A Characterization of Virtually Cyclic Outer Automorphism Groups of Right-Angled Coxeter Groups

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Abstract

Existing research gives conditions for when the outer automorphism group of a graph product of primary cyclic groups W_Γ is finite, virtually abelian, or large. We seek to prove a set of conditions for when this outer automorphism group is virtually cyclic. To this end, we study the finite index subgroup $\mathrm{Out}^0(W_\Gamma)$, which is generated by specific partial conjugations. The presence or absence of Coxeter and non-Coxeter separating intersections of links (SILs), separating triple intersections of links (STILs), and flexible separating intersections of links (FSILs) in Γ determines algebraic properties of $\mathrm{Out}^0(W_\Gamma)$. We identify each SIL with a pair of partial conjugations in $\mathrm{Out}^0(W_\Gamma)$ and place restrictions on the SILs in Γ to ensure that $\mathrm{Out}^0(W_\Gamma)$ is virtually $\mathbb{Z}$ both when Γ is connected or disconnected. In particular, this applies to the study of right-angled Coxeter groups.

Acknowledgments

This thesis would not have been possible without guidance from my advisor, Professor Kim Ruane. Her patience, enthusiasm, expertise, and interesting anecdotes helped me through every week of this research and made me fall in love with right-angled Coxeter groups.

I am also grateful to my peers Léo Delage, Catherine DiLeo, Cisco Hadden, and Hikaru Jitsukawa at Tufts. We began thinking about the questions answered in this thesis as part of a research group in the summer of 2025, so I appreciate what they taught me during that summer project, as well their insights since then. I'm especially grateful to Hikaru for organizing a set of notes of our conjectures as of August 2025 which I referred back to during the course of this research, as well as for letting me include two of their proofs in this document. Thank you also to Emma Rao for proofreading my document.

A special thank you to my parents, who have supported me financially and emotionally throughout my time at Tufts. They have cheered for me after every tiny milestone this year and reassured me through every tiny crisis. Finally, I appreciate every friend who looked at the cryptic TikZ diagrams that took me so long to make, and despite not knowing what they meant, still shared my excitement.

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Chapter 1

Introduction

1.1 Motivation and related efforts

A graph product of groups is a group construction introduced by [Gre90], which interpolates between the direct product and free product of a collection of groups. This thesis studies graph products of primary cyclic groups, denoted W_Γ, which notably include right-angled Coxeter groups. For a graph product of primary cyclic groups, the inner and outer automorphism groups (denoted $\mathrm{Inn}(W_\Gamma)$ and $\mathrm{Out}(W_\Gamma)$, respectively) constitute all but a finite part of the automorphism group $\mathrm{Aut}(W_\Gamma)$. Thus a study of the outer automorphism group correlates to a deeper understanding of the entire automorphism group. Our results do not generalize to graph products of directly indecomposable groups, which include right-angled Artin groups. [CV08] and [GS18], among others, study the structure of the outer automorphism group of a right-angled Artin group.

We build upon the results of [CRSV09] and [SS21] on $\mathrm{Out}(W_\Gamma)$, and the results of [Lau95], [GK08], and [CG09] on $\mathrm{Aut}(W_\Gamma)$. This thesis is sandwiched between the work of Gutierrez, Piggott, and Ruane ([GPR12]) and that of Susse and Sale [SS17]. [GPR12] characterized when $\mathrm{Out}(W_\Gamma)$ is finite, depending on the presence of a separating intersection of links (SIL). In [SS17], Susse and Sale furthered this work by characterizing when $\mathrm{Out}(W_\Gamma)$ is virtually abelian or large (that is, it maps onto a non-abelian free group). This thesis aims to fill the gap between these two papers by describing which properties of Γ occur exactly when $\mathrm{Out}(W_\Gamma)$ is virtually cyclic, furthering the study of how the number of SIL's in Γ impacts structural properties of

$\mathrm{Out}(W_\Gamma)$. We follow the methods of both sets of authors, utilizing in particular the generating set $\mathcal{P}^0$ of $\mathrm{Out}^0(W_\Gamma)$ defined in [GPR12], and following the structure used in [SS17] to prove that our set of conditions is necessary and sufficient.

The broader motivation for studying this particular question is to describe the geometry of $\mathrm{Aut}(W_\Gamma)$ and $\mathrm{Out}(W_\Gamma)$. These are groups that are similar to, yet simpler than, $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ (where F_n is a free group with n generators). While there is an immense amount of literature on these groups, we do not yet have a complete understanding of their structure for all W_Γ. [CRSV09] studied the geometry of $\mathrm{Aut}(W_\Gamma)$ when $\mathrm{Out}(W_\Gamma)$ is finite; the next simplest case is when $\mathrm{Out}(W_\Gamma)$ is a "small" infinite, i.e. virtually cyclic, which we discuss in this thesis. $\mathrm{Out}(W_\Gamma)$ is certainly well-understood geometrically when it is virtually cyclic, but a study of Γ in this case has been missing to get a complete characterization.

A better comprehension of exactly when $\mathrm{Out}(W_\Gamma)$ is virtually cyclic also helps in the study of the geometry of $\mathrm{Aut}(W_\Gamma)$. As we will see in Remark 3.1.1, it follows that $\mathrm{Aut}(W_\Gamma)$ is virtually a group of the form $W_\Gamma \rtimes_\varphi \mathbb{Z}$. Recall that multiplication in such a semi-direct product is defined by specifying a homomorphism $\psi : \mathbb{Z} \to \mathrm{Aut}(W_\Gamma)$. Since the domain of this homomorphism is cyclic, ψ is determined by $\psi(1)$, and so this amounts to simply choosing one element $\psi(1) := \varphi$ in $\mathrm{Aut}(W_\Gamma)$; this choice of φ is shown in the notation $\rtimes_\varphi$. If φ and φ' are automorphisms in $\mathrm{Aut}(W_\Gamma)$ and if α is an inner conjugation in $\mathrm{Inn}(W_\Gamma)$ such that $\varphi' = \alpha \circ \varphi$, then $W_\Gamma \rtimes_\varphi \mathbb{Z} \cong W_\Gamma \rtimes_{\varphi'} \mathbb{Z}$. If φ is an inner automorphism (equivalently, the identity in $\mathrm{Out}(W_\Gamma)$), then the resulting semidirect product is $W_\Gamma \rtimes_\varphi \mathbb{Z} \cong W_\Gamma \rtimes_1 \mathbb{Z} \cong W_\Gamma \times \mathbb{Z}$, the direct product of the two groups. In particular, if $\mathrm{Out}(W_\Gamma)$ is virtually cyclic, then up to finite index, it is generated by a single automorphism φ; this greatly reduces the effort required to check $W_\Gamma \rtimes_\varphi \mathbb{Z}$ for all φ in $\mathrm{Out}(W_\Gamma)$.

We can describe the geometry of W_Γ because there is a natural cubical CAT(0) geometry X on which this group acts. We also understand the geometry of the direct product $W_\Gamma \times \mathbb{Z}$ because we can simply take the metric product $X \times \mathbb{R}$ as a geometry for the direct product. The goal is to find a geometry that reflects the algebra of each semi-direct product. We do not explore geometrical implications in this thesis, but [Dan18] surveys recent research on the geometry of right-angled Coxeter groups.

1.2 Summary

In Chapter 2, we list basic definitions and results from graph theory and group theory which are used in the rest of the thesis. Chapter 3 includes results from Gutierrez, Piggott, and Ruane ([GPR12]), in particular the splitting of $\mathrm{Aut}(W_\Gamma)$, the construction of a generating set $\mathcal{P}^0$ for $\mathrm{Out}^0(W_\Gamma)$, and a set of conditions for when $\mathrm{Out}(W_\Gamma)$ is finite. This chapter also introduces the notion of FSILs and STILs in the graph Γ and a set of conditions from Susse and Sale ([SS17]) for when $\mathrm{Out}(W_\Gamma)$ is large. In Chapter 4, we prove that a pair of distinct SILs in Γ corresponds to four distinct partial conjugations in $\mathrm{Out}^0(W_\Gamma)$ which together generate a subgroup that is not virtually $\mathbb{Z}$. Chapter 5 considers separately the case of a disconnected graph Γ, summarizing methods from [SS17] and proving the main result for the disconnected case. Chapter 6 concludes with a proof of the main result, a condition on Γ so that $\mathrm{Out}(W_\Gamma)$ is virtually $\mathbb{Z}$, for the connected case. We also see a few abstract and concrete examples of what connected graphs satisfy this condition.

Chapter 2

Preliminaries

Recall some basic definitions about graphs.

Definition 2.0.1. A **graph** $\Gamma = (V(\Gamma), E(\Gamma))$ is an ordered pair consisting of a set $V(\Gamma)$ of vertices and a set $E(\Gamma)$ of undirected edges which each connect two vertices.

Γ is **finite** if it has finitely many vertices.

A **loop** in Γ is an edge connecting a vertex $v \in V(\Gamma)$ to itself.

Multiple edges are two or more edges connecting the same pair of vertices.

A **simple** graph is a graph with no loops or multiple edges.

An **induced subgraph** Ω of Γ has as its vertex set a subset $V(\Omega) \subseteq V(\Gamma)$ and as its edge set $E(\Omega) = \{(v_i, v_j) \in E(\Gamma) : v_i, v_j \in V(\Omega)\}$. We may refer to this as just a subgraph, but we always assume a subgraph is induced unless otherwise noted.

Γ is **complete** if every pair of vertices in $V(\Gamma)$ is joined by an edge.

A subgraph $C \subseteq \Gamma$ is a **connected component** of Γ if there is a path in C between any two vertices of C, and if for any vertex $v \in V(\Gamma) \backslash V(C)$, there is no path in the subgraph $C \cup \{v\}$ between v and a vertex in C.

Γ is **disconnected** if it has more than one connected component.

The **center** of Γ, denoted $Z(\Gamma)$, is the subset of $V(\Gamma)$ consisting of those vertices which are adjacent to all other vertices in $V(\Gamma)$.

In this thesis, we will assume graphs are simple, finite, and non-trivial. As in the paper by Gutierrez, Piggott, and Ruane ([GPR12]), we will define two categories of graph products.

Definition 2.0.2. Let Γ be a graph with vertex set $V(\Gamma) = \{v_1, ..., v_n\}$. An **order map** on Γ is a function

$$\mathbf{m} : \{1, 2, ..., n\} \to \{p^\alpha : p \text{ prime}, \alpha \in \mathbb{N}\} \cup \{\infty\}.$$

The pair $(\Gamma, \mathbf{m})$ forms a **labelled graph**. We define the **graph product of directly indecomposable cyclic groups** $W(\Gamma, \mathbf{m})$, usually denoted W_Γ, as the group with the following presentation:

$$W_\Gamma := \langle V(\Gamma) \mid v_i^{\mathbf{m(i)}}, [v_j, v_k] \text{ for } 1 \leq i, j, k \leq n \text{ and } (v_j, v_k) \in E(\Gamma) \rangle.$$

By convention, if $\mathbf{m}(i) = \infty$, then v_i has infinite order.

We know from [Rad01] that given a group with a graph product decomposition into primary cyclic groups, this decomposition is unique; [GP08] extended this result to all graph products of directly indecomposable cyclic groups. So we have a one-to-one correspondence between labelled graphs $(\Gamma, \mathbf{m})$ and graph products of directly indecomposable cyclic groups W_Γ. Given a generator v of W_Γ, we often do not distinguish between v as a group element and v as a vertex of the graph. [GPR12] proved that the graph products of directly indecomposable cyclic groups are exactly the graph products of finitely generated abelian groups, hence some authors refer to these groups with the latter terminology.

Recall that all directly indecomposable cyclic groups have one of three forms: the trivial group $\{\text{id}\}$, $\mathbb{Z}$, or $\mathbb{Z}/p^k\mathbb{Z}$, where p is a prime number and k is a positive integer. Only $\{\text{id}\}$ and $\mathbb{Z}/p^k\mathbb{Z}$ are also primary cyclic groups, and all primary cyclic groups have one of these two forms, so we want to adjust our definition of W_Γ so as to not allow vertex groups of $\mathbb{Z}$.

Definition 2.0.3. We say that $W(\Gamma, \mathbf{m})$ on n vertices is a **graph product of primary cyclic groups** if $\mathbf{m}(i) < \infty$ for $1 \leq i \leq n$.

From now on, W_Γ will refer strictly to a graph product of primary cylic groups.

Definition 2.0.4. A **right-angled Coxeter group** is a graph product of primary cyclic groups $W(\Gamma, \mathbf{m})$ on n vertices with $\mathbf{m}(i) = 2$ for $1 \leq i \leq n$.

We immediately see that for elements a and b in a right-angled Coxeter group W_Γ, either

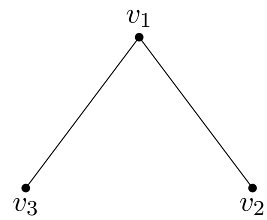


Figure 2.1: Simple example of a graph product of primary cyclic groups

$[a, b] = 1$ or the element ab has infinite order. We will give a simple example of a graph product of primary cyclic groups.

Example 2.0.1. Consider the graph Γ in Figure 2.1. Let $\mathbf{m}$ be the order map on Γ given by $\mathbf{m}(1) = 2$, $\mathbf{m}(2) = 2$, and $\mathbf{m}(3) = 3$. In other words, v_1 and v_2 both correspond to the group $\mathbb{Z}/2\mathbb{Z}$ and v_3 corresponds to $\mathbb{Z}/3\mathbb{Z}$. This determines the graph product of primary cyclic groups

$$W_\Gamma = \langle v_1, v_2, v_3 \mid v_1^2, v_2^2, v_3^3, [v_1, v_2], [v_1, v_3]\rangle.$$

An edge between vertices indicates that their corresponding groups commute, so we clearly see that

$$W_\Gamma \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}).$$

Note that Γ as a graph is independent from the order map $\mathbf{m}$; we could define another order map $\mathbf{m'}$ on this particular graph which defines a different graph product. However, this graph product will always be of the form $\mathbb{Z}/\mathbf{m'}(1)\mathbb{Z} \times (\mathbb{Z}/\mathbf{m'}(2)\mathbb{Z} * \mathbb{Z}/\mathbf{m'}(3)\mathbb{Z})$.

Of course, graph products of primary cyclic groups worth studying are much more complex. A graph theory property that directly impacts the structure of $\mathrm{Out}(W_\Gamma)$ is called a SIL.

Definition 2.0.5. In a simple graph Γ, the **link** of a vertex $v \in V(\Gamma)$ is the set $Lk(v) := \{w : (v, w) \in E(\Gamma)\}$. The **star** of v is the set $St(v) := Lk(v) \cup \{v\}$.

Definition 2.0.6. A **Separating Intersection of Links (SIL)** in a graph is two vertices v_1 and v_2 in $V(\Gamma)$ satisfying:

1. $d(v_1, v_2) \geq 2$; and

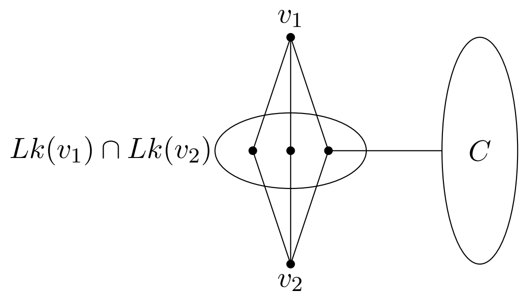


Figure 2.2: SIL (Separating Intersection of Links)

2. There is a connected component $C \subseteq \Gamma \backslash (Lk(v_1) \cap Lk(v_2))$ such that v_1 and v_2 are not in C. (See Figure 2.2).

There are two equivalent notations for a SIL formed by v_1 and v_2: we can write either $\{v_1, v_2|C\}$, where C is the connected component satisfying condition (2) of a SIL, or $\{v_1, v_2|z\}$ for some vertex z in C. We define two SILs $\{v_1, v_2|C\}$ and $\{w_1, w_2|D\}$ as distinct if $C \cap D = \emptyset$ or if $\{v_1, v_2\} \neq \{w_1, w_2\}$. In the second notation, two SILs $\{v_1, v_2|z\}$ and $\{v_1, v_2|y\}$ are distinct if z and y are not in the same connected component of $\Gamma \backslash (Lk(v_1) \cap Lk(v_2))$, or again if $\{v_1, v_2\} \neq \{w_1, w_2\}$.

We call the SIL $\{v_1, v_2|C\}$ a **Coxeter SIL** if $\mathbf{m}(1) = \mathbf{m}(2) = 2$, or a **non-Coxeter SIL** if either $\mathbf{m}(1)$ or $\mathbf{m}(2)$ is greater than 2. We use "SIL" as an umbrella term to encompass both types.

Definition 2.0.7. Let G be a group and P a group property. G is **virtually** P if G contains a finite index subgroup H with the property P.

We will consider when a subgroup of $\mathrm{Aut}(W_\Gamma)$ has one of four algebraic properties: finite, virtually $\mathbb{Z}$ (also called virtually cyclic), virtually abelian, and large. Although the latter three are not cardinalities of the group, we can roughly think of them as descriptions of the "size" of the group. Note that a virtually cyclic group is virtually abelian, but otherwise a group cannot have two or more of these properties. Theorem 3.2.2 will establish the strict dichotomy between virtually abelian and large.

Definition 2.0.8. A group G is **large** if there is a finite index subgroup H of G and a surjective homomorphism $H \to F_2$, the free group of rank 2.

Example 2.0.2. Let $W_\Gamma = W(\Gamma, \mathbf{m})$ be a graph product of primary cyclic groups. Suppose that v_1 and v_2 are non-adjacent vertices and consider the empty subgraph Ω induced by v_1 and v_2. Now, W_Ω has the presentation $\langle v_1, v_2 | v_1^{\mathbf{m}(1)}, v_2^{\mathbf{m}(2)} \rangle$. Since there is no edge between the two vertices, we get the free product

$$W_\Omega \cong (\mathbb{Z}/\mathbf{m}(1)\mathbb{Z}) * (\mathbb{Z}/\mathbf{m}(2)\mathbb{Z}).$$

If $\mathbf{m}(1) = \mathbf{m}(2) = 2$, then

$$W_\Omega \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) \cong D_\infty.$$

If either $\mathbf{m}(1)$ or $\mathbf{m}(2)$ is greater than 2, say $\mathbf{m}(1) > 2$, then W_Ω contains the subgroup $\langle v_1 v_2, v_1^2 v_2 \rangle$, which is isomorphic to F_2. In that case, W_Ω is large.

Now suppose there is another vertex v_3 of Γ and that the subgraph $\Delta \subseteq \Gamma$ induced by v_1, v_2, and v_3 is empty. Then

$$W_\Delta \cong (\mathbb{Z}/\mathbf{m}(1)\mathbb{Z}) * (\mathbb{Z}/\mathbf{m}(2)\mathbb{Z}) * (\mathbb{Z}/\mathbf{m}(3)\mathbb{Z}).$$

For any values of $\mathbf{m}(1), \mathbf{m}(2)$, and $\mathbf{m}(3)$, we have a subgroup $\langle v_1 v_2, v_1 v_3 \rangle$ in W_Δ. This subgroup is isomorphic to F_2, so W_Δ is large.

Note that if a general group G is large, then $\mathrm{Inn}(G)$ is large, and thus $\mathrm{Aut}(G)$ is large.

We conclude this chapter with a few facts that we will be referring to as tools. We won't prove them here, but leave them as exercises to the reader.

Fact 1. $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong D_\infty$.

Fact 2. D_∞ is virtually $\mathbb{Z}$.

Fact 3. $D_\infty \times D_\infty$ is virtually $\mathbb{Z} \times \mathbb{Z}$, hence it is not virtually $\mathbb{Z}$.

Chapter 3

Existing results about $\mathrm{Aut}(W_\Gamma)$ and $\mathrm{Out}(W_\Gamma)$

3.1 The splitting of $\mathrm{Aut}(W_\Gamma)$

To motivate our study of the outer automorphism group $\mathrm{Out}(W_\Gamma)$, we will see how the entire automorphism group $\mathrm{Aut}(W_\Gamma)$ splits as a semidirect product.

Definition 3.1.1. $\mathrm{Inn}(W_\Gamma)$ is the subgroup of $\mathrm{Aut}(W_\Gamma)$ generated by inner conjugations by vertices in Γ. An **inner conjugation** by an element $v \in \Gamma$ is the map $\varphi_v : W_\Gamma \to W_\Gamma$ given by $\varphi_v(x) = vxv^{-1}$ for all $x \in W_\Gamma$.

Note that if v and w are adjacent vertices, meaning they commute as group elements, then the inner conjugation of w by v is $vwv^{-1} = wvv^{-1} = w$. Thus φ_v is trivial on $St(v)$.

Definition 3.1.2. Let v be a vertex of Γ and let C be a connected component of the subgraph $\Gamma \backslash St(v)$. The **partial conjugation** of C by v is the group action denoted $\chi_{v,C}$ and is defined:

$$\chi_{v,C}(w) = \begin{cases} vwv^{-1} & \text{if } w \in C \\ w & \text{if } w \notin C \end{cases}$$

$\chi_{v,C}$ is a **non-trivial partial conjugation** if $C \subsetneqq \Gamma \backslash St(v)$, and it is a **trivial partial conjugation** if $C = \Gamma \backslash St(v)$, that is, $\chi_{v,C}$ is an inner conjugation of Γ by v. We denote the set of all partial conjugations of Γ by $\mathcal{P}$.

Let $C_1, ..., C_n$ be all disjoint connected components in $\Gamma \backslash St(v)$. Then the product of partial conjugations $\chi_{v,C_1} \circ \chi_{v,C_2} \circ ... \circ \chi_{v,C_n}$ is an inner conjugation of Γ by v, thus a trivial partial conjugation.

Definition 3.1.3. We now define relevant subgroups of $\mathrm{Aut}(W_\Gamma)$ as in [GPR12].

- The group $\mathrm{Aut}^0(W_\Gamma)$ is those automorphisms of W_Γ which map each vertex $v \in V(\Gamma)$ to a conjugate of itself.
- $\mathrm{Aut}^1(W_\Gamma)$ is those automorphisms of W_Γ which map each maximal complete subgroup of W_Γ to a maximal complete subgroup; in other words, they map maximal cliques of Γ to maximal cliques.
- $\mathrm{Out}(W_\Gamma)$ is the quotient $\mathrm{Aut}(W_\Gamma)/\mathrm{Inn}(W_\Gamma)$.
- $\mathcal{O}ut^0(W_\Gamma)$ is the quotient $\mathrm{Aut}^0(W_\Gamma)/\mathrm{Inn}(W_\Gamma)$.
- $\mathrm{Out}^0(W_\Gamma)$ is the group generated by the set $\mathcal{P}^0 \subseteq \mathcal{P}$ of partial conjugations of Γ.

We will define the set $\mathcal{P}^0$ later on.

Lemma 3.1.1 (Corollary 4.5 [GPR12])**.** *The set $\mathcal{P}$ of partial conjugations of Γ is a minimal generating set for $\mathrm{Aut}^0(W_\Gamma)$.*

Combining results from [GPR12] and Laurence ([Lau95]), we obtain:

Theorem 3.1.1. *If W_Γ is a graph product of primary cyclic groups, then*

$$\mathrm{Aut}(W_\Gamma) = \mathrm{Aut}^0(W_\Gamma) \rtimes \mathrm{Aut}^1(W_\Gamma) = (\mathrm{Inn}(W_\Gamma) \rtimes \mathrm{Out}^0(W_\Gamma)) \rtimes \mathrm{Aut}^1(W_\Gamma)$$

and $\mathrm{Aut}^1(W_\Gamma)$ is a finite group.

Remark 3.1.1. What can we say about $\mathrm{Inn}(W_\Gamma)$ in this splitting? As remarked by the authors of [CRSV09], $\mathrm{Inn}(W_\Gamma)$ turns out to share roughly the structure of W_Γ. Recall that only vertices in $\Gamma \backslash Z(\Gamma)$ act in a non-trivial inner automorphism of W_Γ. Let Ω be the subgraph induced by $\Gamma \backslash Z(\Gamma)$, then $W_\Omega \cong \mathrm{Inn}(W_\Gamma)$ and $W_\Gamma = W_\Omega \times W_{Z(\Gamma)}$. $W_{Z(\Gamma)}$ is a direct product of finitely many finite groups, so W_Ω is a finite index subgroup of W_Γ. Therefore $\mathrm{Inn}(W_\Gamma)$ is isomorphic to a finite index subgroup of W_Γ and $\mathrm{Aut}(W_\Gamma)$ is, up to finite index, a group of the form $W_\Gamma \rtimes \mathrm{Out}^0(W_\Gamma)$.

To understand $\mathcal{O}ut^0(W_\Gamma)$ as a subgroup of $\mathrm{Out}(W_\Gamma)$, we have the following lemma.

Lemma 3.1.2. *Let Γ be a graph product of directly indecomposable groups. Then $\mathcal{O}ut^0(W_\Gamma)$ is isomorphic to a normal subgroup of* $\mathrm{Out}(W_\Gamma)$ *and* $\mathrm{Out}(W_\Gamma)/\ \mathcal{O}ut^0(W_\Gamma) \cong \mathrm{Aut}^1(W_\Gamma)$.

Proof. By definition, $\mathrm{Out}(W_\Gamma) = \mathrm{Aut}(W_\Gamma)/\mathrm{Inn}(W_\Gamma)$ and $\mathcal{O}ut^0(W_\Gamma) = \mathrm{Aut}^0(W_\Gamma)/\mathrm{Inn}(W_\Gamma)$. We know that $\mathrm{Inn}(W_\Gamma)$ is a normal subgroup of $\mathrm{Aut}^0(W_\Gamma)$ and $\mathrm{Aut}^0(W_\Gamma)$ is a normal subgroup of $\mathrm{Aut}(W_\Gamma)$. By the third isomorphism theorem, the quotient group $\mathrm{Aut}(W_\Gamma)/\mathrm{Inn}(W_\Gamma) = \mathrm{Out}(W_\Gamma)$ has a normal subgroup isomorphic to the quotient group $\mathrm{Aut}^0(W_\Gamma)/\mathrm{Inn}(W_\Gamma) = \mathcal{O}ut^0(W_\Gamma)$. The third isomorphism theorem also implies that

$$(\mathrm{Aut}(W_\Gamma)/\mathrm{Inn}(W_\Gamma))/(\mathrm{Aut}^0(W_\Gamma)/\mathrm{Inn}(W_\Gamma)) \cong \mathrm{Aut}(W_\Gamma)/\mathrm{Aut}^0(W_\Gamma).$$

$$\Rightarrow \mathrm{Out}(W_\Gamma)/\mathcal{O}ut^0(W_\Gamma) \cong \mathrm{Aut}^1(W_\Gamma).$$

□

Only when W_Γ is the graph product of primary cyclic groups do we get that $\mathrm{Aut}^0(W_\Gamma)$ is a finite index subgroup of $\mathrm{Aut}(W_\Gamma)$, which reduces all automorphisms of W_Γ to conjugations of Γ, up to finite index. We emphasize that this is not the case for graph products of directly indecomposable cyclic groups in general. Moreover, the last statement of Theorem 3.2.1 (below) fails if W_Γ is not a graph product of primary cyclic groups, which is why our results hold only for graph products of primary cyclic groups.

By Lemma 3.1.2, if W_Γ is a graph product of primary cyclic groups, then $\mathcal{O}ut^0(W_\Gamma)$ is a finite index subgroup of $\mathrm{Out}(W_\Gamma)$; if $\mathcal{O}ut^0(W_\Gamma)$ is virtually $\mathbb{Z}$, then $\mathrm{Out}(W_\Gamma)$ is virtually $\mathbb{Z}$. And if $\mathrm{Out}(W_\Gamma)$ is virtually $\mathbb{Z}$, then each of its infinite subgroups, including $\mathrm{Out}^0(W_\Gamma)$, must be virtually $\mathbb{Z}$. Similarly, $\mathrm{Out}(W_\Gamma)$ is virtually abelian (or large) if and only if $\mathrm{Out}^0(W_\Gamma)$ is virtually abelian (respectively, large). For this reason, some statements in this paper may discuss $\mathrm{Out}(W_\Gamma)$ being virtually $\mathbb{Z}$, and others use that $\mathcal{O}ut^0(W_\Gamma)$ is virtually $\mathbb{Z}$, but all results

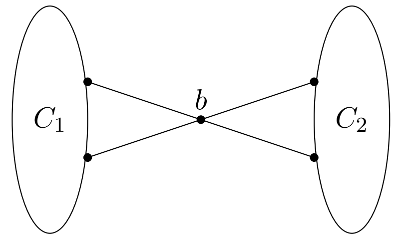


Figure 3.1: Star cut point

tell us about the structure of $\mathrm{Out}(W_\Gamma)$. This fact and the usefulness of the set $\mathcal{P}^0$ motivates our study of the group $\mathcal{O}ut^0(W_\Gamma)$ in particular.

Definition 3.1.4. A vertex $b \in V(\Gamma)$ is a **star cut point** if $\Gamma \backslash St(b)$ contains at least two distinct connected components. (See Figure 3.1).

Note that if Γ contains the SIL $\{v_1, v_2 | C\}$, then both v_1 and v_2 are star star cut points. The set of star cut points is exactly the set of vertices which act in a non-trivial partial conjugation.

Definition 3.1.5. [GPR12] defines the set $\mathcal{P}^0$ for W_Γ as follows. Number all the vertices of Γ in any manner. Note that each star cut point in Γ is the acting vertex of at least two non-trivial partial conjugations of W_Γ. Take a star cut point $v \in V(\Gamma)$ and index the n connected components of $\Gamma \backslash St(v)$ such that the component with the smallest numbered vertex is indexed C_1, the remaining component with the next smallest vertex is C_2, and so on (see an example in Figure 3.2). Then we have the partial conjugations $\chi_{v,C_1}, \chi_{v,C_2}, ..., \chi_{v,C_n}$ in $\mathrm{Out}^0(W_\Gamma)$. Now put all χ_{v,C_i} with $i \geq 2$ in the set $\mathcal{P}^0$. We get $n-1$ partial conjugations with acting vertex v in $\mathcal{P}^0$. Repeat for each star cut point in $V(\Gamma)$. Recall that we define $\mathrm{Out}^0(W_\Gamma)$ as the set generated by $\mathcal{P}^0$. Intuitively, this construction stops short of generating an inner conjugation by any star cut point. If we included all the partial conjugations $\chi_{v,C_1}, ..., \chi_{v,C_n}$ in $\mathcal{P}^0$, then their product would be an inner conjugation, which is trivial as a partial conjugation.

Theorem 1.7 of [GPR12] shows that the isomorphism $\mathrm{Out}^0(W_\Gamma) \cong \mathcal{O}ut^0(W_\Gamma)$ holds regardless of how the vertices are numbered. For simplicity, we will treat them as the same group $\mathrm{Out}^0(W_\Gamma)$, so that we can consider the group generated by $\mathcal{P}^0$ as a finite index subgroup of $\mathrm{Out}(W_\Gamma)$, but we emphasize that they are different groups. Thus if we can find an appropriate numbering of the vertices of Γ, we can make certain partial conjugations be in $\mathcal{P}^0$ and prove structural properties of $\mathrm{Out}^0(W_\Gamma)$ and hence of $\mathrm{Out}(W_\Gamma)$.

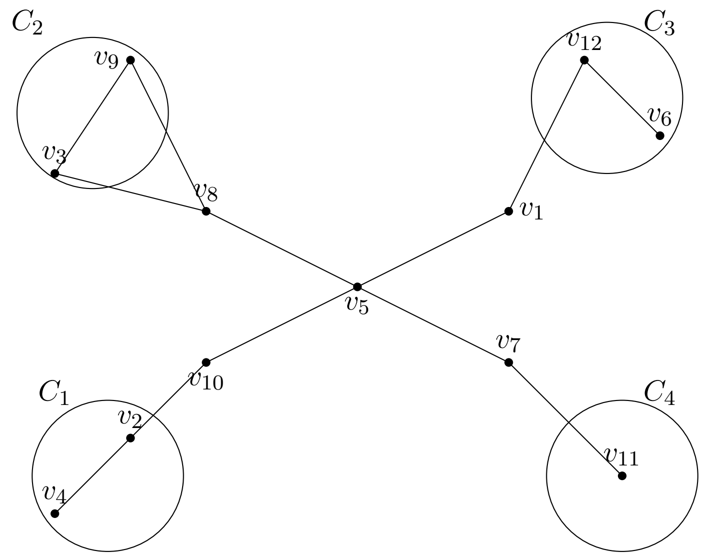


Figure 3.2: An example of the construction of $\mathcal{P}^0$. We consider some indexing of $V(\Gamma)$ and a star cut point $v_5 \in V(\Gamma)$.

3.2 Determining when $\mathrm{Out}(W_\Gamma)$ is finite or large

First, we concretely connect the presence of a SIL with a pair of partial conjugations of Γ. The statement and proof of this lemma are by Hikaru Jitsukawa.

Lemma 3.2.1. *If Γ contains a SIL $\{v_1, v_2|z\}$, then there is a connected component $C \subseteq \Gamma$ such that z is in C, v_1 and v_2 are not in C, and both $\chi_{v_1,C}$ and $\chi_{v_2,C}$ are partial conjugations in $\mathcal{P}$.*

Proof. Suppose there is a vertex $z \in V(\Gamma)$, a connected component $C_1 \subseteq \Gamma \backslash Lk(v_1)$, and a connected component $C_2 \subseteq \Gamma \backslash Lk(v_2)$ such that $z \in C_1 \cap C_2$. For the sake of contradiction, assume $C_1 \neq C_2$, and without loss of generality, assume there is $y \in C_1 \backslash C_2$. Since $y \in C_1$, there is a path γ in Γ between y and z such that no point passes through $St(v_1)$. But since $y \notin C_2$, γ must intersect $St(v_2)$ at some point a. Then there is a path γ' between a and z which does not intersect $St(v_1)$. But this means that v_2 and z are in the same path component of $\Gamma \backslash (Lk(v_1) \cap Lk(v_2))$, contradicting our assumption of a SIL. Therefore we must have that $C_1 = C_2$, and $\chi_{v_1,C}$ and $\chi_{v_2,C}$ are well-defined. (See Figure 3.3).

□

This confirms that each SIL in Γ corresponds to a pair of non-trivial partial conjugations in $\mathcal{P}$. A useful result from ([GPR12]) connects the presence of a SIL with the cardinality of

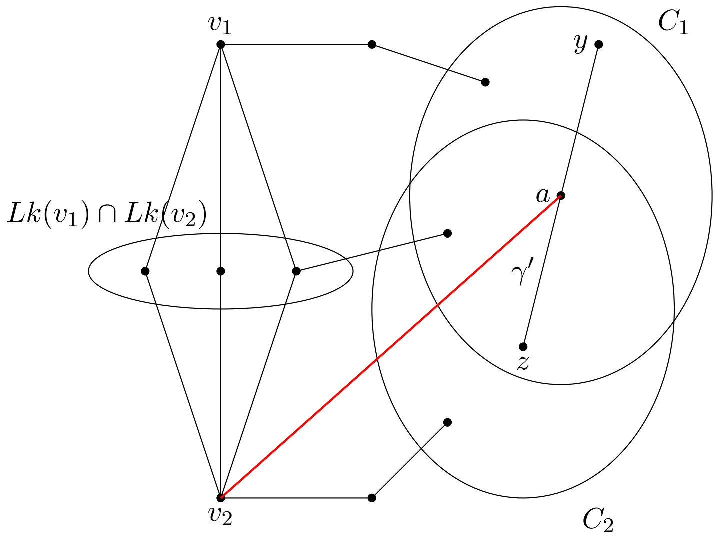


Figure 3.3: Picture for Lemma 3.2.1. We prove that the red edge connecting v_2 and a must exist.

$\mathrm{Out}(W_\Gamma)$.

Theorem 3.2.1 (Corollary 1.11 [GPR12])**.** *If W_Γ is a graph product of primary cyclic groups, then the following are equivalent:*

1. $\mathrm{Out}^0(W_\Gamma)$ *is an abelian group.*

2. Γ *does not contain a SIL.*

3. $\mathrm{Out}(W_\Gamma)$ *is finite.*

Therefore, in our search for graphs Γ with $\mathrm{Out}(W_\Gamma)$ virtually $\mathbb{Z}$, we know that we need Γ to contain at least one SIL. A logical next question is: how many Coxeter SILs can Γ contain before $\mathrm{Out}^0(W_\Gamma)$ is virtually abelian and not virtually $\mathbb{Z}$? We will address the number of SILs more in Chapter 4. Now, there are two related graph properties that Susse and Sale ([SS17]) have proven to make $\mathrm{Out}(W_\Gamma)$ large.

Definition 3.2.1. Three vertices $v_1, v_2, v_3 \in V(\Gamma)$ form a **Separating Triple Intersection of Links (STIL)** if

- The subgraph spanned by $\{v_1, v_2, v_3\}$ contains at most one edge; and
- There is a connected component $C \subseteq \Gamma \backslash (L_{v_1} \cap L_{v_2} \cap L_{v_3})$ such that v_1, v_2, and v_3 are not in C.

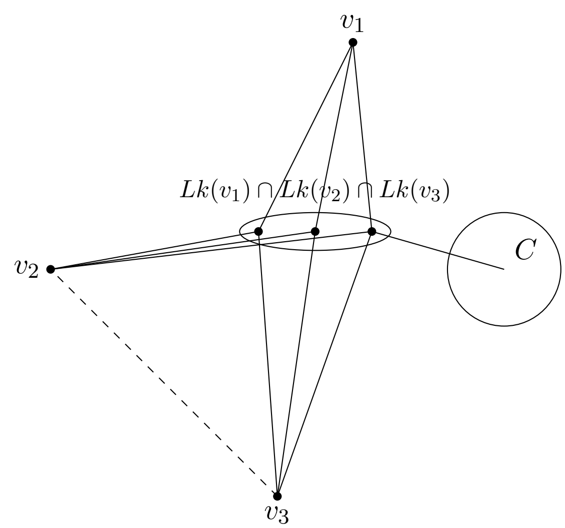


Figure 3.4: STIL (Separating Triple Intersection of Links)

See Figure 3.4.

Definition 3.2.2. Three vertices $v_1, v_2, v_3 \in V(\Gamma)$ form a **Flexible Separating Intersection of Links (FSIL)** if $\{v_i, v_j \mid v_k\}$ forms a SIL for $\{i, j, k\} = \{1, 2, 3\}$.

Theorem 3.2.2 (Theorem 2 [SS17])**.** *Let W_Γ be a graph product of primary cyclic groups. Then the following are equivalent:*

1. $\mathrm{Out}(W_\Gamma)$ *is large;*

2. $\mathrm{Out}(W_\Gamma)$ *is not virtually abelian;*

3. Γ *contains a STIL, an FSIL, or a non-Coxeter SIL.*

In particular, $\mathrm{Out}(W_\Gamma)$ *obeys a strict dichotomy: it is either virtually abelian or large.*

Chapter 4

Connected graphs with two distinct Coxeter SILs

In this chapter, we look at the case where Γ is a connected graph containing two distinct Coxeter SILs. As we will see, the partial conjugations resulting from these SILs generate an infinite subgroup of $\mathrm{Out}^0(W_\Gamma)$ which is not virtually $\mathbb{Z}$. We will need to check many cases to prove that $\mathcal{P}^0$ always generates this subgroup. A connected graph Γ can have two SILs $\{v_1, v_2|C\}$ and $\{w_1, w_2|D\}$ with overlapping acting vertices (say, $v_1 = w_1$ and/or $v_2 = w_2$) and still have $\mathrm{Out}(W_\Gamma)$ be virtually abelian; this is not true in a disconnected graph. For this reason, it is practical to split some of our proofs into connected and disconnected. The layout of this chapter, the statements of Lemma 4.1.1, Lemma 4.1.2, Lemma 4.1.5, and Theorem 4.2.1, as well as the outline of the proof of Theorem 4.2.1, are written by Hikaru Jitsukawa at Tufts. I edited these components and added details where necessary.

4.1 Considering the partial conjugations induced by two SILs

Proposition 4.1.1. *Let W_Γ be a graph product of primary cyclic groups, where Γ is a connected graph containing two distinct Coxeter SILs $\{v, v'|C_v\}$ and $\{w, w'|C_w\}$. Assume also that $\mathrm{Out}(W_\Gamma)$ is virtually abelian. Denote the corresponding partial conjugations $\chi_{v,C_v}, \chi_{v',C_v}, \chi_{w,C_w}$, and χ_{w',C_w}, as in Lemma 3.2.1. Then all four of these partial conjugations are in $\mathrm{Out}^0(W_\Gamma)$.*

We have two proofs for this proposition. In the first, through brute force, we will show that there exists an ordering of the vertices in Γ so these four partial conjugations lie in the generating set $\mathcal{P}^0$. We will prove this statement in three cases, depending on the number of distinct acting vertices in the two SILs. After proving Lemma 4.1.1, Lemma 4.1.2, and Lemma 4.1.5, we can complete the first proof of this proposition.

The second proof of Proposition 4.1.1 is much more succinct and assumes $\mathcal{P}^0$ has already been constructed. We then find the four desired partial conjugations in $\mathrm{Out}^0(W_\Gamma)$.

Lemma 4.1.1. *Let W_Γ be a graph product of primary cyclic groups, where Γ is a connected graph containing two distinct Coxeter SILs $\{v, v' \mid C_v\}$ and $\{w, w' \mid C_w\}$. Assume that $\{v, v'\} \cap \{w, w'\} = \emptyset$. Denote their corresponding partial conjugations as $\chi_{v,C_v}, \chi_{v',C_v}, \chi_{w,C_w}$, and χ_{w',C_w}, as in Lemma 3.2.1. Then there exists an ordering of the vertices in Γ so that all four of these partial conjugations lie in the generating set $\mathcal{P}^0$.*

Proof. Note that Lemma 3.2.1 proves that v and v' both act on the same set C_v by partial conjugation, as do w and w' on C_w. We will prove this lemma by strategically ordering the vertices v, v', w, and w' in order to throw out unwanted partial conjugations from $\mathcal{P}$, one for each acting vertex v, v', w, and w'. Thus, we keep the desired partial conjugations $\chi_{v,C_v}, \chi_{v',C_v}, \chi_{w,C_w}$, and χ_{w',C_w} in $\mathcal{P}^0$.

The easiest case is when neither v nor v' is in $St(w)$ or $St(w')$, neither w nor w' is in $St(v)$ or $St(v')$, neither v nor v' is in C_w, and neither w nor w' is in C_v. In other words, the two SILs "do not interact" with each other within Γ, although Γ is connected (see Figure 4.1).

In this case, since each of v, v', w, and w' acts on the other three vertices by partial conjugation and since neither v nor v' is in C_w and neither w nor w' is in C_v, we can assign the numbering of the vertices: v gets 1 and v' gets 2. Then for v, we throw out a partial conjugation acting on a connected component containing v'. Technically, we could denote this partial conjugation as $\chi_{v,\{\ldots,v',\ldots\}}$, because the action of conjugation is on an entire connected component, not just the one vertex v'. For simplicity we will denote it simply as $\chi_{v,v'}$. For the other vertices, we throw out $\chi_{v',v}, \chi_{w,v}$, and $\chi_{w',v}$. Then we keep the four desired partial conjuagtions $\chi_{v,C_v}, \chi_{v',C_v}, \chi_{w,C_w}$, and χ_{w',C_w} in $\mathcal{P}^0$, and we are done.

Now, two ways in which Γ could not fit this ideal case are:

1. Either v or v' does not act on both w and w' by partial conjugation (or vice versa). For

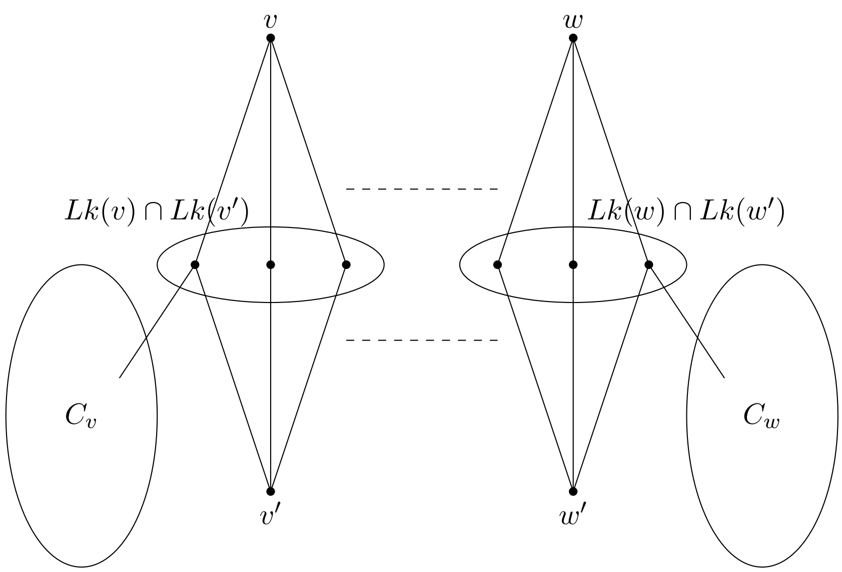


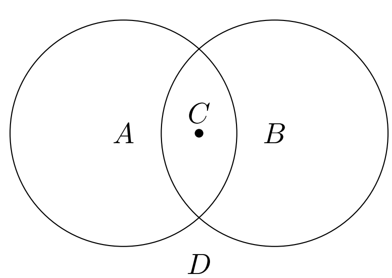
Figure 4.1: The ideal case for Lemma 4.1.1. The graph Γ is connected, but there are no edges connecting the vertices $\{v, v\}$ with $\{w, w'\}$. Neither v nor v' is in C_w and neither w nor w' is in C_v.

example, if v is in $St(w)$, then v does not act on w by partial conjugation. Then we may have a vertex for which our process does not throw out any unwanted partial conjugation.

2. Either v or v' is in C_w (or, either w or w' is in C_v): then a numbering of the vertices may throw out either χ_{w,C_w} or χ_{w',C_w}.

That is, when the SILs do "interact" with each other, we have to be more careful in our numbering. We can further break both of these cases down:

1. Is either v or v' in $St(w)$ or $St(w')$? Note that we can switch the pair v, v' for w, w' in the analogous cases. To make sure we cover all such cases, consider this diagram. We have $A = St(w),\ B = St(w'),\ C = St(w) \cap St(w'),\ D = \Gamma \backslash (St(w) \cup St(w'))$.



(a) v and v' are in D (the "nice" case)

(b) v is in A and v' is in D

(c) v is in C and v' is in D

(d) v is in A and v' is in A

(e) v is in A and v' is in B

(f) v is in C and v' is in A

(g) v and v' are in C

We can check that these are in fact all the cases, up to symmetry of v and v'.

- v' in D
  - v in A: case (b)
  - v in B: case (b)
  - v in C: case (c)
  - v in D: case (a)
- v' in A
  - v in A: case (d)
  - v in B: case (e)
  - v in C: case (f)
  - v in D: case (b)
- v' in B
  - v in A: case (e)
  - v in B: case (d)
  - v in C: case (c)
  - v in D: case (b)
- v' in C
  - v in A: case (f)
  - v in B: case (f)
  - v in C: case (g)
  - v in D: case (c)

Therefore, cases a) through g) are indeed all the cases we need, up to symmetry.

2. Is either v or v' in C_w, or is either w or w' in C_v? The possible cases are listed below; for each case, assume the given inclusion of v, v', w, or w' in C_v or C_w is the only inclusion.

   (i) Neither v nor v' is in C_w and neither w nor w' is in C_v (the "nice" case)

   (ii) One of v and v' is in C_w and neither w nor w' is in C_v

   (iii) Both v and v' are in C_w and neither w nor w' is in C_v

   (iv) Neither v nor v' is in C_w and one of w and w' is in C_v

   (v) Neither v nor v' is in C_w and both w and w' are in C_v

   (vi) One of v and v' is in C_w and one of w and w' is in C_v

   (vii) Both v and v' are in C_w and one of w and w' is in C_v

(viii) One of v and v' is in C_w and both w and w' are in C_v

Note: The case where both v and v' are in C_w and both w and w' are in C_v is not possible. If this case occurred in Γ, then in $\Gamma\backslash(Lk(v) \cap Lk(v'))$ there would exist a path $w \to L(w) \cap L(w') \to C_w \to v$, so $\{w, w'|C_w\}$ does not form a SIL.

We consider the combinations of the cases from lists 1 and 2 in Γ. In each combination of cases, we find an ordering of the vertices that ensures one partial conjugation gets thrown out for each of $\{v, v', w, w'\}$ that is not in $\{\chi_{v,C_v}, \chi_{v',C_v}, \chi_{w,C_w}, \chi_{w',C_w}\}$. In some of the cases, there are multiple orderings of the vertices to achieve this goal, and we only need to name one to prove existence. Additionally, the cases below are listed up to symmetry between v and v' and between w and w'.

(a) v and v' are not in $St(w)$ or $St(w')$.

Each of v, v', w, and w' acts on the other three by partial conjugation.

(i) Neither v nor v' is in C_w and neither w nor w' is in C_v.

This is the simplest case, already addressed above.

(ii) v is in C_w and neither w nor w' is in C_v.

v' gets 1 and v gets 2. Then we remove the partial conjugations $\chi_{v,v'}, \chi_{v',v}, \chi_{w,v'},$, and $\chi_{w',v'}$. Again, this is a slight abuse of notation, where $\chi_{v,v'}$ is the partial conjugation by v of the entire connected component containing v'.

(iii) Both v and v' are in C_w and neither w nor w' is in C_v.

w gets 1 and w' gets 2. Then we remove the partial conjugations $\chi_{v,w}, \chi_{v',w}, \chi_{w,w'}$, and $\chi_{w',w}$.

(iv) Neither v nor v' is in C_w and w is in C_v.

w' gets 1 and w gets 2. Then we remove the partial conjugations $\chi_{v,w'}, \chi_{v',w'}, \chi_{w,w'}$, and $\chi_{w',w}$.

(v) Neither v nor v' is in C_w and both w and w' are in C_v.

v gets 1 and v' gets 2. Then we remove the partial conjugations $\chi_{v,v'}, \chi_{v',v}, \chi_{w,v}$, and $\chi_{w',v}$.

(vi) v is in C_w and w is in C_v.

v' gets 1 and w' gets 2. Then we remove the partial conjugations $\chi_{v,v'}, \chi_{v',w'}, \chi_{w,v'}$, and $\chi_{w',v'}$.

(vii) Both v and v' are in C_w and w is in C_v.

w' gets 1 and w gets 2. Then we remove the partial conjugations $\chi_{v,w'}, \chi_{v',w'}, \chi_{w,w'}$, and $\chi_{w',w}$.

(viii) v is in C_w and both w and w' are in C_v.

v' gets 1 and v gets 2. Then we remove the partial conjugations $\chi_{v,v'}, \chi_{v',v}, \chi_{w,v'}$, and $\chi_{w',v'}$.

(b) v is in $St(w) \backslash St(w')$ and v' is in neither $St(w)$ nor $St(w')$.

v acts on v' and w' by partial conjugation; v' acts on v, w, and w'; w acts on v' and w'; w' acts on v, v', and w.

(i) Neither v nor v' is in C_w and neither w nor w' is in C_v.

v gets 1 and v' gets 2. Then we remove the partial conjugations $\chi_{v,v'}, \chi_{v',v}, \chi_{w,v'}$, and $\chi_{w',v}$.

(ii) v' is in C_w and neither w nor w' is in C_v (we cannot have v in C_w because v is in $St(w)$).

v gets 1 and w' gets 2. Then we remove the partial conjugations $\chi_{v,w'}, \chi_{v',v}, \chi_{w,w'}$, and $\chi_{w',v}$.

(iii) Both v and v' are in C_w and neither w nor w' is in C_v.

w gets 1 and w' gets 2. Then we remove the partial conjugations $\chi_{v,w'}, \chi_{v',w}, \chi_{w,w'}$, and $\chi_{w',w}$.

(iv) Neither v nor v' is in C_w and w' is in C_v (we cannot have w in C_v because w is in $St(v)$).

v gets 1 and v' gets 2. Then we remove the partial conjugations $\chi_{v,v'}, \chi_{v',v}, \chi_{w,v'}$, and $\chi_{w',v}$.

(v) Neither v nor v' is in C_w and both w and w' are in C_v: This case cannot happen because w is in $St(w)$.

(vi) v is in C_w and w is in C_v: If this case occurred in Γ, then we would have $w \in St(v)\backslash St(v')$, so $w, w' \in \Gamma\backslash(Lk(v) \cap Lk(v'))$. w is connected to w' in $\Gamma\backslash(Lk(v) \cap Lk(v'))$, so we have a path $v \to w \to w'$ in $\Gamma\backslash(Lk(v) \cap Lk(v')$, so $\{v, v'|C_v\}$ does not form a SIL. Thus, this case cannot happen.

(vii) Both v and v' are in C_w and w is in C_v: For the same reason as case (vi), this case cannot happen.

(viii) v is in C_w and both w and w' are in C_v: This case cannot happen because v is in $St(w)$.

(c) v is in both $St(w)$ and $St(w')$ and v' is in neither $St(w)$ nor $St(w')$.

v acts on v' by partial conjugation; v' acts on v, w, and w'; w acts on v' and w'; w' acts on v' and w.

(i) Neither v nor v' is in C_w and neither w nor w' is in C_v.

v gets 1 and v' gets 2. Then we remove the partial conjugations $\chi_{v,v'}, \chi_{v',v}, \chi_{w,v'}$, and $\chi_{w',v'}$.

(ii) v' is in C_w and neither w nor w' is in C_v (we cannot have v in C_w).

w gets 1, w' gets 2, and v' gets 3. Then we remove the partial conjugations $\chi_{v,v'}, \chi_{v',w}, \chi_{w,w'}$, and $\theta_{w',w}$.

(iii) Both v and v' are in C_w and neither w nor w' is in C_v: This case cannot happen because v is in $St(w) \cap St(w')$.

(iv) Neither v nor v' is in C_w and one of w and w' is in C_v: This case cannot happen because w and w' are both in $St(v)$. For the same reason, case c) cannot happen at the same time as cases v)-viii).

(d) Both v and v' are in $St(w)\backslash St(w')$.

v acts on v' and w' by partial conjugation; v' acts on v and w'; w acts on w'; w' acts on v, v', and w.

(i) Neither v nor v' is in C_w and neither w nor w' is in C_v.

w' gets 1 and v gets 2. Then we remove the partial conjugations $\chi_{v,w'}, \chi_{v',w'}, \chi_{w,w'}$, and $\chi_{w',v}$.

(ii) One of v and v' is in C_w and neither w nor w' is in C_v: This case cannot happen because both v and v' are in $St(w)$. Similarly, neither w nor w' can be in C_v, so case d) cannot happen at the same time as cases iii) - viii).

(e) v is in $St(w) \backslash St(w')$ and v' is in $St(w') \backslash St(w)$.

v acts on v' and w' by partial conjugation; v' acts on v and w; w acts on v' and w'; w' acts on v and w.

(i) Neither v nor v' is in C_w and neither w nor w' is in C_v.

v gets 1, v' gets 2. Then we remove the partial conjugations $\chi_{v,v'}, \chi_{v',v}, \chi_{w,v'}$, and $\chi_{w',v}$.

(ii) One of v and v' is in C_w and neither w nor w' is in C_v: This case cannot happen because v is in $St(w)$ and v' is in $St(w')$. For the same reason, case e) cannot happen at the same time as cases iii)- viii).

(f) v is in both $St(w)$ and $St(w')$ and v' is in $St(w) \backslash St(w')$.

v acts on v' by partial conjugation; v' acts on v and w'; w acts on w'; w' acts on v' and w.

(i) Neither v nor v' is in C_w and neither w nor w' is in C_v.

v' gets 1 and w' gets 2. Then we remove the partial conjugations $\chi_{v,v'}, \chi_{v',w'}, \chi_{w,w'}$ and $\chi_{w',v'}$.

(ii) One of v and v' is in C_w and neither w nor w' is in C_v: This case cannot happen because both v and v' are in $St(w)$. For the same reason, case f) cannot happen at the same time as cases iii)- viii).

(g) Both v and v' are in both $St(w) \cap St(w')$.

v acts on v' by partial conjugation; v' acts on v; w acts on w'; w' acts on w.

(i) Neither v nor v' is in C_w and neither w nor w' is in C_v.

v gets 1, v' gets 2, w gets 3, and w' gets 4. Then we remove the partial conjugations $\chi_{v,v'}, \chi_{v',v}, \chi_{w,w'}$, and $\chi_{w',w}$.

(ii) One of v and v' is in C_w and neither w nor w' is in C_v: This case cannot happen because both v and v' are in $St(w)$. For the same reason, case g) cannot happen at the same time as cases iii)- viii).

This covers all cases of 2 distinct SILs with completely distinct vertices. In all possible cases, we can order the vertices of Γ in such a way that we keep $\chi_{v,C_v}, \chi_{v',C_v}, \chi_{w,C_w}$, and χ_{w',C_w} in $\mathcal{P}^0$.

□

Lemma 4.1.2. *Let W_Γ be a graph product of primary cyclic groups, where Γ is a connected graph containing two distinct Coxeter SILs $\{v, v'|C\}$ and $\{v, v'|D\}$. Denote their corresponding partial conjugations as $\chi_{v,C}, \chi_{v',C}, \chi_{v,D}$, and $\chi_{v',D}$, as in Lemma 3.2.1. Then there exists an ordering of the vertices in Γ so that all four of these partial conjugations lie in the generating set $\mathcal{P}^0$.*

Proof. We will prove this lemma by strategically ordering the vertices v and v' in order to throw out unwanted partial conjugations from $\mathcal{P}$, one each for v and v', as in the proof of Lemma 4.1.1.

Neither v nor v' can be in C or D, nor in the star of the other vertex. Thus, there exists a connected component containing v' in $\Gamma \setminus St(v)$ which v acts on in some partial conjugation in $\mathrm{Out}^0(W_\Gamma)$. In a slight abuse of notation, we denote this $\chi_{v,v'}$. Give v' a 1 in the ordering of the vertices, then we ensure that $\chi_{v,v'}$ is not in $\mathcal{P}^0$, thus $\chi_{v,C}$ and $\chi_{v,D}$ are in $\mathcal{P}^0$.

Similarly, there exists a connected component containing v in $\Gamma \setminus St(v')$ such that $\chi_{v',v}$ is a partial conjugation in $\mathrm{Out}^0(W_\Gamma)$. Give v a 2 in the ordering of the vertices, then we ensure that $\chi_{v',v}$ is not in $\mathcal{P}^0$, thus $\chi_{v',C}$ and $\chi_{v',D}$ are in $\mathcal{P}^0$. □

Before proving the third case for the proposition, we need to state two auxiliary lemmas from [SS17].

Lemma 4.1.3 (Lemma 1.7 [SS17])**.** *Suppose that Γ is connected and contains the SILs $\{x_1, x_2|Z\}$ and $\{x_1, x_3|Z'\}$. If $z \in Z \cap Z'$, then $\{x_1, x_2, x_3|z\}$ is a STIL.*

Lemma 4.1.4 (Lemma 1.8 [SS17])**.** *Let $x_1, x_2, x_3 \in V(\Gamma)$ be such that there exist vertices y and z with $\{x_1, x_2 | y\}$ and $\{x_1, x_3 | z\}$ both Coxeter SILs. Then one of the following occurs:*

1. *$\{x_1, x_2, x_3\}$ is a Coxeter FSIL;*

2. *There exists $w \in \Gamma$ so that $\{x_1, x_2, x_3 | w\}$ is a Coxeter STIL; or*

3. *x_1, x_2, y are all in the same component of $(\Gamma \backslash St(x_3)) \cup \{x_2\}$, and x_1, x_3, z are all in the same component of $(\Gamma \backslash St(x_2)) \cup \{x_3\}$.*

Lemma 4.1.5. *Let W_Γ be a graph product of primary cyclic groups, where Γ is a connected graph containing two distinct Coxeter SILs $\{v, v' | C\}$ and $\{v, w' | D\}$. Assume that $v' \neq w'$ and that $\mathrm{Out}(W_\Gamma)$ is virtually abelian. Denote their corresponding partial conjugations as $\chi_{v,C}, \chi_{v',C}, \chi_{v,D}$, and $\chi_{w',D}$, as in Lemma 3.2.1. Then there exists an ordering of the vertices in Γ so that all four of these partial conjugations lie in the generating set $\mathcal{P}^0$.*

Proof. If $C \cap D \neq \emptyset$, then $C = D$. By Lemma 4.1.3, $\{v, v', w' | C\}$ forms a STIL, and by Theorem 3.2.2, $\mathrm{Out}(W_\Gamma)$ is not virtually abelian. So we assume that $C \cap D = \emptyset$. Then by Lemma 4.1.4, one of the following holds:

1. $\{v, v', w\}$ forms an FSIL;

2. There is a connected component $Z \subseteq \Gamma$ such that $\{v, v', w | Z\}$ is a STIL; or

3. v, v', and C are all in the same component of $(\Gamma \backslash St(w)) \cup \{v'\}$ and v, w, and D are all in the same component of $(\Gamma \backslash St(v')) \cup \{w\}$.

If the result is either (1) or (2), then Theorem 3.2.2 proves that $\mathrm{Out}^0(W_\Gamma)$ is not virtually abelian, so we may disregard these cases.

If the result is (3), then we need an ordering of the vertices that maintains our desired partial conjugations in $\mathcal{P}^0$. Since v and C are in $\Gamma \backslash St(w)$, w acts on a component containing v and C by partial conjugation. This component cannot contain D because w and v act on the same component containing D by partial conjugation, so w cannot act on a different component containing D (i.e. including v) than v does.

Since v and D are in $\Gamma \backslash St(v')$, then v' acts on a component containing v and D by partial conjugation. Again, since $\{v, v' | C\}$ is a SIL, this component cannot contain C.

Finally, since v is in $\Gamma\backslash St(v')$, then v' is in $\Gamma\backslash St(v)$, so v also acts on a component containing v' but neither C nor D.

In an ordering of the vertices of Γ, give v a 1 and v' a 2. Then we throw out the partial conjugations $\chi_{v,v'}, \chi_{v',v}$, and $\chi_{w,v}$. In the proof of Lemma 4.1.4 in [SS17], Susse and Sale show that case (3) occurs when neither $\{v, v'|w\}$ nor $\{v, w|v'\}$ are SILs, so in this case we do not risk that v' is in D. Thus, this ordering ensures that the desired partial conjugations $\chi_{v,C}, \chi_{v',C}, \chi_{v,D}$, and $\chi_{w,D}$ are in $\mathcal{P}_0$. □

Proof 1 of Proposition 4.1.1. By Lemma 4.1.1, Lemma 4.1.2, and Lemma 4.1.5, there exists an ordering of the vertices in Γ so that $\chi_{v,C_v}, \chi_{v',C_v}, \chi_{w,C_w}$, and χ_{w',C_w} lie in $\mathcal{P}^0$. We assume this numbering on Γ. Since $\mathcal{P}^0$ generates $\mathrm{Out}^0(W_\Gamma)$, these four partial conjugations are naturally included in $\mathrm{Out}^0(W_\Gamma)$. □

Proof 2 of Proposition 4.1.1. Assume $\mathcal{P}^0$ has been constructed with an arbitrary ordering of the vertices of Γ. By definition, $\mathrm{Out}^0(W_\Gamma) = \langle \mathcal{P}^0 \rangle$. Suppose that χ_{v,C_v} is not in $\mathcal{P}^0$. Since v is a star cut point, there is at least one partial conjugation in $\mathcal{P}^0$ with acting vertex v. In fact, every other partial conjugation with acting vertex v lies in $\mathcal{P}^0$, by the construction of $\mathcal{P}^0$; denote these $\chi_{v,C_2}, ..., \chi_{v,C_n}$. Inner conjugations are trivial in $\mathrm{Out}(W_\Gamma)$, so the composition of automorphisms $\chi_{v,C_v} \circ \chi_{v,C_2} \circ ... \circ \chi_{v,C_n} = 1$ in $\mathrm{Out}(W_\Gamma)$. Then $\chi_{v,C_v} = (\chi_{v,C_2} \circ ... \circ \chi_{v,C_n})^{-1}$ in $\mathrm{Out}(W_\Gamma)$. $(\chi_{v,C_2} \circ ... \circ \chi_{v,C_n})^{-1}$ is in $\mathrm{Out}^0(W_\Gamma)$, therefore χ_{v,C_v} is in $\mathrm{Out}^0(W_\Gamma)$.

The same argument shows that $\chi_{v',C_v}, \chi_{w,C_w}$, and χ_{w',C_w} are in $\mathrm{Out}^0(W_\Gamma)$. □

4.2 The impact of two SILs on $\mathrm{Out}^0(W_\Gamma)$

We will see that these four partial conjugations make $\mathrm{Out}^0(W_\Gamma)$ "too big" to be virtually $\mathbb{Z}$. First, we state a few more definitions and auxiliary lemmas.

Definition 4.2.1. Let S be a set. A **word** in S is a product $x_{i_1}^{a_1} x_{i_2}^{a_2} ... x_{i_n}^{a_n}$ for some elements $x_{i_1}, ..., x_{i_n}$ of S, where $x_{i_j} \neq x_{i_{j+1}}$, and some non-zero integers $a_1, ..., a_n$. We call S an **alphabet**.

Lemma 4.2.1 (Lemma 2.4 [GPR12])**.** *Let Ω be a subgraph of Γ. The natural map $W_\Omega \to W_\Gamma$ is an embedding.*

Thus, taking any subgraph $\Omega \subseteq \Gamma$, we get a subgroup W_Ω of W_Γ, which is itself a graph product of primary cyclic groups.

Lemma 4.2.2 (Corollary 1.6 [GPR12])**.** *Let Γ be a graph and let $\Omega \subseteq \Gamma$ be an induced subgraph. Let $\mathcal{P}_\Omega$ be the set of partial conjugations of Γ by a vertex in Ω. If $\varphi \in \mathrm{Aut}^0(W_\Gamma)$ and if there exist words $z_1, ..., z_n \in W_\Omega$ such that $\varphi(v_j) = z_j v_j z_j^{-1}$ for each $1 \le j \le n$, then any word for φ in the alphabet $\mathcal{P}^{\pm 1}$ can be rewritten as a word in the alphabet $\mathcal{P}_\Omega^{\pm 1}$ (a word which still spells φ) by simply omitting those generators not in $\mathcal{P}_\Omega^{\pm 1}$.*

Theorem 4.2.1. *Let W_Γ be a graph product of primary cyclic groups. If Γ is connected and contains two or more distinct Coxeter SILs, then $\mathrm{Out}^0(W_\Gamma)$ is not virtually $\mathbb{Z}$.*

Proof. This is an edited version of Hikaru Jitsukawa's proof. If $\mathrm{Out}(W_\Gamma)$ is not virtually abelian, then clearly $\mathrm{Out}^0(W_\Gamma)$ is not virtually $\mathbb{Z}$. So we assume $\mathrm{Out}(W_\Gamma)$ is virtually abelian. We take any two distinct SILs in Γ and denote them $\{v_1, v_2 | C\}$ and $\{w_1, w_2 | D\}$. By Proposition 4.1.1, $\chi_{v_1,C}, \chi_{v_2,C}, \chi_{w_1,D}$, and $\chi_{w_2,D}$ are in $\mathrm{Out}^0(W_\Gamma)$. Define the subgroups $A = \langle \chi_{v_1,C}, \chi_{v_2,C} \rangle$ and $B = \langle \chi_{w_1,D}, \chi_{w_2,D} \rangle$ of $\mathrm{Out}^0(W_\Gamma)$. Note that $A \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong D_\infty$ and
$B \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong D_\infty$. We will show that A and B have trivial intersection.

Suppose, for the sake of contradiction, that there is a non-trivial element $\varphi \in A \cap B$. First, we consider the case where $\{v_1, v_2\} \cap \{w_1, w_2\} = \emptyset$. Since $\varphi \in A$, there are elements $\chi_1, ..., \chi_n \in A$, with each χ_i equal to either $\chi_{v_1,C}$ or $\chi_{v_2,C}$, such that $\varphi = \chi_1 \circ ... \circ \chi_n$. Then for some $v \in V(\Gamma)$, we can write $\varphi(v) = g_n g_{n-1} ... g_1(v) g_1 ... g_{n-1} g_n$, where each g_i is either v_1 or v_2.

Since $\varphi \in B$, we can also write $\varphi(v) = h_m h_{m-1} ... h_1(v) h_1 ... h_{m-1} h_m$, where each h_i is either w_1 or w_2. By Lemma 4.2.2, the word $h_m ... h_1(v) h_1 ... h_m$ can be rewritten in the alphabet $\mathcal{P}^{\pm 1}_{\{v_1,v_2\}}$, by omitting any generators besides v_1 and v_2. But then we delete every letter of $h_m ... h_1(v) h_1 ... h_m$ and are left with the empty word. Similarly, we can rewrite the word $g_n ... g_1(v) g_1 ... g_n$ in the alphabet $\mathcal{P}^{\pm 1}_{\{w_1,w_2\}}$, and are left with the empty word. Thus $\varphi(v) = v$ for all $v \in V(\Gamma)$, so φ is the identity on Γ.

Now suppose the two SILs share at least one vertex; without loss of generality, suppose $v_1 = w_1$. By Lemma 4.1.3, if $C \cap D \neq \emptyset$, then Γ contains a STIL, so by Theorem 3.2.2, $\mathrm{Out}(W_\Gamma)$ is large. So we may assume $C \cap D = \emptyset$. If the two SILs share both acting vertices, so $v_1 = w_1$ and $v_2 = w_2$, then we also must have $C \cap D = \emptyset$ for the SILs to be distinct.

For some vertex v, if v is in $\Gamma\backslash(C \cup D)$, then $\varphi(v) = v$. If $v \in C$, then since $\varphi \in B$, it is a word in partial conjugations acting on D, so $\varphi(v) = v$. On the other hand, if $v \in D$, then since $\varphi \in A$, it is a word in partial conjugations acting on C, so $\varphi(v) = v$. Thus φ is the identity on Γ.

Then in all cases, we have $A \cap B = \{\mathrm{id}\}$. Now suppose for the sake of contradiction that $\mathrm{Out}^0(W_\Gamma)$ has a finite index subgroup $K \cong \mathbb{Z}$. Since A and B are both infinite subgroups in $\mathrm{Out}^0(W_\Gamma)$, then both $A \cap K$ and $B \cap K$ are infinite subgroups of finite index in $\mathrm{Out}^0(W_\Gamma)$. $A \cap K$ and $B \cap K$ must also have non-trivial intersection. On the other hand, $(A \cap K) \cap (B \cap K) = (A \cap B) \cap K = \{\mathrm{id}\}$, which is a contradiction. Therefore $\mathrm{Out}^0(W_\Gamma)$ is not virtually $\mathbb{Z}$.

$\square$

Chapter 5

Disconnected graphs

We study the disconnected case separately. As we will see, when Γ is disconnected, we have to place stronger restrictions on the structure of Γ to keep $\mathrm{Out}^0(W_\Gamma)$ (or equivalently $\mathrm{Out}(W_\Gamma)$) from being large. We can then describe the family of disconnected graphs with exactly one SIL, and prove that in this case, a more precise statement holds than for the connected case: $\mathrm{Out}^0(W_\Gamma) \cong D_\infty$.

5.1 Summarizing a proof by Susse and Sale

Since our proof of Proposition 5.2.1 will be an application of the methods Susse and Sale use to prove their Proposition 2.3 in [SS17], it is worthwhile to summarize their proof here.

Proposition 5.1.1 (Proposition 2.3 [SS17])**.** *Suppose W_Γ is a right-angled Coxeter group defined by a disconnected graph Γ which contains no FSIL or STIL. Then $\mathrm{Out}^0(W_\Gamma)$ is a virtually abelian right-angled Coxeter group.*

Proof. We summarize the proof that Susse and Sale give of this proposition in [SS17]. Susse and Sale first describe the disconnected graphs Γ which contain no FSIL or STIL. They prove that if Γ contains three (or more) connected components Γ_1, Γ_2, and Γ_3, then we can take $x_i \in \Gamma_i$ and form an FSIL $\{x_1, x_2, x_3\}$. Then by Theorem 3.2.2, $\mathrm{Out}(W_\Gamma)$ is large. So we may assume Γ contains exactly two connected components Γ_1 and Γ_2.

For $\{i, j\} = \{1, 2\}$, each triple of vertices in Γ_i must have at least 2 edges between them to prevent a STIL acting on Γ_j. Thus for a vertex $v \in \Gamma_i$, $\Gamma_i \backslash St(v)$ contains at most one vertex. This means $v \in \Gamma_i$ satisfies either:

- $\Gamma \backslash St(v) = \Gamma_j$, where Γ_j is the component which does not contain v. Then v determines only an inner conjugation of Γ, which is a trivial partial conjugation in $\mathrm{Out}^0(W_\Gamma)$.
- v is a star cut point such that $\Gamma \backslash St(v)$ has exactly two components; then there is a unique partial conjugation with acting vertex v, $\chi_{v,\Gamma_j} \in \mathrm{Out}^0(W_\Gamma)$. In particular, this is the case for any acting vertex in a SIL. This is the only partial conjugation with acting vertex v because if $w = \Gamma_i \backslash St(v)$, then $\chi_{v,w} \circ \chi_{v,\Gamma_j}$ is an inner conjugation, thus trivial in $\mathrm{Out}^0(W_\Gamma)$. Then $\chi_{v,w} = \chi_{v,\Gamma_j}^{-1}$.

See Figure 5.1 and Figure 5.2 for examples of graphs that satisfy these criteria. With this well-defined identification of each vertex in Γ with a partial conjugation in $\mathrm{Out}^0(W_\Gamma)$ (which may be trivial), and since partial conjugations generate $\mathrm{Out}^0(W_\Gamma)$, the authors prove the existence of a surjective homomorphism

$$\iota : W_{\Gamma_1} \times W_{\Gamma_2} \to \mathrm{Out}^0(W_\Gamma).$$

Using the first isomorphism theorem, they prove that

$$\ker(\iota) = Z(W_{\Gamma_1}) \times Z(\Gamma_2)$$

$$\Rightarrow W_{\Gamma_1}/Z(W_{\Gamma_1}) \times W_{\Gamma_2}/Z(W_{\Gamma_2}) \cong \mathrm{Out}^0(W_\Gamma).$$

Then each $W_{\Gamma_i}/Z(W_{\Gamma_i})$ is isomorphic to the right-angled Coxeter group with defining graph $\Gamma_i \backslash Z(\Gamma_i)$, and $\mathrm{Out}^0(W_\Gamma)$ is isomorphic to the right-angled Coxeter group with defining graph $\Gamma_1 \backslash Z(\Gamma_1) \sqcup \Gamma_2 \backslash Z(\Gamma_2)$. Each of W_{Γ_1} and W_{Γ_2} is virtually abelian, because every triple of vertices in each component induces at least two edges. Therefore $\mathrm{Out}^0(W_\Gamma)$ is virtually abelian. $\square$

5.2 Applying these methods to our case

This lemma is the same result as Theorem 4.2.1, only now for the disconnected case.

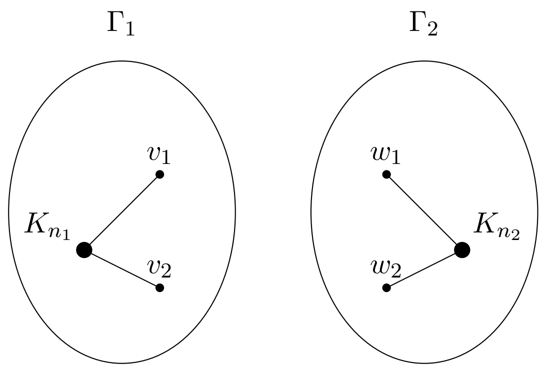


Figure 5.1: Example 1 of a disconnected graph Γ with $\text{Out}^0(W_\Gamma)$ virtually abelian but not virtually $\mathbb{Z}$. Here the SILs are $\{v_1, v_2|\Gamma_2\}$ and $\{w_1, w_2|\Gamma_1\}$. Let $|V(\Gamma_1) - 2| = n_1$ and $|V(\Gamma_2) - 2| = n_2$.

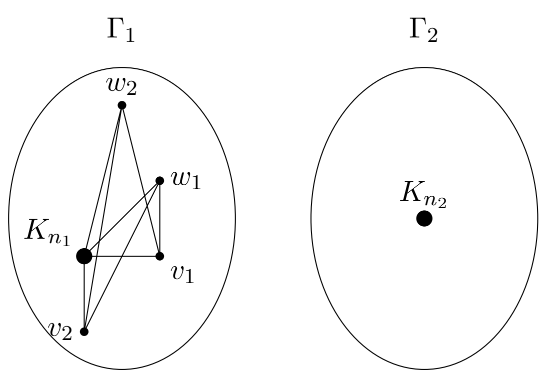


Figure 5.2: Example 2 of a disconnected graph Γ with $\text{Out}^0(W_\Gamma)$ virtually abelian but not virtually $\mathbb{Z}$. Here the SILs are $\{v_1, v_2|\Gamma_2\}$ and $\{w_1, w_2|\Gamma_2\}$. Let $|V(\Gamma_1) - 4| = n_1$ and $|V(\Gamma_2)| = n_2$. Note that even with multiple SILs in Γ_1, for each vertex $v \in \Gamma_1$, there is at most one vertex in $\Gamma_1 \backslash St(v)$.

Lemma 5.2.1. *Let W_Γ be a graph product of primary cyclic groups where Γ is a disconnected graph. If Γ contains two or more distinct Coxeter SILs, then $\mathrm{Out}^0(W_\Gamma)$ is not virtually $\mathbb{Z}$.*

Proof. This proof closely follows that of Proposition 5.1.1. If Γ does not satisfy the characterization given above, then $\mathrm{Out}^0(W_\Gamma)$ is not virtually abelian, hence not virtually $\mathbb{Z}$. If Γ contains a non-Coxeter SIL, then by Theorem 3.2.2, $\mathrm{Out}(W_\Gamma)$ is large. So we may assume that Γ has exactly two connected components Γ_1 and Γ_2, that every triple of vertices in Γ_i has at least three edges, and that any SIL in Γ is a Coxeter SIL.

First, let us consider the case where Γ has at least one Coxeter SIL in each connected component: say $\{v_1, v_2|\Gamma_2\}$ is a SIL with $v_1, v_2 \in \Gamma_1$, and $\{w_1, w_2|\Gamma_2\}$ is a SIL with $w_1, w_2 \in \Gamma_2$. v_1 and v_2 are not adjacent by definition, but they must be adjacent to each other vertex in Γ_1, and each other pair in Γ_1 must be adjacent. Similarly, $\Gamma_2\backslash\{w_1\}$ and $\Gamma_2\backslash\{w_2\}$ must both be complete subgraphs. The center of Γ_1 is $Z(\Gamma_1) = \Gamma_1\backslash\{v_1, v_2\}$, and $Z(\Gamma_2) = \Gamma_2\backslash\{w_1, w_2\}$. See Figure 5.1 for a diagram.

Now, we want to apply the surjection ι defined in the proof of Proposition 5.1.1. Although Susse and Sale assume that W_Γ is a right-angled Coxeter graph, all of the conditions required to define ι still hold: each vertex in $\Gamma\backslash\{v_1, v_2, w_1, w_2\}$ defines only a trivial partial conjugation in $\mathrm{Out}^0(W_\Gamma)$. Each of v_1, v_2, w_1, and w_2 determines a unique, non-trivial partial conjugation of order 2; for example, the partial conjugation determined by v_1 is $\chi_{v_1,\Gamma_2} = \chi^{-1}_{v_1,v_2}$. So even though elements in $\Gamma\backslash\{v_1, v_2, w_1, w_2\}$ may have order larger than 2, the surjection

$$\iota : W_{\Gamma_1} \times W_{\Gamma_2} \to \mathrm{Out}^0(W_\Gamma)$$

holds, as does the isomorphism

$$W_{\Gamma_1}/Z(W_{\Gamma_1}) \times W_{\Gamma_2}/Z(W_{\Gamma_2}) \cong \mathrm{Out}^0(W_\Gamma).$$

In the case that there are no other Coxeter SILs in Γ, we get

$$\langle v_1, v_2\rangle \times \langle w_1, w_2\rangle \cong \mathrm{Out}^0(W_\Gamma)$$

$$\Rightarrow D_\infty \times D_\infty \cong \mathrm{Out}^0(W_\Gamma),$$

which is not virtually $\mathbb{Z}$. If there are additional Coxeter SILs in Γ, then $\langle v_1, v_2\rangle \subseteq W_{\Gamma_1}/Z(W_{\Gamma_1})$ and $\langle w_1, w_2\rangle \subseteq W_{\Gamma_2}/Z(W_{\Gamma_2})$, thus $D_\infty \times D_\infty$ is a subgroup of $\mathrm{Out}^0(W_\Gamma)$.

Now we consider the case where $\Gamma = \Gamma_1 \sqcup \Gamma_2$ contains at least two SILs in one of the Γ_i and no SILs in the other component, as in Figure 5.2. Suppose, without loss of generality, that $\{v_1, v_2|\Gamma_2\}$ and $\{w_1, w_2|\Gamma_2\}$ are SILs with acting vertices in Γ_1. If the set $\{v_1, v_2, w_1, w_2\}$ contains only three distinct vertices, say $v_1 = w_1$, then the triple v_1, v_2 and w_2 has one edge between v_2 and w_2, forming a STIL. Thus we assume that the set $\{v_1, v_2, w_1, w_2\}$ contains four distinct vertices.

We assume first that there are no other Coxeter SILs with acting vertices in Γ_1. Then $W_{\Gamma_1}/Z(W_{\Gamma_1}) = \langle v_1, v_2\rangle \times \langle w_1, w_2\rangle$, because both v_1 and v_2 commute with w_1 and w_2. No SILs with acting vertices in Γ_2 also implies that $Z(W_{\Gamma_2}) = W_{\Gamma_2}$. Applying ι again,

$$\langle v_1, v_2\rangle \times \langle w_1, w_2\rangle \times 1 \cong \mathrm{Out}^0(W_\Gamma)$$

$$\Rightarrow D_\infty \times D_\infty \cong \mathrm{Out}^0(W_\Gamma).$$

As with the first case, if there are more than 2 Coxeter SILs with acting vertices in Γ_1, $D_\infty \times D_\infty$ is still a subgroup of $\mathrm{Out}^0(W_\Gamma)$, and the result holds. □

Here is a lemma that is useful for both the disconnected and connected cases, and proves that the conditions stated in our main result, Theorem 6.0.1 in Chapter 6, agree with the conditions stated in Theorem 3.2.2 for $\mathrm{Out}(W_\Gamma)$ to be virtually abelian.

Lemma 5.2.2. *Let W_Γ be a graph product of primary cyclic groups, where Γ is an arbitrary graph. If Γ contains exactly one SIL, then Γ cannot contain a STIL.*

Proof. We prove the statement by contrapositive: suppose Γ contains the STIL $\{v_1, v_2, v_3|Z\}$ as in Figure 5.3. Since the STIL has at most one edge between the vertices v_1, v_2 and v_3, we suppose, without loss of generality, that there is an edge between v_2 and v_3. We will leverage this to prove Γ contains two SILs. If there are no edges in the subgraph induced by v_1, v_2, and v_3, then we will find three SILs. If Γ is disconnected, it is possible that there is no edge connecting the components $L_1 \cap L_2 \cap L_3$ and Z, but the result still holds.

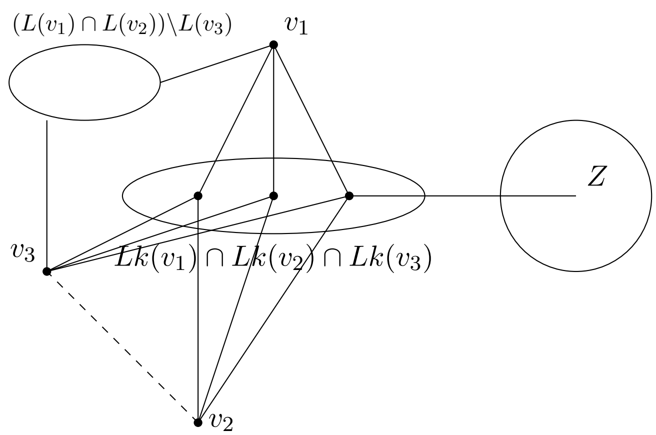


Figure 5.3: STIL for proof of Lemma 5.2.2

Suppose there is no SIL $\{v_1, v_2|Z\}$. Then there must be the paths $v_1 \to Z$ and $v_2 \to Z$, both passing through $(L(v_1) \cap L(v_2))\backslash L(v_3)$. But then v_1, v_2, and Z would be in the same connected component of $\Gamma\backslash(L(v_1) \cap L(v_2) \cap L(v_3))$, contradicting the assumption of a STIL $\{v_1, v_2, v_3|Z\}$. Therefore $\{v_1, v_2|Z\}$ must form a SIL. Similarly, $\{v_1, v_3|Z\}$ must form a SIL because there can be no paths $v_1 \to Z, v_2 \to Z$ in $\Gamma\backslash(L(v_1) \cap L(v_2) \cap L(v_3))$.

Therefore, the existence of a STIL in Γ implies the existence of at least two SILs in Γ. $\square$

Lemma 5.2.1 proves that a disconnected graph Γ must have a unique Coxeter SIL in order for $\mathrm{Out}(W_\Gamma)$ to be virtually $\mathbb{Z}$. We conclude this chapter by showing that this is a sufficient condition for $\mathrm{Out}(W_\Gamma)$ to be virtually $\mathbb{Z}$, and the proof describes the family of graphs satisfying these conditions.

Proposition 5.2.1. *Let W_Γ be a graph product of primary cyclic groups where Γ is a disconnected graph. If Γ has a unique Coxeter SIL $\{v_1, v_2|C\}$ and no non-Coxeter SIL, then* $\mathrm{Out}^0(W_\Gamma) \cong D_\infty$.

Proof. We again follow the structure of the proof of Proposition 5.1.1. By Lemma 5.2.2, since Γ has exactly one SIL, there can be no STIL or FSIL in Γ. As noted in the proof of Proposition 5.1.1, Γ must have exactly 2 connected components Γ_1 and Γ_2. Since there are two components, the SIL must be acting on an entire connected component. Suppose, without loss of generality, that the Coxeter SIL is $\{v_1, v_2|\Gamma_2\}$, where v_1 and v_2 are in Γ_1.

v_1 is not adjacent to v_2, but every other pair in Γ_1 must be adjacent, otherwise we get a

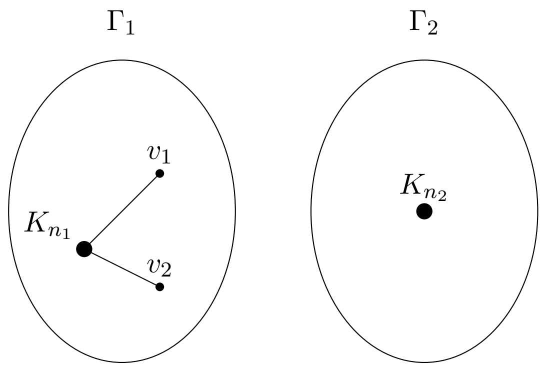


Figure 5.4: A characterization of all disconnected graphs Γ with $\mathrm{Out}^0(W_\Gamma)$ virtually $\mathbb{Z}$. Let $|V(\Gamma_1) - 2| = n_1$ and $|V(\Gamma_2)| = n_2$.

second SIL acting on Γ_2. Γ_2 must be a complete subgraph, otherwise we get a second SIL acting on Γ_1. Therefore $Z(\Gamma_1) = \Gamma_1 \backslash \{v_1, v_2\}$ and $Z(\Gamma_2) = \Gamma_2$. See Figure 5.4 for a diagram.

In Lemma 5.2.1 we saw that the surjection ι defined in Proposition 5.1.1 still holds when Γ contains only Coxeter SILs. We apply the isomorphism determined by ι and get

$$W_{\Gamma_1}/Z(W_{\Gamma_1}) \times W_{\Gamma_2}/Z(W_{\Gamma_2}) \cong \mathrm{Out}^0(W_\Gamma)$$

$$\Rightarrow \langle v_1, v_2 \rangle \times 1 \cong \mathrm{Out}^0(W_\Gamma)$$

$$\Rightarrow D_\infty \cong \mathrm{Out}^0(W_\Gamma).$$

In particular, $\mathrm{Out}^0(W_\Gamma)$ is virtually $\mathbb{Z}$. □

Chapter 6

Proving our main result

This chapter covers a few more necessary lemmas and concludes with a proof of Theorem 6.0.1.

Lemma 6.0.1 (Lemma 1.4 [SS17])**.** *Suppose W_Γ is a graph product of directly indecomposable cyclic groups and let x and y be distinct vertices in Γ. Two partial conjugations $\chi_{x,C}$ and $\chi_{y,D}$ in $\mathrm{Out}(W_\Gamma)$ do not commute if and only if there is a SIL $\{x, y|z\}$ and one of the following holds:*

1. *$z \in C = D$;*
2. *$x \in D$ and $z \in C$;*
3. *$y \in C$ and $z \in D$;*
4. *$x \in D$ and $y \in C$.*

The next lemma shows that the condition of Γ having exactly one SIL is very strong. A similar result was shown in the proof of Proposition 5.1.1 for the disconnected case. Again, this result shows that the identification of a SIL with a pair of partial conjugations in $\mathrm{Out}^0(W_\Gamma)$ is unique when Γ has exactly one Coxeter SIL, a fact that we will prove explicitly for the general case in Theorem 6.0.1.

Lemma 6.0.2. *Let W_Γ be a graph product of primary cyclic groups, where Γ is a connected graph. If Γ contains exactly one SIL $\{v_1, v_2|Z\}$, then each of $\Gamma \backslash St(v_1)$ or $\Gamma \backslash St(v_2)$ has exactly two connected components.*

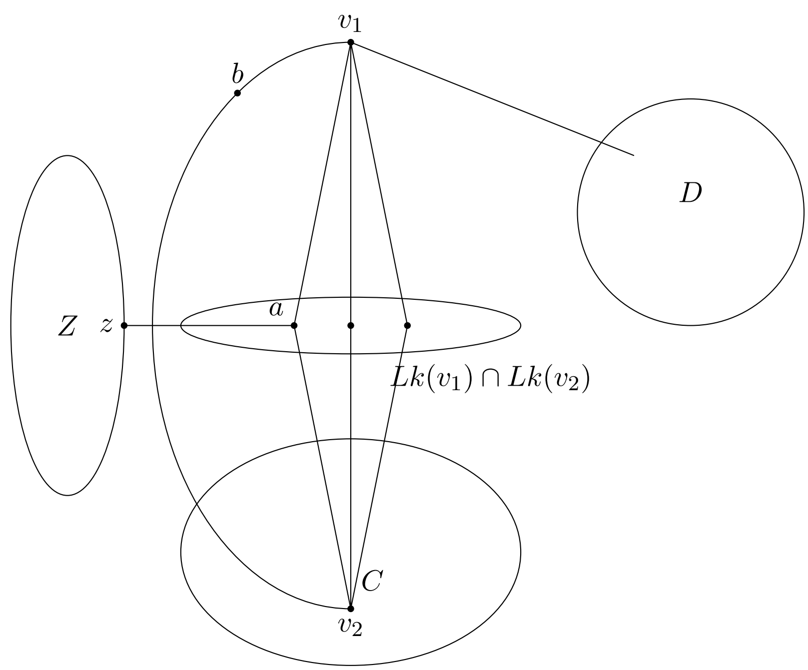


Figure 6.1: SIL for Lemma 6.0.2, where $\Gamma \backslash St(v_1)$ has 3 connected components.

Proof. We prove the statement by contrapositive: suppose $\Gamma \backslash St(v_1)$ has 3 connected components, Z, C, D, labeled so that $v_2 \in C$, as in Figure 6.1.

Suppose no path γ between v_1 and v_2 exists which does not pass through $Lk(v_1) \cap Lk(v_2)$. Then consider $z \in Z$ such that $Lk(z) \cap Lk(v_1) \cap Lk(v_2) \neq \emptyset$. v_1 and v_2 are in separate components of $\Gamma \backslash (Lk(v_1) \cap Lk(v_2))$, so $\{v_1, v_2, z\}$ forms an FSIL, which of course implies that Γ has more than one SIL. Therefore such a path γ must exist. γ has length ≥ 2 as a condition of the SIL; it cannot pass through Z or C, because then these components would not be disconnected from one another. The only path between any two of C, D, and Z must pass through either $Lk(v_1) \cap Lk(v_2)$ or v_1. Let a be a vertex in $Lk(v_1) \cap Lk(v_2)$ and let b be the point in γ adjacent to v_1. Then $\{a, b | D\}$ forms a SIL, so Γ has two distinct SILs.

Similarly, if $\Gamma \backslash St(v_2)$ has at least three connected components, then Γ has more than one SIL. $\square$

The proof of our main theorem, Theorem 6.0.1, has a similar structure to the proof of Proposition 2.2 in [SS17]:

Proposition 6.0.1 (Proposition 2.2 [SS17])**.** *Suppose that W_Γ is a right-angled Coxeter group defined by a connected graph Γ containing no STIL or FSIL. Then the commutator subgroup*

$(\mathrm{Out}^0(W_\Gamma))'$ *is abelian.*

In particular, we will see that the commutator subgroup is virtually cyclic when Γ satisfies our conditions.

Definition 6.0.1. Let G be an arbitrary group. For elements h and g in G, we define their **commutator** as $[g,h] = ghg^{-1}h^{-1}$. The **commutator subgroup** (also called the **derived subgroup**) of G, denoted G', is the subgroup generated by all commutators of elements in G. Note that G' is normal in G.

Finally, we have a necessary and sufficient condition for $\mathrm{Out}(W_\Gamma)$ to be virtually $\mathbb{Z}$.

Theorem 6.0.1. *Let W_Γ be graph product of primary cyclic groups.* $\mathrm{Out}(W_\Gamma)$ *is virtually $\mathbb{Z}$ if and only if Γ contains a unique Coxeter SIL and no non-Coxeter SIL.*

Proof. We prove the ($\Rightarrow$) direction by contrapositive.

If Γ has no Coxeter SIL, then Theorem 3.2.1 shows that $\mathrm{Out}(W_\Gamma)$ is finite. Applying Theorem 4.2.1 for the connected case and Lemma 5.2.1 for the disconnected case, if Γ contains two distinct Coxeter SILs, then $\mathrm{Out}(W_\Gamma)$ is not virtually $\mathbb{Z}$. If Γ contains a non-Coxeter SIL, then by Theorem 3.2.2, $\mathrm{Out}(W_\Gamma)$ is large. Therefore if either of these conditions fails, $\mathrm{Out}(W_\Gamma)$ is not virtually $\mathbb{Z}$, proving the ($\Rightarrow$) direction.

Now we prove the ($\Leftarrow$) direction directly. Apply Proposition 5.2.1 for the case where Γ is disconnected; then $\mathrm{Out}^0(W_\Gamma) \cong D_\infty$, which is virtually $\mathbb{Z}$.

Now assume Γ is connected and let $\{v_1, v_2|Z\}$ be the unique Coxeter SIL. By Lemma 5.2.2, since Γ has exactly one Coxeter SIL, there can be no STIL or FSIL in Γ. Lemma 3.2.1 shows that this SIL corresponds to the pair of partial conjugations $\chi_{v_1,Z}$ and $\chi_{v_2,Z}$ in $\mathrm{Out}^0(W_\Gamma)$. By Lemma 6.0.2, since $\Gamma \backslash St(v_1)$ contains exactly two connected components, $\chi_{v_1,Z}$ is the unique partial conjugation whose acting vertex is v_1 in $\mathrm{Out}^0(W)$: Let $D = \Gamma\backslash(St(v_1) \cup Z)$ be the other connected component that v_1 acts on by partial conjugation. Then $\chi_{v_1,Z} \circ \chi_{v_1,D} = 1$ in $\mathrm{Out}^0(W_\Gamma)$, because this is an inner conjugation of Γ by v_1, and inner conjugations are trivial in $\mathrm{Out}^0(W_\Gamma)$. Thus $\chi_{v_1,Z} = (\chi_{v_1,D})^{-1} = \chi_{v_1,D}$ in $\mathrm{Out}^0(W_\Gamma)$, using the fact that v_1 and v_2 have order 2. Similarly, $\chi_{v_2,Z}$ is the unique partial conjugation in $\mathrm{Out}^0(W_\Gamma)$ whose acting vertex is v_2.

Now consider the commutator $[\chi_{v_1,Z}, \chi_{v_2,Z}]$. By conjugating this commutator by $\chi_{v_1,Z}$, we get

$$\chi_{v_1,Z} \circ (\chi_{v_1,Z} \circ \chi_{v_2,Z} \circ \chi_{v_1,Z} \circ \chi_{v_2,Z}) \circ \chi_{v_1,Z}$$

$$= [\chi_{v_2,Z}, \chi_{v_1,Z}] = [\chi_{v_1,Z}, \chi_{v_2,Z}]^{-1}.$$

On the other hand, if we conjugate by $\chi_{v_2,Z}$, we get

$$\chi_{v_2,Z} \circ (\chi_{v_1,Z} \circ \chi_{v_2,Z} \circ \chi_{v_1,Z} \circ \chi_{v_2,Z}) \circ \chi_{v_2,Z}$$

$$= [\chi_{v_2,Z}, \chi_{v_1,Z}] = [\chi_{v_1,Z}, \chi_{v_2,Z}]^{-1}.$$

This is analogous to the proof of Lemma 2.5 from [SS17], but these are the only two cases, as v_1 and v_2 are each the acting vertex in a unique partial conjugation in $\mathrm{Out}^0(W)$.

Let $b \in \Gamma \backslash \{v_1, v_2\}$ be a star cut point and let $\chi_{b,C}$ be a partial conjugation in $\mathrm{Out}^0(W_\Gamma)$. By Lemma 6.0.1, $\chi_{b,C}$ commutes with $\chi_{v_i,Z}$ (for $i \in \{1,2\}$) if and only if there is a SIL with acting vertices b and v_i satisfying certain conditions. However, since $\{v_1, v_2 | Z\}$ is the unique SIL in Γ, and $b \notin \{v_1, v_2\}$, then the partial conjugations $\chi_{b,C}$ and $\chi_{v_i,Z}$ commute. As a result, $[\chi_{b,C}, \chi_{v_i,Z}] = 1$ for $i \in \{1,2\}$. Similarly, for any other partial conjugation $\chi_{b',C'}$ in $\mathrm{Out}^0(W_\Gamma)$, $[\chi_{b,C}, \chi_{b',C'}] = 1$, so the only non-trivial commutator in $(\mathrm{Out}^0(W_\Gamma))'$ is $[\chi_{v_1,Z}, \chi_{v_2,Z}]$. When we commute $[\chi_{v_1,Z}, \chi_{v_2,Z}]$ by $\chi_{b,C}$, we have

$$\chi_{b,C} \circ [\chi_{v_1,Z}, \chi_{v_2,Z}] \circ \chi_{b,C}^{-1}$$

$$= \chi_{b,C} \circ \chi_{b,c}^{-1} \circ [\chi_{v_1,Z}, \chi_{v_2,Z}]$$

$$= [\chi_{v_1,Z}, \chi_{v_2,Z}],$$

so the subgroup $\langle [\chi_{v_1,Z}, \chi_{v_2,Z}] \rangle$ is closed in $\mathrm{Out}^0(W_\Gamma)$ under conjugation.

Thus, we conclude that $\chi_1[\chi_2, \chi_3]\chi_1 \in \langle [\chi_{v_1,Z}, \chi_{v_2,Z}] \rangle$ for all partial conjugations $\chi_1, \chi_2, \chi_3 \in \mathrm{Out}^0(W_\Gamma)$. Therefore the commutator subgroup is $(\mathrm{Out}^0(W_\Gamma))' \cong \langle [\chi_{v_1,Z}, \chi_{v_2,Z}] \rangle$. $[\chi_{v_1,Z}, \chi_{v_2,Z}]^2 = (\chi_{v_1,Z}\chi_{v_2,Z})^4$, so $[\chi_{v_1,Z}, \chi_{v_2,Z}]$ is an infinite order element, and $(\mathrm{Out}^0(W_\Gamma))' \cong \mathbb{Z}$.

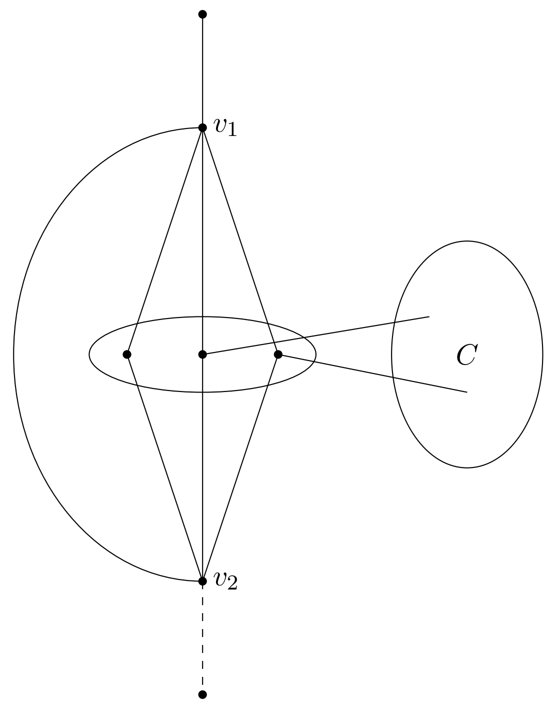


Figure 6.2: One type of connected graph Γ with exactly one Coxeter SIL. v_1 is the endpoint of a path of length 1. The dashed path may or may not be present.

The abelianization $\mathrm{Out}^0(W_\Gamma)/(\mathrm{Out}^0(W_\Gamma))'$ is finite because $\mathrm{Out}^0(W_\Gamma)$ is finitely generated by finite order elements, so $|\mathrm{Out}^0(W_\Gamma) : (\mathrm{Out}^0(W_\Gamma))'|$ is finite. Then we have a finite index normal subgroup isomorphic to $\mathbb{Z}$ in $\mathrm{Out}^0(W_\Gamma)$, thus $\mathrm{Out}^0(W_\Gamma)$ is virtually $\mathbb{Z}$, and hence $\mathrm{Out}(W_\Gamma)$ is virtually $\mathbb{Z}$. $\square$

Now, we describe connected graphs which satisfy the conditions of Theorem 6.0.1.

Example 6.0.1. The condition that Γ must have exactly one Coxeter SIL places strong restrictions on its structure. First, we have a Coxeter SIL $\{v_1, v_2|C\}$. There can be at most one path of length 1 starting at v_1, and similarly for v_2. If we had two or more paths of length 1 starting at v_1, then v_1 would be in the intersection of the links of the endpoints of these paths, and the endpoints would form a second SIL. If there is a path of length 1 at either v_1 or v_2, then there is exactly one path of length at least 2 between v_1 and v_2 which does not pass through $Lk(v_1) \cap Lk(v_2)$, as in Figure 6.2. We cannot have two or more paths between v_1 and v_2 outside of $Lk(v_1) \cap Lk(v_2)$ in this case, otherwise two vertices on these paths would form a second SIL acting on the endpoint of the path of length 1.

If there is no path of length 1 starting at v_1 or v_2, then there is at least one path connecting

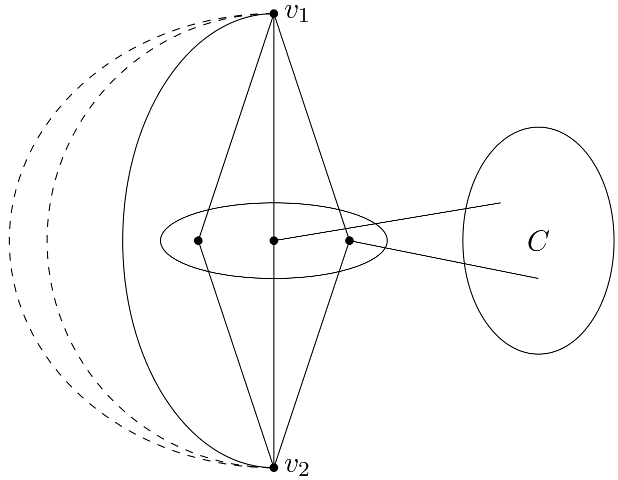


Figure 6.3: The second type of connected graph Γ with exactly one Coxeter SIL. There is no restriction on the number of paths between v_1 and v_2, but there must be at least one outside $Lk(v_1) \cap Lk(v_2)$.

v_1 and v_2 which does not pass through the intersection of links, and there is no maximum to how many such paths exist in Γ. See Figure 6.3.

The connected component C also cannot contain another SIL. Note there is no restriction on the value of the order map $\mathbf{m}$ on any vertices besides v_1 and v_2.

Example 6.0.2. We give a few concrete examples to show how removing or rearranging one or two edges is the difference between $\mathrm{Out}(W_\Gamma)$ being virtually $\mathbb{Z}$ or not. As in Figure 6.4, suppose we have a right-angled Coxeter group W_Γ and that Γ has a SIL $\{v_1, v_2|C\}$, where C is the subgraph induced by d, e, and f. Since C is a complete subgraph, we do not get a second SIL in Γ, so $\mathrm{Out}(W_\Gamma)$ is virtually $\mathbb{Z}$. We can verify this concretely: the star cut points of Γ are v_1, v_2, and c, each acting in two non-trivial partial conjugations. We construct $\mathcal{P}^0$, the generating set of $\mathrm{Out}^0(W_\Gamma)$, by assigning the vertex a a 1 in the ordering of the vertices. Then we get

$$\mathcal{P}^0 = \{\chi_{v_1,C}, \chi_{v_2,C}, \chi_{c,\{e,f\}}\}.$$

Of these partial conjugations, only $\chi_{v_1,C}$ and $\chi_{v_2,C}$ do not commute, by Lemma 6.0.1. Thus

$$\mathrm{Out}^0(W_\Gamma) \cong (\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z} \cong D_\infty \times \mathbb{Z}/2\mathbb{Z},$$

which is virtually $\mathbb{Z}$.

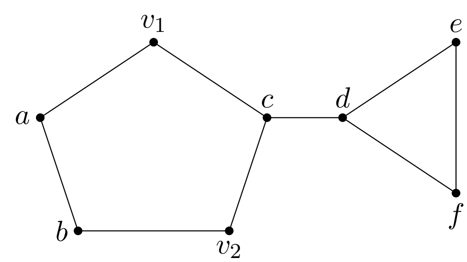


Figure 6.4: Example 1 of a graph satisfying the conditions of Theorem 6.0.1. We assume each vertex is an element of order 2 in W_Γ.

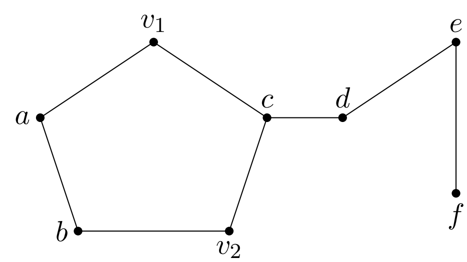


Figure 6.5: Example 2 of a graph satisfying the conditions of Theorem 6.0.1.

Now suppose we remove the edge between the vertices d and f, as in Figure 6.5. Even though C now has one fewer edge, still no pair of vertices in C has an intersection of links that separates them from the subgraph containing v_1 and v_2. Therefore we have no additional SILs and $\mathrm{Out}(W_\Gamma)$ still satisfies the condition of Theorem 6.0.1. Again, we can check this concretely: the star cut points of Γ are v_1, v_2, c, and d, each acting in two non-trivial partial conjugations. As in the previous case, we construct $\mathcal{P}^0$ by assigning the vertex a a 1 in the ordering of the vertices. Then

$$\mathcal{P}^0 = \{\chi_{v_1,C}, \chi_{v_2,C}, \chi_{c,\{e,f\}}, \chi_{d,\{f\}}\}.$$

Because we added no new SILs, we still have only one pair of partial conjugations which do not commute, $\chi_{v_1,C}$ and $\chi_{v_2,C}$. Thus

$$\mathrm{Out}^0(W_\Gamma) \cong D_\infty \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Finally, suppose we return to the first case and remove the edge between the vertices e and f, as in Figure 6.6. Now $\{e, f|D\}$ forms a second distinct SIL in Γ, where $D = \{v_1, a, b, v_2, c\}$. Γ contains no FSIL or STIL, so $\mathrm{Out}(W_\Gamma)$ is virtually abelian but not virtually $\mathbb{Z}$. In this case, the

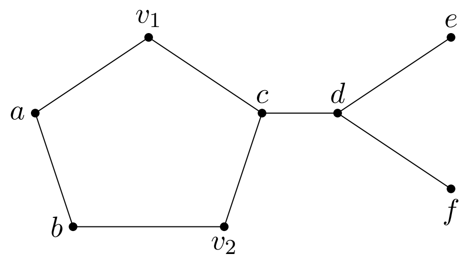


Figure 6.6: An example of a graph which does not satisfy the conditions of Theorem 6.0.1. Even though the vertices d, e, and f induce the same number of edges as in Figure 6.5, this graph contains two distinct SILs.

star cut points are v_1, v_2, c, e, and f; each of these act in two non-trivial partial conjugations, except for c, which acts in three. We construct $\mathcal{P}^0$ by assigning the vertex a a 1 in the ordering of the vertices. Then

$$\mathcal{P}^0 = \{\chi_{v_1,C}, \chi_{v_2,C}, \chi_{c,\{e\}}, \chi_{c,\{f\}}, \chi_{e,\{f\}}, \chi_{f,\{e\}}\}.$$

In $\mathrm{Out}^0(W_\Gamma)$, $\chi_{e,\{f\}} = \chi_{e,D}^{-1} = \chi_{e,D}$, because e is an order 2 element. Similarly, $\chi_{f,\{e\}} = \chi_{f,D}$ in $\mathrm{Out}^0(W_\Gamma)$. Now that we have two SILs, we have two pairs of non-commuting partial conjugations, $\chi_{v_1,C}, \chi_{v_2,C}$ and $\chi_{e,D}, \chi_{f,D}$, each generating a D_∞ in $\mathrm{Out}^0(W_\Gamma)$. Thus

$$\mathrm{Out}^0(W_\Gamma) \cong D_\infty \times D_\infty \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

This shows how both the number and placement of edges in Γ determine the structure of $\mathrm{Out}(W_\Gamma)$.

Finally, we encapsulate Theorem 3.2.1, Theorem 3.2.2, and Theorem 6.0.1 in one statement.

Theorem 6.0.2. *Let Γ be a graph product of primary cyclic groups.*

1. *Γ contains no SIL exactly when $\mathrm{Out}(W_\Gamma)$ is finite.*
2. *Γ contains a SIL exactly when $\mathrm{Out}(W_\Gamma)$ is infinite.*
    - *(a) Γ contains a unique Coxeter SIL and no non-Coxeter SIL exactly when $\mathrm{Out}(W_\Gamma)$ is virtually $\mathbb{Z}$.*

(b) Γ *contains two or more SILs which are Coxeter SILs and do not form an FSIL or STIL exactly when* $\mathrm{Out}(W_\Gamma)$ *is virtually abelian but not virtually* $\mathbb{Z}$*.*

(c) Γ *contains a non-Coxeter SIL, an FSIL, or a STIL exactly when* $\mathrm{Out}(W_\Gamma)$ *is large.*

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