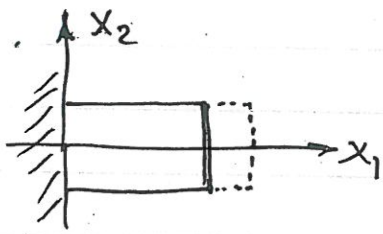
Strains – 2

Example : uniaxial strain



extent of straining

$$x_1 \to \bar{x}_1 = x_1 + \underbrace{k x_1}_{u_1}$$
$$\bar{x}_2 = x_2$$
$$\bar{x}_3 = x_3$$

Displacements :

$$u_1 = k x_1$$
$$u_2 = u_3 = 0$$

Displac. gradient tensor:

$$D_{ij} = \frac{\partial u_i}{\partial x_j} = \begin{vmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad \text{– symmetric (no } \omega \text{ part)}$$

$$= \epsilon_{ij}$$

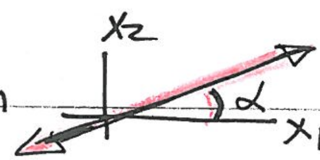
$\epsilon_{11} = k$, other $\epsilon_{ij} = 0$

In dyadic form:

$$\underset{\sim}{\epsilon} = k\, \underline{e}_1 \underline{e}_1$$

Using strain tensor:

find relative elongation in direction



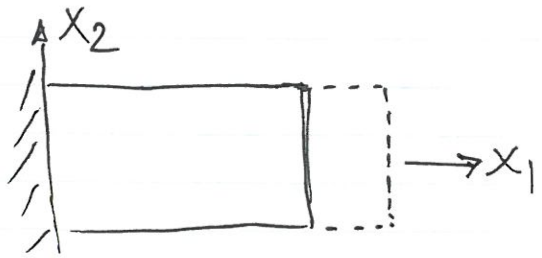
Unit vector : $\underline{n} = \underbrace{\cos\alpha}_{n_1}\, \underline{e}_1 + \underbrace{\sin\alpha}_{n_2}\, \underline{e}_2$

$$\frac{\Delta\bar{S} - \Delta S}{\Delta S} = \epsilon_{ij} n_i n_j = \epsilon_{11} n_1^2 = k \cos^2\alpha$$

Changes from k to 0 as α changes from 0 to 90°

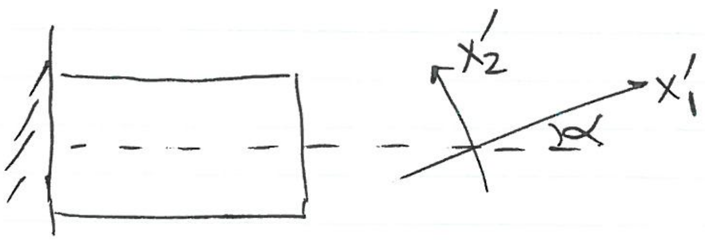
[Note : $\cos^2$ not cos!]

Uniaxial straining - cont'd



No angle change between X_1, X_2 directions : $\epsilon_{12}=0$
(intuitively obvious)

But what about angle between x_1' x_2' ?



$$\underset{\sim}{\epsilon} = k\, \underset{\sim}{e}_1 \underset{\sim}{e}_1$$

$$\underline{e}_1 = \cos\alpha\, \underline{e}_1' - \sin\alpha\, \underline{e}_2'$$

$$\underset{\sim}{\epsilon} = k\left(\cos\alpha\, \underline{e}_1' - \sin\alpha\, \underline{e}_2'\right)\left(\cos\alpha\, \underline{e}_1' - \sin\alpha\, \underline{e}_2'\right)$$

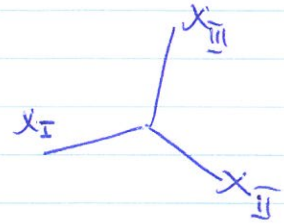
$$= \underbrace{k\cos^2\alpha}_{\text{relative elong in } x_1'\text{-dir.}}\, \underline{e}_1'\underline{e}_1' + \underbrace{k\sin^2\alpha}_{\text{---- in } x_2'\text{-dir.}}\, \underline{e}_2'\underline{e}_2' \overbrace{- k\sin\alpha\cos\alpha}^{\text{“}\epsilon_{1'2'}\text{”}}\left(\underline{e}_1'\underline{e}_2' + \underline{e}_2'\underline{e}_1'\right)$$

recover: relative elong in x_1'-dir.   ---- in x_2'-dir.   ↑ 1/2 angle distortion

Basic formula of strain analysis

$$\frac{\overline{\Delta S} - \Delta S}{\Delta S} = \epsilon_{ij} n_i n_j$$

in <u>principal axes</u> of strain

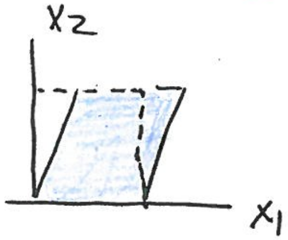


$$\underset{\sim}{\epsilon} = \epsilon_I \, \underline{e}_I \underline{e}_I + \epsilon_{II} \, \underline{e}_{II} \underline{e}_{II} + \epsilon_{III} \, \underline{e}_{III} \underline{e}_{III}$$ no off-diagonals

$$\underset{\sim}{n} = n_I \, \underline{e}_I + n_{II} \, \underline{e}_{II} + n_{III} \, \underline{e}_{III}$$

$$\Rightarrow \quad \frac{\overline{\Delta S} - \Delta S}{\Delta S} = \epsilon_I n_I^2 + \epsilon_{II} n_{II}^2 + \epsilon_{III} n_{III}^2$$

Example : Shear strain



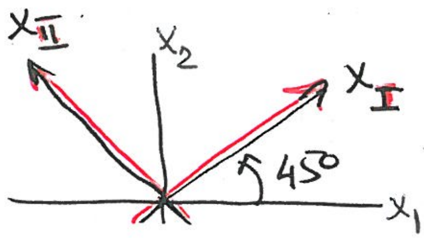
$$\begin{cases} u_1 = k x_2 \\ u_2 = 0 \end{cases}$$

$$D_{ij} = \frac{\partial u_i}{\partial x_j} = \begin{vmatrix} 0 & k \\ 0 & 0 \end{vmatrix}$$ – not symmetric

$$\epsilon_{ij} = \begin{vmatrix} 0 & k/2 \\ k/2 & 0 \end{vmatrix} \qquad \underset{\sim}{\epsilon} = \frac{k}{2}\left(\underline{e}_1 \underline{e}_2 + \underline{e}_2 \underline{e}_1\right)$$ (no volume change $\epsilon_{ii} = 0$

Eigenvalue problem : $\det \begin{vmatrix} -\lambda & k/2 \\ k/2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda_{1,2} = \pm k/2$

Eigenvectors: $\lambda_1 = \frac{k}{2}$ $\quad \left.\begin{array}{l} -\frac{k}{2}x_1 + \frac{k}{2}x_2 = 0 \\ \frac{k}{2}x_1 - \frac{k}{2}x_2 = 0 \end{array}\right\}$

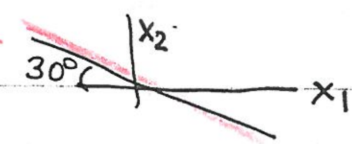


$\Rightarrow x_1 = x_2$

$\lambda_2 = -k/2 \Rightarrow x_1 = -x_2$

Principal axes representation $\underset{\sim}{\epsilon} = \frac{k}{2}\left(\underline{e}_I \underline{e}_I - \underline{e}_{II} \underline{e}_{II}\right)$ – shear strain is equiv. to stretch & contract. at 45° directions

relative elongation in direction



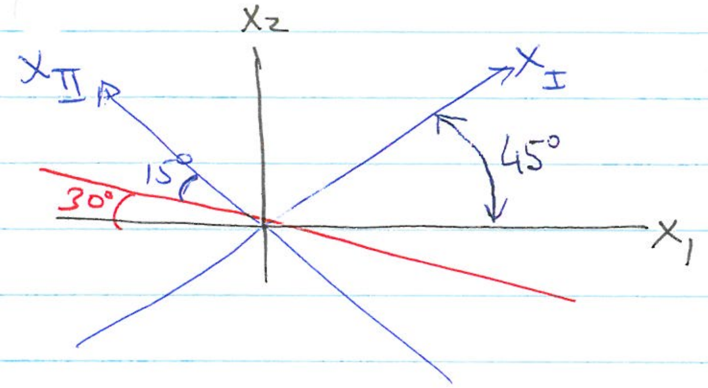
Working in $x_1 x_2$ axes :

$$\underbrace{-\frac{\sqrt{3}}{2}}_{n_1}\underline{e}_1 + \underbrace{\frac{1}{2}}_{n_2}\underline{e}_2 = \underset{\sim}{n}$$ – unit vector of this direction

$$\frac{\overline{\Delta S} - \Delta S}{\Delta S} = \epsilon_{ij} n_i n_j = \frac{k}{2}\left(n_1 n_2 + n_2 n_1\right) = -\frac{\sqrt{3}}{4} k$$

contraction

→ same calculation in the principal axes



$$\epsilon_I = \frac{k}{2} \qquad \epsilon_{II} = -\frac{k}{2}$$

$$\underset{\sim}{\epsilon} = \frac{k}{2}\left(\underline{e}_I\,\underline{e}_I - \underline{e}_{II}\,\underline{e}_{II}\right)$$

Unit vector of the same direction in the princ. coord. system:

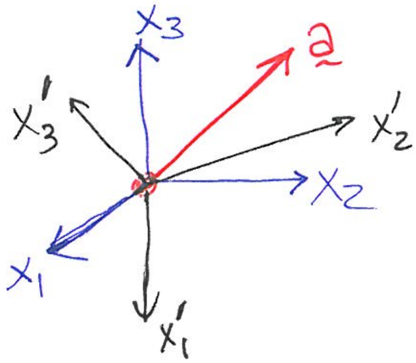
$$\underline{n} = \underbrace{-\sin 15°}_{n_I}\,\underline{e}_I + \underbrace{\cos 15°}_{n_{II}}\,\underline{e}_{II}$$

$$\frac{\overline{\Delta S} - \Delta S}{\Delta S} = \epsilon_I\, n_I^2 + \epsilon_{II}\, n_{II}^2 =$$

$$= \frac{k}{2}\underbrace{\left(\sin^2 15° - \cos^2 15°\right)}_{-\cos 30°} = -\frac{\sqrt{3}}{4}\,k \quad \text{-same}$$

tensor invariants

Components of vectors and tensors change when coord system is rotated



Certain combinations of components do not change $\rightarrow$ invariants

Vector: its length is invariant:

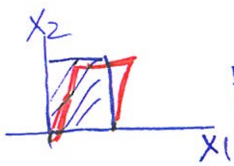
$$a_1^2 + a_2^2 + a_3^2 = a_1'^2 + a_2'^2 + a_3'^2 = |a|^2$$

Tensor: has 3 invariants

One of them: sum of diagonal elements

(called "first", or "linear" invariant)

Strain: ϵ_{ii} = relative volume change = ϵ_{ii}'



pure shear

$$\epsilon_{ij} = \begin{pmatrix} 0 & k/2 \\ k/2 & 0 \end{pmatrix} \text{ in } x_1 x_2 \text{ system}$$

$$= \begin{pmatrix} k/2 & 0 \\ 0 & -k/2 \end{pmatrix} \text{ in the princ. system}$$

Sum $\epsilon_{ii} = 0$ in both

Example: $\underset{\sim}{\epsilon} = d \cdot \underset{\sim}{I}$ $\quad (\epsilon_{ij} = d \cdot \delta_{ij})$

Dilatancy: $\epsilon_{kk} = 3d$

For any material line, with $\underset{\sim}{n}$:

$$\frac{\overline{\Delta S} - \Delta S}{\Delta S} = \epsilon_{ij} n_i n_j = d \left(n_1^2 + n_2^2 + n_3^2\right) = d$$

same for all orientations

uniformly heated mat'l (if $d<0$ - uniformly cooled)

Unit tensor $\underline{I}$ has the same δ_{ij} matrix in any coord system - no angle distortions between any directions

Maximal Relative Elongation

(important for fracture)

$$\max \frac{\overline{\Delta S} - \Delta S}{\Delta S}$$

(with respect to line orientation)

→ • equals to... ?

→ • on which orientation?

In the principal axes

$$\underset{\sim}{\epsilon} = \epsilon_{I}\, \underline{e}_{I}\, \underline{e}_{I} + \epsilon_{II}\, \underline{e}_{II}\, \underline{e}_{II} + \epsilon_{III}\, \underline{e}_{III}\, \underline{e}_{III}$$

ϵ_I — **the largest** (convention, for certainty)

Line Orientation $\underset{\sim}{n} = n_I\, \underline{e}_I + n_{II}\, \underline{e}_{II} + n_{III}\, \underline{e}_{III}$,

$$\frac{\overline{\Delta S} - \Delta S}{\Delta S} = \epsilon_I\, n_I^2 + \epsilon_{II}\, n_{II}^2 + \epsilon_{III}\, n_{III}^2 =$$

$$= \epsilon_I \underbrace{(n_I^2 + n_{II}^2 + n_{III}^2)}_{1} - \underbrace{\left[(\overset{>0}{\epsilon_I - \epsilon_{II}})\, n_{II}^2 + (\overset{>0}{\epsilon_I - \epsilon_{III}})\, n_{III}^2\right]}_{\geq 0}$$

$\Rightarrow$
- max for $n_{II} = n_{III} = 0$ ~ along x_I direction.
- equals ϵ_I

Minimal elongation - similarly, is equal to ϵ_{III} and is along the III-dir.

Note : min / max - with the account of signs

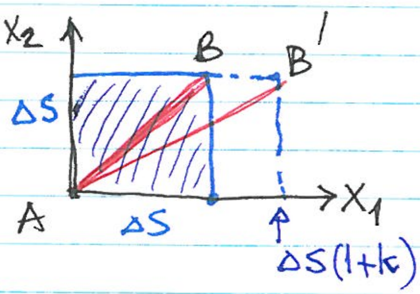
(negative for contraction)

Note on Smallness of Strains & Rotations

We assume all $\left|\frac{\partial u_i}{\partial x_j}\right| \ll 1$ (not only ϵ_{ij}, but ω_{ij} as well)

Only then $\begin{pmatrix}\epsilon_{ij} \\ \omega_{ij}\end{pmatrix}$ have the meanings we identified

Example uniaxial elongation



$$u_1 = kx_1 \qquad \Rightarrow \epsilon_{11} = k$$
$$u_2 = u_3 = 0$$

Elongation of diagonal (Pithagorean theorem):

$$AB' - AB = \left[\sqrt{(1+k)^2+1} - \sqrt{2}\right] \Delta S$$

Relative elong : $\dfrac{\left[\sqrt{(1+k)^2+1} - \sqrt{2}\right] \cancel{\Delta S}}{(\cancel{\Delta S} \cdot \sqrt{2})}$

From strains :

$\epsilon_{11} = k$, other $\epsilon_{ij} = 0$

relative elong: $\epsilon_{ij} n_i n_j = \epsilon_{11} \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} k$ - differs !

But , if k is small - the same

Indeed : $\sqrt{(1+k)^2+1} \overset{\text{small } k}{\approx} \sqrt{2+2k} = \sqrt{2}\sqrt{1+k}$;

recall : $\sqrt{1+k} \approx 1 + \frac{1}{2} k$ (Taylor series)

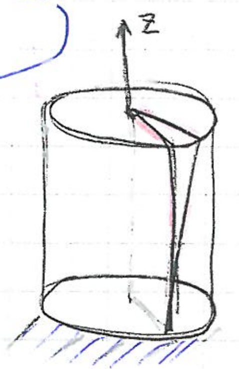
$\Rightarrow$ Pithagorean theorem yields $\frac{1}{2} k$

What if ϵ_{ij} and ω_{ij} are not small ?

then $\frac{\partial u_i}{\partial x_j}$ quantities do not mean anything

Example   Torsion of a circular cylinder – Strains

Notation: $X_1, X_2, X_3 \rightarrow x, y, z$



Kinematics of torsion:

- Each "slice" rotates in its own plane
- The angle of rotation of a given slice is proportional to its distance from the base

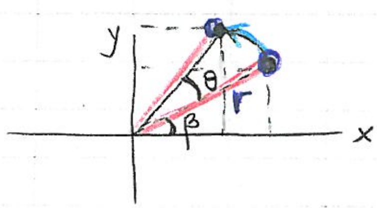
$$\theta = \alpha z$$

(twist per unit length)

Strains?

Displacements: (1) $u_z = 0$

(2) To derive u_x, u_y, consider rotation of a given slice



$$u_x = r\cos(\beta+\theta) - r\cos\beta =$$

$$= r\left[\cos\beta\cos\theta - \sin\beta\sin\theta - \cos\beta\right]$$

for small θ: $\cos\theta \approx 1$

$\sin\theta \approx \theta$

$$u_x = r\left[\cos\beta - \theta\sin\beta - \cos\beta\right] =$$

$$= -\theta \underbrace{r\sin\beta}_{y} = -\theta y \qquad = \underline{-\alpha y z}$$

$$u_y = r\left[\sin(\theta+\beta) - \sin\beta\right] = \theta x = \underline{\alpha x z}$$

$$D_{ij} = \frac{\partial u_i}{\partial x_j} = \begin{vmatrix} 0 & -\alpha z & -\alpha y \\ \alpha z & 0 & \alpha x \\ 0 & 0 & 0 \end{vmatrix}$$

$$\epsilon_{ij} = \begin{vmatrix} 0 & 0 & -\frac{\alpha}{2}y \\ 0 & 0 & \frac{\alpha}{2}x \\ -\frac{\alpha}{2}y & \frac{\alpha}{2}x & 0 \end{vmatrix}$$

- zero strains at centerline $x=y=0$
- non-uniform strains!
  depend on X, Y (not on Z)

- no volume change
- no angle distortions in plane: $\epsilon_{xy}=0$
- there are distortions of angles between the z- and -inplane directions

Eigenvalue problem: at some point (x_0, y_0) ← have to specify - non uniform ϵ

$$\det \begin{vmatrix} -\lambda & 0 & -\frac{\alpha}{2}y_0 \\ 0 & -\lambda & \frac{\alpha}{2}x_0 \\ -\frac{\alpha}{2}y_0 & \frac{\alpha}{2}x_0 & -\lambda \end{vmatrix} = -\lambda^3 + \frac{\alpha^2}{4}\lambda y_0^2 + \frac{\alpha^2}{4}\lambda x_0^2 = 0$$

$$\lambda_{1,3} = \pm\frac{\alpha}{2}\sqrt{x_0^2+y_0^2} = \pm\frac{\alpha}{2}r_0 \qquad \lambda_2 = 0$$

proportional to distance from the axis

this state of strain is shear:

(two equal princ. strains of opposite sign)

$$\epsilon_{IJ} = \begin{pmatrix} \frac{\alpha}{2}r_0 & & 0 \\ & 0 & \\ 0 & & -\frac{\alpha}{2}r_0 \end{pmatrix}$$