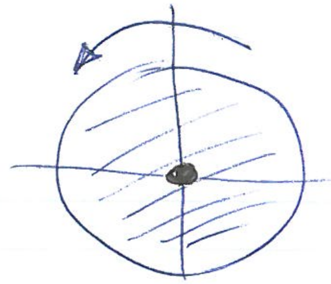
Plane problem - 2



ω - ang. velocity

Rotating Disk

(Axisymm. problem)

Material points have (normal) acceleration (non-rectilinear motion)

$$a_n = \frac{v^2}{r} = \omega^2 r$$

$\Rightarrow$ replace equilibrium in r-direction $\Rightarrow$ eq. of motion:

force in r-direction per unit volume $= -\rho a_n$

towards center

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = -\rho\omega^2 r$$

Express σ's in terms of ϵ's, and ϵ's in displacements.

$$\Rightarrow \boxed{\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = -(1-\nu^2)\frac{\rho\omega^2}{E} r}$$

non-homog. eq-n

General sol'n: (general, homog) + (particular, non-homog)

Homogeneous eq-n, general solution: $u(r) = C_1 r + C_2 \frac{1}{r}$

Non-homog. (particular solution): try $u \sim r^3$ $\Rightarrow$ works

$$u = -(1-\nu^2)\frac{\rho\omega^2}{E}\frac{r^3}{8}$$

∴ General sol'n:

$$u(r) = -\frac{\rho\omega^2(1-\nu^2)}{8E} r^3 + C_1 r + C_2 \frac{1}{r}$$

→ strains → Stresses:

$$\begin{cases} \sigma_{rr} = \frac{E}{1-\nu^2}\left[\frac{-(3+\nu)(1-\nu^2)\rho\omega^2}{8E} r^2 + (1+\nu) C_1 - (1-\nu)\frac{C_2}{r^2} \right] \\ \sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left[-\frac{(1+3\nu)(1-\nu^2)\rho\omega^2}{8E} r^2 + (1+\nu) C_1 + (1-\nu)\frac{C_2}{r^2} \right] \end{cases}$$

Constants C_1, C_2 : from b.c. – they depend on: solid disk or annular disk

Annular disk

$\sigma_{rr} = 0$ at $r = b$ (outer)

$\sigma_{rr} = 0$ at $r = a$ (inner).

they yield

$$C_1 = \rho\omega^2 \frac{a^2+b^2}{E} \frac{(1-\nu)(3+\nu)}{8}$$

$$C_2 = \rho\omega^2 \frac{a^2 b^2}{E} \frac{(1+\nu)(3+\nu)}{8}$$

∴

$$\begin{cases} \sigma_{rr} = \frac{3+\nu}{8}\left(a^2+b^2-r^2-\frac{a^2b^2}{r^2}\right)\rho\omega^2 \\ \sigma_{\theta\theta} = \frac{3+\nu}{8}\left(a^2+b^2-\frac{1+3\nu}{3+\nu}r^2+\frac{a^2b^2}{r^2}\right)\rho\omega^2 \end{cases}$$

<u>Max. σ_{rr}</u>: from $\quad \dfrac{d\sigma_{rr}}{dr} = 0 \Rightarrow$ at $r = \sqrt{ab}$

equals to $\quad \sigma_{rr}^{max} = \dfrac{3+\nu}{8}(a-b)^2 \rho\omega^2$

$\rightarrow 0$ if $b-a$ is small

Solid disk

$C_2 = 0 \quad$ (otherwise $\sigma \rightarrow \infty$ at $r=0$)

$\sigma_{rr} = 0$ at $r = b$

$\Rightarrow C_1 = \dfrac{(1-\nu)(3+\nu)}{8E} b^2 \rho\omega^2$

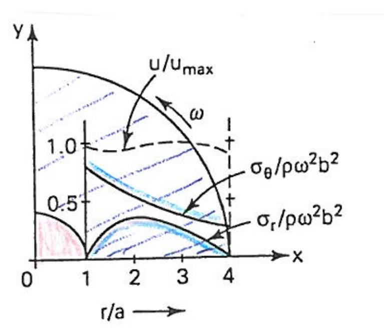
.....

<u>Stresses</u> :

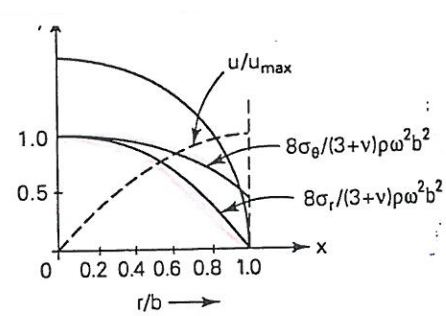
$$\sigma_{rr} = \frac{3+\nu}{8}\left(b^2 - r^2\right)\rho\omega^2 \qquad \text{(max. at center)}$$

$$\sigma_{\theta\theta} = \frac{3+\nu}{8}\left(b^2 - \frac{1+3\nu}{3+\nu} r^2\right)\rho\omega^2$$

- hydrostat. at center
- $\sigma_{\theta\theta} - \sigma_{rr} \nearrow$ with r

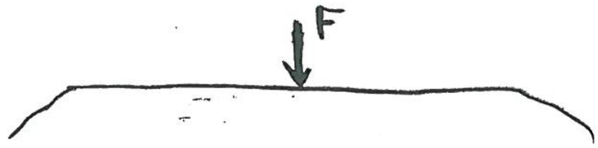


annular $\left(\frac{a}{b} = 0.25\right)$

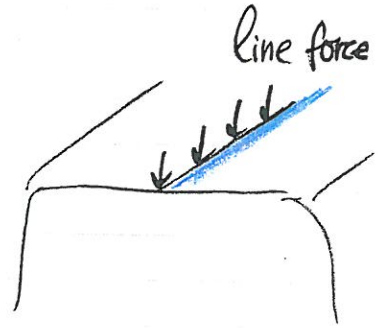


solid

Normal force on boundary of half-plane



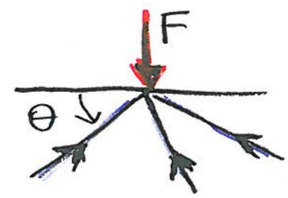
In the context of plane ϵ :



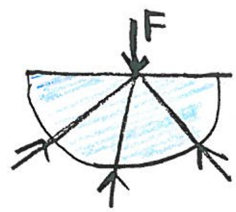
Importance :

by integration of point force sol'n

Guess : σ_{rr} is the only stress : $\sigma_{rr} = \sigma_{rr}(r, \theta)$



Eq-m of half-circle:



compressive

$$\int_0^{\pi} -\sigma_{rr} \sin\theta \, r \, d\theta = F$$

const. when integrating

$\Rightarrow$ Seek σ_{rr} in the form

$$-\sigma_{rr} = \frac{F}{r} f(\theta)$$

to be found

$$-\sigma_{rr} = \frac{F}{r} f(\theta)$$

Check eq-ns

• **Equilibrium** :

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \cancelto{=0}{\sigma_{\theta\theta}}}{r} = 0$$: satisfied for any f(

no restrictions

• **Compatibility** : need to check since our guess is in stresses (not displacements)

Expressed in stresses, with account of eq-m $\Rightarrow$

$$\nabla^2(\sigma_{rr} + \cancelto{0}{\sigma_{\theta\theta}}) = 0$$

$$\frac{\partial^2 \sigma_{rr}}{\partial r^2} + \frac{1}{r}\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \sigma_{rr}}{\partial \theta^2} = 0$$

Laplacian in cylindr. coord

$$\Rightarrow \quad f''(\theta) + f(\theta) = 0$$

($1/r^3$ factors out)

$$\Downarrow$$

$$f(\theta) = A\sin\theta + B\cos\theta$$

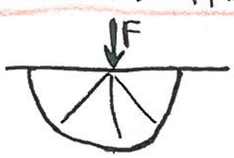
Symmetry:

$$\sigma_{rr}(\theta) = \sigma_{rr}(\pi - \theta) \quad \Rightarrow B = 0$$

$$-\sigma_{rr} = A\,\frac{\sin\theta}{r}\,F$$

from eq. of half-circle

Equilibrates applied force:



$$\int_0^{\pi} -\sigma_{rr}\sin\theta \cdot r\,d\theta = F$$

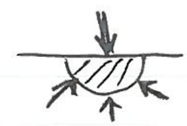
$$\Downarrow$$

$$A = \frac{1}{\int_0^{\pi}\sin^2\theta\,d\theta = \pi/2} \Rightarrow \boxed{\sigma_{rr} = -\frac{2}{\pi}\frac{\sin\theta}{r}F}$$

• <u>Hooke's law</u> : would give strains generated by σ_{rr} (as found

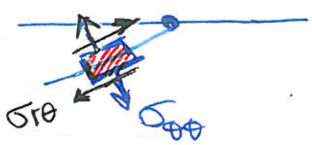
Boundary cond :

traction-free except one point where $\sigma_{rr} \to \infty$



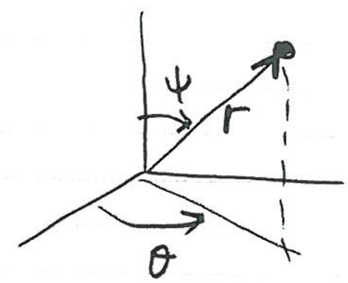
<u>Satisfied</u>

Since $\sigma_{\theta\theta} = \sigma_{r\theta} = 0$ everywhere and, at boundary,



At point of infinite stress, the b.c. is satisfied in the <u>integral</u> sense for the domain

Spherical coordinates: (r, θ, ψ)



Example of application: spherical shell under internal/external pressures

Radially-symmetric case: $u_r \equiv u = u(r)$ is the only displac. component

$$\epsilon_{rr} = \frac{du}{dr}, \quad \epsilon_{\theta\theta} = \epsilon_{\psi\psi} = \frac{u}{r}$$

(similar to cylindrical)

Stresses: no shear stresses (symmetry) $(\sigma_{r\theta} = \sigma_{r\psi} = \sigma_{\theta\phi} = 0)$

σ_{rr}; $\sigma_{\psi\psi} = \sigma_{\theta\theta}$ – normal stresses

Equilibrium:

$$\frac{d\sigma_{rr}}{dr} + 2\,\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

unlike cylindrical

Expressing σ's in terms of u,

$$\frac{d}{dr}\left(\frac{du}{dr} + 2\,\frac{u}{r}\right) = 0$$

General sol'n:

$$u = Ar + B\frac{1}{r^2}$$

rather than $\frac{1}{r}$ in cylindr.

from b.c.