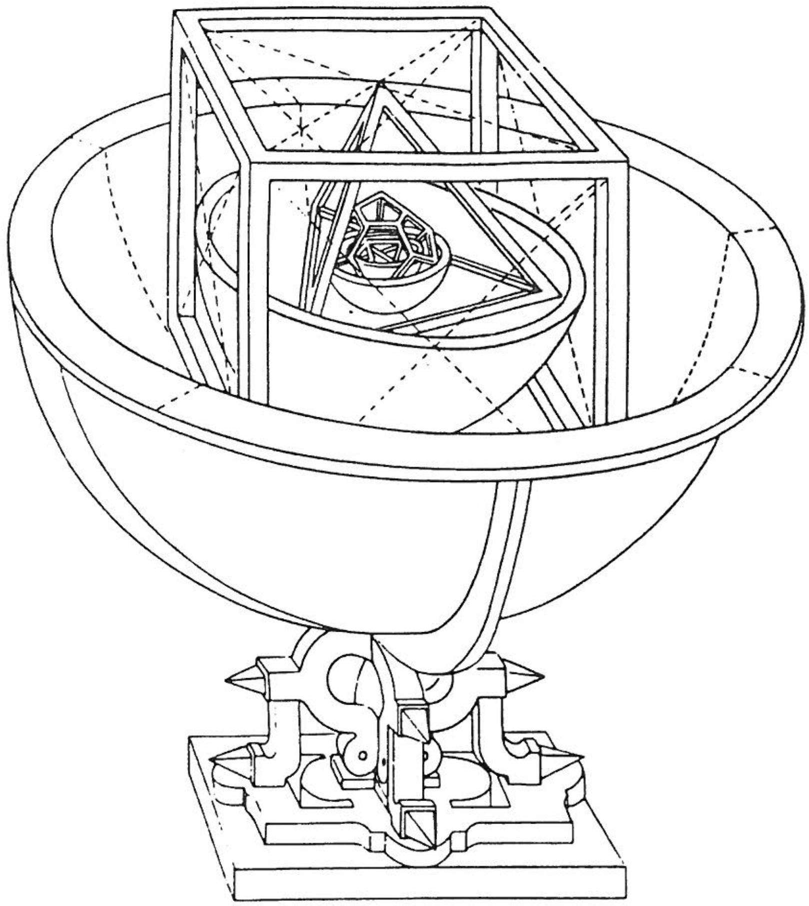
Mysterium Cosmographicum (revised, 1621)

Tabulae Rudolphinae (1627)

Admonitio ad Astronomos rerumque celesti studiosos, de mirisque rarisque anni 1631phaenomensis (1630)

Somnium seu astronomia lunari (1634)

After 1615, Ephemerides announcing astronomical phenomena on a regular basis, including one of 1630 above announcing 1631 transits of Mercury & Venus



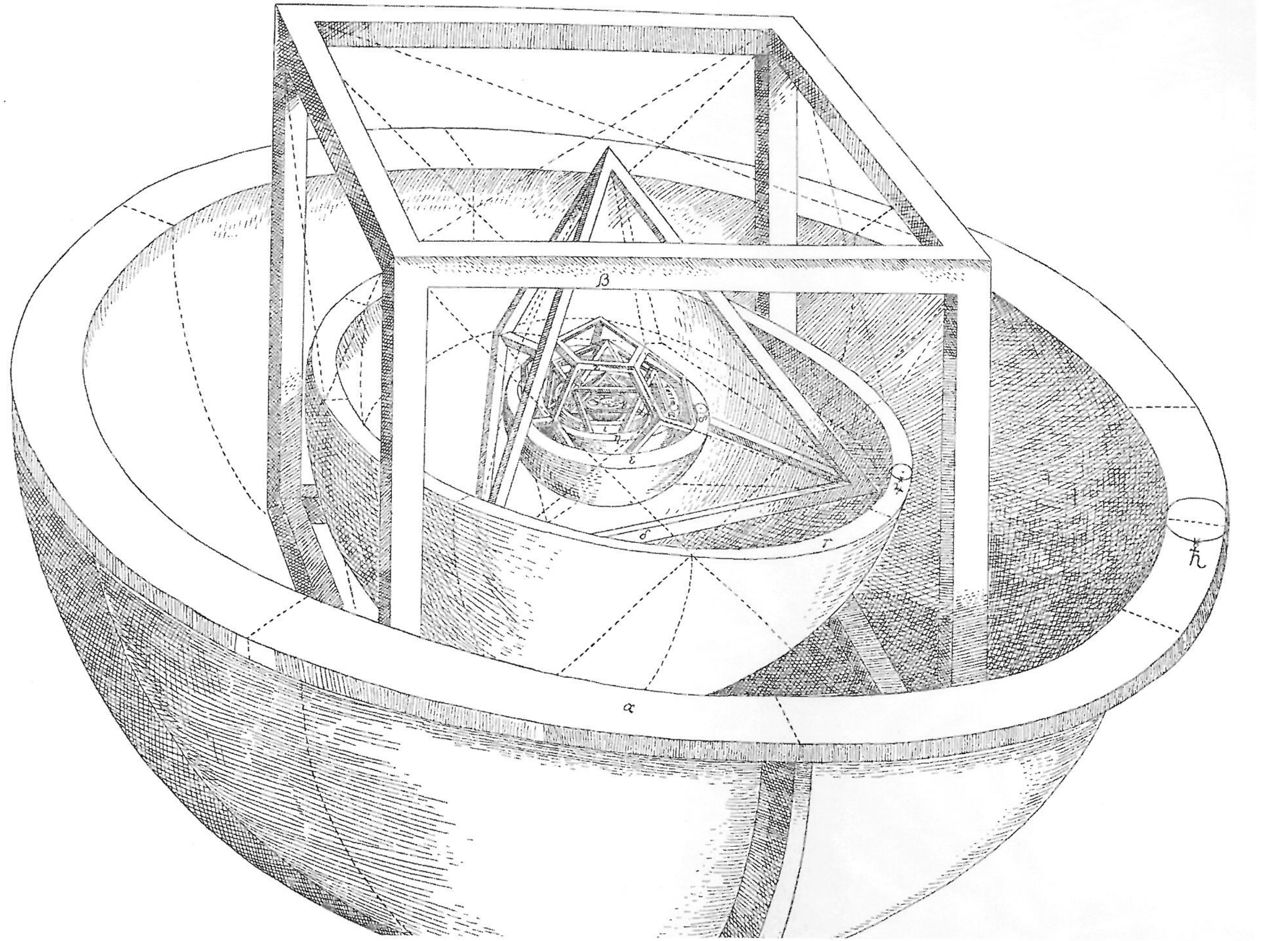
5.2. Kepler's nested polygons and planetary spheres (redrawn for greater clarity in the central portion), from the *Mysterium cosmographicum*.

Table 5.1 *Relative sizes of the adjacent planetary orbits (assuming the innermost part of the outer orbit to be 1000)*

Planet	Intervening regular solid	Computed by Kepler	From Copernicus
Saturn			
	Cube	577	635
Jupiter			
	Tetrahedron	333	333
Mars			
	Dodecahedron	795	757
Earth			
	Icosahedron	795	794
Venus			
	Octahedron	707	723
Mercury			

TABELLA III.

ORBIVM PLANETARVM DIMENSIONES, ET DISTANTIAS PER QVINQVE REGVLARIA CORPORA GEOMETRICA EXHIBENS.



Ioannis Keppleri

HARMONICES MVNDI

LIBRI V. QVORVM

Primus GEOMETRICVS, De Figurarum Regularium, quæ Proportiones Harmonicas constituunt, ortu & demonstrationibus.

Secundus ARCHITECTONICVS, seu ex GEOMETRIA FIGVRATA, De Figurarum Regularium Congruentia in plano vel solido:

Tertius propriè HARMONICVS, De Proportionum Harmonicarum ortu ex Figuris; deque Naturâ & Differentiis rerum ad cantum pertinentium, contra Veteres:

Quartus METAPHYSICVS, PSYCHOLOGICVS & ASTROLOGICVS, De Harmoniarum mentali Essentiâ earumque generibus in Mundo; præsertim de Harmonia radiorum, ex corporibus cœlestibus in Terram descendentibus, eiusque effectu in Natura seu Anima sublunari & Humana:

Quintus ASTRONOMICVS & METAPHYSICVS, De Harmoniis absolutissimis motuum cœlestium, ortuque Eccentricitatum ex proportionibus Harmonicis.

Appendix habet comparationem huius Operis cum Harmonices Cl. Ptolemæi libro III. cumque Roberti de Fluctibus, dicti Flud. Medici Oxoniensis speculationibus Harmonicis, operi de Macrocosmo & Microcosmo insertis.

Cum S. C. Mtis. Priuilegio ad annos XV.

Lincii Austriæ,

Sumptibus GODOFREDI TAMPACHII Bibl. Francof.
Excudebat IOANNES PLANCVS.

ANNO M. DC. XIX.

KEPLER'S THIRD "LAW"

FROM KEPLER'S DATA

	Period (years)	Mean Distance	Period Squared	Distance Cubed
Mercury	0.2408	0.3881	0.0580	0.0585
Venus	0.6152	0.7241	0.3785	0.3797
Earth	1.0000	1.0000	1.0000	1.0000
Mars	1.8808	1.5235	3.5376	3.5361
Jupiter	11.8621	5.200	140.7086	140.61
Saturn	29.4571	9.510	867.7198	860.08

	Period (days)	a from Periods	a from Orbits	Percent Difference
Mercury	87.97	0.3871	0.3881	0.26
Venus	224.70	0.7233	0.7241	0.11
Earth	365.25	1.0000	1.0000	----
Mars	686.98	1.5237	1.5235	0.01
Jupiter	4332.62	5.2012	5.200	0.02
Saturn	10759.20	9.5381	9.510	0.29

Books IV and V of Kepler's *Epitome*: An Overview

> "... all the more strange to them [readers] would be this Fourth Book, which airs so many new and unthought-of things concerning the whole nature of the heavens – so that you might doubt whether you were doing a part of physics or astronomy, unless you recognized that speculative astronomy is one whole part of physics." (p. 5)

Book IV

Part I: System of the world, pp. 13-47 (of Prometheus ed.)

1. On the Principal Parts: p. 16, no solid spheres

2. On the Place of the Sun at the center: true Sun, p. 20

3. On the order of the movable spheres

   With respect to the perfect solids p. 23ff

   The harmonies: p. 31-32

4. On the ratios of the Principal Bodies to one Another

   on imperceptibility of annual stellar parallax: p. 43ff

Part II: On the Movement of Bodies: pp. 47-88

1. How many and what sort of movements

   denial of motion of fixed stars

2. Concerning the Causes of the Movements: pp. 48-55

   "inertia" and the need for a motive force

3. On the Revolution of the Sun about its Axis, and Its Effects: pp. 55-65

solar magnetic emanation: pushing planets

strength varies inversely with distance

p. 63: why not as inverse-square, like light

4. On the Causes of the 3/2 Power Ratio: pp. 65-67

5. On the Annual Movement of the Earth: pp. 67-88

p. 67: five questions

p. 68: appeal to phases of Venus

p. 70: eight reasons for true Sun

pp. 71-76: eighteen reasons to reject Tychonic

pp. 78-88: rotation of Earth, Jupiter; Moons

Part III: On the Real and True Irregularity of the Planets and its Causes (pp. 88-120)

1. The causes of the true irregularities: pp. 88-93

three part answer: p. 89

2. On the Causes of Irregularity in Longitude: pp. 93-94

laws of variation in speed: lever, balance analogy

3. Causes of Irregularity in Altitude (i.e. r or SP): pp. 95-106

p. 98: no interaction between planet and Sun

p. 99: basic account

pp. 101-105: thread displacement, restoration; precession of apsides

4. On the Movement of the Latitude: pp. 106-112

    basic answer: pp. 106-107

    pp. 107-112: why inclination at all; why different from one to another

5. On the Twofold Irregularities of the Moon and Their Causes: pp. 112-120

    basic answer: pp. 113-114

    precession of the line of apsides: p. 117

<u>Book V</u> (pp. 124-164)

Part I: On the Eccentric Circles or Schemata of the Planets (pp. 124-146)

    p. 125: basic answer (figure)

    p. 127: seven tasks

1. Concerning the Increment in Libration: p. 128

    why sinusoidal (p. 129); balance (p. 131)

2. On the Sum of the Libration Gone Through With (p. 133)

    (i.e. the cumulative or progressive change in *r*)

3. On the Figure of the Orbit: pp. 135-139

    i.e. circumscribed circle, eccentric anomaly, and varying *r*

4. On the Measure of Time (with apology for statement in *Astronomia Nova*): pp. 139-143

5. On the Equivalence of the Circle of the Plane and the Plane of the Ellipse in Measuring Delays: pp. 143-145

    "Kepler's problem"

6. On the Regularity of the Digression in Latitude: pp. 145-146

   i.e. motion of the nodes

Part II: On the Astronomical Terms Arising From Calculations and the Eccentric Orbit (pp. 146-164)

   (an explanation of important terms)

1. Concerning Designation
2. Concerning Libration
3. Concerning the Delay of the Planet in any Arc
4. Concerning the Angle of the Sun
5. On the Digression of the Planets Away from the Ecliptic
6. On the Movement of the Apsides and Nodes

For the contents of the first three books, see p. 4; for the contents of Books VI, on the individual orbits, and VII, see p. 164; the following is from Book VI:

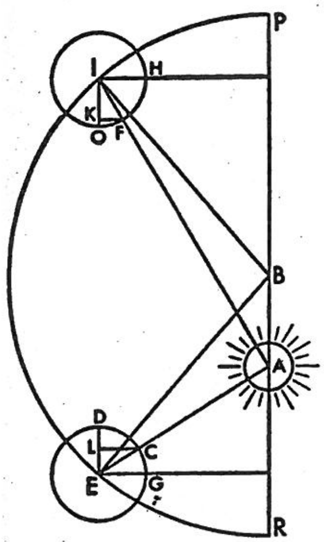
	Aphelium.	Medium.	Perihelium.	Eccentricitates qualium semidiameter est 100000.
Saturni	1005207.	951000.	896793.	5700
Iouis	544708.	519650.	494592.	4822
Martis	166465.	152350.	138235.	9263
Orbis Magni . .	101800.	100000.	982000.	1800

Epitome on Diametral Distance Physics

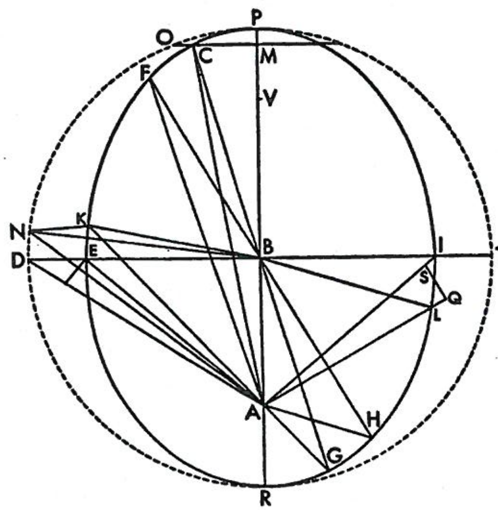
Now therefore state the measure of the portion of the quantity of the solar forces which the planet admits into itself in any posture with respect to the sun.

We must consider the angle which the rays [*radii*] of the sun make with the magnetic threads of the planetary globe. For the sine of the angle complementary to this measures this portion of forces admitted. For since the efficient causes of the libration are the ray of the sun and the magnetic threads of the planet's body, two physical lines; it is right to seek the measure of the strength of the libration from the angle between these lines and from its sine.

[649] For example, let *A* be the sun; and *I*, *E*, the centre of the planetary body; *RP* the line drawn through *A* the sun and through *B* the centre of the orbit; *EG* and *IH* the magnetic threads will be practically perpendicular to *RP*—at least if the counterbalancing of the semicircles is considered—and *H* and *F* are the solipetal termini. For it was laid down in Book IV page 95 that during the revolution of the body the magnetic threads stay practically parallel to themselves and at points *P* and *R* provide no occasion of attraction or repulsion, because there the positions of both the solipetal and the solifugal termini are equally distant from the sun at *A*. But in the intermediate positions, where the solipetal or the solifugal termini regard the sun directly, the strength of the libration is greatest of all. *AE* and *AI* are the rays of the sun. Let lines *ED* and *IO* be drawn parallel to line *RP*, and let perpendiculars be dropped upon them from points *F* and *C*, wherein the rays of the sun cut the middle circles of the planetary globe; and let the perpendiculars be *CL* and *FK*. Here the angles made by the rays of the sun with the threads are *AEG* and *AIH*; the complementary angles are *CED* and *FIO*, or the arcs *CD* and *FO*, and *CL* and *FK* are the sines of these angles, according as *IH* or *EG* is the total sine of 100,000. Therefore it is established that as *EG* and *IH* are to *LC* and *KF*, so the total quantity of the forces from the sun which are present at *I* or *E* is to the portion which the planet admits at the postures of the threads *EG* and *IH*.



Epitome on Area Rule



Then in this way let the area of the ellipse be rightly distributed between the pairs of opposite arcs; now demonstrate that the single triangles are separately the most exact measures of the single delays?

The demonstration is easy by means of the foregoing.

For since by our axiom the delay of the planet in arc *PC* is to the delay in the equal arc *RG*, as *AP* the distance of arc *PC* from the source of motion is to *AR* the distance of arc *RG*; but since also the area of triangle *PCA* is to the area of triangle *RGA*—which has its base *RG* equal to *PC* the base of the first triangle—as *PA* the altitude of triangle *PCA* is to *RA* the altitude of triangle *RGA:* Wherefore the delay of the planet in arc *PC* is to the delay in the equal arc *RG* as the area of triangle *PCA* is to the area of triangle *RGA*.

In the same way it will be demonstrated that the delay of the planet in *CF* —which is equal in power (*potestate*) to *CP*—is to the delay of the same in *GH*, as the area *ACF* is to the area *AGH*—where the sum of each pair of areas is equal to the sum of the [two] prior areas, and so on in order. Therefore the total area of the ellipse—which has been cut up at *A* into triangles—[671] is distributed among the arcs in the same proportion wherein the total periodic time has been distributed among them. Therefore the single triangles are proportionally the most exact measures of their single arcs.

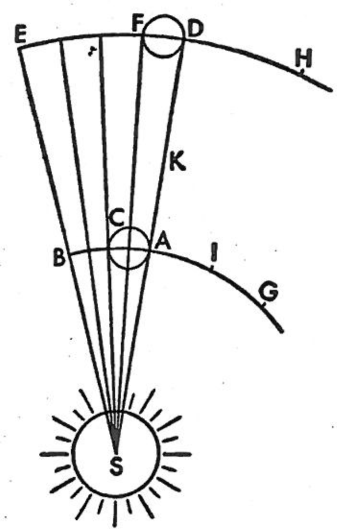
A demonstration of this full equivalence is given in my *Commentaries on Mars*, Chapter 59, page 291. On that page at the line *Apsis longiorem*, one word, *erit*, has brought in great obscurity; and if you change it to *computaretur*, everything will be clearer. Although I confess that the thing is given rather obscurely there, and most of the trouble comes from the fact that there the distances are not considered as triangles, but as numbers and lines.

Epitome on 3/2 Power Rule Physics

4. On the Causes of the Ratio of the Periodic Times

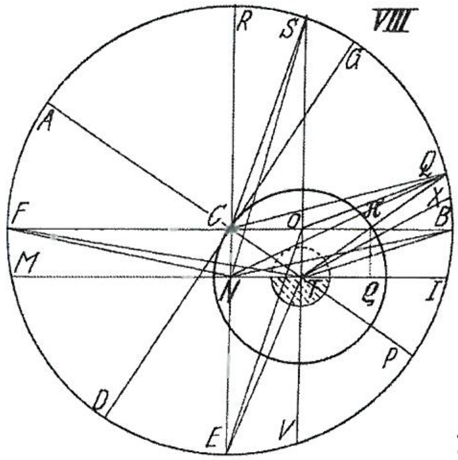
In the beginning of this consideration of movement you said that the periodic times of the planets are found to be quite exactly in the ratio of the $\frac{3}{2}$th powers of their orbits or circles. I ask what the cause of this thing is.

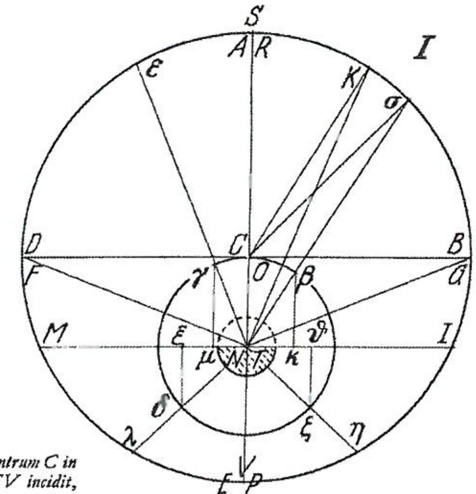
Four causes come together in establishing the length of the periodic time. The first is the length of the route; the second is the weight or the amount [*copia*] of matter to be transported; the third is the strength of the motor virtue; the fourth is the bulk [*moles*] or space in which the matter to be transported is unrolled. For as is the case in a mill, where the wheel is turned by the force of the stream, so that, the wider and longer the wings, planks, or oars which you fasten to the wheel, the greater the force of the stream pouring through the width and depth which you will divert into the machine; so too that is the case in this celestial vortex of the solar form moving rapidly in a gyro—and this form causes the movement. [531] Consequently the more space the body—*A* or *D* in this case—occupies, the more widely and deeply it occupies the motor virtue, as in this case *BCA* understood according to its width; and the more swiftly, other things being equal, is it borne forward; and the more quickly does it complete its periodic journey.



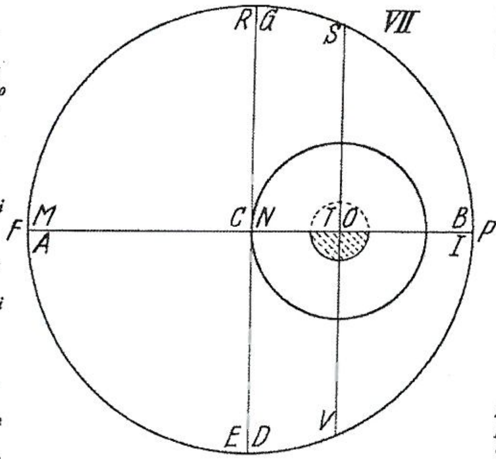
But the circular journeys of the planets are in the simple ratio of the intervals. For as *SA* is to *SD*, so too is the whole circle *BA* to the whole circle *ED*. But the weight, or the amount of matter in the different planets, is in the ratio of the $\frac{1}{2}$th powers of the intervals, as was proved above, so that always the higher planet has more matter and is moved the more slowly and piles up the more time in its period, since even before now by reason of its journey it would have wanted more time. For with *SK* taken as a mean proportional between *SA* and *SD* the intervals of the two planets; as *SK* is to the greater distance *SD*, so the amount of matter in the planet *A* is to the amount in the planet *D*. But the third and fourth causes balance one another in the comparison of the different planets. But the simple ratio of the intervals plus the ratio of the $\frac{1}{2}$th powers constitute the ratio of the $\frac{3}{2}$th powers of the same. Therefore the periodic times are in the ratio

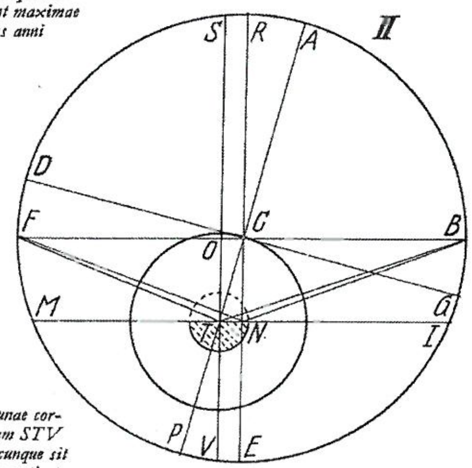
A. Apogaeum Solutae.
B. Quadraturae secundae locus in Eccentrico medius.
C. Centrum Eccentrici.
D. Longitudo Media Eccentrici prima.
E. Oppositionis cum Sole locus in Eccentrico medius.
F. Quadraturae primae locus in Eccentrico medius.
G. Longitudo media Eccentrici secunda.
I. Quadraturae secundae locus in Eccentrico verus.
K. est respectu situs Lunae in σ, locus suo sensu medius, & vicissim.
M. Quadraturae primae locus in Eccentrico verus.
N. est punctum centri Terrae vicarium in Eccentricitate menstrua, dicitur punctum Menstruum.
O. est punctum centri Eccentrici vicarium in Eccentricitate Menstrua.
P. Perigaeum Solutae.
Q. Prope verus Lunae locus resp. X.
R. Conjunctionis cum Sole locus in Eccentrico medius.
S Punctum subsolare, Apogaeum vel Perigaeum menstruae alternis. Conjunctionis cum Sole locus in Eccentrico verus.
T. Centrum Terrae.
V. Vmbrae seu Oppositionis cum Sole locus in Eccentrico verus. Perigaeum vel Apogaeum menstruae alternis.
X. verus Lunae locus respectu Q.
$\beta \cdot \gamma \cdot \delta \cdot \zeta \cdot \pi$. puncta in quibus circellus, à Centro Eccentrici circa Terram descriptus, secatur à lineis motus Lunae prope veri.
$\varepsilon \cdot \eta \cdot \lambda \cdot \sigma$. Locus Lunae Eccentricus prope verus.
$\vartheta \cdot \varkappa \cdot \mu \cdot \xi \cdot \varrho$. puncta in linea circuli illuminationis, Variationi indagandae servientia.



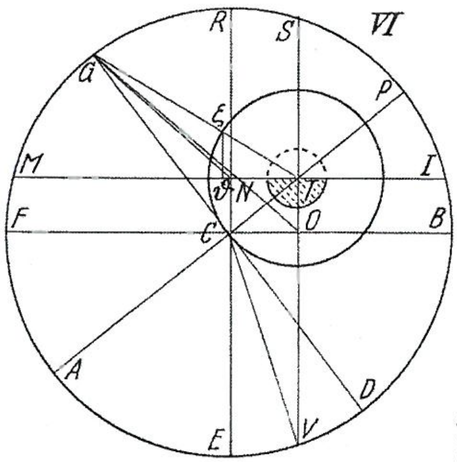


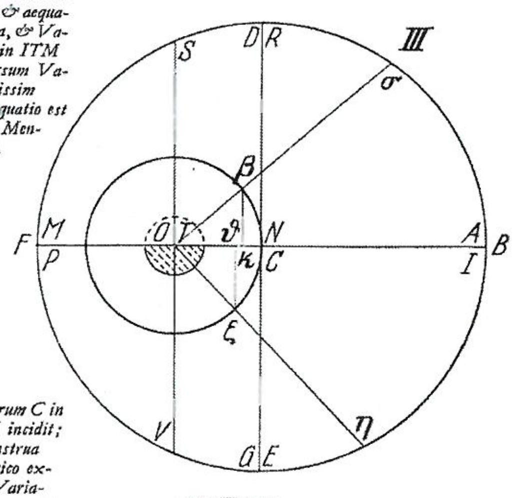
Quando centrum C in lineam STV incidit, menstruae aequationes eveniunt maximae inter omnes anni menses.



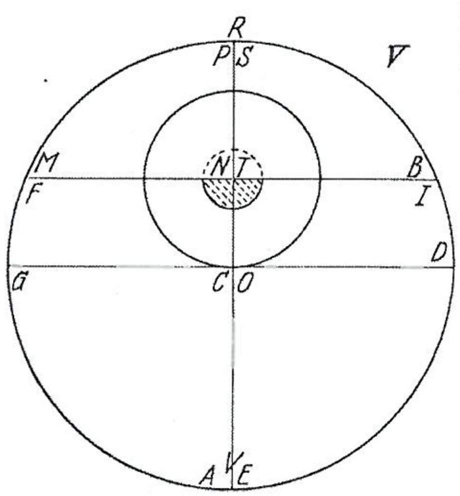


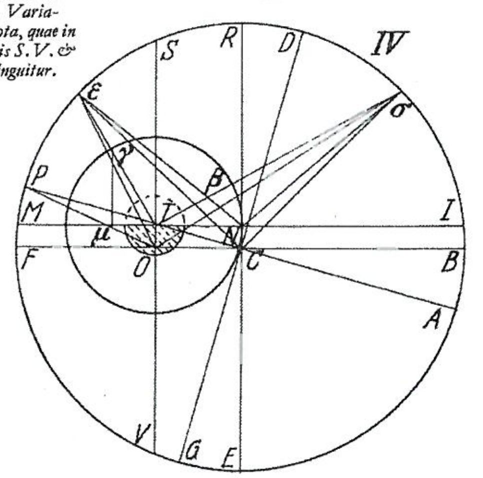
Quoties Lunae corpus in lineam STV incidit, ubicunque sit centrum C: extinctae sunt ambae, & aequatio menstrua, & Variatio. At in ITM extincta rursum Variatione, vicissim menstrua aequatio est totius illius Mensis maxima.





Quoties centrum C in lineam ITM incidit; aequatio menstrua toto Eccentrico extincta est, Variatione excepta, quae in solis punctis S. V. & I. M. extinguitur.





Questions of "Projectability"

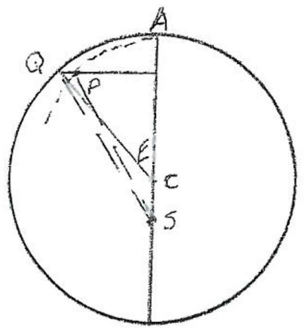
The evidence in *Astronomia Nova* shows that the four key claims – diametral distance rule, area rule, ellipse, and plane of fixed inclination through the true Sun – hold at least to high approximation for Mars over the course of its twelve revolutions of the zodiac between 1580 and 1604.

***Question*: What grounds could Kepler have offered for extrapolating – i.e. "*projecting*" – this result beyond that period of 24 years to the motion of Mars into the indefinite past and indefinite future?**

***Question*: What grounds could Kepler have offered for extending – i.e. *projecting* – this result to the motions of Jupiter, Saturn, Venus and Mercury?**

In other words, to what extent and on what grounds were the chief results of *Astronomia Nova* directly "projectable" – to use Nelson Goodman's term – into broader "nomological" generalizations?

Eccentric Circle vs. Ellipse, Both with Area Rule



< ASQ: heliocentric longitude, circle

< ASP: heliocentric longitude, ellipse

< E:	30 deg	45 deg	60 deg	75 deg	90 deg
e=0.02					
< ASQ	29.437	44.201	59.018	73.899	88.854
< ASP	29.432	44.195	59.013	73.896	88.854
diff	17.6 sec	20.7 sec	18.1 sec	11.1 sec	0.7 sec
e=0.05					
< ASQ	28.63	43.04	57.58	72.27	87.14
< ASP	28.60	43.01	57.55	72.25	87.13
diff	108.5 sec	128.8 sec	116.9 sec	74.9 sec	12.9 sec
e=0.09					
< ASQ	27.61	41.58	55.73	70.14	84.86
< ASP	27.51	41.46	55.63	70.01	84.84
diff	5.73 min	6.94 min	6.51 min	4.47 min	1.25 min

For purposes of comparison, the eccentricity of the Earth is a little less than 0.02; the eccentricity of Jupiter a little less and that of Saturn a little greater than 0.05; and the eccentricity of Mars a little greater than 0.09. In short, without Tycho's data, the distinction between circle and ellipse would not have been detectable, and even with it this difference was not detectable for Venus, Earth, Jupiter and Saturn.

Finally, a circle with equant and bisected eccentricity gives, for e=0.09 and E=45 deg, an angle ASP′ of 41.36 deg and hence a difference between the angle ASP of the ellipse plus area rule of 5.84 min. So, asking astronomers before Tycho to have noticed the difference between a circle with bisected eccentricity and an equant and a Keplerian ellipse is asking for something they could not have done.

TABULÆ

RUDOLPHINÆ,

QUIBUS ASTRONOMICÆ SCIENTIÆ, TEMPOrum longinquitate collapsæ RESTAURATIO *continetur;*

A Phœnice illo Aſtronomorum

TYCHONE

Ex Illuſtri & Generoſa BRAHEORUM *in Regno Daniæ familiâ oriundo Equite,*

PRIMUM ANIMO CONCEPTA ET DESTINATA ANNO CHRISTI MDLXIV: EXINDE OBSERVATIONIBUS SIDERUM ACCURATISSIMIS, POST ANNUM PRÆCIPUE MDLXXII, QUO SIDUS IN CASSIOPEJÆ CONSTELLATIONE NOVUM EFFULSIT, SERIÒ AFFECTATA; VARIISQUE OPERIBUS, cùm mechanicis, tùm librariis, impenſo patrimonio ampliſſimo, accedentibus etiam ſubſidiis FRIDERICI II. DANIÆ REGIS, regali magnificentia dignis, tracta per annos XXV, potiſſimùm in Inſula freti SUNDICI HUENNA, & arce URANIBURGO, in hos uſus à fundamentis extructâ:

TANDEM TRADUCTA IN GERMANIAM, INQUE AULAM ET Nomen RUDOLPHI IMP. *anno MDIIC.*

TABULAS IPSAS, JAM ET NUNCUPATAS, ET AFFECTAS, SED MORTE AUTHORIS SUI ANNO MDCI DESERTAS,

JUSSU ET STIPENDIIS FRETUS TRIUM IMPPP.

RUDOLPHI, MATTHIÆ, FERDINANDI,

ANNITENTIBUS HÆREDIBUS BRAHEANIS; EX FUNDAMENTIS obſervationum relictarum; ad exemplum ferè partium jam exſtructarum; continuis multorum annorum ſpeculationibus, & computationibus, primùm PRAGÆ *Bohemorum continuavit; deindè* LINCII, *ſuperioris Auſtriæ Metropoli, ſubſidiis etiam Ill. Provincialium adjutus, emendavit, perfecit, abſolvit; adq; cauſarum & calculi perennis formulam traduxit*

IOANNES KEPLERUS,

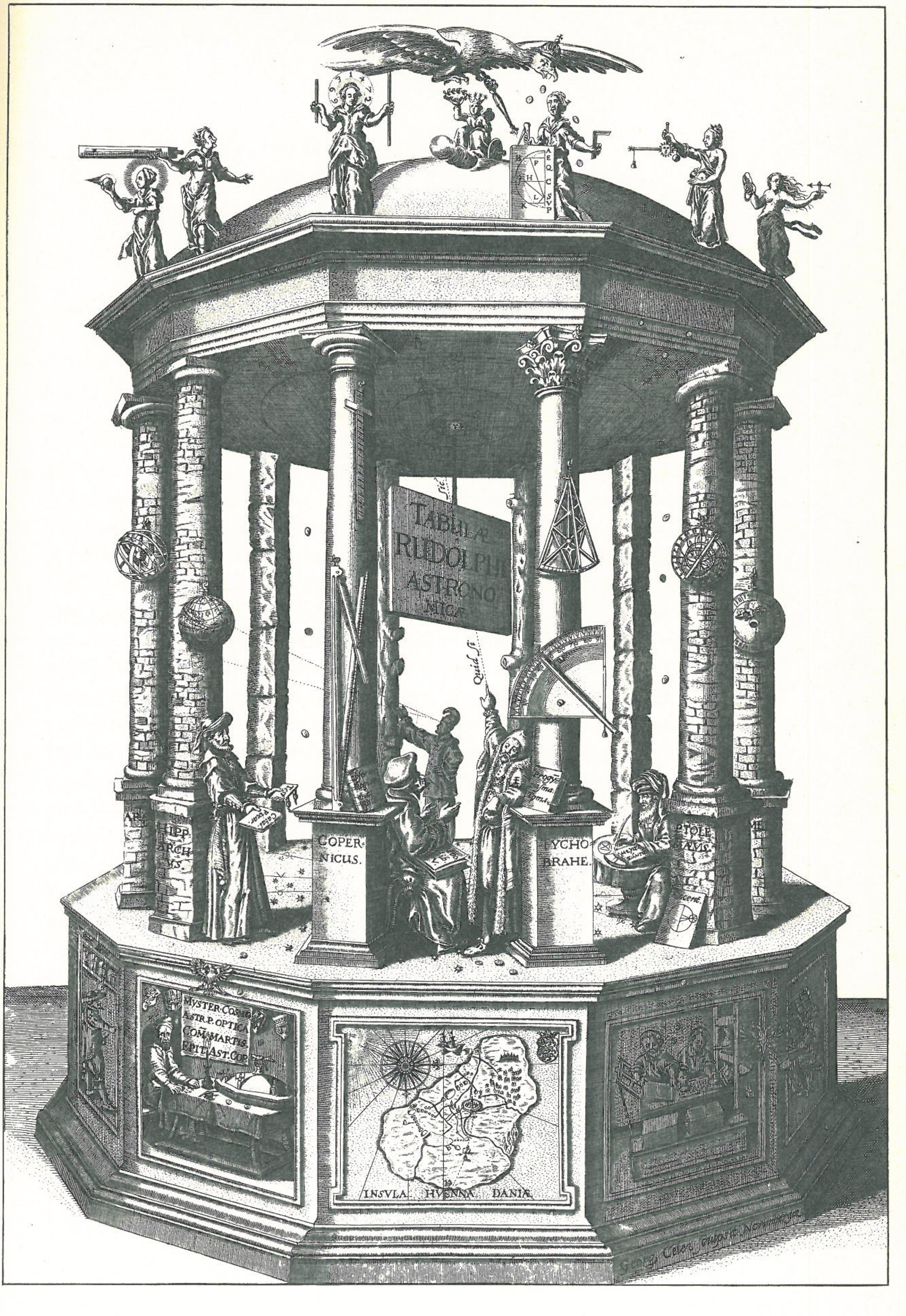
TYCHONI *primùm à* RUDOLPHO *II. Imp. adjunctus calculi miniſter; indéq; trium ordine Imppp. Mathematicus:*

Qui idem de ſpeciali mandato FERDINANDI II. IMP. petentibus inſtantibúsq; Hæredibus,

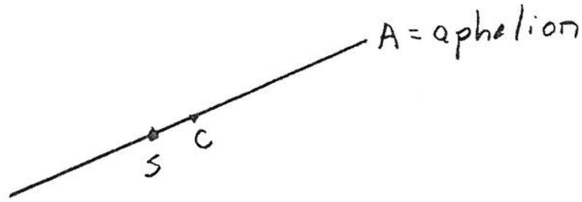
Opus hoc ad uſus præſentium & poſteritatis, typis, numericis propriis, cæteris & prælo JONÆ SAURII, *Reip. Ulmanæ Typographi, in publicum extulit, & Typographicis operis* ULMÆ *curator affuit.*

Cum Privilegiis, IMP. & Regum Rerúmq; publ. vivo TYCHONI ejusq; Hæredibus, & ſpeciali Imperatorio, ipſi KEPLERO conceſſo, ad annos XXX.

ANNO M. DC. XXVII.


TABULÆ RUDOLPHI ASTRONOMICA
COPERNICUS.
TYCHO BRAHE.
HIPPARCHUS.
PTOLEMÆUS.
MYSTER. COSMO.
ASTR. P. OPTICA
COM. MARTIS
EPIT. AST. COP.
INSULA HVENNA DANIÆ

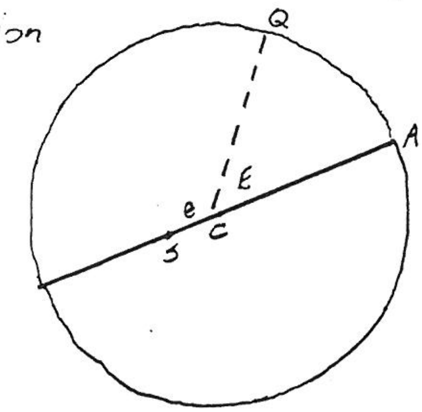
The Keplerian Ellipse



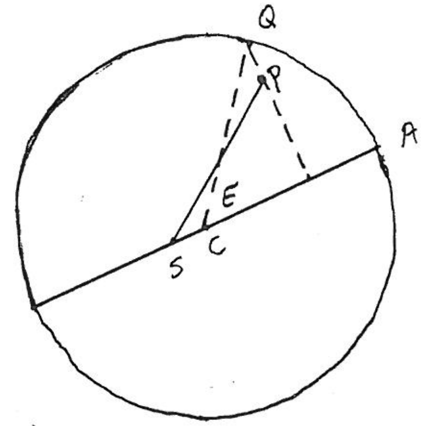
CA = mean distance $= a$

$\frac{CS}{CA}$ = eccentricity $= e$

Start with the eccentricity, the length of line of apsides ($2CA$), and its orientation, as given by the (heliocentric) longitude of its aphelion



Next draw a circle of radius $r = CA$, with the line of apsides as a diameter. Then, for any angle $ACQ = E$, the so-called "eccentric anomaly," the location of the planet on the elliptic trajectory is given as follows



$$SP = a\,(1 + e\cos E)$$

$$\angle ASP = \sin^{-1}\left(\frac{a\sqrt{1-e^2}\,\sin E}{SP}\right)$$

$$= \sin^{-1}\left(\frac{\sqrt{1-e^2}\,\sin E}{1 + e\cos E}\right)$$

The elliptic trajectory is then given by the locus of points P (as E goes from 0 to 360 deg)

Elements of Keplerian Orbits

Orientation of the line of apsides – i.e. longitude of aphelion: *ώ*

Length of the semi-major axis – i.e. mean distance of planet from Sun: *a*

Eccentricity: *e*

Period: *P* – or mean motion in longitude *n*

Orientation of the line of nodes – i.e. longitude of ascending node: *Ω*

Inclination: *i*

Mean longitude at epoch: *L*, or time last at aphelion, *T*

An Assessment of Kepler's Orbital Elements

Based on Those Inferred for 1600 from Newcomb

		Eccentricity	Apehelion	Mean Distance
Mercury	N	0.20555	251°14′09″	0.38710
	K	0.21005	252°49′58″	0.38808
Venus	N	0.00697	305°55′51″	0.72333
	K	0.00692	301°14′22″	0.72413
Earth	N	0.01688	276°04′02″	1.00000
	K	0.01800	275°44′08″	1.00000
Mars	N	0.09304	148°41′58″	1.52369
	K	0.09265	148°59′54″	1.5235
Jupiter	N	0.04784	187°53′27″	5.2027
	K	0.04822	186°52′01″	5.20
Saturn	N	0.05693	265°13′24″	9.546
	K	0.05693	264°57′36″	9.51

$\tilde{\omega}$ - longitude of perihelion
Ω - longitude of ascending node

Appendix C

Numerical Values

Numerical values continue to be refined, and further objects will be found that might be included in tables such as these. Don't be surprised if quite different values appear in other references (and even within the same reference). In some instances there are "official" values, so that different computations share a common set, making comparison possible. The following values are included partly for their interest, and partly so that you can include them, if you wish, in your computations.

C.1 Orbital Elements of Planets

The data for the planets (excluding Pluto) were kindly supplied by L. E. Doggett from data of P. Bretagnon (Reference 72). They are mean elements, referred to the equinox J2000.0. T is measured in Julian centuries from that epoch. The semimajor axes have been derived from the terms in L (the mean longitude at epoch) factored by T, with allowance first made for the precession in longitude. (See Appendix C7.)

For detailed information on osculating elements, refer to the *Astronomical Almanac*. For low-precision formulas, see Reference 38.

Mercury ☿

$$\tilde{\omega} = 77.4561\ 19 + 1.5564\ 77 \times T + 0.0002\ 96 \times T^2$$
$$\Omega = 48.3308\ 93 + 1.1861\ 88 \times T + 0.0001\ 76 \times T^2$$
$$i = 7.0049\ 86 + 0.0018\ 21 \times T - 0.0000\ 18 \times T^2$$
$$e = 0.2056\ 3175 + 0.0000\ 2041 \times T - 0.0000\ 0003 \times T^2$$
$$L = 252.2509\ 06 + 149474.0722\ 49 \times T + 0.0003\ 04 \times T^2$$
$$a = 0.3871\ 0353$$

P= 87.969 days

Venus ♀

$$\tilde{\omega} = 131.5637\ 07 + 1.4022\ 21 \times T - 0.0010\ 73 \times T^2 - 0.0000\ 05 \times T^3$$
$$\Omega = 76.6799\ 20 + 0.9011\ 20 \times T + 0.0004\ 07 \times T^2$$
$$i = 3.3946\ 62 + 0.0010\ 04 \times T - 0.0000\ 01 \times T^2$$
$$e = 0.0067\ 7188 - 0.0000\ 4777 \times T$$
$$L = 181.9798\ 01 + 58519.2130\ 30 \times T + 0.0003\ 11 \times T^2$$
$$a = 0.7233\ 074$$

P= 224.701 days

Earth ♁

$$\tilde{\omega} = 102.9373\ 48 + 1.7195\ 39 \times T + 0.0004\ 60 \times T^2$$
$$e = 0.0167\ 0862 - 0.0000\ 4204 \times T$$
$$L = 1000.4664\ 49 + 36000.7698\ 23 \times T + 0.0003\ 04 \times T^2$$
$$a = 1.0000\ 1161$$

P= 365.256 days

Mars ♂

$$\tilde{\omega} = 336.0602\ 34 + 1.8410\ 44 \times T + 0.0001\ 35 \times T^2$$
$$\Omega = 49.5580\ 93 + 0.7720\ 96 \times T + 0.0000\ 16 \times T^2$$
$$i = 1.8497\ 26 - 0.0006\ 01 \times T + 0.0000\ 13 \times T^2$$
$$e = 0.0934\ 0062 + 0.0000\ 9048 \times T - 0.0000\ 0008 \times T^2$$
$$L = 355.4332\ 75 + 19141.6964\ 75 \times T + 0.0003\ 11 \times T^2$$
$$a = 1.5237\ 1069$$

P= 686.980 days

Jupiter ♃

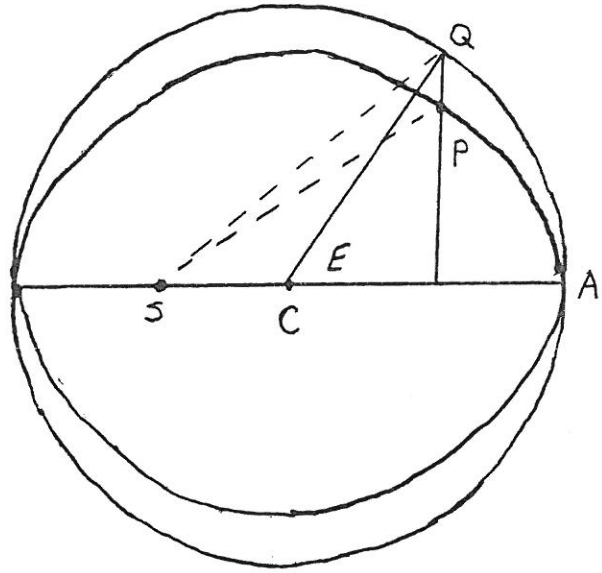
$$\tilde{\omega} = 374.3313\ 09 + 1.6126\ 38 \times T + 0.0010\ 31 \times T^2 - 0.0000\ 04 \times T^3$$
$$\Omega = 100.4644\ 41 + 1.0209\ 54 \times T + 0.0004\ 01 \times T^2$$
$$i = 1.3032\ 70 - 0.0054\ 97 \times T + 0.0000\ 05 \times T^2$$
$$e = 0.0484\ 9485 + 0.0001\ 6324 \times T - 0.0000\ 0047 \times T^2$$
$$L = 34.3514\ 84 + 3036.3027\ 89 \times T + 0.0002\ 24 \times T^2$$
$$a = 5.2102\ 1558$$

P= 4332.589 days

Saturn ♄

$$\tilde{\omega} = 93.0567\ 87 + 1.9637\ 61 \times T + 0.0008\ 38 \times T^2 + 0.0000\ 05 \times T^3$$
$$\Omega = 113.6655\ 24 + 0.8770\ 94 \times T - 0.0001\ 21 \times T^2 - 0.0000\ 02 \times T^3$$
$$i = 2.4888\ 78 - 0.0037\ 36 \times T - 0.0000\ 15 \times T^2$$
$$e = 0.0555\ 0862 - 0.0003\ 4682 \times T - 0.0000\ 01 \times T^2$$
$$L = 50.0774\ 71 + 1223.5110\ 14 \times T + 0.0005\ 20 \times T^2$$
$$a = 9.5380\ 7012$$

P= 10759.22 days



$CA = a$

$\frac{CS}{CA} = e$

$$\text{fraction of period} = \frac{n(t-T)}{2\pi} = \frac{M}{2\pi}$$

where n is mean (angular) motion, T is time of aphelion, and M is called "mean anomaly"

$$\frac{M}{2\pi} = \frac{\text{area of sector } ASP}{\text{area of ellipse}} = \frac{\text{area of sector } ASQ}{\text{area of circle}}$$

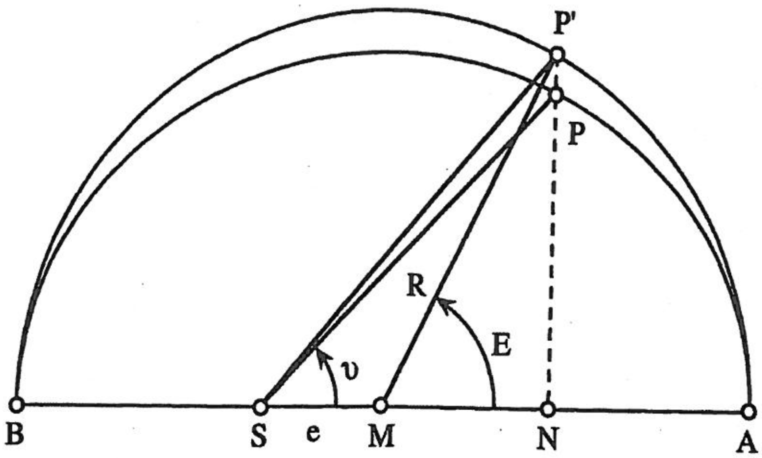
$$= \frac{\frac{a^2}{2}E + \frac{a^2 e}{2}\sin E}{\pi a^2}$$

$$= \frac{E + e\sin E}{2\pi}$$

Therefore

$$n(t-T) = M = E + e\sin E \quad \text{(Kepler's equation)}$$

Where E is called the "eccentric anomaly"



You have shown how to compute the mean anomaly and the coequated (true) anomaly from a given eccentric anomaly. But more frequently use requires that for a given mean anomaly, as from a given time, we find the others. Explain that also.

Here there is no direct way, but one who wishes to compute this without tables must employ the 'rule of assumptions', namely, as shown in the following figure, assuming the eccentric anomaly (arc) *AP′* as such and such an amount, and for the eccentric anomaly so assumed, computing its mean anomaly (area) *ASP′*. And if the result is the amount (of the mean anomaly) that was proposed, the eccentric anomaly (arc) *AP′* will have been assumed correctly. But if the result is not such an amount, the assumption will have to be corrected by means of the result and the work repeated.

Kepler's "Laws"

----

1. **(The) planets describe elliptical trajectories about the Sun (or central body) at a focus**

2. **The radius vectors of (the) planets sweep out equal areas in equal times about the Sun (or central body)**

3. **The periods of (the) planets are in sesquialter-nate proportion to their mean distances from the Sun (or central body)**

4. **The distances of (the) planets from the Sun (or central body) vary as $a(1 + e \cdot \cos E)$, where a is the mean distance, e is the eccentricity, and E is the "eccentric anomaly"**

5. **The trajectories defined by (the) planets are confined to single planes through the Sun (or central body) at a fixed inclination to the plane of the ecliptic**

Keplerian Motion: Projectability

At least to high approximation, the five planets move along ellipses, sweeping out equal areas in equal times with respect to the (true) Sun located at a focus common to all, on planes passing through the Sun at fixed angles of inclination, in periods proportional to the 3/2 power of their mean distances from the Sun.

Questions about this group of generalizations:

1. What grounds were there for extending – i.e. projecting – each of the generalizations beyond the five planets to support claims about:
   a. Any ("possible") body orbiting the Sun?
   b. Any body engaged in celestial orbital motion – e.g. Moon, satellites of planets?
   c. Any celestial body moving within our planetary system – e.g. comets?
2. What grounds were there for concluding that the specific statement of each generalization was properly suited for any such projection?
3. What, if any, further qualifications – e.g. tacit *ceteris paribus* conditions – needed to be noted with each generalization before projecting it?

Status of the Generalizations

---

1) Exact, essentially exact, merely approximate

2) If not exact, then which combination of:

   a. Holds in the mean or not in the mean (skewed) – e.g. ideal gas law does not hold in the mean

   b. Idealization or mere approximation

where idealizations can be:

   i. Generalizations that would hold exactly in certain specifiable circumstances

   ii. Generalizations that serve only to provide simpler mathematics – e.g. linear elasticity

Question: Assuming, as Kepler had by 1625, that his generalizations are probably not exact, what, if any evidence could have indicated for each which of the above categories it belonged in?

Kepler's Three Lines of Evidential Reasoning

Astronomia Nova (1609)

Start from Tycho's observations and use past practices in mathematical astronomy, including most notably Tycho's solar theory, to leverage reforms in those practices and a new orbit for Mars, invoking plausible physical considerations and precision of inferences from observation to conclude that the reforms and new orbit amount to something more than merely a representation of Tycho's data to (surprisingly) high approximation; thereby justify the claim that, absent the emergence of any substantial discrepancies with observation, the reforms and hence the orbital schema for Mars hold for the other planets as well.

Epitome Astronomiae Copernicanae (1618-21)

Start from proposed physical principles, defended in part through analogy with known physics and in part by anticipatory claims about their consequences agreeing with observation, to derive, with minimal auxiliary assumptions, the reforms and orbital schema as holding exactly (where no exteernal, secondary physical effects involved) for all the planets; and then (attempt to) show that the far more complicated motion of the Moon, theretofore never successfully represented to within observational precision, can be derived from the same physical principles without need of further auxiliary assumptions.

Tabulae Rudolphinae (1627)

Start from the reforms and orbital schema, taken as given, and use appropriate observations to derive values of the elements for each planet, opening the way to comparisons between calculation and all available observations, with provision for refining the values of the elements on the basis of these and further observations; success in reducing the discrepancies between calculation and observation through such refinements is evidence that the reforms and orbital schema amount to something more than merely one among many possible representations of Tycho's data to high approximation.

Fundamental Evidence Problems in Orbital Astronomy

1. How to determine the distances of celestial objects from the Earth -- in particular, the distance of Mercury, Venus, Mars, Jupiter, and Saturn as well as the Sun and Moon (and even comets) -- not only on average, but also over the course of time.

2. How to determine which apparent celestial motions and changes of motion, as observed from the Earth, are merely apparent and which are real, at least relative to the fixed stars.

3. How to determine whether such calculational rules as Kepler's area rule and ellipse, strikingly accurate as they seem to be, are merely one among many comparably accurate, though still only approximate, ways of representing Tycho's observations, or instead are something more than this.

    i. Whether they hold exactly or at least would hold exactly were it not for various "external" factors.

    ii. Whether they project beyond the given observations indefinitely into the past for the celestial objects in question and whether they project beyond these objects to other possible celestial objects.

4. How to determine what physical principles are governing the motions of the various celestial objects -- especially the planets, but also the Sun and Moon and maybe comets as well.

The first two of these problems go back at least as far as Ptolemy. The third also could be asked of Ptolemy's calculational rules, though asking it becomes much more appropriate after the factor of 50 or so improvement in accuracy achieved by Kepler. The fourth was asked in ancient times as well, though answered largely by philosophical appeals to the perfection of circular motion. Taken as requesting empirical evidence, it comes to the fore more strongly after Kepler turns to physical principles as the appropriate way of settling the other three evidence problems.

SIX ASPECTS OF THE EVIDENCE PROBLEM IN PLANETARY ASTRONOMY RECOGNIZED BY KEPLER

Whether Kepler should be regarded as the first to take the problem of planetary astronomy to be that of establishing a physically correct account of orbital motion is unclear. Nevertheless, he was almost certainly the first to realize the magnitude of this problem, for he did appreciate the challenge posed by the following issues:

1. The possibility not just of systematic error in the raw data, but of the corrections introduced to compensate for systematic error being inadequate.

2. The possibility of being led down a garden path by the need to use theoretical assumptions of some sort or other to extract conclusions from astronomical data, returning only much later to assess and, if need be, to refine these assumptions.

3. The fact that any number of distinct trajectories can, in principle, agree with imprecise data to more or less the same level of accuracy -- no matter how small the imprecision is -- so that the choice of any one of these trajectories over the others cannot be dictated by astronomical data alone.

4. The possibility of being led down a garden path by the need to appeal to underlying physics in one way or another in reaching conclusions about the physically true trajectory, while at the same time having to use conclusions about this trajectory as the principal basis for evidence for the conjectures about the underlying physics.

5. How to decide whether residual discrepancies between theory and observation should be dismissed, at least for the moment, as arising from observational error or extraneous secondary physical effects, or whether they should instead be regarded as falsifying the theory.

6. How to reach conclusions -- especially in the face of the above difficulties, but even without them -- about orbital motion in the remote past and in the long term future and about whether other orbiting bodies would have the same trajectories as those now accessible to investigation.

Sources for the Material in the Handout

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