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PRINCIPAL INDECOMPOSABLE CHARACTERS AND NORMAL SUBGROUPS

William F. Reynolds

We present an analogue of Clifford's tensor-factorization theorem for principal indecomposable (projective indecomposable) characters, i.e. the Brauer characters afforded by indecomposable projective modules over group algebras over splitting fields of prime characteristic and discuss some related results.

1. The Statement

The Brauer characters of a finite group X for a prime p are usually defined on the set $X_{p'}$ of p-regular elements of X. Instead, I prefer to define them as functions on all of X having their usual values on $X_{p'}$ and vanishing on $X-X_{p'}$ (cf. [8]). (This superficial change lets us use the same inner product for Brauer characters as for ordinary characters, and makes the principal indecomposable characters be ordinary characters and Brauer characters at once.) Let $\mathrm{IBr}(X)$ denote the set of all irreducible Brauer characters of X in this sense

If N is a normal subgroup of a finite group G, each class-function τ on G/N vanishing outside $(G/N)_{p'}$ determines a class-function $\inf_p\tau$ on G vanishing outside $G_{p'}$ by

$$(1.1)\qquad (\inf_p\tau)(g) = \begin{cases} \tau(gN) & \text{if } g \in G_{p'}, \\ 0 & \text{if } g \in G - G_{p'}. \end{cases}$$

The map

$$(1.2)\qquad \psi_j \mapsto \inf_p\psi_j : \qquad \mathrm{IBr}(G/N) \to \mathrm{IBr}(G)$$

is an injection, corresponding to the regarding of representations of G/N as representations of G. If $\theta \in \mathrm{IBr}(N)$, let $\mathrm{IBr}(G|\theta)$ be the set of those $\phi_j \in IBr(G)$ whose restrictions $(\phi_j)_N$ to N contain θ as an irreducible constituent. With this terminology we can state the character-theoretic part of a special case of a theorem of Clifford [2, Theorem 3 and 5] as follows.

Theorem 1 (Clifford). Let $\theta \in IBr(N)$ for a normal subgroup N of G. Assume that $\theta = \xi_N$ for some $\xi \in IBr(G)$. Then there is a bijection

$$(1.3) \qquad \psi_j \mapsto \phi_j: \qquad IBr(G/N) \rightarrow IBr(G|\theta)$$

given by

$$(1.4) \qquad \phi_j = \xi \, inf_p \psi_j.$$

Observe that the Brauer 1-character ψ_1 of G/N is mapped on $\phi_1 = \xi$.

For any X there is a canonical bijection of $IBr(X)$ to the set $PI(X)$ of principal indecomposable characters of X. In particular we have bijections

$$(1.5) \qquad \psi_j \mapsto \Psi_j \qquad IBr(G/N) \rightarrow PI(G/N),$$

$$(1.6) \qquad \phi_j \mapsto \Phi_j: \qquad IBr(G|\theta) \rightarrow PI(G|\theta),$$

where (1.6) defines $PI(G\ \theta)$ as a subset of $PI(G)$. Then Theorem 1 yields a bijection

$$(1.7) \qquad \Psi_j \mapsto \Phi_j: \qquad PI(G/N) \rightarrow PI(G|\theta)$$

It is natural to ask whether (1.7) can be obtained from a formula like (1.4) An answer to this question is given by our main theorem, as follows.

Theorem 2. Let $\theta \in IBr(N)$ for a normal subgroup N of G. Assume that $\theta = \xi_N$ for some $\xi \in IBr(G)$. Then:

(a) There is a unique class-function Ω on G vanishing outside $G_{p'}$ such that in the bijection (1.7) of $PI(G/N)$ to $PI(G|\theta)$,

(1.8) $$\Phi_j = \Omega \inf_p \Psi_j.$$

(b) Ω is the unique function on G vanishing outside $G_{p'}$ such that whenever $N \leq H \leq G$ and $p \nmid |H:N|$, Ω_H is the element of $PI(H)$ corresponding to ξ_H.

(c) $$\Omega(g) = \frac{1}{|C_{G/N}(gN)|} \sum_j \psi_{j*}(gN)\Phi_j(g), \qquad g \in G,$$

summed over $IBr(G/N)$, where ψ_{j*} is the complex conjugate of ψ_j.

(d) If G is replaced by any H, $N \leq H \leq G$, and ξ by ξ_H (so that the hypotheses still hold), Ω is replaced by its restriction Ω_H.

(e) If $p \nmid |G:N|$, then Ω is the element Ξ of $PI(G)$ corresponding to ξ

(f) Ω_N is the element Θ of $PI(N)$ corresponding to θ, and

$$(\Phi_j)_N = \Psi_j(1)\Theta.$$

It is important to note that Ω is not in general a principal indecomposable character, any more than $\inf_p \Psi_j$ is; in fact I cannot even prove that it is a Brauer character.

2. The Proof

The following lemma contains a central part of Theorem 2 in a special case

Lemma. If $p \nmid |G:N|$ under the hypotheses of Theorem 2, then in (1.7) we have

(2.1) $$\Phi_j = \Xi \inf_p \Psi_j$$

where Ξ is the element of $PI(G)$ corresponding to ξ.

Proof. For general G, the orthogonality relations for Brauer characters imply that Φ_j is the only class-function vanishing outside $G_{p'}$ such that $(\Phi_j, \phi_k) = \delta_{jk}$ for all $\phi_k \in IBr(G)$. Hence (1.8) is equivalent to the statement that

$$(\Omega\ \inf_p \Psi_j, \phi_k) = \delta_{jk} \tag{2.2}$$

for all $\psi_j \in IBr(G/N)$ and $\phi_k \in IBr(G)$, suitably indexed. The left side of (2.2) equals $(\Omega, \phi_k \inf_p \Psi_{j*})$.

If G/N is a p'-group, $IBr(G/N)$ and $PI(G/N)$ both coincide with the set of ordinary irreducible characters of G/N; taking $\Omega = \Xi$ in (1.8) we find that $(\Omega, \phi_k \inf_p \Psi_{j*}) = (\Xi, \phi_k \inf_p \psi_{j*})$ is the multiplicity of ξ as a constituent of $\phi_k \inf_p \psi_{j*}$. So (2.1) is equivalent to the assertion that

$$\begin{cases} \phi_k \inf_p \psi_{j*} \text{ contains } \xi \text{ once if } \phi_k = \xi \inf_p \psi_j, \\ \phi_k \inf_p \psi_{j*} \text{ does not contain } \xi \text{ otherwise,} \end{cases} \tag{2.3}$$

with ψ_j and ϕ_k as in (2.2).

Since the restriction to N of $\phi_k \inf_p \psi_{j*}$ is $\psi_{j*}(1)(\phi_k)_N$, this product cannot contain ξ unless $(\phi_k)_N$ contains θ. Hence in proving (2.3) we may assume that $\phi_k = \xi \inf_p \psi_k$ for some $\psi_k \in IBr(G/N)$. We can write

$$\psi_k\ \psi_{j*} = \sum_m a_m \psi_m, \tag{2.4}$$

summed over the irreducible characters of G/N. By Clifford s result (1.4),

$$\phi_k \inf_p \psi_{j*} = \xi\ \inf_p \psi_k\ \inf_p \psi_{j*} = \xi \sum_m a_m \inf_p \psi_m = \sum_m a_m \phi_m, \tag{2.5}$$

the last sum being over $IBr(G|\theta)$. For the 1-character ψ_1, $\phi_1 = \xi$ as in the comment after Theorem 1. By (2.5), the multiplicity of ξ in $\phi_k \inf_p \psi_{j*}$ is a_1; by (2.4), this equals

$(\psi_k\psi_{j^*}, \psi_1) = (\psi_k, \psi_j\psi_1) = (\psi_k, \psi_j) = \delta_{kj}$, proving (2.3) and the lemma.

We now prove Theorem 2. Under the hypotheses of the theorem, consider any group H_i, $N \leq H_i \leq G$. Denote correponding elements of $IBr(H_i/N)$ and $PI(H_i/N)$ by ψ^i_m and Ψ^i_m respectively. By Nakayama reciprocity [4, (III.2.6)] there exist nonnegative integers b^i_{mj} such that

$$(2.6) \qquad (\psi^i_m)^{G/N} = \sum_j b^i_{mj}\ \psi_j,$$

$$(2.7) \qquad (\Psi_j)_{H_i/N} = \sum_m b^i_{mj}\ \Psi^i_m.$$

By Theorem 1 for H_i and [6, (V.16.8)], each $\phi^i_m \in IBr(H_i|\theta)$ satisfies

$$(\phi^i_m)^G = (\xi_{H_i} \inf_p \psi^i_m)^G = \xi\ (\inf_p \psi^i_m)^G = \xi\ \inf_p((\psi^i_m)^{G/N})$$

By (2.6) and Theorem 1,

$$(\phi^i_m)^G = \xi \sum_j b^i_{mj} \inf_p \psi_j = \sum_j b^i_{mj}\ \phi_j,$$

summed over $IBr(G|\theta)$. Nakayama reciprocity again shows that

$$(2.8) \qquad (\Phi_j)_{H_i} = \sum_m b^i_{mj}\ \Phi^i_m,$$

summed over all $\Phi^i_m \in PI(H_i|\theta)$, for all $\Phi_j \in PI(G|\theta)$. Thus the decompositions (2.7) and (2.8) correspond.

From now on, assume that H_i satisfies the condition $p \nmid |H_i : N|$. Since $\xi_N = \theta$, ξ_{H_i} is an element ξ^i of $IBr(H_i)$ with $(\xi^i)_N = \theta$ and with corresponding $\Xi^i \in PI(H_i)$. By the lemma, $\Phi^i_m = \Xi^i \inf_p \Psi^i_m$. By (2.8) and (2.7),

$$(\Phi_j)_{H_i} = \Xi^i \sum_m b^i_{mj} \inf_p \Psi^i_m = \Xi^i \inf_p (\Psi_j)_{H_i/N} = \Xi^i (\inf_p \Psi_j)_{H_i}.$$

In other words, for each H_i and for every $\Phi_j \in PI(G|\theta)$,

$$(2.9) \qquad \Phi_j(g) = \Xi^i(g)\Psi_j(gN)$$

for all $g \in H_i$

We need to know that the functions Ξ^i fit together consistently. To show this, fix any $g \in G_{p'}$. At least one of the groups H_i, namely $\langle g \rangle N$, contains g. For any H_i that contains g, (2.9) and column-orthogonality imply that

$$(2.10)\quad \frac{1}{|C_{G/N}(gN)|} \sum_j \psi_{j*}(gN)\Phi_j(g) = \frac{\Xi^i(g)}{|C_{G/N}(gN)|} \sum_j \psi_{j*}(gN)\Psi_j(gN) = \Xi^i(g).$$

Since the first member is independent of i, we can call it $\Omega(g)$ as in (c) of the theorem. This determines a unique function Ω on G vanishing outside $G_{p'}$ with $\Omega_{H_i} = \Xi^i$ for all i; this is (b). By (2.9) Ω satisfies (1.8), and by (c) it is a class-function. This gives (a) except for uniqueness, but that holds since (1.8) implies (c) just as (2.9) implies (2.10). Finally (b) implies (d), (b) or the lemma implies (e), and (a) and (b) imply (f).

3. Related Results

If the assumption that $\theta = \xi_N$ is weakened to the assumption that θ is stable under conjugation by G, Theorem 1 still holds using projective Brauer characters (with respect to a 2-cocycle); the same is to be expected for Theorem 2, but I haven't completed the proof.

In the case that θ is the Brauer 1-character of N, Alperin, Collins, and Sibley have strengthened Theorem 2 greatly to a result on modules [1, Theorem]. Sibley has studied this for more general θ (unpublished). It is worth noting that the character formula [1, p. 419] that these authors derive from their theorem and use in a proof concerning the Cartan determinant can be proved more easily: see Harris [5, Corollary 2] and his references.

If G/N is a p'-group, Theorem 2 (or the lemma) implies that if θ extends to ξ, then θ extends to the corresponding principal indecomposable Ξ. This follows from a result of Rukolaĭne [9, Proposition 3] and has recently been proved by Schmid [10, Remark 1] and in a more general form by Willems [11, Proposition 2.8]

We conclude by collecting some results on principal indecomposables analogous to Clifford's other theorems [2, Theorems 1 and 2].

Theorem 3. For a normal subgroup N of G, let $\theta \in \mathrm{IBr}(N)$. Let S be the stabilizer of θ in G. Then induction gives bijections

$$(3.1) \qquad \phi_j' \mapsto (\phi_j')^G = \phi_j: \qquad \mathrm{IBr}(S\ \theta) \to \mathrm{IBr}(G|\theta),$$

$$(3.2) \qquad \Phi_j' \mapsto (\Phi_j')^G = \Phi_j: \qquad \mathrm{PI}(S|\theta) \to \mathrm{PI}(G|\theta).$$

For each $\phi \in \mathrm{IBr}(G|\theta)$ there are positive integers e_j and E_j such that

$$(3.3) \qquad (\phi_j')_N = e_j\theta, \qquad (\phi_j)_N = e_j \sum_u \theta^u,$$

$$(3.4) \qquad (\Phi_j')_N = E_j\theta, \qquad (\Phi_j)_N = E_j \sum_u \theta^u,$$

where $\tau^u(g) = \tau(ugu^{-1})$ and u ranges over right coset representatives of S in G.

Proof. The statements about irreducibles are due to Clifford; this gives a bijection $\Phi_j' \mapsto \Phi_j$. (3.2) follows from a result of Conlon [3, p. 162], [7, Satz 2.2] and the fact that projective modules induce to projective modules. We give another proof. For $\Phi_j' \in \mathrm{PI}(S|\theta)$, we have $(\Phi_j')^G = \sum_k m_{jk}\Phi_k$, summed over $\mathrm{PI}(G)$; Nakayama reciprocity gives

$$(3.5) \qquad (\phi_k)_S = \sum_j m_{jk}\phi_j' + \alpha_k,$$

where all the irreducible constituents of α_k are outside $\mathrm{IBr}(S|\theta)$. If $\phi_k \notin \mathrm{IBr}(G|\theta)$, restricting (3.5) to N shows that $m_{jk} = 0$ for all $\phi_j' \in \mathrm{IBr}(S|\theta)$; so suppose that $\phi_k \in \mathrm{IBr}(G|\theta)$. Since $(\phi_k)_S = ((\phi_k')^G)_S$, Mackey decomposition [6, (V.16.9)] implies that $m_{kk} \geq 1$ Restricting (3.5) to N, using (3.3), and equating multiplicities of θ, we get $e_k = \sum_j m_{jk}e_j$; since $m_{kk} \geq 1$, $m_{jk} = \delta_{jk}$ and (3.2) is proved.

As for (3.4), by Nakayama reciprocity, Φ_j is a component of θ^G. By the formula for induction, $(\theta^G)_N = |G:S| \sum_u \theta^u$. The separate statements of (3.4), due to Rukolaĭne [9, Proposition 2], follow. The same number E_j occurs in both statements by (3.2); this completes the proof.

Statements (3.2) and (3.4) imply the corresponding statements for modules, by projectivity.

Applying (3.4) to all $\theta \in \mathrm{IBr}(N)$ shows that for each θ, $\mathrm{PI}(G\ \theta)$ consists of those principal indecomposables Φ_k such that θ is a component of $(\Phi_k)_N$; hence it would be appropriate to call it $\mathrm{PI}(G|\theta)$ Note that (3.4) overlaps (f) of Theorem 2.

Theorem 3 establishes some statements I left without proof in [8, p. 341]

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Department of Mathematics
Tufts University
Medford, MA 02155, USA

A remark on "Principal indecomposable characters and normal subgroups" (June 2018)

M. E. Harris, *Clifford theory and filtrations,* J. Algebra 132 (1990) 205-218 answered some questions raised in this paper. Therefore I never wrote up anything else on this topic.