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- We demonstrate duality in the phase behavior of an asset-exchange model of wealth distribution.
- The duality is associated with a second-order phase transition between a classical wealth distribution and a partially wealth-condensed state.
- The solution to the model is derived analytically, both from the microscopic statistical process describing the model, and from the Fokker-Planck equation governing the agent density function.

Duality in an Asset Exchange Model for Wealth Distribution [1]

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Abstract

Asset exchange models are agent-based economic models with binary transactions. Previous investigations have augmented these models with mechanisms for wealth redistribution, quantified by a parameter χ, and for trading bias favoring wealthier agents, quantified by a parameter ζ. By deriving and analyzing a Fokker-Planck equation for a particular asset exchange model thus augmented, it has been shown that it exhibits a second-order phase transition at $\zeta/\chi = 1$, between regimes with and without partial wealth condensation. In the "subcritical" regime with $\zeta/\chi < 1$, all of the wealth is classically distributed; in the "supercritical" regime with $\zeta/\chi > 1$, a fraction $1 - \chi/\zeta$ of the wealth is condensed. Intuitively, one may associate the supercritical, wealth-condensed regime as reflecting the presence of "oligarchy," by which we mean that an infinitesimal fraction of the total agents hold a finite fraction of the total wealth in the continuum limit.

In this paper, we further elucidate the phase behavior of this model – and hence of the generalized solutions of the Fokker-Planck equation that describes it – by demonstrating the existence of a remarkable symmetry between its supercritical and subcritical regimes in the steady-state. Noting that the replacement $\{\zeta \to \chi,\ \chi \to \zeta\}$, which clearly has the effect of inverting the order parameter ζ/χ, provides a one-to-one correspondence between the subcritical and supercritical states, we demonstrate that the wealth distribution of the subcritical state is identical to that of the corresponding supercritical state when the oligarchy is removed from the latter. We demonstrate this result analytically, both from the microscopic agent-level model and from its macroscopic Fokker-Planck description, as well as numerically. We argue that this symmetry is a kind of *duality*, analogous to the famous Kramers-Wannier duality between the subcritical and supercritical states of the Ising model, and to the Maldacena duality that underlies AdS/CFT theory.

Keywords: Fokker-Planck equation, Asset Exchange Model, Yard-Sale Model, phase transitions, phase coexistence, wealth condensation
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1. Introduction

1.1. Background and prior work

Asset-exchange models are agent-based models of economies with binary transactions amongst agents, first proposed by Angle [1]. In the late 1990s, it was demonstrated that

Boltzmann equations can be derived to describe such models [2], and this was an important step toward the realization that macroeconomic behavior can be derived by kinetic-theoretical analysis of agent-based models [3].

A particularly useful asset-exchange model of agent transaction was proposed by Chakraborti in 2002 [4], in which each of N agents has a single positive scalar attribute, which we call wealth w. In each transaction, a certain amount of wealth is moved from one transacting agent to another. The amount of wealth moved is proportional to the wealth of the poorer of the two agents, with positive proportionality constant $\beta < 1$. The direction in which it moves is decided by the flip of a fair coin.

In 2014, Chakraborti's model generated renewed interest when a Boltzmann equation was derived for it, and it was additionally demonstrated that when β scales like the square root of the transaction time Δt, and when the latter tends to zero, that Boltzmann equation reduces to a certain nonlinear, integrodifferential Fokker-Planck equation [5]. A year later, it was furthermore shown that the Gini coefficient is a Lyapunov functional of both the Boltzmann and the Fokker-Planck equations for this model, and hence that the time-asymptotic state of the model will be one of total concentration of wealth to a single agent, regardless of initial condition and of the value of β used [6].

We may define an "oligarchical" wealth distribution to be one in which a vanishingly small number of agents hold a finite fraction of the total wealth, even in the continuum limit [2]. If that finite fraction is equal to one, the state is said to be a "total oligarchy." The phase transition from a classical to an oligarchical wealth distribution is called "wealth condensation." This transition was first described in a 2000 paper by Bouchaud and Mézard [7], and analyzed further in subsequent work [8], always in the context of a first-order phase transition. The above considerations show that Chakraborti's model results in a total oligarchy.

Because the complete concentration of wealth is unrealistic, and because the suggestion that the direction of wealth movement in a transaction between two agents is independent of their relative wealths is also questionable to say the least, recent efforts have focused on modifying Chakraborti's model to make it more realistic. In 2017, two new features were added to the model [9]:

(i) First, redistribution was accounted for by imposing a flat "wealth tax" at rate $\chi\Delta t$ on all agents on a per-transaction basis, and redistributing the revenues thereby collected to all agents uniformly. A similar model of redistribution had been employed in earlier studies of asset-exchange models [7], and it can be shown to result in the appearance of an Ornstein-Uhlenbeck [10] term in the Fokker-Planck equation [9]. This has the effect of arresting the concentration of wealth, resulting in a steady-state distribution with a depleted region near the origin, approximate power-law behavior for very low χ, and a gaussian cutoff at very large values of wealth [9]. The first two of these features (but not the third) may be recognized as consistent with the famous Pareto Law of wealth distribution.

(ii) Second, motivated by the substantial evidence that wealthier economic agents have an advantage over poorer agents in real economic activities, the direction of wealth move-

[2]In the continuum limit, one is forced to consider the concentration of wealth to an infinitesimal "fraction of an agent" – a concept that can be made rigorous using standard methods of functional analysis.

ment was biased in favor of the wealthier agent. This "Wealth-Attained Advantage" (WAA) was accomplished by biasing the coin flip by an amount proportional to the difference in wealth between the richer and the poorer agents, times $\zeta\sqrt{\Delta t}$, where ζ is a coefficient that quantifies the level of WAA present [9]. Because the bias is proportional to the wealth difference, it naturally reduces to zero when the transacting agents have equal wealth.

When the Fokker-Planck equation for this extended model was derived and its steady-state solutions were analyzed using both analytic arguments and numerical evidence [9], it was shown that the new model exhibits a second-order version of the wealth condensation phase transition to a partial oligarchy, with criticality occurring when the order parameter ζ/χ is equal to unity. More specifically, when the redistribution coefficient exceeds the WAA coefficient, we have a "subcritical" regime where $\zeta/\chi < 1$ and wealth is classically distributed. Conversely, when the WAA coefficient exceeds the redistribution coefficient, we have a "supercritical" regime where $\zeta/\chi > 1$ and a fraction $1 - \chi/\zeta$ of the population's total wealth is "condensed to the oligarchy," while the remainder of the wealth is classically distributed.

1.2. Purpose of this paper

In this paper, we shall demonstrate the existence of a remarkable symmetry between the supercritical and subcritical regimes of the above-described model in the steady state. This symmetry has to do with the replacement $\{\zeta \to \chi,\ \chi \to \zeta\}$, which obviously has the effect of inverting the order parameter ζ/χ, and thereby providing a one-to-one correspondence between subcritical and supercritical states. We shall show that the wealth distribution of the subcritical state is *identical* to that of the corresponding supercritical state *when the oligarchy is removed from the latter*. We shall argue that this symmetry is an example of the notion of *duality*, which appears in other subfields of the physics of critical phenomena.

The first and perhaps best known example of duality in physics is that which was discovered by Kramers and Wannier in 1941 in the context of the two-dimensional square-lattice Ising model of ferromagnetism, and which they used to make the first prediction of the critical temperature of that model [11]. They did this by comparing the high-temperature and low-temperature expansions of the partition function of the model, and supposing that the partition function has a singularity at the critical point and nowhere else. This forces a mathematical identity from which one can back out the critical temperature. When Onsager presented an exact solution for the two-dimensional Ising model in 1944 [12], Kramers' and Wannier's prediction for the critical temperature was verified, and moreover it became clear that their approach could be understood as establishing a deep one-to-one correspondence between the subcritical and supercritical states of the model [13]. If the temperature is scaled so that the critical temperature is equal to unity, the associated subcritical and supercritical temperatures are multiplicative inverses of one another, just as they are in our economic model. This correspondence is not the least bit obvious, especially because it associates highly disordered states with highly ordered ones.

The notion of duality in physics exploded in importance after Maldacena's 1997 conjecture that there is a duality between the anti-de Sitter (AdS) spaces used in theories of quantum gravity, and conformal field theories (CFT) which are quantum field theoretical descriptions of elementary particles on the boundaries of those AdS spaces [14]. This conjectured association is sometimes called the AdS/CFT correspondence. Because this association relates strongly coupled field theories which can not be treated perturbatively with weakly coupled field theories which can, the methodology has the potential to enhance our understanding of strongly coupled field theories.

Given the importance attached to duality in modern physics, we find it fascinating that the very same concept appears in a simple agent-based model of the economy. In the following sections, we shall demonstrate this duality analytically, both from the microscopic agent-level model and from its macroscopic Fokker-Planck description, as well as numerically, using both Monte Carlo simulations and numerical solutions of the Fokker-Planck equation.

2. Mathematical description of model

2.1. Microscopic, agent-level description

We suppose that we have a population of N economic agents, each possessing a single positive scalar attribute, which we call *wealth*. The sum of the wealth of all the agents will be denoted by W. The average wealth of an agent in the population is then W/N.

We first consider the agent-level transaction between one agent A who has wealth w, and another agent B who has wealth x. We suppose that this interaction takes place in a time increment Δt and results in the transfer of wealth Δw from B to A [3]. Hence, after the transaction, the wealth of A will be $w + \Delta w$, while that of B will be $x - \Delta w$.

In what follows, we shall write

$$\Delta w = \Delta w_t + \Delta w_r, \tag{1}$$

where Δw_t is the contribution to Δw due to the basic transaction model, and Δw_r is that due to the redistribution model. We consider these two contributions separately.

In the basic transaction model proposed by Chakraborti in 2002 [4] and described in the Introduction to this paper, the magnitude of the wealth increment is proportional to the minimum of the wealths of the two agents, and so we write

$$\Delta w_t = \sqrt{\gamma \Delta t}\ \min(w, x)\eta. \tag{2}$$

Here the quantity $\sqrt{\gamma \Delta t}$ is what was called β in the Introduction, and we allow the constant γ to control the proportionality between β and $\sqrt{\Delta t}$. Then the quantity $\eta \in \{-1, +1\}$ is a binary random variable that models the coin flip. In Chakraborti's original model [4], the coin was fair so $E[\eta] = 0$, where we have used E to denote the expectation value of functions of the random variable η.

To model WAA, we will need to modify $E[\eta]$ so that it is no longer zero, but rather favors the wealthier of the two agents. Again following the discussion presented in the Introduction, and in much more detail in reference [9], we suppose that $E[\eta]$ is proportional to $w - x$. Hence $E[\eta] > 0$, which favors agent A, if $w > x$; likewise $E[\eta] < 0$, which favors agent B, if $x > w$. We normalize the difference $w - x$ by dividing it by the average wealth W/N, so that it does

[3] Note that Δw is a signed quantity, so in the event that it is negative, an amount of wealth $|\Delta w|$ actually moves from agent A to agent B.

not depend on the units chosen for wealth. If we then multiply by $\sqrt{\Delta t}$, as mentioned in the Introduction, and introduce a new parameter ζ to quantify WAA, we have

$$E[\eta] = \zeta\sqrt{\frac{\Delta t}{\gamma}}\left(\frac{w-x}{W/N}\right). \tag{3}$$

Here, with forethought given to simplifying the resulting Fokker-Planck equation, and without any loss of generality, we have divided Δt under the square root by the parameter γ introduced earlier.

We next consider the redistribution model. We apply a flat tax rate $\chi\Delta t$ to every agent at each time step, combine all of the tax thereby collected, and redistribute an equal share of this total to each of the N agents. So an agent with wealth w would pay tax $\chi w\Delta t$. Hence the total tax collected from the entire society would be $\chi W\Delta t$, and each agent would receive a share equal to $\chi W\Delta t/N$. The net amount added to the wealth of agent A from this redistribution process is then

$$\Delta w_r = \chi\left(\frac{W}{N} - w\right)\Delta t. \tag{4}$$

Note that agents with wealth less than the average experience a net benefit from the redistribution, while those with wealth greater than the average experience a net loss.

The full microscopic model is then given by inserting Eqs. (2) and (4) into Eq. (1) to obtain

$$\Delta w = \sqrt{\gamma\Delta t}\ \min(w,x)\eta + \chi\left(\frac{W}{N} - w\right)\Delta t, \tag{5}$$

which, along with Eq. (3), defines the microscopic agent-level model. Henceforth, we shall refer to χ as the *redistribution coefficient* and ζ as the *WAA coefficient*, respectively. In the steady-state, these are the only two relevant parameters of the model; as we shall show, the other parameter γ can be absorbed into the time scale, which is irrelevant in the steady state [9].

2.2. Macroscopic, Fokker-Planck description

We denote the agent density function by $P(w,t)$, so the total number of agents and the total amount of wealth are given by

$$N := \int_0^\infty dx\ P(x,t) \tag{6}$$

and

$$W := \int_0^\infty dx\ P(x,t)x, \tag{7}$$

respectively. For the dynamics described by Eq. (5), we expect N and W to be constants of the motion.

In the limit as $\Delta t \to 0$, for which we suppose that there are many small transactions, the dynamics of $P(w,t)$ can be described by a Fokker-Planck equation [15, 16, 17] which has the general form

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial w}(\sigma P) + \frac{1}{2}\frac{\partial^2}{\partial w^2}(DP), \tag{8}$$

where σ and D are called the *drift coefficient* and the *diffusivity*, respectively, and are given by:

$$\sigma = \lim_{\Delta t \to 0} \mathcal{E}\left[\frac{\Delta w}{\Delta t}\right] \tag{9}$$

and

$$D = \lim_{\Delta t \to 0} \mathcal{E}\left[\frac{(\Delta w)^2}{\Delta t}\right]. \tag{10}$$

Here we have used $\mathcal{E}[f]$ to denote the expected value of a bivariate function $f(\eta, x)$ of the two random variables η and x over the distribution $P(x,t)$, which is given by:

$$\mathcal{E}[f] = \frac{1}{N}\int_0^\infty \mathrm{d}x\ P(x,t)E[f(\eta,x)]. \tag{11}$$

Using Δw given by Eq. (5), and applying Eqs. (3), (9), (10) and (11), we find

$$\sigma = \chi\left(\frac{W}{N} - w\right) - 2\zeta\left[\frac{N}{W}\left(B - \frac{w^2}{2}A\right) + \left(\frac{1}{2} - L\right)w\right] \tag{12}$$

and

$$D = 2\gamma\left(B + \frac{w^2}{2}A\right), \tag{13}$$

where we have defined the *Pareto-Lorenz potentials*,

$$A(w,t) := \frac{1}{N}\int_w^\infty dx\ P(x,t), \tag{14}$$

$$L(w,t) := \frac{1}{W}\int_0^w dx\ P(x,t)x, \tag{15}$$

and

$$B(w,t) := \frac{1}{N}\int_0^w dx\ P(x,t)\frac{x^2}{2}. \tag{16}$$

Inserting our results for the drift coefficient and diffusivity, Eqs. (12) and (13), into Eq. (8), we obtain the Fokker-Planck equation for the agent density distribution $P(w,t)$,

$$\begin{aligned}\frac{\partial P}{\partial t} = &-\frac{\partial}{\partial w}\left[\chi\left(\frac{W}{N} - w\right)P\right] \\ &+\frac{\partial}{\partial w}\left\{2\zeta\left[\frac{N}{W}\left(B - \frac{w^2}{2}A\right) + \left(\frac{1}{2} - L\right)w\right]P\right\} \\ &+\frac{\partial^2}{\partial w^2}\left[\gamma\left(B + \frac{w^2}{2}A\right)P\right].\end{aligned} \tag{17}$$

Note that this is a nonlinear, integrodifferential equation, because the Pareto-Lorenz potentials, Eqs. (14), (15) and (16), depend on the unknown function $P(w,t)$, and do so in an essentially nonlocal way.

The microscopic model can never change the sign of the wealth of an agent, and this property is inherited by the macroscopic Fokker-Planck description, so if the initial condition $P(w,0)$ has support only for $w>0$, that will remain true at later times. Hence we suppose that the appropriate domain for an initial-value problem is $w\in\mathbb{R}^+$ and $t\in\mathbb{R}^+$.

We can now adopt natural "transactional" units of time t, by absorbing the constant γ into the time scale t, the redistribution coefficient χ, and the WAA coefficient ζ. Physically, if we imagine that γ is the transaction rate, then our new time units for t are measured in "transactions," while χ becomes the redistribution rate per transaction, and ζ becomes the WAA per transaction. Mathematically, the net effect of adopting these units is to set $\gamma=1$, and that is what we shall do henceforth.

To analyze the distribution in steady-state, we set the time derivative to zero in Eq. (17), and integrate once with respect to w to obtain the nonlinear, integrodifferential, ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}w}\left[\left(B+\frac{w^2}{2}A\right)P\right]=\chi\left(\frac{W}{N}-w\right)P-2\zeta\left[\frac{N}{W}\left(B-\frac{w^2}{2}A\right)+\left(\frac{1}{2}-L\right)w\right]P. \tag{18}$$

Henceforth, we focus on discussing the weak solutions of this equation, and we ignore time dependence throughout, writing $P(w)$ instead of $P(w,t)$, etc.

3. Wealth condensation

3.1. Oligarchical distributions

In prior work [9, 18, 19], solutions to Eq. (18) with the boundary conditions

$$P(0) = 0 \tag{19}$$

and

$$\lim_{w\to\infty} A(w) = 1 \tag{20}$$

have been studied. From Eqs. (14) and (15), it is clear that Eq. (20) implies that

$$\lim_{w\to\infty} P(w)=0, \tag{21}$$

but remarkably it does not imply that $\lim_{w\to\infty} L(w)=1$. In fact, it has been shown that Eq. (18), with boundary conditions given by Eqs. (19) and (20), admits distributional solutions of the form:

$$P(w)=p(w)+c_\infty W\Xi(w), \tag{22}$$

where $p(w)$ is a classical function defined on $w\in[0,\infty)$, where $c_\infty\in[0,1]$, and where $\Xi(w)$ is a distribution with vanishing zeroth moment and unit first moment,

$$\int_0^\infty \Xi(w)=0 \tag{23}$$

$$\int_0^\infty \Xi(w)w=1, \tag{24}$$

and divergent higher moments. While a formal definition of $\Xi(w)$ can be provided using Sobolev-Schwarz distribution theory [9] or by distribution theory [19], it can be envisioned more intuitively as, for example, the limit of a sequence of functions,

$$\Xi(w) = \lim_{\epsilon \to 0} \Xi_\epsilon(w), \tag{25}$$

where we have defined

$$\Xi_\epsilon(w) = \begin{cases} 1 & \text{for } \frac{1}{\epsilon} - \frac{\epsilon}{2} \leq w \leq \frac{1}{\epsilon} + \frac{\epsilon}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

It is not difficult to see that this definition is consistent with Eqs. (23) and (24). This definition also makes clear that any incomplete moment of Ξ vanishes, i.e.,

$$\int_0^w \Xi(x) x^k = 0, \text{for k} = 0,\ 1,\ 2,\ \ldots \tag{27}$$

for any finite $w \in \mathbb{R}$. Further intuition for and a more detailed mathematical treatment of the distribution $\Xi(w)$ is provided in Appendix A of [9].

Intuitively, we may think of the second term of Eq. (22) as corresponding to the presence of an "oligarchy" – a vanishingly small fraction of the total number of economic agents who nevertheless possess a finite fraction of the total wealth. To see this, note that the second term of Eq. (22) contributes nothing to N because of Eq. (23), whereas it contributes $W_\Xi := c_\infty W$ to the total wealth W because of Eq. (24). We may therefore surmise that the zeroth and first moments of the classical part of the distribution are given by

$$\int dw\ p(w) = N \tag{28}$$

$$\int dw\ p(w) w = (1 - c_\infty)\, W. \tag{29}$$

A distribution with the form of Eq. (22) with $c_\infty \neq 0$ will be called an *oligarchical distribution.*

To summarize, the contributions of the classical and oligarchical terms to N and W can be written

$$N = \int_0^\infty \mathrm{d}w\ P(w) = \underbrace{\int_0^\infty \mathrm{d}w\ p(w)}_{N_p := N} + \underbrace{c_\infty W \int_0^\infty \mathrm{d}w\ \Xi(w)}_{N_\Xi := 0} \tag{30}$$

and

$$W = \int_0^\infty \mathrm{d}w\ P(w) w = \underbrace{\int_0^\infty \mathrm{d}w\ p(w) w}_{W_p := (1-c_\infty)W} + \underbrace{c_\infty W \int_0^\infty \mathrm{d}w\ \Xi(w) w}_{W_\Xi := c_\infty W}, \tag{31}$$

where we have used Eqs. (23) and (24) to evaluate the integrals over Ξ.

3.2. The Lorenz curve

It is important to note that in any graph of $P(w)$ versus w, the second, oligarchical term in Eq. (22) will be essentially invisible – an imperceptible, measure-zero adjustment to the extreme tail of the distribution. This is because this term contributes nothing to the total amount of economic agents present, which is, after all, the zeroth moment of the distribution. This term can not be neglected, however, because it contributes significantly to the total amount of wealth present, which is the first moment of the distribution.

If you had a way of knowing N and W in advance, you might be able to *infer* the presence of the oligarchical term from the portion of the distribution p that you can see as follows: If you take the zeroth moment of p, and you confirm that the result is N, you can be sure that you have accounted for the full measure of the classical distribution p. If you then take the first moment of p and find that the result is $W_p < W$, you can infer the existence of a second, oligarchical term in Eq. (22) with $c_\infty = 1 - {}^{W_p}/_{W}$.

A more direct way of recognizing the presence of an oligarchical term in Eq. (22) is to employ the *Lorenz curve*, first introduced by Max O. Lorenz in 1905 [20] as a way to represent inequality in wealth (or income). For a given distribution, the Lorenz curve plots the cumulative share of wealth against the cumulative share of economic agents. Hence a point (x, y) on Lorenz curve can be interpreted as the bottom $x\%$ of the population of the society possessing $y\%$ of the total wealth of the society.

The culmulative share of economic agents is given by

$$F(w) := \frac{1}{N}\int_0^w \mathrm{d}x\ P(x) = 1 - A(w), \tag{32}$$

while the cumulative share of wealth is just the function $L(w)$ introduced in Eq. (15). Hence, the Lorenz curve is a parametric plot of $L(w)$ versus $F(w)$, where the parameter w runs from zero to infinity. Going forward, we shall refer to this functional form as $\mathcal{L}(\mathcal{F})$, defined so that $\mathcal{L}(\mathcal{F}) = L(w)$ when $\mathcal{F} = F(w) \in [0, 1]$. Three important properties of the function $\mathcal{L}(\mathcal{F})$ follow from this definition:

1. It is easy to see that the graph of $\mathcal{L}(\mathcal{F})$ must include the points $(0, 0)$ and $(1, 1)$, and must necessarily lie below the straight diagonal line connecting those two points. In fact, twice the area between the Lorenz curve and the diagonal is a commonly used measure of wealth (or income) inequality called the *Gini coefficient* [21], introduced by Corrado Gini in 1912 [22].
2. It is also clear that if one is given the distribution $P(w)$, one can straightforwardly compute N, W and the Lorenz curve $\mathcal{L}(\mathcal{F})$. It is less clear that the opposite is true: If one knows N, W and $\mathcal{L}(\mathcal{F})$, one can recover the distribution $P(w)$. To see this, note that the slope of the Lorenz curve is

$$\mathcal{L}'(\mathcal{F}) = \frac{L'(w)}{F'(w)} = \frac{w}{W/N}, \tag{33}$$

which is the wealth, normalized to the average wealth. If you are given the Lorenz curve, you can suppose that you know $\mathcal{L}'(\mathcal{F})$ as a function of $\mathcal{F}$, and hence this function of $\mathcal{F}$ must be equal to $w/(W/N)$. Knowing W/N, you can invert this relation to obtain

F as a function of w. Differentiating that, recalling that $F'(w) = P(w)/N$, and knowing N, you can recover $P(w)$. Hence the triplet $\{N, W, \mathcal{L}(\mathcal{F})\}$ is equivalent in information content to $P(w)$.

3. Finally, by examining the second derivative $\mathcal{L}''(\mathcal{F})$ is possible to show that the graph of $\mathcal{L}(\mathcal{F})$ must be concave up.

From the above observations, it follows that Lorenz curves corresponding to subcritical solutions are continuous and concave up, extending from point $(0, 0)$ to point $(1, 1)$. In the supercritical case, however, when the oligarchical term $c_\infty \Xi(w)$ is present, the graph of the Lorenz curve on the clopen domain $\mathcal{F} \in [0, 1)$, though still concave up, approaches the point $(1, 1-c_\infty)$ on the right-hand boundary of its domain, instead of $(1, 1)$; it then discontinuously jumps to the point $(1, 1)$ when $\mathcal{F} = 1$.

A property of Lorenz curves closely related to Property 2 above is that they are invariant under scaling of both the abscissa and the ordinate of $P(w)$. Both $F(w)$ and $L(w)$ are invariant under scaling of the ordinate of P, because their definitions in Eqs. (32) and (15) include division by N and W, respectively. Scaling of the abscissa involves a scaling of w, which changes the parametrization of the Lorenz curve, but not the functional form of the curve itself. This scale invariance is reflected in the fact that W and N are needed along with the Lorenz curve $\mathcal{L}(\mathcal{F})$ to recover the distribution $P(w)$. This means that to each Lorenz curve there corresponds an entire equivalence class of possible distributions, and that this equivalence class is isomorphic to $(\mathbb{R}^+)^2$ (since N and W are both positive). Conversely, each equivalence class can be specified by a single representative distribution, obtained by taking $W = N = 1$, which we refer to as the *canonical form*.

It has been shown [9] that the Fokker-Planck equation, Eq. (18), is invariant under this equivalence relation. That is, for any agent density function $P(w)$ that is a solution to Eq. (18) with $W = N = 1$, the distribution $\overline{P}(w)$ is also a solution with total wealth $\overline{W}$ and total number of agents $\overline{N}$, given by the transformation:

$$\overline{P}(w) = \frac{\overline{N}^2}{\overline{W}} P\left(\frac{w}{\overline{W}/\overline{N}}\right), \tag{34}$$

where $\overline{W}$ and $\overline{N}$ can be any positive real numbers. Eq. (34) is therefore a two-parameter relation between every member of the equivalence class and its canonical form representative. Because P and $\overline{P}$ are in the same equivalence class, in the sense described above, they will have exactly the same Lorenz curve. The Lorenz curve may thus be thought of as a property of the entire equivalence class, rather than any single representative thereof, which makes it the most appropriate metric for comparison with empirical results.

In the following section, we shall investigate the Lorenz curves for a subcritical solution with parameters χ and ζ, with $\zeta < \chi$, and that of its dual supercritical solution with the two parameters swapped. We shall demonstrate that the two Lorenz curves are identical to within an overall scale factor. Specifically, the supercritical Lorenz curve is $\frac{\zeta}{\chi} < 1$ times the subcritical Lorenz curve. The proof of this assertion will involve using Eqs. (44) and (46) to show that for a specific supercritical solution, $P(w)$ with total wealth W, its non-oligarchical population $p(w)$ corresponds to its dual subcritical solution with total wealth W_p, and the Lorenz curve of the previous one is just a scaling of the latter. Together with the above-described invariance property of the Lorenz curve, this will establish that it is true between all dual solution pairs, even though their total wealths, W and W_p, may differ. This establishes the one-to-one correspondence between the subcritical and supercritical solutions that is the hallmark of duality. After establishing this for the macroscopic Fokker-Planck description, we shall trace the origin of the duality back to the underlying microscopic process, where it is slightly more difficult to recognize.

4. Duality

4.1. Duality in the macroscopic, Fokker-Planck description

It has been demonstrated [9] that Eq. (18) exhibits a second-order phase transition, in that the character of its solutions abruptly changes at the critical value $\zeta = \chi$. When $\zeta < \chi$, the solutions are subcritical, and when $\zeta > \chi$ they are supercritical. In the framework of Eq. (22), we can write these two types of solutions in a unified way by writing c_∞ as follows:

$$c_\infty = \begin{cases} 0 & \text{for } \chi \geq \zeta \\ \left(1 - \frac{\chi}{\zeta}\right) W & \text{for } \chi < \zeta. \end{cases} \tag{35}$$

This phase transition corresponding to the sudden appearance of oligarchy as ζ is increased is an example of a phenomenon sometimes known as *wealth condensation* [7].

To investigate the duality between the supercritical and subcritical solutions, we begin by considering the supercritical case (i.e. $\zeta > \chi$). Then, as shown in Eq. (31), the total wealth of the society W can be written

$$W = W_p + W_\Xi, \tag{36}$$

then by applying Eq. (31) and Eq. (35) to the above equation in the supercritical case, we have the wealth of the classical part of the distribution,

$$W_p = \frac{\chi}{\zeta} W, \tag{37}$$

and that of the oligarchical part,

$$W_\Xi = \left(1 - \frac{\chi}{\zeta}\right) W. \tag{38}$$

That is, a fraction of $1 - \chi/\zeta$ of the total wealth of the entire society is held by the oligarchy, while the rest of the population holds the remaining fraction χ/ζ. In the subcritical case, by contrast, the oligarchy vanishes and the non-oligarchical population holds the entire wealth of the society, i.e., $W_p = W$.

Now, let us focus on $p(w)$, the non-oligarchical population part of a supercritical solution, and see if we can write an integrodifferential equation for it alone. Analogous to W_p, we can define variants of the Pareto potentials in Eqs. (14) through (15), for $p(w)$ alone,

$$N_p = \int_0^\infty \mathrm{d}x\; p(x) \tag{39}$$

$$A_p = \frac{1}{N_p}\int_w^\infty \mathrm{d}x\; p(x) \tag{40}$$

$$B_p = \frac{1}{N_p}\int_0^w \mathrm{d}x\; p(x)\frac{x^2}{2} \tag{41}$$

$$L_p = \frac{1}{W_p}\int_0^w \mathrm{d}x\; p(x)x. \tag{42}$$

Using Eqs. (23) through (27), we can insert Eq. (22) back into Eqs. (14) through (15) and derive the relationships between the Pareto potentials for $p(w)$ and those for $P(w)$ as follows,

$$N_p = N \tag{43}$$

$$A_p = A \tag{44}$$

$$B_p = B \tag{45}$$

$$L_p = \frac{\zeta}{\chi}L \tag{46}$$

Now, with the above equations established, we can insert Eq. (22) back into Eq. (18) and rewrite the steady-state equation of the Fokker-Planck equation as:

$$\frac{\mathrm{d}}{\mathrm{d}w}\left[\left(B_p + \frac{w^2}{2}A_p\right)p\right] + \underbrace{c_\infty \frac{\mathrm{d}}{\mathrm{d}w}\left[\left(B_p + \frac{w^2}{2}A_p\right)\Xi\right]}_{I} \tag{47}$$

$$= \zeta\left(\frac{W_p}{N_p} - w\right)p + \underbrace{c_\infty \zeta\left(\frac{W_p}{N_p} - w\right)\Xi}_{II} - 2\chi\left[\frac{N_p}{W_p}\left(B_p - \frac{w^2}{2}A_p\right) + w\left(\frac{1}{2} - L_p\right)\right]p$$

$$\underbrace{+\,2c_\infty\chi\left[\frac{N_p}{W_p}\left(B_p - \frac{w^2}{2}A_p\right) + w\left(\frac{1}{2} - L_p\right)\right]\Xi}_{III}. \tag{48}$$

When considering distributional solutions to the above equation, we can choose any smooth test function $\phi_\alpha(w)$ with compact support. By multiplying both sides of Eq. (48) by $\phi_\alpha(w)$, and integrating with respect to w, the contribution of the terms I, II and III are all zero, leaving the equation:

$$\frac{\mathrm{d}}{\mathrm{d}w}\left[\left(B_p + \frac{w^2}{2}A_p\right)p\right] = \zeta\left(\frac{W_p}{N_p} - w\right)p - 2\chi\left[\frac{N_p}{W_p}\left(B_p - \frac{w^2}{2}A_p\right) + w\left(\frac{1}{2} - L_p\right)\right]p \tag{49}$$

The principal observation of this paper can be seen in the above equation. Notice that Eq. (49) is of exactly the same form as the original Fokker-Planck equation in steady-state Eq. (18), but with the redistribution and the WAA coefficients swapped. Recalling that $p(w)$ is the non-oligarchical part of a supercritical solution $P(w)$, we know from Eq. (49) that $p(w)$ would correspond to a subcritical solution of Eq. (18). Hence we have established the promised one-to-one correspondence between subcritical and supercritical solutions, which we recognize as the key feature of duality.

4.2. Duality in the microscopic random process

The duality between the supercritical and the subcritical solutions can be also understood at the microscopic process level. Again, we begin by considering the supercritical case when $\chi < \zeta$, and we return to the random walk process described in Eqs. (5) and (3). We rewrite these two equations in terms of W_p instead of W by applying Eq. (37),

$$
\begin{aligned}
\Delta w &= \sqrt{\gamma \Delta t}\min(w,x)\eta + \zeta\left(\frac{W_p}{N} - w\right)\Delta t + (\zeta-\chi)w\Delta t && (50)\\
E[\eta] &= \chi\sqrt{\frac{\Delta t}{\gamma}}\frac{N}{W_p}(w-x) && (51)
\end{aligned}
$$

Notice that the above two equations are of the very same form as Eqs. (5) and (3), aside from redefinitions of parameters. If we were to apply Eqs. (9) through (11) to compute the drift coefficient and diffusivity of the corresponding Fokker-Planck equation, we would again end up with Eq. (17). To gain some insight into the relationship between the non-oligarchical population and the oligarchy, however, instead of directly averaging over $P(w)$, we apply Eq. (22) to break up the average defined in Eq. (11) into two contributions,

$$
\begin{aligned}
\mathcal{E}[f(\eta,x)] &= \frac{1}{N}\int_0^\infty \mathrm{d}x\; P(x,t)E[f(\eta,x)]\\
&= \underbrace{\frac{1}{N}\int_0^\infty \mathrm{d}x\; p(x,t)E[f(\eta,x)]}_{\mathcal{E}_p[f(\eta,x)]} + \underbrace{\frac{1}{N}\int_0^\infty \mathrm{d}x\; c_\infty \Xi(x)E[f(\eta,x)]}_{\mathcal{E}_\Xi[f(\eta,x)]}. && (52)
\end{aligned}
$$

By breaking up the averages over the non-oligarchical population $p(w)$ and the oligarchy $\Xi(w)$ in this way, we can likewise break up the drift coefficient and the diffusivity into contributions from the non-oligarchical population and the oligarchy separately, i.e., $D = D_p + D_\Xi$ and $\sigma = \sigma_p + \sigma_\Xi$, respectively. Straightforward calculation yields

$$
\begin{aligned}
D_p &= \lim_{\Delta t\to 0}\mathcal{E}_p\left[\frac{\Delta w^2}{\Delta t}\right] = 2\left(B_p + \frac{w^2}{2}A_p\right) && (53)\\
D_\Xi &= \lim_{\Delta t\to 0}\mathcal{E}_\Xi\left[\frac{\Delta w^2}{\Delta t}\right] = 0 && (54)\\
\sigma_p &= \lim_{\Delta t\to 0}\mathcal{E}_p\left[\frac{\Delta w}{\Delta t}\right] = \zeta\left(\frac{W_p}{N} - w\right) + (\zeta-\chi)w\\
&\quad -2\chi\left[\frac{N}{W_p}\left(B_p - \frac{w^2}{2}A_p\right) + w\left(\frac{1}{2} - L_p\right)\right] && (55)\\
\sigma_\Xi &= \lim_{\Delta t\to 0}\mathcal{E}_\Xi\left[\frac{\Delta w}{\Delta t}\right] = (\chi-\zeta)w && (56)
\end{aligned}
$$

If we write the Fokker-Planck equation in steady state using Eqs. (53) through Eq. (56), we obtain Eq. (49) exactly.

From the above derivation, we can see that the random process within the non-oligarchical population $p(w)$ is equivalent to a subcritical random process with redistribution coefficient

χ and WAA coefficient ζ swapped, plus an extra term $(\zeta - \chi)w$. Remembering that we are considering the case $\zeta > \chi$, we see that this extra term is positive. This extra term is the wealth flow into the non-oligarchical population due to the tax on the oligarchy. To see this, we can compute the $\mathcal{E}_p$ average of the extra term $(\chi - \zeta)w$, obtaining $\chi(\zeta/\chi - 1)W_p$, which is exactly the amount of the tax per unit time paid by the oligarchy at tax rate χ.

Conversely, when we compute the $\mathcal{E}_\Xi$ average of the extra term, we obtain a negative term $(\chi - \zeta)w$, balancing the above-described wealth flow into the non-oligarchical population. This is due to the transaction between the non-oligarchical population and the oligarchy. Because the oligarchy is a vanishingly small fraction of an agent possessing infinite wealth in the continuum limit, the WAA model guarantees that the oligarchy wins in every such transaction. Therefore, the WAA acts like a "effective tax" on the non-oligarchical population which is balanced only by the actual redistributive tax on the oligarchy.

The above argument provides a heuristic explanation of why there exists a symmetry between the redistribution and WAA coefficients. When the wealth flow between the two systems balance each other, the steady-state is reached. The distribution of the non-oligarchical population $p(w)$ would then satisfy Eq. (49), and would correspond to a subcritical solution of Eq. (18).

5. Numerical Results

Figure 1 confirms the presence of the duality by comparing two numerical solutions of Eq. (18) with different parameters, corresponding subcritical and supercritical cases. These numerical solutions are found using a shooting method, as described in Appendix B of [9]. We can see that the wealth distribution of a subcritical solution when $\chi = 0.03$, $\zeta = 0.02$ and $W = 1$ (plotted in green) is identical to the wealth distribution for the non-oligarchical population of a supercritical solution when $\chi = 0.02$, $\zeta = 0.03$ and $W = 1.5$ (plotted in red).

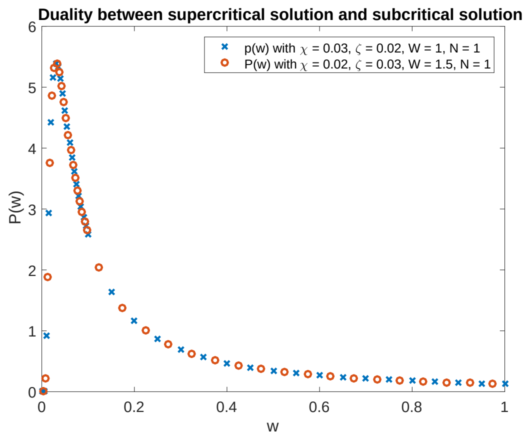

Figure 1: Comparison of the wealth distribution of a subcritical solution $P(w)$ and the wealth distribution of the non-oligarchical population of a supercritical solution $p(w)$, with the swapping of the redistribution coefficient and WAA coefficient.

From a practical point of view, the duality provides an effective way to solve for a supercritical distributional solution to Eq. (18). If we want to solve for a supercritical distributional solution with redistribution coefficient χ and WAA coefficient ζ and total wealth W, we can instead solve for the dual subcritical classical solution with the two parameters swapped and with total wealth equal to $\frac{\chi}{\zeta}W$. Then we can simply augment this solution by the addition of an oligarchy with wealth $(1 - \frac{\chi}{\zeta})W$.

Figure 2 shows the Lorenz curves between the subcritical solution and the supercritical solution by swapping the two parameters. We can see that while the Lorenz curve for the subcritical solution is a curve from $(0, 0)$ to $(1, 1)$, the Lorenz curve for the supercritical solution is just a scaling of the previous one by a factor of $\frac{\chi}{\zeta}$ and hence intersecting the right boundary at $(1, \frac{\chi}{\zeta})$

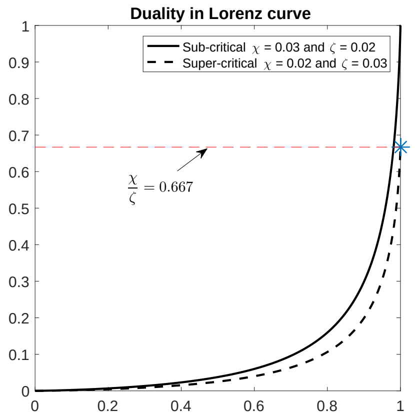

Figure 2: Comparison of the Lorenz curves of a subcritical solution (solid curve) and its dual supercritical solution (dotted curve). When scaled vertically, the curves perfectly coincide.

From a practical point of view, the duality provides a numerical advantage in solving for the Lorenz curve. To compute the Lorenz curve for a supercritical distributional solution, we can just compute that for its dual subcritical classical solution and scale it appropriately.

6. Discussion and Conclusion

We have demonstrated a very non-trivial one-to-one correspondence between two classes of steady-state solutions of the agent-based asset-exchange model considered in [9]. The first are distributional solutions of the corresponding Fokker-Planck equation, which are characterized by wealth condensation and oligarchy, and which we refer to as *supercritical* solutions. The second are classical solutions of the corresponding Fokker-Planck equation, which exhibit neither wealth condensation nor oligarchy, and which we refer to as *subcritical solutions*. We have identified this one-to-one correspondence as an example of the phenomenon of duality.

More specifically, we have shown that the wealth distribution of the non-oligarchical part of a supercritical distribution is precisely equal to the subcritical solution obtained by swapping the redistribution and WAA coefficients. If we think of the ratio of these two coefficients as the order parameter $z = {}^{\zeta}/_{\chi}$, then the swapping of the two coefficients is equivalent to taking the inverse of z. As noted earlier, this is very similar to the Krammer-Wannier duality where the free energy of an Ising model with a high temperature is "dual" to an Ising model with the inverse of the temperature of the previous one. Hence, the order parameter z in this context plays the role of temperature in the Ising model.

We presented two mathematical arguments explaining the origin of the above-described duality, one based on the macroscopic Fokker-Planck description of the agent steady state, and the other based on the microscopic process-level description thereof. From the microscopic description, we were able to identify the crucial balance between the effects of taxation and redistribution on the oligarchy on one hand, and those of biased transaction outcomes on the non-oligarchical population on the other.

It should be noted that hints of the existence of this duality were present in earlier work. In [9], for example, it was noted that for very large values of w, the distribution exhibits an asymptotic Gaussian tail of the form:

$$\exp\left(-a|\zeta-\chi|w^2-bw\right). \tag{57}$$

With hindsight, it is evident that the above equation is indicative of the symmetry described in this paper – if you swap χ and ζ, the Gaussian tail would decay at exactly the same rate [4]. From the result of this paper, however, we can see this symmetry is indicative of a much deeper exact symmetry between the subcritical and supercritical steady-state solutions to the model.

As noted above, the presence of duality has already had practical benefit in reducing the numerical work involved in finding supercritical solutions of the model. We hope that it will also have theoretical benefit in allowing us to understand and analyze the mathematical properties of this fascinating model of wealth distribution.

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