Nilpotent Orbits in the Symplectic and Orthogonal Groups

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Abstract

Let K be an algebraically closed field of arbitrary characteristic and consider a linear algebraic group G over K and its Lie algebra $\mathfrak{g}$. For $X \in \mathfrak{g}$, denote by $\mathcal{O}_X$ the orbit of X under the action of G on $\mathfrak{g}$ defined by the adjoint representation. The Zariski closure $\overline{\mathcal{O}_X}$ is then a subvariety of $\mathfrak{g}$. For $G = \mathrm{O}_n$ or Sp_n, the orthogonal and symplectic groups, Kraft and Procesi showed that $\overline{\mathcal{O}_X}$ is a normal variety for certain nilpotent $X \in \mathfrak{g}$ when $\operatorname{char} K = 0$. We begin to generalize their result for $\operatorname{char} K = p \neq 2$, concluding that an orbit closure $\overline{\mathcal{O}_X}$ of a nilpotent element $X \in \mathfrak{g}$ is normal if and only if it is normal in the union of $\mathcal{O}_X$ with all orbits $\mathcal{O}$ of codimension 2 contained in the boundary of $\overline{\mathcal{O}_X}$. In particular, if $\overline{\mathcal{O}_X} \setminus \mathcal{O}_X$ does not contain any orbits of codimension 2, then $\overline{\mathcal{O}_X}$ is normal.

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Chapter 1

Introduction

1.1 Historical context

Over the past 50 years, many people have studied the question of which elements of a Lie algebra have normal orbit closure. Kostant first proved in [Kos63] that when char $K = 0$, regular nilpotent elements in a semisimple Lie algebra have normal orbit closure, i.e. that the full nilpotent variety is normal. Veldkamp extended this result to most positive characteristics, Demazure later proving this result for semisimple and simply connected Lie algebras when K has characteristic p that is "good" [Jan04, 8.5]. Additionally, the closure of the "minimal" nilpotent orbit is normal, and is equal to the union of the orbit and the zero element, for simply connected, almost simple algebraic groups [Jan04, 8.6]. This was first proved by Vinberg and Popov in [VP72] in characteristic 0, and is implied by results of Ramanan and Ramanathan in [RR85] in prime characteristic.

There are also many results for orbits of arbitrary nilpotent elements of simple Lie algebras that make up a growing classification of normal nilpotent orbit closures. Kraft and Procesi proved that for K with characteristic 0, closures of orbits are normal in the case of $G = \mathrm{GL}_n$ acting on its Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ [KP79]. Generalizing their method for the general linear case, they also obtained a partial classification of normal nilpotent orbit closures for G of type B, C, or D, the symplectic and orthogonal groups [KP82]. Sommers then completed this classification in the symplectic and orthogonal cases, over K with characteristic still zero [Som05]. Other results in some of the exceptional groups when $\operatorname{char} K = 0$ include those of Broer in F_4[Bro98], Kraft in G_2[Kra89], and Sommers in E_6[Som03].

Some of these results have been generalized to positive characteristic. Donkin generalized the general linear results of Kraft and Procesi to positive characteristic in [Don90], Thomsen generalized in [Tho00] those results of Broer in F_4, and Christophersen identified the normal orbit closures of E_6 in [Chr06], generalizing the results of Sommers. The generalization of Kraft and Procesi's results, and those of Sommers, in symplectic and orthogonal cases would complete the classification of normal nilpotent orbit closures in the classical cases. The majority of Kraft and Procesi's proof is still valid in positive characteristic, and we follow their method closely, adjusting the proofs of those results that no longer hold for $\operatorname{char} K > 0$.

As we will see in section 1.3, the classification of normal nilpotent orbit closures extends to arbitrary orbit closures, and thus can answer the question of which orbits have normal closure.

1.2 Symplectic and orthogonal definitions

To examine the symplectic and orthogonal cases, we begin with a finite dimensional vector space V over an algebraically closed field K of characteristic $p \neq 2$. Endow V with a non-degenerate bilinear form $\langle\ ,\ \rangle_V$ such that $\langle u, v\rangle_V = \varepsilon\langle v, u\rangle_V$ for all $u, v \in V$, $\varepsilon = \pm 1$.

Denote by $\mathrm{GL}(V)$ the linear algebraic group of all invertible linear transformations of V and let $\mathrm{G}(V)$ denote the subgroup of $\mathrm{GL}(V)$ that leaves the form $\langle\ ,\ \rangle_V$ invariant:

$$\mathrm{G}(V) = \{g \in \mathrm{GL}(V) \mid \langle gu, gv\rangle_V = \langle u, v\rangle_V\}.$$

When $\varepsilon = +1$, $\mathrm{G}(V)$ is the *orthogonal group* O_n, $n = \dim V$. When $\varepsilon = -1$, $\mathrm{G}(V)$ is the *symplectic group* Sp_n.

The Lie algebra $\mathfrak{g}(V)$ of $\mathrm{G}(V)$ consists of those elements of $\mathfrak{gl}(V)$ that are *skew* with respect to the form:

$$\mathfrak{g}(V) = \{X \in \mathfrak{gl}(V) \mid \langle Xu, v\rangle_V = -\langle u, Xv\rangle_V\}.$$

The adjoint representation of $\mathrm{G}(V)$ is given by conjugation in $\mathrm{End}(V)$ of elements in $\mathfrak{g}(V)$, so that the adjoint orbit $\mathcal{O}_X$ of an element $X \in \mathfrak{g}(V)$ is the conjugacy class

$$\mathrm{Ad}(\mathrm{G}(V))X = \{gXg^{-1} \mid g \in \mathrm{G}(V)\}.$$

Our question then becomes: is the closure of a conjugacy class normal?

It is worth noting that $\mathcal{O}_X$ is a locally closed subvariety of $\mathfrak{g}(V)$ isomorphic to the quotient space $\mathrm{G}(V)/\,\mathrm{G}(V)_X$, where $G(V)_X = \{g \in \mathrm{G}(V) \mid gX = X\}$ is the centralizer of X in $\mathrm{G}(V)$ [Jan04] section 2.1. This relies on our restriction $p \neq 2$, so that the character of K is good for $\mathrm{G}(V)$, the symplectic or orthogonal group. This then implies that $\mathcal{O}_X$ is *separable*, meaning that $\mathfrak{g}(V)_X = \mathrm{Lie}(\mathrm{G}(V)_X)$ [Jan04, 2.5]. (For K with positive characteristic, the centralizer of X in the Lie algebra $\mathfrak{g}(V)$ is defined as $\mathfrak{g}_X = \{Z \in \mathfrak{g} \mid [Z, X] = 0\}$.) For $\operatorname{char} K = 0$, one always has that $\mathrm{Lie}(\mathrm{G}(V)_X) = \mathfrak{g}(V)_X$.

1.3 Reducing to the Nilpotent Case

In this paper, we consider only those orbits arising as conjugacy classes of nilpotent elements $X \in \mathfrak{g}(V)$. These are, in some instances, sufficient to treat the general case. For arbitrary $X \in \mathfrak{g}(V)$, $\mathcal{O}_X$ is open in its closure (locally closed) and $\overline{\mathcal{O}_X}$ is the union of finitely many conjugacy classes [Spr98], Lemma 2.3.3. There is then a unique closed class $\mathcal{O}'$ contained in $\overline{\mathcal{O}_X}$, which is necessarily the conjugacy class of a semisimple element. By [Lun73], one can fiber $\overline{\mathcal{O}_X}$ over $\mathcal{O}'$, associating to each element Y of $\overline{\mathcal{O}_X}$ its semisimple part S in $\mathcal{O}'$, where $Y = SU$ is the Jordan decomposition of Y, and U is a unipotent element of $\mathrm{G}(V)$. If $Y \in \mathcal{O}_X$ has Jordan decomposition $Y = SU$ with $S \in \mathcal{O}'$, the fiber over S is isomorphic to the closure of the conjugacy class of U in the centralizer $Z_{\mathrm{G}(V)}(S)$ of S in $\mathrm{G}(V)$. The group

$Z_{\mathrm{G}(V)}(S)$ is reductive, and since $\mathrm{G}(V)$ is one of the classical groups, so is $Z_{\mathrm{G}(V)}(S)$. Hence we restrict our study to unipotent elements. Since char K is good as long as $p \neq 2$, the unipotent variety of $\mathrm{G}(V)$ is isomorphic to the nilpotent variety of $\mathfrak{g}(V)$ [Jan04] Remark 6.1. Thanks to the above filtration, one can study the normality of the closure of the original orbit by studying the closures of nilpotent orbits in the various centralizers. In the case $G = \mathrm{GL}(V)$ where all orbit closures $\overline{\mathcal{O}_X}$ of nilpotent elements are found to be normal, this is enough to show that orbit closures are normal for arbitrary $X \in \mathfrak{g}(V)$ [Don90] Theorem 2.2(a).

1.4 Some Key Theorems and Definitions

Given an affine variety $\mathcal{V}$, it is irreducible when its coordinate ring $K[\mathcal{V}]$ is an integral domain. In this case, we denote the field of fractions of $K[\mathcal{V}]$ by $K(\mathcal{V})$. An irreducible affine variety $\mathcal{V}$ is then *normal* if $K[\mathcal{V}]$ is integrally closed in $K(\mathcal{V})$. It is *smooth* if, for each $x \in \mathcal{V}$, the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a K-vector space is equal to the dimension of the local ring $K[\mathcal{V}]_{\mathfrak{m}}$ of regular functions at x, where $\mathfrak{m}$ is the maximal ideal of functions vanishing at x. Equivalently, $\mathcal{V}$ is nonsingular at x if $K[\mathcal{V}]_{\mathfrak{m}}$ is regular [Har77, page 32], [Ser00, page 76] Theorem 9. As a consequence of Corollary 3 of [Ser00, page 77], a smooth affine variety is necessarily normal.

The orbits $\mathcal{O}_X$ for $X \in \mathfrak{g}(V)$ are themselves smooth, and in general it is the boundary of the orbit closure, $\overline{\mathcal{O}_X} \backslash \mathcal{O}_X$, that may contain singularities. The following consequence of Serre's Criterion for Normality, [Mat80] Theorem 116, will be our main tool in proving the normality of orbit closures.

Theorem 1.1. *The affine variety $\mathcal{V}$ is normal under both of the following two assumptions:*

(a) $\mathcal{V}$ is a complete intersection in some affine space $\mathbb{A}^n$.

(b) The singular locus of $\mathcal{V}$ has codimension at least two in $\mathcal{V}$.

The concept of a (local) complete intersection is defined for subschemes of nonsingular varieties: If Y is a closed subscheme of a nonsingular variety X over K, then Y is a *local complete intersection* in X if the ideal sheaf $\mathscr{I}_Y$ of Y in X can be locally generated by $r = \operatorname{codim}_X Y$ elements at every point. A complete intersection subscheme is Cohen-Macaulay [Har77] Proposition 8.23. Applying this to $\mathcal{V}$ as in Theorem 1.1, condition (a) implies that $\mathcal{V}$ is Cohen-Macaulay, i.e. satisfies (S_k) for all $k \geq 0$ [Mat80] Proposition 113. Theorem 1.1(b) is equivalent to condition (R_1) in Serre's Criterion.

To use the above theorem, we also need the concept of a *quotient* for affine varieties, referred to as a *categorical quotient* in [Bor91, 6.16], where it is defined in more generality. If $\mathcal{V}$ and $\mathcal{W}$ are affine G-varieties and $\pi : \mathcal{V} \to \mathcal{W}$ is a morphism of varieties, then π is a *G-quotient* if it is closed, surjective and

$$\pi^*(K[\mathcal{W}]) = K[\mathcal{V}]^G.$$

We will also refer (somewhat imprecisely) to $\mathcal{W}$ itself as a G-quotient of $\mathcal{V}$. The upshot of this is that if $\mathcal{V}$ is a normal variety and $\mathcal{W}$ is a quotient of $\mathcal{V}$ for some linear algebraic group G, then $\mathcal{W}$ is also a normal variety.[1]

Following the method of Kraft and Procesi, our over-arching strategy will be to define an affine variety Z for each nilpotent $X \in \mathfrak{g}(V)$ such that $\overline{\mathcal{O}_X}$ is a quotient of Z under an appropriate linear algebraic group. If we can then show that Z satisfies (a) and (b) of Theorem 1.1, we have that $\overline{\mathcal{O}_X}$ is a normal variety. Initially, Z is defined as a scheme, and we must show that it is indeed an affine variety by showing that its associated coordinate ring is reduced, i.e. that it satisfies (R_0) and (S_1) [Mat80] Proposition 115. This is done by showing that Z is a complete intersection and nonsingular in codimension 0. Kraft and Procesi remark that, in general, Z is not normal and can have singular locus Z_{sing} with codimension two. If $\mathcal{O}'$ is an orbit contained in the boundary $\overline{\mathcal{O}_X} \setminus \mathcal{O}_X$, we shall see that $\operatorname{codim}_Z Z_{sing} \geq \frac{1}{2} \operatorname{codim}_{\overline{\mathcal{O}_X}} \mathcal{O}'$. Our approach then leads rather to a reduction of our original question to one involving the analysis of certain orbits contained in the boundary of $\overline{\mathcal{O}_X}$.

[1] Suppose that $f \in K(\mathcal{W})$ is integral over $K[\mathcal{W}]$. Then f viewed as an element of $K(\mathcal{V})$ is integral over $K[\mathcal{V}]$ since the comorphism $\pi^* : K[\mathcal{W}] \to K[\mathcal{V}]$ is injective. This implies that $f \in K[\mathcal{V}]$ since $\mathcal{V}$ is normal. Since $K[W] = K[V]^G$, $f \in K(\mathcal{W})$ is fixed by G, hence $f \in K[\mathcal{V}]^G = K[\mathcal{W}]$.

1.5 Summary

In Chapter 2, we give a rigorous treatment of the background needed for stating the main result of the paper in Theorem 2.13. Section 2.1 contains foundational results about orthogonal and symplectic orbits, and section 2.2 introduces spaces of linear homomorphisms from an ε-space V to a $-\varepsilon$-space U, from which Z is built and which serve to unite the orthogonal and symplectic cases. In particular, section 2.3 contains the explicit construction of Z, which is not *a priori* an affine variety.

Chapter 3 contains the background and proof of part (a) of Theorem 2.13. It generalizes the proof given in [KP82] for K with characteristic zero. The key additions here are an explicit treatment of the classification of orthosymplectic ab-diagrams, and the existence of a cocharacter associated to the element $\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \in \mathrm{End}(U \oplus V)$ that takes values in $\mathrm{O}(U) \times \mathrm{Sp}(V)$. The associated cocharacter replaces the use of $\mathfrak{sl}_2$-triples in [KP82] in the proof of a dimension formula relating the dimension of an orthosymplectic orbit to its associated orthogonal and symplectic orbits (sections 3.3.2, 3.3.3, and 3.3.4).

In section 3.1, we introduce nilpotent pairs and go on to classify them via ab-diagrams. These serve as a classification of the orbits of elements in $\mathrm{Hom}(V, U)$ under $\mathrm{O}(U) \times \mathrm{Sp}(V)$, introduced in Chapter 2. Section 3.2 uses this classification to prove the existence of cocharacters associated to elements of $\mathrm{Hom}(V, U)$. Section 3.3 contains results on the dimension of such "orthosymplectic" orbits, relating them to the dimensions of symplectic and orthogonal groups. The main result of this section is Proposition 3.10, which generalizes the characteristic zero result in [KP82]. In section 3.4, we first describe Z as a fiber product, and then use this to relate the dimension of Z_{sing} to the dimension of orbits in our nilpotent orbit closure. Section 3.4.3 contains the proof of part (a) of Theorem 2.13, and shows that Z is a complete intersection in an affine space and is nonsingular in codimension 1, showing that Z is an affine variety, as well as a complete intersection.

Chapter 4 introduces the concept of good pairs of varieties and gives the proof of part (b) of Theorem 2.13. In [KP82], this analysis was not necessary since quotient maps necessarily restrict to quotient maps on a stable subvariety when working over characteristic zero. We follow the method of [Don90], which generalizes [KP79] to arbitrary characteristic, and utilize a property of good pairs of varieties to restrict a quotient map to $Z \to \overline{\mathcal{O}_X}$.

Section 4.1 introduces the concept of good filtrations, as well as good pairs of varieties, and presents general results, mainly taken directly from [Don90]. Section 4.2 contains results specific to the orthosymplectic case, in particular that V is a good $\mathrm{G}^0(V)$-variety [AJ84] section 4.9, and that $(\mathfrak{g}(V), \mathfrak{g}(V)_e)$ is a good pair of $\mathrm{G}^0(V)$-varieties, where $\mathfrak{g}(V)_e$ is the subvariety of matrices of rank less than or equal to e, and $\mathrm{G}^0(V)$ is the connected identity component of $\mathrm{G}(V)$, which differs from $\mathrm{G}(V)$ only in the orthogonal case. Section 4.3 completes the proof of Theorem 2.13 where we prove that the map $Z \to \overline{\mathcal{O}_D}$ is indeed a quotient for a suitable algebraic group.

In Chapter 5 we restate our results and their implications. In particular, we prove the final result of the paper: that $\overline{\mathcal{O}_X}$ is normal if and only if it is normal in classes of codimension 2. We also discuss further results in [KP82] and [Som05] that complete the classification of normal nilpotent orbit closures in characteristic zero for the symplectic and orthogonal groups.

Chapter 2

Foundational Material

We begin by introducing the background necessary for stating the main result of the paper, Theorem 2.13. We also give the classification of nilpotent orbits in the orthogonal and symplectic groups.

2.1 Nilpotent Orbits and their Classification

2.1.1 Quadratic Spaces

Let K be an algebraically closed field of characteristic $p \neq 2$ and V a finite dimensional vector space over K, where $\dim V = n$ for $n \in \mathbb{Z}_{>0}$. For ε either $+1$ or -1, a *quadratic space of type* ε is such a space V endowed with a non-degenerate bilinear form, denoted by $\langle\ ,\ \rangle$, such that $\langle u, v\rangle = \varepsilon\langle v, u\rangle$ for all $u, v \in V$. In the case that $\varepsilon = +1$, V is called an *orthogonal space*, and if $\varepsilon = -1$, V is called a *symplectic space*. We will often refer to such a quadratic space by V alone when the specific form is understood, though the notation $(V, \langle\ ,\ \rangle)$ is more precise and will be used when necessary. There is a more general definition of orthogonal spaces when $\operatorname{char} K = 2$, but we will not be concerning ourselves with it here.

Denote by $\mathrm{G}(V)$ the subgroup of the linear algebraic group $\mathrm{GL}(V)$ leaving the form invariant:

$$\mathrm{G}(V) = \{g \in \mathrm{GL}(V) \mid \langle gu, gv\rangle = \langle u, v\rangle \text{ for all } u, v \in V\}.$$

$G(V) \cong O_n$ or $G(V) \cong Sp_n$ when $\varepsilon = +1$ or -1 respectively. We will often use the notation $O(V)$ and $Sp(V)$ when $G(V)$ is ambiguous. When $\varepsilon = +1$, $G(V) = O(V)$ is not connected, and has connected component $O(V)^0 = SO(V)$, the subgroup of elements of $O(V)$ with determinant 1. (Elements of $O(V) \setminus SO(V)$ have determinant -1. When $\varepsilon = -1$, $Sp(V)$ is connected and all elements have determinant 1.) $SO(V)$ will play a role in what follows, though we will largely work with the full orthogonal group.

The Lie algebra $\mathfrak{g}(V)$ of $G(V)$ consists of the elements of $\mathfrak{gl}(V) = \operatorname{Lie} GL(V)$ that are *skew* with respect to the form:

$$\mathfrak{g}(V) := \operatorname{Lie} G(V) = \{X \in \mathfrak{gl}(V) \mid \langle Xu, v\rangle = -\langle u, Xv\rangle \text{ for all } u, v \in V\}$$

and we have that $\mathfrak{g}(V) = \mathfrak{so}(V) \cong \mathfrak{so}_n$ when $\varepsilon = +1$, and $\mathfrak{g}(V) = \mathfrak{sp}(V) \cong \mathfrak{sp}_n$ when $\varepsilon = -1$.

For any $X \in \operatorname{End} V$, there exists a unique $X^* \in \operatorname{End} V$ called the *adjoint* of X, defined by

$$\langle Xu, v\rangle = \langle u, X^* v\rangle \text{ for } u, v \in V.$$

We have that $(XY)^* = Y^* X^*$ and that $(X^*)^* = X$, hence the map taking $X \in \operatorname{End} V$ to its adjoint is an *involution*. From the above definitions, $G(V)$ is the set of all $g \in GL(V)$ such that $g^* = g^{-1}$, and $\mathfrak{g}(V)$ is the set of all $X \in \mathfrak{gl}(V) = \operatorname{End}(V)$ such that $X^* = -X$. Choosing an appropriate basis for V, one can compute that

$$\dim G(V) = \dim \mathfrak{g}(V) = \frac{n^2 - \varepsilon n}{2}.$$

2.1.2 Classification of Orbits

$G(V)$ acts on $\mathfrak{g}(V)$ via the adjoint representation, i.e. by conjugation in $\operatorname{End} V$: $\operatorname{Ad}(g)X = gXg^{-1}$ for $g \in G(V)$, $X \in \mathfrak{g}(V)$. Denote by $\mathcal{O}_X$ the adjoint orbit of $X \in \mathfrak{g}(V)$:

$$\mathcal{O}_X = \operatorname{Ad}(G)X = \{gXg^{-1} \mid g \in G(V)\}.$$

If $X \in \mathfrak{g}(V)$ is nilpotent, the theory of Jordan canonical form says that there exists a basis for V such that the matrix of X with respect to this basis has block form

$$\begin{pmatrix} J_{\lambda_1} & 0 & \dots & 0 \\ 0 & J_{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{\lambda_r} \end{pmatrix}$$

for suitable integers $\lambda_i > 0$. For $m \in \mathbb{Z}_{\geq 1}$, J_m is the $(m \times m)$-matrix with the $(i, i+1)$-entries equal to one $(1 \leq i < m)$ and all remaining entries equal to zero (the $m \times m$ Jordan block).

We may insist that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$, in which case the Jordan canonical form of X is unique. The λ_i form a partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of n, which we can also write $\lambda = [1^{r_1}\, 2^{r_2}\, 3^{r_3} \ldots]$ where r_1 is the number of i with $\lambda_i = 1$, r_2 is the number of i with $\lambda_i = 2$, etc. Since the r_i are uniquely determined by X, we may refer to λ as *the partition of* X. It is a basic result of the theory of the Jordan canonical form that two nilpotent elements in $\operatorname{End} V$ belong to the same adjoint $\mathrm{GL}(V)$-orbit if and only if they have the same partition. The following theorems from [Jan04, 1.4 and 1.6] classify the orbits of nilpotent elements of $\mathfrak{g}(V)$ under the adjoint action of $\mathrm{G}(V)$:

Theorem 2.1. *Two elements in* $\mathfrak{g}(V)$ *belong to the same* $\mathrm{G}(V)$*-orbit if and only if they belong to the same* $\mathrm{GL}(V)$*-orbit.*

Theorem 2.2. *Let* $\lambda = [1^{r_1}\, 2^{r_2}\, 3^{r_3} \ldots]$ *be a partition of* $n = \dim V$*,* V *a quadratic space of type* ε*. Then there exists a nilpotent element in* $\mathfrak{g}(V)$ *with partition* λ *if and only if* r_i *is even for* $i \equiv \frac{1-\varepsilon}{2} \bmod 2$*, i.e.*

(a) if V *is an orthogonal space, then there exists a nilpotent element in* $\mathfrak{g}(V)$ *with partition* λ *if and only if* r_i *is even for all even* i *(even parts occur an even number of times).*

(b) if V *is a symplectic space, then there exists a nilpotent element in* $\mathfrak{g}(V)$ *with*

partition λ if and only if r_i is even for all odd i (odd parts occur an even number of times).

A partition corresponding to a nilpotent element in $\mathfrak{g}(V)$ as described in Theorem 2.2 is called an *ε-partition* or an *ε-diagram* (the terminology will become clear in section 2.1.3).

In addition to classifying the open nilpotent orbits, ε-diagrams provide a classification for the orbit closures, since the closure of the $\mathrm{G}(V)$-orbit of an element of $\mathrm{End}\, V$ is equal to the intersection of $\mathfrak{g}(V)$ with the closure of the $\mathrm{GL}(V)$-orbit of that element. This follows from Theorems 2.1 and 2.2, which imply that $\mathrm{Ad}(\mathrm{G}(V))X = \mathrm{Ad}(\mathrm{GL}(V))X \cap \mathfrak{g}(V)$ for $X \in \mathfrak{g}(V)$ (the right-hand side is empty otherwise), and the characterization of an orbit closure in Proposition 2.6 as a finite union of orbits. In other words,

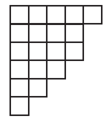
$$\overline{\mathcal{O}_X} := \overline{\mathrm{Ad}(\mathrm{G}(V))X} = \overline{\mathrm{Ad}(\mathrm{GL}(V))X} \cap \mathfrak{g}(V) \quad \text{for } X \in \mathrm{End}\, V.$$

Remark 2.3. A conjugacy class $\mathcal{O}_X$ under the orthogonal group is connected if and only if $\mathcal{O}_X$ is also a conjugacy class under the special orthogonal group $\mathrm{SO}(V) = \mathrm{O}(V)^0$. Jantzen discusses this distinction in [Jan04, 1.12], and concludes that a conjugacy class $\mathcal{O}_X$ under $\mathrm{G}(V)$ is disconnected if and only if V is an orthogonal space ($\varepsilon = +1$), and the partition of X is *very even*, i.e. if $r_i = 0$ for all odd i (all rows are of even length) and r_i is even for all even i (because we are considering the partition of an element in the orthogonal group). In this case, $n = \dim V \equiv 0 \bmod 4$, and $\mathcal{O}_X$ splits into two conjugacy classes, $\mathcal{O}_1$ and $\mathcal{O}_2$, with respect to $\mathrm{SO}(V)$. The normality of orbit closures under the action of the special orthogonal group was established for $\mathrm{char}\, K = 0$ in [Som05].

2.1.3 Young Diagrams

A useful way of representing partitions visually is as *Young diagrams* or *Young tableaux*. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of n with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$, the Young diagram of λ is an array with $l(\lambda) = r$ rows of boxes and λ_i boxes in the i^{th}

row. For example, the partition $(5,4,4,3,2,1) = [1\,2\,3\,4^2\,5]$ has Young diagram



The *dual partition* $\hat{\lambda}$ of λ is the partition $(\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_s)$ of $|\lambda| := n = \dim V$ with $\hat{\lambda}_i$ equal to the number of boxes in the i^{th} column of the Young diagram of λ (alternately, $\hat{\lambda}_i = \sum_{j \geq i} r_j$). The dual partition provides a concise way of describing the nullspace of a nilpotent endomorphism: $\dim \operatorname{Ker} X^j = \sum_{i=1}^{j} \hat{\lambda}_i$ for $X \in \operatorname{End} V$ with partition λ. Equivalently, $\operatorname{rk} X^j = \sum_{i>j} \hat{\lambda}_i$.

The following proposition describes the dimensions of a conjugacy class $\mathcal{O}_X$ in terms of the dimension of the centralizer $\mathrm{G}(V)_X = \{g \in \mathrm{G}(V) \mid gX = Xg\}$ of X. These are both given explicitly in terms of the Young diagram λ of X.

Proposition 2.4. *Let V be a quadratic space of type ε and $X \in \mathfrak{g}(V)$ a nilpotent element with associated Young diagram λ. Then*

$$\dim \mathrm{G}(V)_X = \frac{1}{2}\Big(\sum_i \hat{\lambda}_i^2 - \varepsilon \sum_{i \text{ odd}} r_i\Big)$$

and

$$\dim \mathcal{O}_X = \frac{1}{2}\Big(|\lambda|^2 - \varepsilon|\lambda| - \sum_i \hat{\lambda}_i^2 + \varepsilon \sum_{i \text{ odd}} r_i\Big)$$

Proof. The first equation follows from [Jan04] Theorem 2.5 and equation 3.2(4) rewritten in terms of the dual partition. The second equation follows from the first since $\dim \mathcal{O}_X = \dim \mathrm{G}(V) - \dim \mathrm{G}(V)_X$, where $|\lambda| = n = \dim V$. □

Remark 2.5. We also have from [Jan04, 2.5, 3.1(3)] that $\dim \mathrm{GL}(V)_X = \sum_i \hat{\lambda}_i^2$, hence

$$\dim \mathrm{GL}(V)_X = 2 \dim \mathrm{G}(V)_X + \varepsilon \sum_{i \text{ odd}} r_i$$

and

$$\dim(\mathrm{Ad}(\mathrm{GL}(V))X) = 2\dim \mathcal{O}_X + \varepsilon(|\lambda| - \sum_{i \text{ odd}} r_i)$$

2.1.4 ε-Degenerations

Let λ be an ε-diagram, i.e. the partition of a nilpotent element $X \in \mathfrak{g}(V)$, V a space of type ε. Denote by $\mathcal{O}_\lambda$ the orbit of such an element. Note that this is well defined, since two elements $X, Y \in \mathfrak{g}(V)$ have the same partition λ if and only if they belong to the same conjugacy class $\mathcal{O}_X = \mathcal{O}_Y = \mathcal{O}_\lambda$ under conjugation by $\mathrm{G}(V)$.

Given an ε-diagram, we define an ε-*degeneration* of λ to be an ε-diagram μ such that $|\mu| = |\lambda|$ and $\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_\lambda}$. We write $\mu \leq \lambda$.

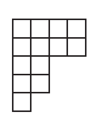
The following proposition gives a description of those conjugacy classes contained in the closure of another in terms of their ε-diagrams.

Proposition 2.6. *Given two ε-diagrams μ and λ with $|\mu| = |\lambda|$, we have $\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_\lambda}$ if and only if $\sum_{i=1}^{j} \mu_i \leq \sum_{i=1}^{j} \lambda_i$ for all j. Equivalently, $\sum_{k>j} \hat{\mu}_k \leq \sum_{k>j} \hat{\lambda}_k$ or $\mathrm{rk}\, Y^j \leq \mathrm{rk}\, X^j$ for all j, Y with partition μ, and X with partition λ.*

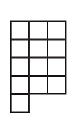
Proof. This is Theorem 3.10 of [Hes76], which holds for $\mathrm{char}\, K \neq 2$. □

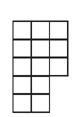
An ε-degeneration $\mu \leq \lambda$ is called *minimal* if $\mu \neq \lambda$ and there is no ε-diagram ν such that $\mu < \nu < \lambda$. The conjugacy class $\mathcal{O}_\mu$ is then open in the complement $\overline{\mathcal{O}_\lambda} \setminus \mathcal{O}_\lambda$ ([KP82] section 3.1).

In terms of Young tableaux, an ε-degeneration of λ is obtained by moving boxes down to another row so that the result is another ε-diagram. For example, when $\varepsilon = 1$, let $\lambda = (4, 4, 2, 2, 1)$:

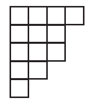


Two ε-degenerations are $\mu = (3, 3, 3, 3, 1)$ and $\nu = (3, 3, 3, 2, 2)$:





μ is obtained from λ by moving the boxes on the ends of the top two rows down to the third and fourth rows. It is a minimal ε-degeneration since the diagram $(4, 3, 3, 2, 1)$ obtained by moving down one box from row two to row three is not an ε-diagram:



ν is obtained from λ by moving the boxes on the ends of the top two rows down to the third and last rows. ν is a minimal ε-degeneration of μ.

Kraft and Procesi classify the *irreducible minimal ε-degenerations* in [KP82] Section 3. They are used to classify those conjugacy classes $\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_\lambda} \setminus \mathcal{O}_\lambda$ of codimension two, the importance of which will become clear later.

2.2 Uniting the Orthogonal and Symplectic Cases

2.2.1 Hom(V,U)

Broadening our setting slightly, if V and U are quadratic spaces of type ε and ε' respectively, denote by $\mathrm{L}(V, U)$ the set of linear maps from V to U. $\mathrm{L}(V, U)$ identifies in a natural way (by choosing bases) with the set of $(m \times n)$ matrices, where $n := \dim V, m := \dim U$. Here, we define the adjoint of $X \in \mathrm{L}(V, U)$ to be the (linear) map $X^* : U \to V$ such that

$$\langle Xv, u\rangle_U = \langle v, X^*u\rangle_V \text{ for } v \in V, u \in U$$

where $\langle\ ,\ \rangle_U$ and $\langle\ ,\ \rangle_V$ are the bilinear forms on U and V respectively. $(X^*)^* \in \mathrm{L}(V, U)$ again, and we can easily compute that $(X^*)^* = \varepsilon \cdot \varepsilon' \cdot X$:

$$\langle Xv, u\rangle_U = \langle v, X^*u\rangle_V = \varepsilon\langle X^*u, v\rangle_V = \varepsilon\langle u, (X^*)^*v\rangle_U = \varepsilon \cdot \varepsilon'\langle (X^*)^*v, u\rangle_U.$$

Let $\varepsilon' = -\varepsilon$, i.e. suppose that V and U are of opposite types, and let $X \in \mathrm{L}(V, U)$. Using the characterization of elements of $\mathfrak{g}(V)$ and $\mathfrak{g}(U)$ from section

2.1.1, we have that $XX^* \in \mathfrak{g}(U)$ and $X^*X \in \mathfrak{g}(V)$:

$$(XX^*)^* = (X^*)^*X^* = -XX^* \quad \text{and} \quad (X^*X)^* = X^*(X^*)^* = -X^*X$$

since here $\varepsilon \cdot \varepsilon' = \varepsilon \cdot -\varepsilon = -1$. We can therefore define maps $\pi : \mathrm{L}(V,U) \to \mathfrak{g}(U)$ and $\rho : \mathrm{L}(V,U) \to \mathfrak{g}(V)$ by $\pi(X) := XX^*$ and $\rho(X) := X^*X$.

$$\begin{array}{ccc} \mathrm{L}(V,U) & \xrightarrow{\pi} & \mathfrak{g}(U) \\ {\scriptstyle\rho}\downarrow & & \\ \mathfrak{g}(V) & & \end{array}$$

π and ρ are equivariant with respect to the action of $\mathrm{G}(U) \times \mathrm{G}(V)$ on $\mathrm{L}(V,U)$, given by $(g,h)X := gXh^{-1}$ for $g \in \mathrm{G}(U), h \in \mathrm{G}(V), X \in \mathrm{L}(V,U)$, and the adjoint operation of $\mathrm{G}(U)$ and $\mathrm{G}(V)$ on $\mathfrak{g}(U)$ and $\mathfrak{g}(V)$ respectively:

$$\pi((g,h)X) = \pi(gXh^{-1}) = gXh^{-1}(gXh^{-1})^*$$

The elements g and h are in $\mathrm{G}(U)$ and $\mathrm{G}(V)$, respectively, and so $h^* = h^{-1}$ and $g^* = g^{-1}$ as in section 2.1.1. Hence

$$gXh^{-1}(gXh^{-1})^* = gXh^{-1}hX^*g^{-1} = gXX^*g^{-1} = g.\pi(X)$$

where the action of $\mathrm{G}(U)$ on $\mathfrak{g}(U)$ is conjugation. Likewise for the map ρ we have

$$\rho((g,h)X) = (gXh^{-1})^*gXh^{-1} = hX^*g^{-1}gXh^{-1} = h.\rho(X).$$

The space $\mathrm{L}(V,U)$ is essential in what follows since it provides a way to unify the symplectic and orthogonal cases. The $\mathrm{G}(V) \times \mathrm{G}(U)$-equivariant maps π and ρ will help us translate back "down" to orbits in $\mathfrak{g}(V)$ and $\mathfrak{g}(U)$. We will always use the setting above, where V and U are of opposite types. For our purposes, it is only necessary to start with a space V of type ε and build an appropriate U. Also essential is the following observation:

Theorem 2.7. *Let V be an ε-space and U a $-\varepsilon$-space with $m = \dim(U) \leq \dim(V)$. The map $\rho : \mathrm{L}(V, U) \to \mathfrak{g}(V)_m$ given by $\rho(X) = X^*X$ is a $\mathrm{G}(U)$-quotient for the action $g.X = gX$ of $\mathrm{G}(U)$ on $\mathrm{L}(V, U)$, where $\mathfrak{g}(V)_m$ is the subvariety of matrices of rank at most m. The map $\pi : \mathrm{L}(V, U) \to \mathfrak{g}(U)$ given by $\pi(X) = XX^*$ is a $\mathrm{G}(V)$-quotient map for the action $h.X = Xh^{-1}$ of $\mathrm{G}(V)$ on $\mathrm{L}(V, U)$.*

Proof. This is Theorem 1.2 in [KP82], which is reformulated from the characteristic free proof due to De Concini-Procesi, [dCP76] Theorem 5.6(i) and Theorem 6.6. □

2.2.2 Decomposing a Nilpotent Element

We begin with a quadratic space V of type ε and a nilpotent element $D \in \mathfrak{g}(V)$ with conjugacy class $\mathcal{O}_D$ as in section 2.1.2. Define $|u, v| := \langle u, Dv \rangle$ a bilinear form on V, which is degenerate with kernel $\ker D$. If we define $U := \operatorname{Im} D \cong V/\ker D$, $|\,,\,|$ defines a non-degenerate form on U of type $-\varepsilon$:

$$|v, u| = \langle v, Du \rangle = \varepsilon \langle Du, v \rangle = \varepsilon \langle u, D^* v \rangle = \varepsilon \langle u, -Dv \rangle = -\varepsilon \langle u, Dv \rangle = -\varepsilon |u, v|.$$

Decompose $D \in \mathfrak{g}(V)$ via $D = I \circ X$ where $X \in \mathrm{L}(V, U)$ is the composition

$$V \xrightarrow{proj} V/\ker D \xrightarrow{D} \operatorname{Im} D = U$$

of D with the projection, the second map being an isomorphism. In other words, X is just the map D with the codomain replaced with the image U of D. I is then the inclusion $U \cong V/\ker D \hookrightarrow V$. I and X are adjoint as in section 2.2.1, i.e. $X^* = I$:

$$\langle Xv, u \rangle_U = \langle [v], [w] \rangle_{V/\ker D} = |v, w| = \langle v, Iu \rangle_V$$

where $v, w \in V, u \in U = \operatorname{Im} D, w$ such that $Dw = Iu \in V$, and $[v]$ and $[w]$ are the class of v, w in $V/\ker D$ respectively. Then $D = IX = X^*X$ and $D' := D|_U = XI = XX^* \in \mathfrak{g}(U)$.

If D has partition λ of $n = \dim V$, D' has partition λ' of $m := n - l(\lambda)$ where

$l(\lambda)$ is the *length* of the partition, i.e. $\sum r_i$ if we write $\lambda = [1^{r_1} 2^{r_2} \ldots]$. We have $m = \dim U$ since D has kernel with dimension $l(\lambda)$. In fact, λ' can be obtained from λ by erasing the first column of the Young diagram of λ. To see this, write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. Then there exists a basis for V of the form

$$\{D^j v_i \mid 1 \leq i \leq r, 0 \leq j \leq \lambda_i - 1\} = \{v_1, Dv_1, \ldots, D^{\lambda_1 - 1} v_1, \ldots, v_r, \ldots, D^{\lambda_r - 1} v_r\}.$$

By definition, $D^{\lambda_i} v_i = 0$. After applying D, U then has basis

$$\{Dv_1, D^2 v_1, \ldots, D^{\lambda_1 - 1} v_1, \ldots, Dv_r, \ldots, D^{\lambda_r - 1} v_r\},$$

with respect to which $D' = D|_U$ has matrix in Jordan canonical form described by the partition $(\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_r - 1) =: \lambda'$.

2.2.3 Orbits in Hom(V,U)

Recall from section 2.2.1 the maps π and ρ:

$$\begin{array}{ccc} \mathrm{L}(V,U) & \xrightarrow{\pi} & \mathfrak{g}(U) \\ \rho \downarrow & & \\ \mathfrak{g}(V) & & \end{array}$$

We are interested in orbits and orbit closures, so we examine the behavior of orbits under π and ρ.

Using the notation from section 2.1.4, let $N_\lambda := \pi^{-1}(\overline{\mathcal{O}_{\lambda'}})$, where $\mathcal{O}_{\lambda'} \subset \mathfrak{g}(U)$. Here U a space of type $-\varepsilon$ constructed from a space V of type ε and a nilpotent element $D \in \mathfrak{g}(V)$ as in section 2.2.2, so that λ' is the $-\varepsilon$-diagram obtained from the ε-diagram λ by removing the first column, where λ is the ε-diagram associated to the orbit of the element D.

$D \in \mathfrak{g}(V)$ determines the surjective map X from V to U by the construction in section 2.2.2. Define $\mathrm{L}'(V,U) := \{Y \in \mathrm{L}(V,U) \mid Y \text{ is surjective}\}$.

Proposition 2.8. *(a)* $\rho(N_\lambda) = \overline{\mathcal{O}_\lambda}$.

(b) $\rho^{-1}(\mathcal{O}_\lambda)$ *is a single orbit under* $\mathrm{G}(U) \times \mathrm{G}(V)$ *contained in* $N_\lambda \cap \mathrm{L}'(V,U)$.

(c) $\pi(\rho^{-1}(\mathcal{O}_\lambda)) = \mathcal{O}_{\lambda'}$.

$$\text{i.e.}\quad \begin{array}{ccc} N_\lambda & \xrightarrow{\pi} & \overline{\mathcal{O}_{\lambda'}} \\ {\scriptstyle \rho}\downarrow & & \\ \overline{\mathcal{O}_\lambda} & & \end{array}$$

We begin first with a lemma regarding surjective maps $V \to U$.

Lemma 2.9. *For any map* $Y \in \mathrm{L}'(V,U)$*, the stabilizer of* Y *in* $\mathrm{G}(U)$ *is trivial and* $\rho^{-1}(\rho(Y))$ *is an orbit under* $\mathrm{G}(U)$.

Proof. Recall the action of $\mathrm{G}(U) \times \mathrm{G}(V)$ on $\mathrm{L}(V,U)$: $(g,h)X = gXh^{-1}$ for $g \in \mathrm{G}(U), h \in \mathrm{G}(V), X \in \mathrm{L}(V,U)$. So the action of $\mathrm{G}(U)$ on $\mathrm{L}(V,U)$ is left multiplication and $\mathrm{Stab}_{\mathrm{G}(U)}(X) = \{g \in \mathrm{G}(U) \mid gX = X\}$. For such g we have a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{X} & U \\ & {\scriptstyle X}\searrow & \uparrow{\scriptstyle g} \\ & & U \end{array}$$

and for Y surjective, this necessitates $g = id$ for $g \in \mathrm{Stab}_{\mathrm{G}(U)}(Y)$. Hence the stabilizer of $y \in \mathrm{L}'(V,U)$ is trivial.

For the second claim, let $X \in \rho^{-1}(\rho(Y))$, meaning that $X^*X = Y^*Y$. Y is surjective and so Y^*Y has maximal rank $m = \dim U$. This forces X to be surjective and X^* injective. Hence $\ker X = \ker X^*X = \ker Y^*Y = \ker Y$ and we can find $g \in \mathrm{GL}(U)$ such that $gX = Y$. We have then $X^*X = Y^*Y = X^*g^*gX$. The injectivity and surjectivity of X^* and X respectively implies that $g^*g = 1$, hence $g \in \mathrm{G}(U)$ and X is in the $\mathrm{G}(U)$-orbit of Y. □

Remark 2.10. It is worth noting that the preimage $\rho^{-1}(\mathcal{O}_\mu)$ for $\mu \leq \lambda$ need not be a single orbit under the action of $\mathrm{G}(U) \times \mathrm{G}(V)$. These will be described in more detail in Section 3.4.2 and Appendix B in terms of *ab*-diagrams.

Proof of Proposition 2.8. Let $(g,h) \in \mathrm{G}(U) \times \mathrm{G}(V)$ and $X \in N_\lambda$, i.e. $X \in \mathrm{L}(V,U)$ such that $XX^* \in \overline{\mathcal{O}_{\lambda'}}$. Then

$$gXh^{-1}(gXh^{-1})^* = gXh^{-1}(h^{-1})^*X^*g^* = gXX^*g^* = gXX^*g^{-1}$$

since $h \in \mathrm{G}(V), g \in \mathrm{G}(V)$ and so $h^* = h^{-1}, g^* = g^{-1}$. Now $\overline{\mathcal{O}_{\lambda'}}$ is a union of conjugacy classes under $\mathrm{G}(U)$, and so $XX^* \in \overline{\mathcal{O}_{\lambda'}}$ implies $gXX^*g^{-1} \in \overline{\mathcal{O}_{\lambda'}}$. Hence N_λ is stable under $\mathrm{G}(U) \times \mathrm{G}(V)$. N_λ is closed since π is open.

If we fix $X \in \mathrm{L}(V,U)$ as constructed in section 2.2.2 from a nilpotent element $D \in \mathfrak{g}(V)$ with partition λ, we have that $\rho(X) = D \in \mathcal{O}_\lambda$ and $\pi(X) = D|_U \in \mathcal{O}_{\lambda'}$. Thus $\mathcal{O}_\lambda \subseteq \rho(N_\lambda)$ and $\overline{\mathcal{O}_\lambda} \subseteq \rho(N_\lambda)$, the image $\rho(N_\lambda)$ being closed as ρ is a quotient map. Conversely, for each $X \in N_\lambda$, $\pi(X) = XX^* \in \overline{\mathcal{O}_{\lambda'}}$, and combining sections 2.1.3 and 2.1.4, we have

$$\mathrm{rk}(XX^*)^{h-1} \leq \sum_{j \geq h} \hat{\lambda}'_j = \sum_{j > h} \hat{\lambda}_j,$$

$\hat{\lambda}'_i = \hat{\lambda}_{i+1}$ by construction. We then have

$$\mathrm{rk}(X^*X)^h = \mathrm{rk}\, X^*(XX^*)^{h-1}X \leq \mathrm{rk}(XX^*)^{h-1} \leq \sum_{j>h} \hat{\lambda}_j.$$

Thus $\rho(X) = X^*X \in \overline{\mathcal{O}_\lambda}$, hence (a).

As stated, X as determined by D is surjective and by Lemma 2.9, $\rho^{-1}(\rho(X))$ is the orbit of X under $\mathrm{G}(U)$. It follows that $\rho^{-1}(\mathcal{O}_\lambda)$ is the orbit of X under $\mathrm{G}(U) \times \mathrm{G}(V)$, the elements of $\mathcal{O}_\lambda$ having the form $hX^*Xh^{-1} = (Xh^{-1})^*Xh^{-1}$ for $h \in \mathrm{G}(V)$, Xh^{-1} surjective. This then implies (b) and (c). □

2.3 The Variety Z

2.3.1 Defining the V_i

Iterating the process in section 2.2.2, we now construct a variety Z sitting "above" the closure of a given nilpotent orbit. The goal is to find an affine variety that is normal and such that there is a quotient map from it to $\overline{\mathcal{O}_D}$.

Start with a nilpotent element $D \in \mathfrak{g}(V)$ with conjugacy class $\mathcal{O}_D = \mathcal{O}_\lambda$, i.e. λ the partition of $n = \dim V$ associated to D, V a quadratic space of type ε. D decomposes (canonically) as $D = I \circ X : V \to D(V) \hookrightarrow V$, and there exists a non-degenerate form of type $-\varepsilon$ on $D(V)$ such that X and I are adjoint, and such that $D|_{D(V)} = X \circ I = XX^*$ is skew (see section 2.2.2 for further details).

Let $V_0 = V$ and $X_1 = X$. Define the space V_i to be the image $D(V_{i-1})$, so that

$$V_0 := V, V_1 := D(V), \ldots, V_i := D^i(V), \ldots, V_t := D^t(V).$$

Each V_i is a quadratic space of type $(-1)^i\varepsilon$. We have $V_{t+1} = 0$ for some minimal $t \in \mathbb{N}$ since D is nilpotent. In the previous paragraph, $D = D|_{V_0} = X_1^*X_1$ and $D|_{V_1} = X_1X_1^*$. We then decompose each $D|_{V_i} = X_iX_i^*$, $X_i \in \mathrm{Hom}(V_{i-1}, V_i)$, as above into the composition $X_{i+1}^*X_{i+1}$, $X_{i+1} \in \mathrm{Hom}(V_i, V_{i+1})$. Note that we naturally have $X_iX_i^* = X_{i+1}^*X_{i+1}$, both equal to the skew endomorphism $D|_{V_i}$ belonging to the conjugacy class $\mathcal{O}_{\lambda^i}$, where

$$\lambda^0 := \lambda, \lambda^1 := \lambda', \ldots, \lambda^i := (\lambda^{i-1})', \ldots, \lambda^t = [1^{\dim V_t}].$$

If we denote by n_i the dimension of V_i, it is easy to see that $n_i = n_{i-1} - l(\lambda_{i-1}) = |\lambda_i|$, where $l(\lambda_i)$ is the *length* (number of rows) of λ_i.

2.3.2 Definition of Z

Define $\mathrm{L}(V_{i-1}, V_i) = \mathrm{Hom}(V_{i-1}, V_i)$, the variety of $(n_i \times n_{i-1})$-matrices for $1 \leq i \leq t$. Let $M := \mathrm{L}(V_0, V_1) \times \mathrm{L}(V_1, V_2) \times \ldots \times \mathrm{L}(V_{t-1}, V_t)$ be their direct product.

Consider the affine subscheme $Z \subseteq M$ of tuples $(X_1, \dots, X_t)$ satisfying the equations $(*)$ below. We will prove that the equations $(*)$ define a reduced ideal, so that Z can be viewed as an affine variety.

$$(*) \qquad \begin{aligned} X_1 X_1^* &= X_2^* X_2 \\ X_2 X_2^* &= X_3^* X_3 \\ &\vdots \\ X_{t-1} X_{t-1}^* &= X_t^* X_t \\ X_t X_t^* &= 0 \end{aligned}$$

Here $X_i \in \mathrm{L}(V_{i-1}, V_i), \pi(X_i) = X_i X_i^* \in \mathfrak{g}(V_i), \rho(X_i) = X_i^* X_i \in \mathfrak{g}(V_{i-1})$. The following diagram helps to understand the equations in $(*)$:

$$V_{i-1} \underset{X_i^*}{\overset{X_i}{\rightleftarrows}} V_i \underset{X_{i+1}^*}{\overset{X_{i+1}}{\rightleftarrows}} V_{i+1}$$

where the two compositions $X_i X_i^*$ and $X_{i+1}^* X_{i+1}$ from V_i to V_i are both equal as in the previous section. We will see below that they are equal to the restriction $\widetilde{D}|_{V_i}$ of some nilpotent element $\widetilde{D} \in \mathfrak{g}(V)$ with $\operatorname{rk} \widetilde{D}^h \leq \operatorname{rk} D^h$ for all $h \in \mathbb{Z}_{\geq 0}$, i.e. some $\widetilde{D} \in \overline{\mathcal{O}_D}$.

2.3.3 About Z

As stated above, each V_i is a quadratic space of type $(-1)^i \varepsilon$, so we have $\mathrm{G}(V) = \mathrm{G}(V_0), \mathrm{G}(V_1), \mathrm{G}(V_2) \dots, \mathrm{G}(V_t)$, symmetric and orthogonal groups of alternating type. Define the action of $\mathrm{G}(V_0) \times \mathrm{G}(V_1) \times \dots \times \mathrm{G}(V_t)$ on M via

$$(g_0, g_1, \dots, g_t)(X_1, \dots, X_t) = (g_1 X_1 g_0^{-1}, g_2 X_2 g_1^{-1}, \dots, g_t X_t g_{t-1}^{-1}).$$

Z is then a stable subset of M. Indeed, if $(X_1, X_2, \dots, X_t) \in Z$,

$$g_i X_i g_{i-1}^{-1} (g_i X_i g_{i-1}^{-1})^* = g_i X_i g_{i-1}^{-1} (g_{i-1}^{-1})^* X_i^* g_i^* = g_i X_i X_i^* g_i^{-1} = g_i X_{i+1}^* X_{i+1} g_i^{-1} = \\ g_i X_{i+1}^* (g_{i+1}^{-1} g_{i+1}) X_{i+1} g_i^{-1} = (g_{i+1} X_{i+1} g_i^{-1})^* g_{i+1} X_{i+1} g_i^{-1}.$$

Once we know that the scheme Z is reduced, it follows that Z is $\mathrm{G}(V_0)\times\ldots\times\mathrm{G}(V_t)$-stable.

Moreover, we can define a map $\vartheta : Z \to \overline{\mathcal{O}_D}$ by $\vartheta(X_1, X_2, \ldots, X_t) = X_1^* X_1$. Indeed, for any $(X_1, X_2, \ldots, X_t) \in Z$,

$$\operatorname{rk}(X_1^*X_1)^h = \operatorname{rk} X_1^*(X_1X_1^*)^{h-1}X_1 \leq \operatorname{rk}(X_1X_1^*)^{h-1} = \operatorname{rk}(X_2^*X_2)^{h-1} =$$
$$\operatorname{rk} X_2^*(X_2X_2^*)^{h-2}X_2 \leq \ldots \leq \operatorname{rk}(X_{h+1}^*X_{h+1})^0 = \operatorname{rk}(id|_{V_h})$$

i.e.

$$\operatorname{rk}(X_1^*X_1)^h \leq \dim V_h = \operatorname{rk} D^h,$$

implying by section 2.1.4 that $X_1^*X_1 \in \overline{\mathcal{O}_D}$. Heuristically, the elements in Z are the tuples that arise from the X_i as in the earlier construction (without altering the V_i's), beginning with a nilpotent element with ε-diagram less than or equal to that of the original D, meaning those elements contained in the closure $\overline{\mathcal{O}_D}$ (see section 2.1.4).

To see that $\mathcal{O}_D$ is contained in the image of ϑ, consider the element $(X_1, X_2, \ldots, X_t)$ of M with X_i the linear map $V_{i-1} \to V_i$ defined in the decomposition of $D|_{V_{i-1}}$. It is in Z by construction and $D = X_1^*X_1$. ϑ is $\mathrm{G}(V_0)$-equivariant by construction, and so $\mathcal{O}_D \subseteq \vartheta(Z)$. We will see in the proof of Theorem 2.13(b) that $\vartheta : M \to \mathfrak{g}(V)$ restricts to a quotient map on Z. Then $\vartheta(Z)$ is a closed subvariety of $\mathfrak{g}(V)$ and since $\mathcal{O}_D \subseteq \vartheta(Z)$, $\overline{\mathcal{O}_D} = \vartheta(Z)$.

Remark 2.11. An alternate method of defining Z as a scheme is as the fiber over 0 of the map $\zeta : M \to N := \mathfrak{g}(V_1) \times \mathfrak{g}(V_2) \times \ldots \times \mathfrak{g}(V_t)$ given by

$$\zeta(X_1, X_2, \ldots, X_t) = (X_1X_1^* - X_2^*X_2, X_2X_2^* - X_3^*X_3, \ldots, X_tX_t^*).$$

Remark 2.12. Note that the i^{th} equation in $(*)$ generates $\dim \mathfrak{g}(V_i)$ equations in $n_{i-1} \times n_i$ affine coordinates. This can be seen inductively, starting with the last (the t^{th}) equation

$$X_tX_t^* = 0.$$

We have $X_tX_t^* \in \mathfrak{g}(V_t)$, and the map $\pi : \mathrm{L}(V_{t-1}, V_t) \to \mathfrak{g}(V_t)$ given by $X \mapsto$

XX^* is onto by Theorem 2.7, so $X_tX_t^*$ gives $\dim \mathfrak{g}(V_t)$ quadratic expressions in the $n_{t-1} \times n_t$ affine coordinates corresponding to the entries in an arbitrary X_t, each equal to 0.

In the i^{th} equation, $X_iX_i^* \in \mathfrak{g}(V_i)$ and $\pi : \mathrm{L}(V_{i-1}, V_i) \to \mathfrak{g}(V_i)$ is onto, so there are $\dim \mathfrak{g}(V_i)$ expressions in the $n_{i-1} \times n_i$ affine coordinates of the entries of X_i, each equal to the corresponding entry in $X_{i+1}^*X_{i+1} \in \mathfrak{g}(V_i)$.

2.3.4 Statement of the Main Theorem

We can now formulate our key theorem, which implies that the variety $\overline{\mathcal{O}_D}$ is normal provided that Z is nonsingular in codimension 1, which is not true for all nilpotent $D \in \mathfrak{g}(V)$. Specific instances where Z is normal will be discussed in Chapter 5.

Theorem 2.13. *Let* $\mathrm{G}(V)$ *be a symplectic or orthogonal group for a finite dimensional vector space* V *over a field* K *of prime characteristic* $p \neq 2$*. For a nilpotent element* $D \in \mathfrak{g}(V)$*, construct* M *and* Z *as in sections 2.3.1 and 2.3.2 above.*

(a) The affine scheme Z *is reduced, hence* Z *is an affine variety. Moreover,* Z *is a complete intersection in* M *with respect to the equations* $(*)$*.*

(b) The map $\vartheta : Z \to \overline{\mathcal{O}_D}$ *given by* $\vartheta(X_1, \dots, X_t) = X_1^*X_1$ *is* $\mathrm{G}(V_0) \times \dots \times \mathrm{G}(V_t)$*-equivariant and a* $\mathrm{G}(V_1) \times \dots \times \mathrm{G}(V_t)$*-quotient map.*

The proof of part (a) is in section 3.4.3. It follows the first half of the proof of the analogous result in section 5.5 of [KP82], which relies on the characteristic of K insofar as it relies on Proposition 7.1 of [KP82], proven there for $\operatorname{char} K = 0$ only. We generalize this result in Proposition 3.10, which is proven in section 3.3 for $\operatorname{char} K \neq 2$ after a lengthy discussion on orthosymplectic nilpotent pairs and associated cocharacters.

Part (b) follows from Theorem 4.22(b) in section 4.3.

Chapter 3

Part A

We begin by stating the classification of orbits of *nilpotent pairs* of linear maps under the general linear group, followed by a careful treatment of their classification under the action of a symplectic and an orthogonal groups. The explicit description of these *orthosymplectic* nilpotent pairs is then used to show that cocharacters $\phi : K^\times \to \mathrm{GL}(U \oplus V)$ exist associated to nilpotent elements of the form $\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \in \mathrm{End}(U \oplus V)$, which naturally identify with the nilpotent orthosymplectic pairs (X^*, X), where V a ε-space and U a $-\varepsilon$-space. These associated cocharacters are then used in the generalized proof of Proposition 3.10 relating the dimension of the orthosymplectic orbits to the dimension of the associated orthogonal and symplectic orbits. We finish by analyzing the dimension of the preimage of $\mathcal{O} \subset \overline{\mathcal{O}_D}$ in Z, concluding that Z_{sing} has codimension at least 1, and that Z is a complete intersection, hence an affine variety, proving part (a) of Theorem 2.13.

3.1 Orthosymplectic Orbits

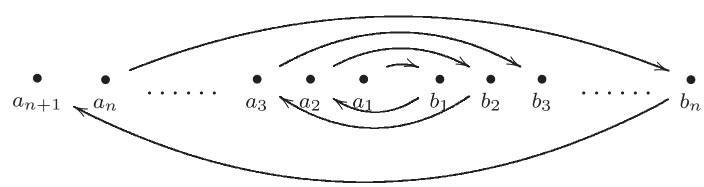
We digress for a moment in order to introduce the concept of a *nilpotent pair*. After establishing the basic notation, we will limit our focus to *orthosymplectic* nilpotent pairs.

3.1.1 Nilpotent Pairs

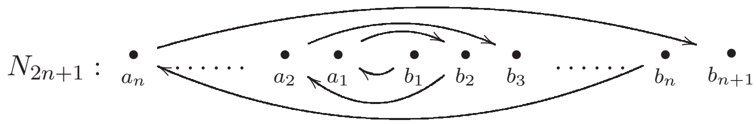
Let U and V be two finite dimensional vector spaces over K. Denote as before $\mathrm{L}(V,U) = \mathrm{Hom}(V,U)$ and consider the product $L := \mathrm{L}(U,V) \times \mathrm{L}(V,U)$ of pairs of maps $U \underset{B}{\overset{A}{\rightleftarrows}} V$. Define the action of $\mathrm{GL}(U) \times \mathrm{GL}(V)$ on L by $(g,h)(A,B) = (hAg^{-1}, gBh^{-1})$, $g \in \mathrm{GL}(U), h \in \mathrm{GL}(V)$. The theory of orbits and the invariant theory for this representation are known [DF73, Naz73, Gab75, KR71, Hes76]. Such pairs can be thought of as objects in a category of modules over a suitable K-algebra, with objects $\mathcal{P} = \{(A,B),(U,V)\}$ and morphisms $\mathcal{P} \to \mathcal{P}'$ given by a pair of linear maps $\phi_U : U \to U'$ and $\phi_V : V \to V'$ such that $A' \circ \phi_U = \phi_V \circ A$ and $B' \circ \phi_V = \phi_U \circ B$, where $\mathcal{P}' = \{(A',B'),(U',V')\}$. The classification is through indecomposable modules, in much the same way as one classifies endomorphisms of vector spaces via Jordan blocks. This is explored in more detail in Appendix A. We are specifically interested in those pairs (A,B) for which the composition $BA \in \mathrm{End}\, U$ (or equivalently $AB \in \mathrm{End}\, V$) is nilpotent, calling such a pair a *nilpotent pair*.

We classify the indecomposable nilpotent pairs by *indecomposable ab-diagrams*. One such is

M_{2n+1} :



meaning that U has basis $\{a_1, a_2, \ldots, a_{n+1}\}$ and V has basis $\{b_1, b_2, \ldots, b_n\}$ and $Aa_i = b_i$, $Bb_j = a_{j+1}$. The diagrams are referred to by a string of a's and b's, hence the name ab-diagram. Type M_{2n+1} above is indicated by the string $ababab\ldots ba$ where the number of a's is $n+1$ and the number of b's is n. Another indecomposable diagram is



where U has basis $\{a_1, a_2, \dots, a_n\}$, V has basis $\{b_1, b_1, \dots, b_{n+1}\}$, and $Aa_i = b_{j+1}$, $Bb_j = a_j$. N_{2n+1} is denoted by the string $bababa \dots ab$ with n a's and $n+1$ b's.

For indecomposable nilpotent pairs (A, B), there are two other indecomposable ab-diagrams under the action of $\mathrm{GL}(U) \times \mathrm{GL}(V)$, $M_{2n} = ababab \dots ab$ and $N_{2n} = bababa \dots ba$, defined similarly to M_{2n+1} and N_{2n+1}.

An arbitrary nilpotent pair (A, B) is a direct sum of indecomposables, and so it is determined by a finite set of indecomposable ab-diagrams to which we associate an array with each row an ab-string. We call this array the *ab-diagram of the pair*. For example a pair may have ab-diagram

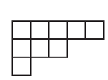
$$\tau = \begin{array}{l} ababababa \\ ababa \\ bab \end{array}$$

consisting of three indecomposable ab-diagrams of types M_9, M_5, N_3.

3.1.2 Young Diagrams of Nilpotent Pairs

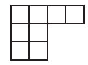
The nilpotent endomorphisms $BA \in \operatorname{End} U$ and $AB \in \operatorname{End} V$ necessarily have associated Young diagrams. Given a nilpotent pair (A, B) and its ab-diagram τ, it is simple to recover the Young diagrams of BA and AB as follows: for the diagram of BA suppress all the b's in the ab-strings of τ, and replace the a's by boxes. For the diagram of AB, suppress the a's in the ab-strings of τ. For example, if the pair (A, B) has ab-diagram τ from above, then BA has Young diagram

$$\begin{array}{l} aaaaa \\ aaa \\ a \end{array} \quad =$$



and AB has Young diagram

$bbbb$

bb = 

bb

Remark 3.1. Not all pairs of Young diagrams describing a nilpotent orbit in End U and one in End V are associated to some nilpotent pair. Furthermore, there can be different nilpotent pairs giving rise to the same pair of Young diagrams. In other words, the "map" taking ab-diagrams to Young diagrams is neither one-to-one, nor onto.

Remark 3.2. For a nilpotent pair (A, B) with ab-diagram τ, we have that

(i) A is injective if and only if every ab-string in τ ends with b.

(ii) A is surjective if and only if every ab-string in τ starts with a.

(iii) B is injective if and only if every ab-string in τ ends with a.

(iv) B is surjective if and only if every ab-string in τ starts with b.

Indeed, if A is injective, $\ker A = 0$, so $Aa_i \neq 0$ for all basis elements $a_i \in U$, and so no string can end with an a. Conversely, if an ab-string ends with a, then there exists some $a_i \in U$ such that $Aa_i = 0$, and A is not injective. If A is surjective, each b in the ab-string corresponds to a basis element $b_i \in V$, which must be the image under A of some $a_i \in U$, which can be taken to be an element of a basis for U. Hence there must be an a preceding each b, and each ab-string in τ starts with an a. If each ab-string in τ starts with a, each basis element b_i of V is the image of some basis element $a_i \in U$, and so each $v \in V$ is the image of a linear combination of the a_i, and A is surjective. The same argument applies to B reversing a's and b's.

3.1.3 Orthosymplectic Nilpotent Pairs

Assume for the remainder of this section that U is an *orthogonal* space, V is a *symplectic* space and let $X \in \mathrm{L}(V, U)$. From now on we restrict our attention to

Table 3.1: Indecomposable Orthosymplectic ab-Diagrams

Type	α_n	β_n	γ_n	δ_n	ϵ_n
ab-diagram	$aba \dots ba$	$bab \dots ab$	$aba \dots ba$ $aba \dots ba$	$bab \dots ab$ $bab \dots ab$	$aba \dots ab$ $bab \dots ba$
n	—	—	odd	even	—
$\#a$	$2n+1$	$2n-1$	$2(n+1)$	$2n$	$2n$
$\#b$	$2n$	$2n$	$2n$	$2(n+1)$	$2n$

nilpotent pairs of the form $(A, B) = (X^*, X)$ where X^* is the adjoint of X as in section 2.2.1. The action of $\mathrm{GL}(U) \times \mathrm{GL}(V)$ above restricts to an action of $\mathrm{G}(U) \times \mathrm{G}(V) = \mathrm{O}(U) \times \mathrm{Sp}(V)$ on $(X^*, X) \in L = \mathrm{L}(U, V) \times \mathrm{L}(V, U)$. So $(g, h)(X^*, X) = (hX^*g^{-1}, gXh^{-1})$. We have then that if O_X denotes the $\mathrm{O}(U) \times \mathrm{Sp}(V)$-orbit of (X^*, X) and P_X denotes the $\mathrm{GL}(U) \times \mathrm{GL}(V)$-orbit of (X^*, X), then

$$P_X \cap L = O_X.$$

Hence the orbit O_X is determined by the ab-diagram of the pair as defined above.

We call nilpotent pairs of the form $(A, B) = (X^*, X)$ *orthosymplectic nilpotent pairs* and the $\mathrm{O}(U) \times \mathrm{Sp}(V)$-orbit O_X of such a pair an *orthosymplectic orbit*. We saw in section 2.2.1 that $\pi(X) = XX^* \in \mathfrak{g}(U) = \mathfrak{so}(U)$ and $\rho(X) = X^*X \in \mathfrak{g}(V) = \mathfrak{sp}(V)$, hence the M_r and N_r no longer classify the orthosymplectic orbits. In particular, the Young diagrams of the compositions $\pi(X)$ and $\rho(X)$, denoted by $\pi(\tau)$ and $\rho(\tau)$ respectively, must satisfy Theorem 2.2. Such an ab-diagram is then called an *orthosymplectic ab-diagram*. The remainder of this section gives an explanation and proof of existence of the five types of indecomposable orthosymplectic ab-diagrams, summarized in Table 3.1. The proof is inspired by the treatment in [Jan04] of the classification of nilpotent orbits in the symplectic and orthogonal groups.

3.1.4 Type α_n

The irreducible string $M_{2r+1} = ababab \dots ba$ is an orthosymplectic ab-diagram only if r is even. Otherwise, for r odd, $\pi(M_{2r+1}) = aaa \dots a$ and $\rho(M_{2r+1}) = bbb \dots b$ (an even number $r+1$ of a's and an odd number r of b's) do not satisfy Theorem

2.2. We define then the indecomposable orthosymplectic ab-diagram of type α_n for $n \in \mathbb{Z}^+$ to be the string $ababa \dots ba$ of $2n+1$ a's and $2n$ b's. The α_n are those M_{2r+1} where $r = 2n$ is even.

We would like to see that such a string exists for each n, i.e. that there exists a pair (X^*, X) with ab-diagram α_n for each n. Let U be a $2n+1$ dimensional vector space and V a $2n$ dimensional vector space, both over K, and choose bases $\{e_1, e_2, \dots, e_{2n+1}\}$ and $\{f_1, f_2, \dots f_{2n}\}$ of U and V respectively. Define a linear map $X : V \to U$ by $X(f_i) = e_{i+1}$ and a map $X' : U \to V$ by $X'(e_j) = f_j$. The pair $(X', X) \in L$ is clearly nilpotent and has ab-diagram α_n. We would like to see that (X', X) is an *orthosymplectic* nilpotent pair.

Denote by $\widetilde{X}$ the endomorphism of $U \oplus V$ given by $\begin{pmatrix} 0 & X \\ X' & 0 \end{pmatrix}$, i.e. corresponding to the pair (X', X). The $4n+1$ dimensional vector space $U \oplus V$ has basis $\{e_1, X'(e_1), XX'(e_1), \dots, X'(XX')^{2n}(e_1)\}$ which we will rewrite using the $\widetilde{X}$ notation as $\{e, \widetilde{X}e, \widetilde{X}^2 e, \dots, \widetilde{X}^{4n} e\}$ where $e = e_1$ viewed as an element of $U \oplus V$. On $U \oplus V$, define a nondegenerate bilinear form β by

$$\beta(\widetilde{X}^j e, \widetilde{X}^h e) = \begin{cases} (-1)^{j/2} & j, h \text{ even}, j+h = 4n, \\ (-1)^{(j+1)/2} & j, h \text{ odd}, j+h = 4n, \\ 0 & \textit{otherwise}. \end{cases}$$

On U, i.e. for j and h even, β restricts to a nondegenerate form, which is symmetric. We have $\beta(\widetilde{X}^j e, \widetilde{X}^h e) = 0$ unless $j+h = 4n$. Assuming $j+h = 4n$ with j and h both even, we have

$$\beta(\widetilde{X}^j e, \widetilde{X}^h e) = (-1)^{j/2} = (-1)^{(4n-h)/2} = (-1)^{h/2} = \beta(\widetilde{X}^h e, \widetilde{X}^j e).$$

Hence U is an orthogonal space with nondegenerate bilinear form given by the restriction of β to U.

On V, i.e. for j and h odd, β restricts to a nondegenerate form, which is alternating. We have $\beta(\widetilde{X}^j e, \widetilde{X}^h e) = 0$ unless $j+h = 4n$. Therefore assuming that

$j + h = 4n$ with j, h odd,

$$\beta(\widetilde{X}^j e, \widetilde{X}^h e) = (-1)^{(j+1)/2} = (-1)^{(4n-h+1)/2} = (-1)^{(h-1)/2}$$
$$= -(-1)^{(h+1)/2} = -\beta(\widetilde{X}^h e \widetilde{X}^j e).$$

Hence V is a symplectic space with nondegenerate bilinear form given by the restriction of β to V.

It remains to be seen that X' is adjoint to X. This is a straightforward calculation once we recall that for odd powers i, $\widetilde{X}\widetilde{X}^i e = X\widetilde{X}^i e$ and for even powers i, $\widetilde{X}\widetilde{X}^i e = X'\widetilde{X}^i e$. We have then for j odd, h even and $j + h = 4n - 1$,

$$\beta(X\widetilde{X}^j e, \widetilde{X}^h e)|_U = \beta(\widetilde{X}\widetilde{X}^j e, \widetilde{X}^h e)|_U = (-1)^{(j+1)/2}$$
$$= \beta(\widetilde{X}^j e, \widetilde{X}\widetilde{X}^h e)|_V = \beta(\widetilde{X}^j e, X'\widetilde{X}^h e)|_V$$

with the left- and right-most terms equal to zero otherwise. Hence $X' = X^*$ and (X^*, X) is an orthosymplectic nilpotent pair with ab-diagram α_n.

3.1.5 Type β_n

The irreducible string $N_{2r+1} = bababa \ldots ab$ is an orthosymplectic ab-diagram only if r is odd, with $\pi(N_{2r+1}) = aaa \ldots a$ and $\rho(N_{2r+1}) = bbb \ldots b$ of odd and even lengths respectively. This leads us to define the indecomposable orthosymplectic ab-diagram of type β_n to be the string $babab \ldots ab$ of $2n - 1$ a's and $2n$ b's (N_{2r+1} where $r = 2n - 1$).

We now show that such strings exist. Let U be a $2n-1$ dimensional vector space and let V be a $2n$ dimensional vector space over K, $\{e_1, e_2, \ldots, e_{2n-1}\}$ a basis for U and $\{f_1, f_2, \ldots, f_{2n}\}$ a basis for V. Define a linear map $X : V \to U$ by $X(f_i) = e_i$ and $X' : U \to V$ by $X'(e_j) = f_{j+1}$. The pair (X', X) is nilpotent and has ab-diagram β_n. As in section 3.1.4, we show that U is an orthogonal space, V is a symplectic space, and that X' is the adjoint of X, hence that we have an orthosymplectic pair with ab-diagram β_n.

The $4n - 1$ dimensional space $U \oplus V$ has basis $\{f = f_1, \widetilde{X}f, \widetilde{X}^2 f, \ldots, \widetilde{X}^{4n-2} f\}$ with $\widetilde{X} = \begin{pmatrix} 0 & X \\ X' & 0 \end{pmatrix}$ as in section 3.1.4. Define the nondegenerate bilinear form β

by

$$\beta(\widetilde{X}^j v, \widetilde{X}^h v) = \begin{cases} (-1)^{(j-1)/2} & j, h \text{ odd}, j+h = 4n-2 \\ (-1)^{j/2} & j, h \text{ even}, j+h = 4n-2 \\ 0 & otherwise, \end{cases}$$

On U, where j and h are odd, β restricts to a nondegenerate form, which is symmetric. Indeed, $\beta(\widetilde{X}^j f, \widetilde{X}^h f) = 0$ unless $j+h = 4n-2$. Assuming $j+h = 4n-2$ with j and h both odd, we have

$$\beta(\widetilde{X}^j f, \widetilde{X}^h f) = (-1)^{(j-1)/2} = (-1)^{(4n-2-h-1)/2} = (-1)^{(-h-3)/2} = (-1)^{(h+3)/2-2} = (-1)^{(h-1)/2} = \beta(\widetilde{X}^h f, \widetilde{X}^j f).$$

Hence U is an orthogonal space with nondegenerate bilinear form given by the restriction of β to U.

On V, where j and h are even, β restricts to a nondegenerate form, which is alternating. Indeed, $\beta(\widetilde{X}^j f, \widetilde{X}^h f) = 0$ unless $j+h = 4n-2$. Therefore assuming that $j+h = 4n-2$ with j, h even,

$$\begin{aligned} \beta(\widetilde{X}^j f, \widetilde{X}^h f) = (-1)^{j/2} = (-1)^{(4n-2-h)/2} = (-1)^{(h+2)/2} \\ = -(-1)^{h/2} = -\beta(\widetilde{X}^h f, \widetilde{X}^j f). \end{aligned}$$

Hence V is a symplectic space with nondegenerate bilinear form given by the restriction of β to V.

Here we have that for even powers i, $\widetilde{X}$ applied to $\widetilde{X}^i f$ is the map $X \in \mathrm{L}(V, U)$, and for odd powers i, $\widetilde{X}$ applied to $\widetilde{X}^i f$ is $X' \in \mathrm{L}(U, V)$. Then for j even and h odd and $j+h = 4n-3$,

$$\beta(X\widetilde{X}^j f, \widetilde{X}^h f)|_U = \beta(\widetilde{X}\widetilde{X}^j f, \widetilde{X}^h f)|_U = (-1)^{(j+1-1)/2} = (-1)^{j/2} = \beta(\widetilde{X}^j f, \widetilde{X}\widetilde{X}^h f)|_V = \beta(\widetilde{X}^j f, X'\widetilde{X}^h f)|_V$$

with the right- and left-most terms equal to zero otherwise. Hence $X' = X^*$ and (X^*, X) is an orthosymplectic nilpotent pair with ab-diagram β_n.

3.1.6 Type γ_n

If n is odd, the ab-diagram consisting of two strings of type M_{2n+1} is an orthosymplectic diagram. Indeed, let $\tau = {aba...ba \atop aba...ba}$, each row having $n+1$ a's and n b's. Then $\pi(\tau) = {aa...a \atop aa...a}$ has all odd-length rows occurring an even number of times and $\rho(\tau) = {bb...b \atop bb...b}$ has all even-length rows occurring an even number of times. Since the ab-diagram consisting of just one M_{2n+1} is not orthosymplectic for n odd as shown in section 3.1.4, $\gamma_n = {aba...ba \atop aba...ba}$ where the number of a's is $2(n+1)$ and the number of b's is $2n$ is an *indecomposable orthosymplectic* ab-diagram for n an odd, positive integer.

To see that γ_n exists as the ab-diagram of an orthosymplectic nilpotent pair for n odd, let U be a $2(n+1)$ dimensional vector space over K with basis $\{e_1, e_2, \ldots, e_{n+1}, e'_1, e'_2, \ldots, e'_{n+1}\}$ and let V be a $2n$ dimensional vector space over K with basis $\{f_1, f_2, \ldots, f_n, f'_1, f'_2, \ldots, f'_n\}$. Define a linear map $X : V \to U$ by $X(f_i) = e_{i+1}$ and $X(f'_i) = e'_{i+1}$ and define $X' : U \to V$ by $X'(e_j) = f_j$ and $X'(e'_j) = f'_j$. The pair (X', X) is nilpotent and has ab-diagram γ_n. We show that (X', X) is orthosymplectic.

$U \oplus V$ is a $4n+2$ dimensional vector space with basis

$$\{e_1, X'(e_1), XX'(e_1), \ldots, (XX')^n(e_1), e'_1, X'(e'_1), XX'(e'_1), \ldots, (XX')^n(e'_1)\}$$

which we rewrite letting $\widetilde{X} = \begin{pmatrix} 0 & X \\ X' & 0 \end{pmatrix}$, $u_1 = e_1$ and $u_2 = e'_1$ as

$$\{u_1, \widetilde{X}u_1, \widetilde{X}^2u_1, \ldots, \widetilde{X}^{2n}u_1, u_2, \widetilde{X}u_2, \widetilde{X}^2u_2, \ldots, \widetilde{X}^{2n}u_2\}.$$

Define the nondegenerate bilinear form β by

$$\beta(\widetilde{X}^j u_i, \widetilde{X}^h u_{i_*}) = \begin{cases} (-1)^{j/2+i} & j, h \text{ even}, j+h = 2n, i \neq i_* \in \{1,2\}, \\ (-1)^{(j+1)/2+i} & j, h \text{ odd}, j+h = 2n, i \neq i_* \in \{1,2\}, \\ 0 & \textit{otherwise}. \end{cases}$$

In particular, $\beta(\widetilde{X}^j u_1, \widetilde{X}^h u_1) = 0 = \beta(\widetilde{X}^j u_2, \widetilde{X}^h u_2)$ for all j and h. The condition $i \neq i_*$ is equivalent to requiring $i - i_* \equiv 1 \bmod 2$ since $i, i_* \in \{1, 2\}$.

On U, i.e. for j and h even, β restricts to a nondegenerate form, which is symmetric. We have $\beta(\widetilde{X}^j u_i, \widetilde{X}^h u_{i_*}) = 0$ unless $j + h = 2n$ and $i \neq i_*$. Assuming $j + h = 2n$ with j and h both even, $i - i_* \equiv 1 \bmod 2$ and recalling that n is odd, we have

$$\beta(\widetilde{X}^j u_i, \widetilde{X}^h u_{i_*}) = (-1)^{j/2+i} = (-1)^{(2n-h)/2+1+i_*} = (-1)^{h/2+i_*+n+1} = (-1)^{h/2+i_*} = \beta(\widetilde{X}^h u_{i_*}, \widetilde{X}^j u_i).$$

Hence U is an orthogonal space with nondegenerate bilinear form given by the restriction of β to U.

On V, i.e. for j and h odd, β restricts to a nondegenerate form, which is alternating. We have $\beta(\widetilde{X}^j u_i, \widetilde{X}^h u_{i_*}) = 0$ unless $j + h = 2n$ and $i \neq i_*$. Therefore assuming that $j + h = 2n$ with j, h odd and $i - i_* \equiv 1 \bmod 2$,

$$\beta(\widetilde{X}^j u_i, \widetilde{X}^h u_{i_*}) = (-1)^{(j-1)/2+i} = (-1)^{(2n-h-1)/2+1+i_*} = (-1)^{(h+1)/2+n+1+i_*} = -(-1)^{(h-1)/2+i_*} = -\beta(\widetilde{X}^h u_{i_*}, \widetilde{X}^j u_i).$$

Hence V is a symplectic space with nondegenerate bilinear form given by the restriction of β to V.

We need now that X' is adjoint to X. In order to see this we unpack some of our identifications. For k odd, $\widetilde{X}^k u_i$ is an element of V and we have $\widetilde{X}^k u_1 = X'(XX')^{(k-1)/2} e_1$ and $\widetilde{X}^k u_2 = X'(XX')^{(k-1)/2} e_1'$. Thus $\widetilde{X}\widetilde{X}^k u_i = X\widetilde{X}^k u_i$ since $\widetilde{X}$ acts on V by X. For k even, $\widetilde{X}^k u_i$ is an element of U with $\widetilde{X}^k u_1 = (XX')^{k/2} f_1$ and $\widetilde{X}^k u_2 = (XX')^{k/2} f_1'$, hence $\widetilde{X}\widetilde{X}^k u_i = X'\widetilde{X}^k u_i$.

For j odd, h even and $j + h = 2n - 1$,

$$\begin{aligned} \beta(X\widetilde{X}^j u_i, \widetilde{X}^h u_{i_*})|_U &= \beta(\widetilde{X}\widetilde{X}^j u_i, \widetilde{X}^h u_{i_*})|_U = (-1)^{(j+1)/2+i} \\ &= \beta(\widetilde{X}^j u_i, \widetilde{X}\widetilde{X}^h u_{i_*})|_V = \beta(\widetilde{X}^j u_i, X'\widetilde{X}^h u_{i_*})|_V. \end{aligned}$$

The left- and right-most terms are equal to zero otherwise, and so $X' = X^*$.

3.1.7 Type δ_n

If n is even, the ab-diagram $\delta_n = {}^{bab...ab}_{bab...ab}$ consisting of two strings of type N_{2n+1} ($2n$ a's and $2(n+1)$ b's) is an indecomposable orthosymplectic ab-diagram. Indeed, $\pi(\delta_n) = {}^{aa...a}_{aa...a}$ has an even number of rows of even length n, and $\rho(\delta_n) = {}^{bb...b}_{bb...b}$ has an even number of rows of odd length $n+1$. Since the ab-diagram consisting of just one N_{2n+1} is not orthosymplectic for n odd as shown in section 3.1.5, δ_n is an *indecomposable orthosymplectic* ab-diagram for n an even, positive integer.

We construct an orthosymplectic nilpotent pair with associated ab-diagram δ_n for n even. Let U be a $2n$ dimensional and V a $2(n+1)$ dimensional vector space over K, with bases $\{e_1, e_2, \ldots, e_n, e'_1, e'_2, \ldots, e'_n\}$ and $\{f_1, f_2, \ldots, f_{n+1}, f'_1, f'_2, \ldots, f'_{n+1}\}$ respectively. Define a linear map $X : V \to U$ by $X(f_i) = e_i$ and $X(f'_i) = e'_i$, and define $X' : U \to V$ by $X'(e_j) = f_{j+1}$ and $X'(e'_j) = f'_{j+1}$. The pair (X', X) is nilpotent and has ab-diagram δ_n. We show that (X', X) is orthosymplectic.

$U \oplus V$ is a $4n+2$ dimensional vector space with basis

$$\{f_1, X(f_1), X'X(f_1), \ldots, (X'X)^n(f_1), f'_1, X(f'_1), X'X(f'_1), \ldots, (X'X)^n(f'_1)\}.$$

Letting $v_1 = f_1$ and $v_2 = f'_1$ and writing $\widetilde{X} = \begin{pmatrix} 0 & X \\ X' & 0 \end{pmatrix}$, this basis becomes

$$\{v_1, \widetilde{X}v_1, \widetilde{X}^2 v_1, \ldots, \widetilde{X}^{2n} v_1, v_2, \widetilde{X}v_2, \widetilde{X}^2 v_2, \ldots, \widetilde{X}^{2n} v_2\}.$$

Define the nondegenerate bilinear form β by

$$\beta(\widetilde{X}^j v_i, \widetilde{X}^h v_{i_*}) = \begin{cases} (-1)^{(j-1)/2+i} & j, h \text{ odd}, j+h = 2n, i \neq i_* \in \{1,2\}, \\ (-1)^{j/2+i} & j, h \text{ even}, j+h = 2n, i \neq i_* \in \{1,2\}, \\ 0 & \textit{otherwise}. \end{cases}$$

On U, where j and h are odd, β restricts to a nondegenerate form, which is symmetric. Indeed, $\beta(\widetilde{X}^j v_i, \widetilde{X}^h v_{i_*}) = 0$ unless $j+h = 2n$ and $i \neq i_*$ (equivalently

$i - i_* \equiv 1 \bmod 2$). Assuming $j + h = 2n$ with j and h both odd, $i - i_* \equiv 1 \bmod 2$, and recalling that n is even, we have

$$\begin{aligned}\beta(\widetilde{X}^j v_i, \widetilde{X}^h v_{i_*}) &= (-1)^{(j-1)/2+i} = (-1)^{(2n-h-1)/2+i_*+1} \\ &= (-1)^{(h+1)/2+i_*+1} = (-1)^{(h-1)/2+i_*} = \beta(\widetilde{X}^h v_{i_*}, \widetilde{X}^j v_i)\end{aligned}$$

Hence U is an orthogonal space with nondegenerate bilinear form given by the restriction of β to U.

On V, where j and h are even, β restricts to a nondegenerate form, which is alternating. Indeed, $\beta(\widetilde{X}^j v_i, \widetilde{X}^h v_{i_*}) = 0$ unless $j + h = 2n$ and $i - i_* \equiv 1 \bmod 2$. Therefore assuming these,

$$\begin{aligned}\beta(\widetilde{X}^j v_i, \widetilde{X}^h v_{i_*}) &= (-1)^{j/2+i} = (-1)^{(2n-h)/2+1+i_*} \\ = (-1)^{h/2+n+1+i_*} &= -(-1)^{h/2+i_*} = -\beta(\widetilde{X}^h v_{i_*}, \widetilde{X}^j v_i).\end{aligned}$$

Hence V is a symplectic space with nondegenerate bilinear form given by the restriction of β to V.

We show now that X' is adjoint to X. For k even, $\widetilde{X}\widetilde{X}^k v_i = X\widetilde{X}^k v_i$ since $\widetilde{X}^k v_i \in V$. For k odd, $\widetilde{X}\widetilde{X}^k v_i = X'\widetilde{X}^k v_i$ since $\widetilde{X}^k v_i \in U$. Thus for j even and h odd and $j + h = 2n - 1$,

$$\begin{aligned}\beta(X\widetilde{X}^j u_i, \widetilde{X}^h u_{i_*})|_U = \beta(\widetilde{X}\widetilde{X}^j u_i, \widetilde{X}^h u_{i_*})|_U = (-1)^{(j-1+1)/2+i} = (-1)^{j/2+i} = \\ \beta(\widetilde{X}^j u_i, \widetilde{X}\widetilde{X}^h u_{i_*})|_V = \beta(\widetilde{X}^j u_i, X'\widetilde{X}^h u_{i_*})|_V.\end{aligned}$$

The left- and right-most terms are equal to zero otherwise, and so $X' = X^*$ and $(X', X) = (X^*, X)$ is orthosymplectic.

3.1.8 Type ϵ_n

The ab-strings of type M_{2n} and N_{2n} present more of a challenge. Neither is orthosymplectic if taken by itself as an ab-diagram τ, since then either $\pi(\tau)$ or $\rho(\tau)$ will violate Theorem 2.2. Indeed,

- if n is even and $\tau = M_{2n}$ or $\tau = N_{2n}$, then $\pi(\tau)$ has one row of even length.

- if n is odd and $\tau = M_{2n}$ or $\tau = N_{2n}$, then $\rho(\tau)$ has one row of odd lenth.

While they do satisfy the criteria for being orthosymplectic, ab-diagrams with two M_{2n}'s or two N_{2n}'s do not occur as the ab-diagrams of orthosymplectic nilpotent pairs. To see this, assume the opposite and let (X^*, X) be an orthosymplectic nilpotent pair with ab-diagram $\tau = \substack{abab...ab \\ abab...ab}$ (consisting of two M_{2n}'s). By remark 3.2, $X^* : U \to V$ is both injective and surjective. By the definition of M_{2n}, there exist a basis

$$\{u_1, XX^*u_1, \ldots, (XX^*)^{n-1}u_1, u_2, XX^*u_2, \ldots, (XX^*)^{n-1}u_2\}$$

of U and a basis

$$\{X^*u_1, X^*(XX^*)u_1, \ldots, X^*(XX^*)^{n-1}u_1, X^*u_2, X^*(XX^*)u_2, \ldots, X^*(XX^*)^{n-1}u_2\}$$

of V. Then

$$\langle X^*(XX^*)^{n-1}u_i, X^*(u)\rangle|_V = \langle X(X^*(XX^*)^{n-1}u_i), u\rangle|_U = \langle 0, u\rangle|_U = 0$$

for all $u \in U$, since by the definition of M_{2n}, $(XX^*)^n u_i = 0$ for $i = 1, 2$. Hence

$$\langle X^*(XX^*)^{n-1}u_i, X^*(u)\rangle|_V = 0$$

for all $u \in U$ and

$$\langle X^*(XX^*)^{n-1}u_i, v\rangle|_V = 0$$

for all $v \in V$ since X^* is surjective. The form is nondegenerate, which forces $X^*(XX^*)^{n-1}u_i = 0$. This contradicts the existence of the above basis for V, and so (X^*, X) cannot have ab-diagram τ. The same argument can be used to show that there is no orthosymplectic pair with ab-diagram consisting of two N_{2n}'s, reversing the roles of U and V. (In this case, $X : V \to U$ is injective and surjective.)

Finally, the ab-diagrams consisting of one M_{2n} and one N_{2n} are orthosymplectic and indecomposable, and also exist as the ab-diagrams of orthosymplectic nilpotent pairs. Let $\epsilon_n = \substack{aba...ab \\ bab...ba}$ for $n \in \mathbb{Z}^+$ with the number of a's and b's both equal to $2n$.

Then $\pi(\epsilon_n)$ and $\rho(\epsilon_n)$ both satisfy Theorem 2.2 for n even and odd. However, we must prove existence for the cases n even and n odd separately.

For arbitrary $n \in \mathbb{Z}^+$ take U and V to be $2n$ dimensional vector spaces over K with respective bases $\{e_1, e_2, \ldots, e_n, e'_1, e'_2, \ldots, e'_n\}$ and $\{f_1, f_2, \ldots, f_n, e'_1, f'_2, \ldots, f'_n\}$. Define $X : V \to U$ by $X(f_i) = e_{i+1}$ and $X(f'_i) = e'_i$ and define $X' : U \to V$ by $X'(e_i) = f_i$ and $X'(e'_i) = f'_{i+1}$ (note the antisymmetry not present in previous cases). Letting $w_1 = f_1 \in U$ and $w_2 = e'_1 \in V$, $U \oplus V$ has basis

$$\{w_1, \widetilde{X}w_1, \widetilde{X}^2 w_1, \ldots, \widetilde{X}^{2n-1}w_1, w_2, \widetilde{X}w_2, \widetilde{X}^2 w_2, \ldots, \widetilde{X}^{2n-1}w_2\}$$

where $\widetilde{X} = \begin{pmatrix} 0 & X \\ X' & 0 \end{pmatrix} \in \operatorname{End}(U \oplus V)$.

First, let n be even. We define nondegenerate bilinear form β by

$$\beta(\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*}) = \begin{cases} (-1)^{(j+i-1)/2} & j+i, h+i_* \text{ odd}, j+h = 2n-1, i \neq i^* \\ (-1)^{(j+i)/2} & j+i, h+i_* \text{ even}, j+h = 2n-1, i \neq i^* \\ 0 & \textit{otherwise.} \end{cases}$$

On U, i.e. for $j+i$ and $h+i_*$ odd, β restricts to a nondegenerate form, which is symmetric. We have $\beta(\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*}) = 0$ unless $j+h = 2n-1$ and $i \neq i_*$ (equivalently $i - i_* \equiv 1 \bmod 2$). We first observe that if $i - i_* \equiv 1 \bmod 2$ with $i, i_* \in \{1, 2\}$, then $i \equiv -1 - i_* \bmod 4$ (this is easily checked explicitly for all possible combinations of i and i_*). Assuming then $j+h = 2n-1$ with $j+i$ and $h+i_*$ both odd, $i \equiv -1 - i_* \bmod 4$, and recalling that n is even, we have

$$\beta(\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*}) = (-1)^{(j+i-1)/2} = (-1)^{(2n-1-h-i_*-1-1)/2} = (-1)^{(2n-3-h-i_*)/2} = (-1)^{(h+i_*+3)/2-2} = (-1)^{(h+i_*-1)/2} = \beta(\widetilde{X}^h w_{i_*}, \widetilde{X}^j w_i).$$

Hence U is an orthogonal space with nondegenerate bilinear form given by the restriction of β to U.

On V, i.e. for $j+i$ and $h+i_*$ even, β restricts to a nondegenerate form, which is alternating. We have $\beta(\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*}) = 0$ unless $j+h = 2n-1$ and $i - i_* \equiv 1 \bmod 2$.

Therefore assuming these and $i \equiv -1 - i_* \bmod 4$ as above,

$$\beta(\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*}) = (-1)^{(j+i)/2} = (-1)^{(2n-1-h-1-i_*)/2} = (-1)^{(-h-i_*-2)/2} = -(-1)^{(h+i_*)/2} = -\beta(\widetilde{X}^h w_{i_*}, \widetilde{X}^j w_i).$$

Hence V is a symplectic space with nondegenerate bilinear form given by the restriction of β to V.

On U ($j+i$ and $h+i_*$ odd), $\widetilde{X}$ acts by X', while on V ($j+i$ and $h+i_*$ even), $\widetilde{X}$ acts by X. Thus for $j+i$ even and $h+i_*$ odd and $j+h = 2n-2$,

$$\beta(X\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*})|_U = \beta(\widetilde{X}\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*})|_U = (-1)^{(j+i-1+1)/2} = (-1)^{(j+i)/2} = \beta(\widetilde{X}^j w_i, \widetilde{X}\widetilde{X}^h w_{i_*})|_V = \beta(\widetilde{X}^j w_i, X'\widetilde{X}^h w_{i_*})|_V.$$

The left- and right-most terms are equal to zero otherwise, and so $X' = X^*$ and $(X', X) = (X^*, X)$ is orthosymplectic.

Lastly, let n be odd, and define nondegenerate bilinear form β by

$$\beta(\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*}) = \begin{cases} (-1)^{(j-i-1)/2} & j+i, h+i_* \text{ odd}, j+h = 2n-1, i \neq i^* \\ (-1)^{(j-i)/2} & j+i, h+i_* \text{ even}, j+h = 2n-1, i \neq i^* \\ 0 & \textit{otherwise}. \end{cases}$$

On U, i.e. for $j+i$ and $h+i_*$ odd, β restricts to a nondegenerate form, which is symmetric. We have $\beta(\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*}) = 0$ unless $j+h = 2n-1$ and $i \neq i_*$ (equivalently $i - i_* \equiv 1 \bmod 2$). As in the previous case, $i - i_* \equiv 1 \bmod 2$ with $i, i_* \in \{1, 2\}$ implies $i \equiv -1 - i_* \bmod 4$. Assuming then $j+h = 2n-1$ with $j+i$ and $h+i_*$ both odd, $i \equiv -1 - i_* \bmod 4$, and recalling that n is odd, we have

$$\beta(\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*}) = (-1)^{(j-i-1)/2} = (-1)^{(2n-1-h+1+i_*-1)/2} = (-1)^{(-h+i_*-1)/2+n} = -(-1)^{(h-i_*+1)/2} = (-1)^{(h-i_*-1)/2} = \beta(\widetilde{X}^h w_{i_*}, \widetilde{X}^j w_i).$$

Hence U is an orthogonal space with nondegenerate bilinear form given by the restriction of β to U.

On V, i.e. for $j+i$ and $h+i_*$ even, β restricts to a nondegenerate form, which is alternating. We have $\beta(\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*}) = 0$ unless $j+h = 2n-1$ and $i - i_* \equiv 1 \bmod 2$. Therefore assuming these and $i \equiv -1 - i_* \bmod 4$ as above,

$$\beta(\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*}) = (-1)^{(j-i)/2} = (-1)^{(2n-1-h+1+i_*)/2} = (-1)^{(-h+i_*)/2+n} = \\ -(-1)^{(h-i_*)/2} = -\beta(\widetilde{X}^h w_{i_*}, \widetilde{X}^j w_i).$$

Hence V is a symplectic space with nondegenerate bilinear form given by the restriction of β to V.

On U ($j+i$ and $h+i_*$ odd), $\widetilde{X}$ acts by X', while on V ($j+i$ and $h+i_*$ even), $\widetilde{X}$ acts by X. Thus for $j+i$ even and $h+i_*$ odd and $j+h = 2n-2$,

$$\beta(X\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*})|_U = \beta(\widetilde{X}\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*})|_U = (-1)^{(j-i-1+1)/2} = (-1)^{(j-i)/2} = \\ \beta(\widetilde{X}^j w_i, \widetilde{X}\widetilde{X}^h w_{i_*})|_V = \beta(\widetilde{X}^j w_i, X'\widetilde{X}^h w_{i_*})|_V.$$

The left- and right-most terms are equal to zero otherwise, and so $X' = X^*$ and $(X', X) = (X^*, X)$ is orthosymplectic.

3.2 Associated Cocharacters

If $\operatorname{char} K = 0$, one can use a semisimple element $H \in \mathfrak{g}$ to define a grading of $\mathfrak{g}$ by letting $\mathfrak{g}_i$ be the subset of all $Z \in \mathrm{G}$ such that $[H, Z] = iZ$. When $\operatorname{char} K > 0$, this method of defining a grading clearly breaks down. One can instead use a cocharacter, i.e. a homomorphism of algebraic groups $\phi : K^\times \to G$, and define $\mathfrak{g}_i = \mathfrak{g}(i; \phi) = \{Z \in \mathfrak{g} \mid \mathrm{Ad}(\phi(t))Z = t^i Z \text{ for all } t \in K^\times\}$. Just as the H arising as semisimple elements of $\mathfrak{sl}_2$-triples have certain properties that make them "nice" to work with, some cocharacters have analogous properties that make them preferable to others. Such cocharacters are called *associated cocharacters* and will be explored in this section.

3.2.1 Definition and Properties

Let G be a connected reductive algebraic group with Lie algebra $\mathfrak{g}$. In order to define an associated cocharacter, we first need to define what it means for a nilpotent element in $\mathfrak{g}$ to be *distinguished*. Let $G_X = \{Z \in G \mid ZX = XZ\}$ be the centralizer of X in G. By a *torus* we mean an algebraic subgroup isomorphic to a direct product of copies of $K^\times$. A nilpotent element $X \in \mathfrak{g}$ is called *distinguished* if each torus contained in G_X is contained in the center of G. Equivalently, X is distinguished if each homomorphism $K^\times \to G_X$ of algebraic groups takes values in the center of G. We will need the following result from [Jan04, 4.1] classifying the distinguished nilpotent elements of $\mathfrak{gl}(V)$.

Lemma 3.3. *Let $X \in \mathfrak{gl}(V)$ be nilpotent. Then X is distinguished if and only if X has partition* $[\dim V]$.

For a nilpotent element $X \in \mathfrak{g}$, a cocharacter $\phi : K^\times \to G$ is *associated* to X if

$$X \in \mathfrak{g}(2;\phi) = \{Z \in \mathfrak{g} \mid \mathrm{Ad}(\phi(t))(Z) = t^2 Z \text{ for all } t \in K^\times\}$$

and if there exists a Levi subgroup L in G such that X is a distinguished nilpotent element in $\mathrm{Lie}\, L$ and such that $\phi(K^\times) \subset \mathcal{D}L$, where $\mathcal{D}L$ is the derived group of L [Jan04, 5.3]. By a *Levi subgroup* of G we mean a Levi factor of a parabolic subgroup of G as in [Hum75, 30.2], i.e. the centralizer in G of a maximal torus of the radical of a parabolic subgroup $P \subseteq G$. For $G = \mathrm{GL}(V)$, the subgroups $L = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \ldots \times \mathrm{GL}(V_r)$ are the Levi subgroups in G, with $V = V_1 \oplus V_2 \oplus \ldots \oplus V_r$ a direct sum decomposition and $\dim V_i = d_i$, $d_1 \geq d_2 \geq \ldots \geq d_r > 0$ a partition of $\dim V$ [Jan04, 4.4].

Let $X \in \mathfrak{g}$ be nilpotent and let τ be a cocharacter associated to X. Associate to τ a parabolic subgoup $P = P_\tau$ such that

$$\mathfrak{p} := \mathrm{Lie}(P_\tau) = \bigoplus_{i \geq 0} \mathfrak{g}_i = \bigoplus_{i \geq 0} \mathfrak{g}(i;\tau).$$

One possible definition of P_τ is as the set of all $g \in G$ such that $\lim_{t \to 0} \tau(t) g \tau(t)^{-1}$ exists [Spr98, 8.4.5]. $\mathfrak{p} \subseteq \mathfrak{g}$ is a parabolic subalgebra with nilpotent radical $\mathfrak{n} := \bigoplus_{i>0} \mathfrak{g}_i$ and Levi-decomposition $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n}$ [Jan04, 5.1]. Define $\mathfrak{n}_2 := \bigoplus_{i \geq 2} \mathfrak{g}_i$.

Proposition 3.4. *Suppose that* $\mathrm{char}\, K$ *is good for G. Then*

(a) The group P depends only on X, not on the choice of τ.

(b) $G_X = P_X$.

(c) $\overline{\mathrm{Ad}(P)(X)} = \mathfrak{n}_2$.

Proof. This is Proposition 5.9 in [Jan04]. □

Proposition 3.5. *Two cocharacters associated to X are conjugate under G_X^0, the identity component of the centralizer of X.*

Proof. This is Lemma 5.3b in [Jan04]. □

Proposition 3.6. $X \in \mathfrak{n}_2 = \bigoplus_{i \geq 2} \mathfrak{g}_i$ *and the map* $\operatorname{ad} X : \mathfrak{p} \to \mathfrak{n}_2$ *is surjective.*

Proof. The first part follows from the definition of an associated cocharacter, and the second assertion from the proof of Prop. 5.9c in [Jan04]. □

Let O_X be the conjugagy class of X in $\mathfrak{g}$. Propositions 3.4 and 3.6 imply that $G \times^P \mathfrak{n}_2 \to \overline{O_X}$ is a resolution of singularities [Jan04, 6.10 and 8.7], and that $\dim \overline{O_X} = \dim \mathfrak{n} + \dim \mathfrak{n}_2$.

3.2.2 Results in the Orthosymplectic Setting

Return to the setting of section 3.1.3 where U is an *orthogonal* space, V is a *symplectic* space and $X \in \mathrm{L}(V, U)$. Let $\widetilde{G}$ denote the group of invertible linear transformations of the sum $U \oplus V$ and let $\widetilde{X}$ be the element $\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \in \mathrm{End}(U \oplus V)$ for $X \in \mathrm{L}(V, U)$ such that (X^*, X) is a nilpotent pair.

Note that $\widetilde{X}$ is nilpotent. Indeed, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{End}(U \oplus V)$,

$$\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} Xc & Xd \\ X^*a & X^*b \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} XX^*a & XX^*b \\ X^*Xc & X^*Xd \end{pmatrix}.$$

Hence $\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}^2 = \begin{pmatrix} XX^* & 0 \\ 0 & X^*X \end{pmatrix}$. Since we are assuming that (X^*, X) is a nilpotent pair, we have $\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}^{2l} = \begin{pmatrix} (XX^*)^l & 0 \\ 0 & (X^*X)^l \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for some suitably large $l \in \mathbf{Z}^+$.

Theorem 3.7. *There is a cocharacter $\phi : K^\times \to \widetilde{G}$ associated to nilpotent $\widetilde{X} = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$ in $\widetilde{G} = \mathrm{GL}(U \oplus V)$ with $\phi(t) \in \mathrm{O}(U) \times \mathrm{Sp}(V)$ for all $t \in K^\times$.*

To prove this, we utilize the classification via ab-diagram of the pairs $(X^*, X) \in \mathrm{L}(U, V) \times \mathrm{L}(V, U)$, which naturally identify with $\widetilde{X}$. We begin by showing that such a cocharacter exists for $\widetilde{X} = (X^*, X)$ with indecomposable orthosymplectic ab-diagram as given in Table 3.1. We go on to show that $U \oplus V$ can be decomposed in such a way that $\widetilde{X}$ restricted to each component corresponds to a pair with indecomposable ab-diagram, and conclude that a cocharacter associated to $\widetilde{X}$ exists by combining the cocharacters associated to the restrictions.

Lemma 3.8. *A cocharacter as in Theorem 3.7 exists for $\widetilde{X} = (X^*, X)$ with indecomposable orthosymplectic ab-diagram.*

Proof. We examine each indecomposable diagram in turn.

Case 1. Let $\widetilde{X}$ have ab-diagram $\alpha_n = abab\ldots ba$ where the number of a's is $2n+1$ and the number of b's is $2n$. Then there exists $u \in U$ such that

$$\{u, \widetilde{X}u, \widetilde{X}^2u, \ldots, \widetilde{X}^{4n}u\}$$

is a basis for $U \oplus V$ (see proof in section 3.1.4).

Define the cocharacter $\phi : K^\times \to \widetilde{G} = \mathrm{GL}(U \oplus V)$ by

$$\phi(t)(\widetilde{X}^k u) = t^{4n-2k}\widetilde{X}^k u.$$

Let $T = \mathrm{diag}(t^{4n}, t^{4n-2}, \ldots, t^{2-4n}, t^{-4n})$ be the matrix associated to $\phi(t)$ with respect to the given basis for $U \oplus V$. Then $\widetilde{X} \in \mathfrak{g}(2; \phi)$ since

$$\mathrm{Ad}(\phi(t))\widetilde{X} = TJ_{4n+1}T^{-1} = t^2J_{4n+1} = t^2\widetilde{X},$$

where J_{4n+1} is the $(4n+1) \times (4n+1)$ Jordan matrix associated to $\widetilde{X}$.

Let $L = \mathrm{GL}(U \oplus V) = \mathrm{GL}_{4n+1}(K)$. Since $\widetilde{X}$ has partition $[4n+1]$ in $\widetilde{G}$, $\widetilde{X}$ is distinguished in $\operatorname{Lie} L = \mathfrak{gl}(U \oplus V)$ by Lemma 3.3. We have $\mathcal{D}L = \mathrm{SL}(U \oplus V)$. The image $\phi(t)$ has matrix representation $\operatorname{diag}(t^{4n}, t^{4n-2}, \dots, t^0, \dots, t^{2-4n}, t^{-4n})$, hence $\operatorname{Det}(\phi(t)) = 1$ and it is immediate that $\phi(t) \in \mathcal{D}L$ for all $t \in K^\times$. Thus ϕ is associated to $\widetilde{X}$ in $\widetilde{G}$.

We would like to see also that $\phi(t) \in \mathrm{O}(U) \times \mathrm{Sp}(V)$ for all $t \in K^\times$. To do this, we may use the bilinear form β defined in section 3.1.4, since β defines symplectic and orthogonal groups $\mathrm{Sp}(V, \beta)$ and $\mathrm{O}(U, \beta)$ conjugate to $\mathrm{Sp}(V)$ and $\mathrm{O}(V)$. In particular, there exist $g \in \mathrm{GL}(U)$ such that $\mathrm{O}(U) = g\,\mathrm{O}(U, \beta) g^{-1}$ with

$$\beta(h(u), h(u'))|_U = \langle u, u' \rangle_U$$

for $u, u' \in U$ and $h \in \mathrm{GL}(V)$ such that $\mathrm{Sp}(V) = h\,\mathrm{Sp}(V, \beta) h^{-1}$ with

$$\beta(h(v), h(v'))|_V = \langle v, v' \rangle_V$$

for $v, v' \in V$ [Jan04, 1.3].

$\phi(t) \in \mathrm{GL}(U) \times \mathrm{GL}(V)$ since $\phi(t)$ leaves U and V invariant and $\det(\phi(t)|_U) = \det(\phi(t)|_V) = 1$. For $t \in K^\times$, $\beta(\phi(t)\widetilde{X}^j u, \widetilde{X}^h u) = t^{4n-2j}\beta(\widetilde{X}^j u, \widetilde{X}^h u) = 0$ unless $j + h = 4n$. In this case, $t^{4n-2j} = t^{-(2(4n-h)-4n)} = (t^{-1})^{4n-2h}$ and so

$$\begin{aligned} \beta(\phi(t)\widetilde{X}^j u, \widetilde{X}^h u) &= t^{4n-2j}\beta(\widetilde{X}^j u, \widetilde{X}^h u) \\ = (t^{-1})^{4n-2h}\beta(\widetilde{X}^j u, \widetilde{X}^h u) &= \beta(\widetilde{X}^j u, \phi(t^{-1})\widetilde{X}^h u). \end{aligned}$$

Thus $\phi(t)^* = \phi(t)^{-1}$ with respect to β on either U or V, so $\phi(t) \in \mathrm{O}(U) \times \mathrm{Sp}(V)$. (In fact, we showed that $\phi(t) \in \mathrm{SO}(U) \times \mathrm{Sp}(V)$.)

Case 2. Let $\widetilde{X}$ have ab-diagram $\beta_n = bab \dots ab$ where the number of a's is $2n-1$ and the number of b's is $2n$. Then there exists $v \in V$ such that $\{v, \widetilde{X}v, \dots, \widetilde{X}^{4n-2}v\}$ is a basis for $U \oplus V$.

Define $\phi : K^\times \to \widetilde{G} = \mathrm{GL}(U \oplus V)$ by $\phi(t)\widetilde{X}^k v = t^{4n-2k-2}\widetilde{X}^k v$. We have then that $\widetilde{X} \in \mathfrak{g}(2; \phi)$. Indeed, $\phi(t)$ has matrix $T = \operatorname{diag}(t^{4n-2}, t^{4n-4}, \dots, t^{4-4n}, t^{2-4n})$

with respect to the above basis, thus $\mathrm{Ad}(\phi(t))\widetilde{X} = TJ_{4n-1}T^{-1} = t^2\widetilde{X}$.

Consider the Levi subgroup $L = \mathrm{GL}(U \oplus V) = \mathrm{GL}_{4n-1}(K)$ of $\widetilde{G}$. The element $\widetilde{X}$ has partition $[4n-1]$ in $\widetilde{G}$, and so $\widetilde{X}$ is a distinguished nilpotent element in $\mathrm{Lie}(L) = \mathfrak{gl}(U \oplus V)$ by Lemma 3.3. The derived group $\mathcal{D}L$ is equal to $\mathrm{SL}(U \oplus V)$, and thus $\phi(t) \in \mathcal{D}L$ since $\mathrm{Det}(T) = 1$.

Using the bilinear form defined in section 3.1.5, we compute that $\phi(t)^* = \phi(t)^{-1}$ for $t \in K^\times$, and thus $\phi(t) \in \mathrm{O}(U) \times \mathrm{Sp}(V)$. Assuming that $j + h = 4n - 2$,

$$\beta(\phi(t)\widetilde{X}^j v, \widetilde{X}^h v) = t^{4n-2j-2}\beta(\widetilde{X}^j v, \widetilde{X}^h v) = t^{4n-2(4n-2-h)-2}\beta(\widetilde{X}^j v, \widetilde{X}^h v) =$$
$$t^{-4n+2+2h}\beta(\widetilde{X}^j v, \widetilde{X}^h v) = (t^{-1})^{4n-2h-2}\beta(\widetilde{X}^j v, \widetilde{X}^h v) = \beta(\widetilde{X}^j v, \phi(t^{-1})\widetilde{X}^h v),$$

the right and left-hand sides equal to zero otherwise. Hence $\phi(t)^* = \phi(t)^{-1}$ with respect to β and $\phi(t) \in \mathrm{O}(U) \times \mathrm{Sp}(V)$.

ϕ is therefore an associated cocharacter of X with $\phi(t) \in \mathrm{O}(U) \times \mathrm{Sp}(V)$.

Case 3. Let $\widetilde{X}$ have ab-diagram $\gamma_n = \substack{aba...b...ba \\ aba...b...ba}$ with n odd, and $\#a = 2(n+1)$ and $\#b = 2n$. Then there exist $u_i \in U, i \in \{1, 2\}$ such that

$$\{u_1, \widetilde{X}u_1, \widetilde{X}^2u_1, \ldots, \widetilde{X}^{2n}u_1, u_2, \widetilde{X}u_2, \widetilde{X}^2u_2, \ldots, \widetilde{X}^{2n}u_2\}$$

is a basis for $U \oplus V$.

Define $\phi : K^\times \to \widetilde{G} = \mathrm{GL}(U \oplus V)$ by $\phi(t)\widetilde{X}^k u_i = t^{2n-2k}u_i$. Then $\widetilde{X} \in \mathfrak{g}(2;\phi)$. Indeed, $\phi(t)$ has matrix T equal to the block-diagonal matrix $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ with $T_i = \mathrm{diag}(t^{2n}, t^{2n-2}, \ldots, t^{2-2n}, t^{-2n})$. Hence

$$\mathrm{Ad}(\phi(t))\widetilde{X} = T \begin{pmatrix} J_{2n+1} & 0 \\ 0 & J_{2n+1} \end{pmatrix} T^{-1} = t^2 \begin{pmatrix} J_{2n+1} & 0 \\ 0 & J_{2n+1} \end{pmatrix} = t^2\widetilde{X}$$

Consider Levi subgroup $L = \mathrm{GL}(W_1) \times \mathrm{GL}(W_2) = \mathrm{GL}_{2n+1}(K) \times \mathrm{GL}_{2n+1}(K)$ where $W_i = \mathrm{span}\{\widetilde{X}^j u_i \mid 0 \leq j \leq 2n\}$, $i = 1, 2$. We have $U \oplus V = W_1 \oplus W_2$ and $\widetilde{X} = X_1 + X_2$ where $X_i = \widetilde{X}|_{W_i}$ has partition $[2n+1]$. Each X_i a distinguished nilpotent element of $\mathrm{Lie}(W_i)$, and so $\widetilde{X}$ is a distinguished nilpotent element of $\mathrm{Lie}(L)$

by Lemma 3.3 and [Jan04, 4.3]. The derived group $\mathcal{D}L$ is equal to $\mathrm{SL}(W_1)\times\mathrm{SL}(W_2)$ and $\phi(t)\in\mathcal{D}L$ for $t\in K^\times$. Indeed, $\phi(t)$ has associated matrix $T=\begin{pmatrix}T_1 & 0\\ 0 & T_2\end{pmatrix}$, which has $\det(T_i)=1$ and so $T\in\mathrm{SL}(W_1)\times\mathrm{SL}(W_2)$.

Using the bilinear form β defined in section 3.1.6, we assume $j+h=2n$. Then

$$\beta(\phi(t)\widetilde{X}^j u_i,\widetilde{X}^j u_{i_*})=t^{2n-2j}\beta(\widetilde{X}^j u_i,\widetilde{X}^h u_{i_*})=t^{-2n+2h}\beta(\widetilde{X}^j u_i,\widetilde{X}^h u_{i_*})=$$
$$(t^{-1})^{2n-2h}\beta(\widetilde{X}^j u_i,\widetilde{X}^h u_{i_*})=\beta(\widetilde{X}^j u_i,\phi(t^{-1})\widetilde{X}^h u_{i_*})$$

and thus $\phi(t)^*=\phi(t)^{-1}$ and $\phi(t)\in\mathrm{O}(U)\times\mathrm{Sp}(V)$. Hence ϕ is a cocharacter associated to $\widetilde{X}$ with $\phi(t)\in\mathrm{O}(U)\times\mathrm{Sp}(V)$.

Case 4. Let $\tau=\delta_n=\frac{bab...b...ab}{bab...b...ab}$ with n even, $\#a=2n$ and $\#b=2(n+1)$. Then $\{\widetilde{X}^i v_1,\widetilde{X}^j v_2\mid 0\le i,j\le 2n\}$ is a basis for $U\oplus V$ with $v_i\in V, i\in\{1,2\}$.

Define $\phi:K^\times\to\widetilde{G}=\mathrm{GL}(U\oplus V)$ by $\phi(t)\widetilde{X}^k v_i=t^{2n-2k}v_i$. Then $\widetilde{X}\in\mathfrak{g}(2;\phi)$: $\phi(t)$ has matrix T equal to the block-diagonal matrix $\begin{pmatrix}T_1 & 0\\ 0 & T_2\end{pmatrix}$ where $T_i=\mathrm{diag}(t^{2n},t^{2n-2},\ldots,t^{2-2n},t^{-2n})$ with respect to the above basis. Hence

$$\mathrm{Ad}(\phi(t))\widetilde{X}=T\begin{pmatrix}J_{2n+1} & 0\\ 0 & J_{2n+1}\end{pmatrix}T^{-1}=t^2\begin{pmatrix}J_{2n+1} & 0\\ 0 & J_{2n+1}\end{pmatrix}=t^2\widetilde{X}$$

Consider Levi subgroup $L=\mathrm{GL}(W_1)\times\mathrm{GL}(W_2)=\mathrm{GL}_{2n+1}(K)\times\mathrm{GL}_{2n+1}(K)$ where $W_i=\mathrm{span}\{\widetilde{X}^j v_i\mid 0\le j\le 2n\}$, $i=1,2$. We have $U\oplus V=W_1\oplus W_2$ and $\widetilde{X}=X_1+X_2$ where $X_i=\widetilde{X}|_{W_i}$ has partition $[2n+1]$. Each X_i is a distinguished nilpotent element of $\mathrm{Lie}(W_i)$, and so $\widetilde{X}$ is a distinguished nilpotent element of $\mathrm{Lie}(L)$ by Lemma 3.3 and [Jan04, 4.3]. $\mathcal{D}L=\mathrm{SL}(W_1)\times\mathrm{SL}(W_2)$ and $\phi(t)\in\mathcal{D}L$. Indeed, for $t\in K^\times$, the matrix $T=\begin{pmatrix}T_1 & 0\\ 0 & T_2\end{pmatrix}$ of $\phi(t)$ has $\det(T_i)=1$ and so $T\in\mathrm{SL}(W_1)\times\mathrm{SL}(W_2)$.

Using the bilinear form β defined in section 3.1.7, we assume $j+h=2n$. Then

$$\beta(\phi(t)\widetilde{X}^j v_i,\widetilde{X}^j v_{i_*})=t^{2n-2j}\beta(\widetilde{X}^j v_i,\widetilde{X}^h v_{i_*})=t^{-2n+2h}\beta(\widetilde{X}^j v_i,\widetilde{X}^h v_{i_*})=$$
$$(t^{-1})^{2n-2h}\beta(\widetilde{X}^j v_i,\widetilde{X}^h v_{i_*})=\beta(\widetilde{X}^j v_i,\phi(t^{-1})\widetilde{X}^h v_{i_*})$$

Thus $\phi(t)^* = \phi(t)^{-1}$ and $\phi(t) \in \mathrm{O}(U) \times \mathrm{Sp}(V)$. Hence ϕ is a cocharacter associated to $\widetilde{X}$ with $\phi(t) \in \mathrm{O}(U) \times \mathrm{Sp}(V)$.

Case 5. Let $\tau = \epsilon_n = \frac{abab...bab}{baba...aba}$, where there are $2n$ a's, and $2n$ b's. $U \oplus V$ has basis $\{w_1, \widetilde{X}w_1, \widetilde{X}^2 w_1, \ldots, \widetilde{X}^{2n-1}w_1, w_2, \widetilde{X}^2 w_2, \ldots, \widetilde{X}^{2n-1}w_2\}$ with $w_1 \in U$ and $w_2 \in V$.

Define $\phi : K^\times \to \widetilde{G} = \mathrm{GL}(U \oplus V)$ by $\phi(t)\widetilde{X}^k w_i = t^{2n-2k-1}w_i$. Then $\widetilde{X} \in \mathfrak{g}(2;\phi)$. Indeed, $\phi(t)$ has matrix T equal to the block-diagonal matrix $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ where $T_i = \mathrm{diag}(t^{2n-1}, t^{2n-3}, \ldots, t^{3-2n}, t^{1-2n})$ with respect to the above basis. Hence

$$\mathrm{Ad}(\phi(t))\widetilde{X} = T \begin{pmatrix} J_{2n} & 0 \\ 0 & J_{2n} \end{pmatrix} T^{-1} = t^2 \begin{pmatrix} J_{2n} & 0 \\ 0 & J_{2n} \end{pmatrix} = t^2 \widetilde{X}$$

Consider Levi subgroup $L = \mathrm{GL}(W_1) \times \mathrm{GL}(W_2) = \mathrm{GL}_{2n}(K) \times \mathrm{GL}_{2n}(K)$ where $W_i = \mathrm{span}\{\widetilde{X}^j w_i \mid 0 \leq j \leq 2n-1\}$, $i = 1, 2$. We have $U \oplus V = W_1 \oplus W_2$ and $\widetilde{X} = X_1 + X_2$ where $X_i = \widetilde{X}|_{W_i}$ has partition $[2n]$. Each X_i is a distinguished nilpotent element of $\mathrm{Lie}(W_i)$, and so $\widetilde{X}$ is a distinguished nilpotent element of $\mathrm{Lie}(L)$ by Lemma 3.3 and [Jan04, 4.3]. $\mathcal{D}L = \mathrm{SL}(W_1) \times \mathrm{SL}(W_2)$ and $\phi(t) \in \mathcal{D}L$. Indeed, for $t \in K^\times$, the matrix $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ of $\phi(t)$ has $\det(T_i) = 1$ and so $T \in \mathrm{SL}(W_1) \times \mathrm{SL}(W_2)$.

In section 3.1.8, we defined two possible bilinear forms, one when n is even and the other when n is odd. The forms are nonzero for the same pairing of basis elements, and so we assume $j + h = 2n - 1$ and proceed identically for either n even or odd. We have then

$$\beta(\phi(t)\widetilde{X}^j w_i, \widetilde{X}^j w_{i_*}) = t^{2n-2j-1}\beta(\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*}) =$$
$$t^{2n-2(2n-1-h)-1}\beta(\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*}) = (t^{-1})^{2n-2h-1}\beta(\widetilde{X}^j w_i, \widetilde{X}^h w_{i_*}) =$$
$$\beta(\widetilde{X}^j w_i, \phi(t^{-1})\widetilde{X}^h w_{i_*})$$

Thus $\phi(t)^* = \phi(t)^{-1}$ and $\phi(t) \in \mathrm{O}(U) \times \mathrm{Sp}(V)$. Hence ϕ is a cocharacter associated to $\widetilde{X}$ with $\phi(t) \in \mathrm{O}(U) \times \mathrm{Sp}(V)$. □

Lemma 3.9. *Given $X \in \mathrm{L}(V,U)$, U and V quadratic spaces of types +1 and -1 respectively, there exist decompositions $V = \bigoplus_{i=1}^{m} V_i$ and $U = \bigoplus_{i=1}^{m} U_i$ such that $X|_{V_i} : V_i \to U_i$ and $X^*|U_i : U_i \to V_i$, with each pair $(X_i^*, X_i) \in \mathrm{L}(U_i, V_i) \times \mathrm{L}(V_i, U_i)$ having indecomposable orthosymplectic ab-diagram τ_i. Furthermore, the restriction to U_i or V_i of the bilinear form on U or V is nondegenerate.*

Proof. This is the content of the *ab*-diagrams. □

Proof of Theorem 3.7. For (X^*, X) with arbitrary *ab*-diagram τ, construct β_i, ϕ_i, and L_i as in Lemma 3.8 for each pair (X_i^*, X_i) with indecomposable *ab*-diagram τ_i as in Lemma 3.9. Taking ϕ such that $\phi|_{U_i \oplus V_i} = \phi_i$ and Levi subgroup $L = \prod_i L_i$, we have that $\phi : K^\times \to \widetilde{G}$ is an associated cocharacter to X and $\phi(t) \in \prod_i \mathrm{O}(U_i) \times \mathrm{Sp}(V_i) \subseteq \mathrm{O}(U) \times \mathrm{Sp}(V)$ for all $t \in K^\times$. □

3.3 Dimensions of Orthosymplectic Orbits

The content of this section is the proof of the following result relating the dimension of such an orbit to the dimensions of its images under the maps ρ and π. This result was originally proven in [KP82] for char $K = 0$, the argument from which is generalized here for char $K \neq 2$. Our proof differs from that in [KP82] in that we use the associated cocharacter in Theorem 3.7 to replace $H \in \widetilde{\mathfrak{g}}^{(0)}$ from [KP82] Lemma 7.3, the semisimple element of an $\mathfrak{sl}_2$-triple. We reproduce the remainder of the proof here to demonstrate its lack of dependence on the character of K, giving references to needed characteristic free results from [Jan04]. Additionally, the method for computing the dimension of the weight spaces of the associated cocharacter ϕ follows from the explicit construction of ϕ in the proof of Lemma 3.8.

Once again, U and V are quadratic spaces of types +1 and −1, respectively. Let $\widetilde{X} = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \in \mathrm{L}(U,V) \times \mathrm{L}(V,U)$ such that (X^*, X) is an orthosymplectic nilpotent pair with associated ab-diagram τ and let $\mathcal{O}_X$ be the $\mathrm{O}(U) \times \mathrm{Sp}(V)$-orbit of $X \in \mathrm{L}(V,U)$ with action as in section 3.1.1. Denote by π and ρ the maps

$$\begin{array}{ccc} \mathrm{L}(V,U) & \xrightarrow{\pi} & \mathfrak{g}(U) \\ \downarrow{\scriptstyle \rho} & & \\ \mathfrak{g}(V) & & \end{array}$$

as in section 2.2.1.

Proposition 3.10.

$$\dim \mathcal{O}_X = \frac{1}{2}(\dim \pi(\mathcal{O}_X) + \dim \rho(\mathcal{O}_X) + \dim U \cdot \dim V - \Delta_\tau)$$

where $\Delta_\tau = \sum_{i odd} a_i b_i$*,* a_i *the number of rows of* τ *of length* i *starting with* a *and* b_i *the number of rows of* τ *of length* i *starting with* b*.*

Note that $\Delta_\tau = 0$ when $X : V \to U$ is either injective or surjective.

3.3.1 $\widetilde{G}$ as a Θ-group

We begin by describing $\mathrm{L}(U,V) \times \mathrm{L}(V,U)$ as a Θ-group as in [Vin76] (see [Lev09] for characteristic $p > 2$). Consider the group $\widetilde{G} := \mathrm{GL}(U \oplus V)$ and the automorphism $\Theta : \widetilde{G} \to \widetilde{G}$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \stackrel{\Theta}{\longmapsto} \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}^{-1}.$$

We observe that Θ^2 is conjugation with the matrix $J := \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}$. Indeed,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \stackrel{\Theta}{\longmapsto} \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}^{-1} \stackrel{\Theta}{\longmapsto} \begin{pmatrix} (A^*)^* & (B^*)^* \\ (C^*)^* & (D^*)^* \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$$

since $A \in \mathrm{L}(U,U)$, $D \in \mathrm{L}(V,V)$, $C \in \mathrm{L}(U,V)$ and $D \in \mathrm{L}(V,U)$, with U and V of type 1 and -1, respectively (see section 2.2.1). This then implies that Θ^4 is the identity.

$\widetilde{G}^{\Theta}$, the fixed point group of Θ, is $\mathrm{O}(U) \times \mathrm{Sp}(V)$ and $\widetilde{G}^{\Theta^2}$, the fixed point group of Θ^2, is $\mathrm{GL}(U) \times \mathrm{GL}(V)$. To see this, we first compute $\widetilde{G}^{\Theta^2}$ since $\widetilde{G}^{\Theta} \subseteq \widetilde{G}^{\Theta^2}$. If $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \widetilde{G}^{\Theta^2}$,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \Theta^2 \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix},$$

which implies $B = C = 0$. Hence any element of $\widetilde{G}^{\Theta^2}$ has block form $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$. Since $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ is invertible, $A \in \mathrm{GL}(U)$ and $D \in \mathrm{GL}(V)$. Now if $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \widetilde{G}^{\Theta}$, we have $B = C = 0$ from before, and

$$\Theta \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} A^* & 0 \\ 0 & D^* \end{pmatrix}^{-1} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

i.e. $\begin{pmatrix} A^* & 0 \\ 0 & D^* \end{pmatrix}$ is the inverse of $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$. Multiplying

$$\begin{pmatrix} A^* & 0 \\ 0 & D^* \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} I_U & 0 \\ 0 & I_V \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} A^* & 0 \\ 0 & D^* \end{pmatrix},$$

we get that $AA^* = I_U = A^*A$ and $DD^* = I_V = D^*D$, hence $A^* = A^{-1}$ and $D^* = D^{-1}$. Thus $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathrm{O}(U) \times \mathrm{Sp}(V)$ (section 2.1.1). The reverse inclusions are clear. Define $G := \widetilde{G}^{\Theta} = \mathrm{O}(U) \times \mathrm{Sp}(V)$ and $G' := \widetilde{G}^{\Theta^2} = \mathrm{GL}(U) \times \mathrm{GL}(V)$.

We denote by $d\Theta$ the automorphism of order four induced by Θ on the Lie algebra $\widetilde{\mathfrak{g}} := \operatorname{Lie}\widetilde{G} = \operatorname{End}(U \oplus V)$. It is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \stackrel{d\Theta}{\longmapsto} - \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}.$$

Since we have been assuming $p \neq 2$, Θ is a semisimple automorphism automorphism of $\widetilde{G}$. Hence the Lie algebra of the centralizer G, above, of Θ is the centralizer in $\operatorname{Lie}(G)$ of $d\Theta$ [Jan04, 2.5, 2.6].

Fixing a primitive fourth root of unity, ζ_4, define $\widetilde{\mathfrak{g}}^{(i)} := \{X \in \widetilde{\mathfrak{g}} \mid d\Theta X = \zeta_4^i X\}$. Then

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}^{(0)} \oplus \widetilde{\mathfrak{g}}^{(1)} \oplus \widetilde{\mathfrak{g}}^{(2)} \oplus \widetilde{\mathfrak{g}}^{(3)}$$

is a G-stable $\mathbb{Z}/4\mathbb{Z}$-graduation of $\widetilde{\mathfrak{g}}$ where, by definition,

$$\widetilde{\mathfrak{g}}^{(0)} = \mathfrak{g} := \operatorname{Lie} G \qquad \text{and} \qquad \widetilde{\mathfrak{g}}^{(0)} \oplus \widetilde{\mathfrak{g}}^{(2)} = \mathfrak{g}' := \operatorname{Lie} G'.$$

Additionally, we can identify $\widetilde{\mathfrak{g}}^{(1)} \cong \mathrm{L}(V, U)$ as G-modules. If

$$d\Theta \begin{pmatrix} A & B \\ C & D \end{pmatrix} = - \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \zeta_4 \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Then $-A^* = \zeta_4 A$, $-D^* = \zeta_4 D$, $-C^* = \zeta_4 B$, and $-B^* = \zeta_4 C$. The last two equations yield $C = \zeta_4 B^*$, recalling that $(C^*)^* = -C$ and $(B^*)^* = -B$. If we apply * to the first two equations, we get $A = (A^*)^* = (-\zeta_4 A)^* = \zeta_4^2 A = -A$, likewise $D = -D$, implying that $A = D = 0$. Hence

$$\widetilde{\mathfrak{g}}^{(1)} = \left\{ \begin{pmatrix} 0 & B \\ \zeta_4 B^* & 0 \end{pmatrix} \mid B \in \mathrm{L}(V, U) \right\}.$$

3.3.2 Grading of $\widetilde{G}$ via Cocharacters

Recall from section 3.2.1 the means of grading the Lie algebra of a reductive group via a cocharacter. Let $\phi : K^\times \to \widetilde{G}$ be a cocharacter associated to $\widetilde{X}$ in $\widetilde{G}$

as in Theorem 3.7 and define $\widetilde{\mathfrak{g}} = \bigoplus_{i\in\mathbb{Z}} \widetilde{\mathfrak{g}}_i$ where $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}(i;\phi) = \{Z \in \widetilde{\mathfrak{g}} \mid \mathrm{Ad}(\phi(t))Z = t^i Z \text{ for all } t \in K^\times\}$.

It is clear from the proof of Lemma 3.8 how to calculate the dimensions of the weight spaces $\widetilde{\mathfrak{g}}_i$ of ϕ in terms of the the ab-diagram, τ, associated to $X \in \mathrm{L}(V,U)$. Namely, count the number of basis vectors $\widetilde{X}^k w$, $w \in U$ or V, such that $\mathrm{Ad}(\phi(t))\widetilde{X}^k w = t^i \widetilde{X}^k$. This is most easily done by labeling each row of the diagram of length r by $(r-1, r-3, \ldots, 2, 0, -2, \ldots, 3-r, 1-r)$ when r is odd, and by $(r-1, r-3, \ldots, 1, -1, \ldots, 3-r, 1-r)$ when r is even. These dimensions depend only on $\widetilde{X}$, thus only on X, and not on the choice of cocharacter since we have defined ϕ to be associated to $\widetilde{X}$ and any associated cocharacter will result in the same weight space decomposition by Proposition 3.5.

The cocharacter ϕ as in Theorem 3.7 takes values in $\mathrm{O}(U) \times \mathrm{Sp}(V) \subset \mathrm{GL}(U) \times \mathrm{GL}(V)$, and so each $\widetilde{\mathfrak{g}}_i$ has the form $U_i \oplus V_i$. The dimensions of each U_i and V_i can be calculated by keeping track of the labeling of the a's (corresponding to basis vectors belonging to U) and b's (corresponding to basis vectors belonging to V) in the ab-diagram of X. In particular,

$$\dim U_0 = (\#\alpha_\tau) + (\#\beta_\tau)$$

and

$$\dim V_0 = 2(\#\delta_\tau) + 2(\#\gamma_\tau)$$

where $\#\alpha_\tau$ denotes the number of irreducible diagrams of type α_k in the ab-diagram τ of (X^*, X), $k \in \mathbb{Z}$, etc.

For example, by labeling the ab-diagram

$$\tau = \begin{array}{l} ababab a \\ bababab \\ ababa \\ ab \\ ba \\ a \end{array} = \begin{array}{l} a_6 b_4 a_2 b_0 a_{-2} b_{-4} a_{-6} \\ b_6 a_4 b_2 a_0 b_{-2} a_{-4} b_{-6} \\ a_4 b_2 a_0 b_{-2} a_{-4} \\ a_1 b_{-1} \\ b_1 a_{-1} \\ a_0 \end{array}$$

and we can see that $\dim(U_0) = 3$ (there are three a_0's) and $\dim(V_0) = 1$ (there is one b_0).

3.3.3 More on Orbit Dimensions

For $X \in \widetilde{\mathfrak{g}}^{(1)} \cong \mathrm{L}(V, U)$ a nilpotent element with associated orthosymplectic ab-diagram τ, choose a cocharacter $\phi : K^\times \to \widetilde{G}$ associated to $\widetilde{X}$ with image in $\widetilde{G}^{(0)} = \mathrm{O}(U) \times \mathrm{Sp}(V)$ as in Theorem 3.7. Let $\widetilde{P} = \widetilde{P}_\phi$ be the associated parabolic subgroup of ϕ in $\widetilde{G}$ with $\mathrm{Lie}(\widetilde{P}) = \widetilde{\mathfrak{p}} := \bigoplus_{i \geq 0} \widetilde{\mathfrak{g}}_i$. The Lie algebra $\widetilde{\mathfrak{p}}$ is a parabolic subalgebra of $\widetilde{\mathfrak{g}}$ and has nilpotent radical $\widetilde{\mathfrak{n}} := \bigoplus_{i > 0} \widetilde{\mathfrak{g}}_i$ and Levi-decomposition $\widetilde{\mathfrak{p}} = \widetilde{\mathfrak{g}}_0 \oplus \widetilde{\mathfrak{n}}$. We have from Proposition 3.6 that $\mathrm{ad}\, X : \widetilde{\mathfrak{p}} \to \widetilde{\mathfrak{n}}_2$ is surjective.

ϕ defines a $\mathbb{Z}$-grading of $\mathfrak{g}' = \mathfrak{gl}(U) \times \mathfrak{gl}(V)$ and $\mathfrak{g} = \mathfrak{so}(U) \times \mathfrak{sp}(V)$ induced by the grading of $\widetilde{\mathfrak{g}}$ in section 3.3.2. $\mathfrak{p}' := \widetilde{\mathfrak{p}} \cap \mathfrak{g}'$ and $\mathfrak{p} := \widetilde{\mathfrak{p}} \cap \mathfrak{g}$ are parabolic subalgebras of $\mathfrak{g}'$ and $\mathfrak{g}$ with Levi decompositions

$$\mathfrak{p}' = \mathfrak{g}_0' \oplus \widetilde{\mathfrak{n}}', \quad \text{where } \mathfrak{g}_0' := \widetilde{\mathfrak{g}}_0 \cap \mathfrak{g}', \quad \widetilde{\mathfrak{n}}' := \widetilde{\mathfrak{n}} \cap \mathfrak{g}'$$

and

$$\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n}, \quad \text{where } \mathfrak{g}_0 := \widetilde{\mathfrak{g}}_0 \cap \mathfrak{g}, \quad \mathfrak{n} := \widetilde{\mathfrak{n}} \cap \mathfrak{g}\,.$$

Let P' and P be the parabolic subgroups of $G' = \mathrm{GL}(U) \times \mathrm{GL}(V)$ and $G = \mathrm{O}(U) \times \mathrm{Sp}(V)$, respectively, with $\mathrm{Lie}(P') = \mathfrak{p}'$ and $\mathrm{Lie}(P) = \mathfrak{p}$. By Proposition

3.4(b), it follows that $G'_X \subset P'$ and $G_X \subset P$. From the surjectivity of the map $\operatorname{ad} X : \widetilde{\mathfrak{p}} \to \widetilde{\mathfrak{n}}_2$, we have that the maps

$$\operatorname{ad} X : \mathfrak{p}' \to \mathfrak{n}'_2 \quad \text{and} \quad \operatorname{ad} X : \mathfrak{p} \to \mathfrak{n}_2$$

are surjective, where $\mathfrak{n}'_2 := \widetilde{\mathfrak{n}}_2 \cap (\widetilde{\mathfrak{g}}^{(1)} \oplus \widetilde{\mathfrak{g}}^{(3)})$ and $\mathfrak{n}_2 := \widetilde{\mathfrak{n}}_2 \cap \widetilde{\mathfrak{g}}^{(1)}$.

Lemma 3.11. *Let $\mathcal{O}'_X$, $\mathcal{O}_X$ and $\mathcal{O}^0_X$ denote the orbits of X under G', G and $G^0 = \mathrm{SO}(U) \times \mathrm{Sp}(V)$ respectively.*

(a) The canonical maps $G' \times^{P'} \mathfrak{n}'_2 \to \overline{\mathcal{O}'_X}$ and $G^0 \times^P \mathfrak{n}_2 \to \overline{\mathcal{O}^0_X}$ are resolutions of singularities.

(b) $\dim \mathcal{O}'_X = \dim \mathfrak{n}' + \dim \mathfrak{n}'_2$ *and* $\dim \mathcal{O}_X = \dim \mathcal{O}^0_X = \dim \mathfrak{n} + \dim \mathfrak{n}_2$*.*

(c) $\dim \mathfrak{n}'_2 = 2 \dim \mathfrak{n}_2$*.*

Proof. (a) and (b) follow from the above discussion and that in section 3.2.1. For (c), we recall from earlier that the automorphism $\Theta^2 : \widetilde{G} \to \widetilde{G}$ is conjugation with $J = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix} \in G$. Then since $\phi(t) \in \widetilde{G}^{\Theta}$, we have $\Theta^2\phi(t) = \phi(t)$, and so J and $\phi(t)$ commute. Thus $J\widetilde{\mathfrak{g}}_i = \widetilde{\mathfrak{g}}_i$ for all i. Additionally, we observe that $J\widetilde{\mathfrak{g}}^{(1)} = \widetilde{\mathfrak{g}}^{(3)}$, so that $J(\widetilde{\mathfrak{g}}^{(1)} \cap \widetilde{\mathfrak{g}}_i) = \widetilde{\mathfrak{g}}^{(3)} \cap \widetilde{\mathfrak{g}}_i$. This implies that $J(\widetilde{\mathfrak{g}}^{(1)} \cap \widetilde{\mathfrak{n}}_2) = \widetilde{\mathfrak{g}}^{(3)} \cap \widetilde{\mathfrak{n}}_2$, and in particular, $\dim(\widetilde{\mathfrak{g}}^{(1)} \cap \widetilde{\mathfrak{n}}_2) = \dim(\widetilde{\mathfrak{g}}^{(3)} \cap \widetilde{\mathfrak{n}}_2)$. Since $\mathfrak{n}'_2 = (\widetilde{\mathfrak{g}}^{(1)} \cap \widetilde{\mathfrak{n}}_2) \oplus (\widetilde{\mathfrak{g}}^{(3)} \cap \widetilde{\mathfrak{n}}_2)$ and $\mathfrak{n}_2 = \widetilde{\mathfrak{g}}^{(1)} \cap \widetilde{\mathfrak{n}}_2$, $\dim \mathfrak{n}_2 = 2 \dim \mathfrak{n}'_2$. □

3.3.4 Proof of Proposition 3.10

Proof. We begin by describing the dimension of the orbit $\mathcal{O}_X$ of a nilpotent element $X \in \widetilde{\mathfrak{g}}^{(1)} \cong \mathrm{L}(V, U)$ under $G = \mathrm{O}(U) \times \mathrm{Sp}(V)$ in terms of the dimension of the orbit $\mathcal{O}'_X$ of X under $G' = \mathrm{GL}(U) \times \mathrm{GL}(V)$. Choose an associated cocharacter ϕ of X with $\phi(t) \in G = \widetilde{G}^{(0)}$ (Theorem 3.7) and consider the associated parabolic subalgebras

$$\mathfrak{p}' = \mathfrak{g}'_0 \oplus \mathfrak{n}' \subseteq \mathfrak{g}' \quad \text{and} \quad \mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n} \subseteq \mathfrak{g}$$

as in section 3.3.3. By definition, the Levi factors $\mathfrak{g}'_0$ and $\mathfrak{g}_0$ are the stabilizers of $\phi(t), t \in K^\times$, in $\mathfrak{g}'$ and $\mathfrak{g}$ respectively. If $U = \bigoplus_i U_i$ and $V = \bigoplus_j V_j$ are the weight space decompositions of U and V with respect to ϕ (i.e. $U_i := \{u \in U \mid \phi(t)u = t^i u \text{ for all } t \in K^\times\}$ and $V_j := \{v \in V \mid \phi(t)v = t^i v \text{ for all } t \in K^\times\}$), we have from [Jan04, 4.4] that

$$\mathfrak{g}'_0 = \bigoplus_i \mathfrak{gl}(U_i) \oplus (\bigoplus_j \mathfrak{gl}(V_j)).$$

We can see from the construction in Lemma 3.8 that the subspaces $U_i + U_{-i}$ are nondegenerate orthogonal subspaces of U, and $V_j + V_{-j}$ are nondegenerate symplectic subspaces of V. So by section 4.5 in [Jan04]

$$\mathfrak{g}_0 \cong (\bigoplus_{i>0} \mathfrak{gl}(U_i)) \oplus \mathfrak{so}(U_0) \oplus (\bigoplus_{i>0} \mathfrak{gl}(V_j)) \oplus \mathfrak{sp}(V_0).$$

Let $m_0 := \dim U_0$ and $n_0 := \dim V_0$. Then by the above and equation (#) in section 2.1.1, we get

$$\begin{aligned} 2\dim \mathfrak{g}_0 - \dim \mathfrak{g}'_0 &= (2\dim \mathfrak{so}(U_0) - m_0^2) + (2\dim \mathfrak{sp}(V_0) - n_0^2) \\ &= (m_0(m_0-1) - m_0^2) + (n_0(n_0+1) - n_0^2) \\ &= n_0 - m_0. \end{aligned}$$

Let $m = \dim U$ and $n = \dim V$. By lemma 3.11(b),(c), and equation (#), we have

$$\begin{aligned} 4\dim \mathcal{O}_X - 2\dim \mathcal{O}'_X &= 4\dim \mathfrak{n} - 2\dim \mathfrak{n}' \\ &= 2(\dim \mathfrak{g} - \dim \mathfrak{g}_0) - (\dim \mathfrak{g}' - \dim \mathfrak{g}'_0) \\ &= (2\dim \mathfrak{g} - \dim \mathfrak{g}') - (n_0 - m_0) \\ &= (m(m-1) + n(n+1) - m^2 - n^2) - (n_0 - m_0). \end{aligned}$$

For the second equality, note that $2\dim \mathfrak{n} = \dim \mathfrak{g} - \dim \mathfrak{g}_0$ and $2\dim \mathfrak{n}' = \dim \mathfrak{g}' - \dim \mathfrak{g}'_0$ by definition since $\dim \widetilde{\mathfrak{g}}_i = \dim \widetilde{\mathfrak{g}}_{-i}$ for all $i \in \mathbb{Z}_{>0}$. Combining these,

$$4\dim \mathcal{O}_X = 2\dim \mathcal{O}'_X + (n - m) - (n_0 - m_0).$$

Next, consider the conjugacy classes $\mathrm{Ad}(\mathrm{O}(U))X = \pi(\mathcal{O}_X) \subseteq \mathfrak{so}(U)$ and $\mathrm{Ad}(\mathrm{Sp}(V))X = \rho(\mathcal{O}_X) \subseteq \mathfrak{sp}(V)$ and denote by $\pi(\mathcal{O}_X)'$ and $\rho(\mathcal{O}_X)'$ the conjugacy classes $\mathrm{Ad}(\mathrm{GL}(U))X \subseteq \mathfrak{gl}(U)$ and $\mathrm{Ad}(\mathrm{GL}(V))X \subseteq \mathfrak{gl}(V)$ generated by $\pi(\mathcal{O}_X)$ and $\rho(\mathcal{O}_X)$ under the action of $\mathrm{GL}(U)$ and $\mathrm{GL}(V)$, respectively. We recall the dimension formula in the linear case from proposition 5.3 in [KP79], which is given in characteristic 0 but works in any characteristic ([Don90] proof of Corollary 2.1(d)):

$$\dim \mathcal{O}'_X = \frac{1}{2}(\dim \pi(\mathcal{O}_X)' + \dim \rho(\mathcal{O}_X)') + nm - \Delta_\tau.$$

From Remark 2.5 in section 2.1.3, we have

$$\dim \pi(\mathcal{O}_X)' = 2 \dim \pi(\mathcal{O}_X) + m - r_a$$

and

$$\dim \rho(\mathcal{O}_X)' = 2 \dim \rho(\mathcal{O}_X) - n + r_b,$$

where r_a, r_b are the number of *odd* rows in the Young diagrams of $\pi(\mathcal{O}_X)$ and $\rho(\mathcal{O}_X)$ respectively.

Combining all of these,

$$\begin{aligned} 4 \dim \mathcal{O}_X &= \dim \pi(\mathcal{O}_X)' + \dim \rho(\mathcal{O}_X)' + 2nm - 2\Delta_\tau + (n - m) - (n_0 - m_0) \\ &= 2 \dim \pi(\mathcal{O}_X) + 2 \dim \rho(\mathcal{O}_X) + 2nm - 2\Delta_\tau + (r_b - r_a) - (n_0 - m_0). \end{aligned}$$

We use our previous analysis of ab-diagrams to show that $r_b - r_a = n_0 - m_0$. Denote by $\tau_{i,a}$ the number of a's in the ith row of τ and by $\tau_{i,b}$ the number of b's. From the discussion in section 3.3.2, we have $m_0 = \#\{i \mid \tau_{i,a} \text{ odd and } \tau_{i,b} \text{ even}\}$ and $n_0 = \#\{i \mid \tau_{i,b} \text{ odd and } \tau_{i,a} \text{ even}\}$. Hence $n_0 - m_0 = \#\{i \mid \tau_{i,b} \text{ odd}\} - \#\{j \mid \tau_{j,a} \text{ odd}\} = r_b - r_a$. Thus

$$2 \dim \mathcal{O}_X = \dim \pi(\mathcal{O}_X) + \dim \rho(\mathcal{O}_X) + nm - \Delta_\tau.$$

□

3.4 Dimensions in Z

Recall the definition of Z from section 2.2.1. It is the subset of tuples $(X_1, \ldots, X_t)$ of $M = \mathrm{L}(V_0, V_1) \times \ldots \times \mathrm{L}(V_{t-1}, V_t)$ defined by the equations

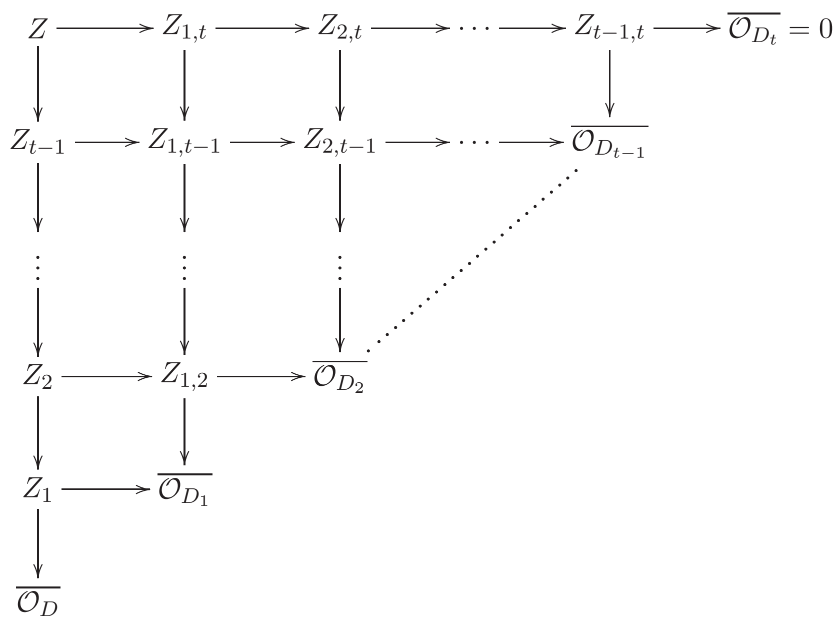
$$
(*) \qquad \begin{aligned}
X_1 X_1^* &= X_2^* X_2 \\
X_2 X_2^* &= X_3^* X_3 \\
&\vdots \\
X_{t-1} X_{t-1}^* &= X_t^* X_t \\
X_t X_t^* &= 0
\end{aligned}
$$

with each $X_i \in \mathrm{L}(V_{i-1}, V_i)$. We have $n_0 = \dim(V_0) \geq n_1 = \dim(V_1) \geq \ldots \geq n_t = \dim(V_t) > 0 = \dim(V_{t+1})$, the V_i quadratic spaces of alternating types, constructed as the images of $D_{i-1} = D|_{V_{i-1}} \in \mathfrak{g}(V_{i-1})$ for some nilpotent $D \in \mathfrak{g}(V)$.

Before proving part (a) of Theorem 2.13, we first need a lemma relating the dimension of the singular locus in Z to the dimension of the orbits in $\overline{\mathcal{O}_D}$. This will use the formula in Proposition 3.10, as well as a new realization of Z as a fiber product of the orbit closures $\overline{\mathcal{O}_{D_i}}$. The proof is identical to that in [KP79] 8.1,8.2, which depends on the character of K only insofar as it relies on the dimension result from Proposition 3.10, proven in [KP79] for $\mathrm{char}\, K = 0$ only.

3.4.1 Z as a Fiber Product

Z can be realized as the iterated fiber product:



We have for $0 \leq i < j \leq t$, $Z_{i,j} = \{(X_{i+1}, \ldots, X_j) \in \mathrm{L}(V_i, V_{i+1}) \times \ldots \times \mathrm{L}(V_{j-1}, V_j) \mid \rho(X_k) \in \overline{\mathcal{O}_{D_{k-1}}}, \pi(X_k) \in \overline{\mathcal{O}_{D_k}}, \rho(X_k) = \pi(X_{k-1}), i+1 \leq k \leq j\}$. For $1 \leq j \leq t$, $Z_j = Z_{0,j}$ (these will reappear in section 4.3). Note that $Z_t = Z$. The last condition, $\rho(X_k) = \pi(X_{k-1})$, is the same as the equations in $(*)$.

Each coordinate X_k of the element $(X_{i+1}, \ldots, X_j) \in Z_{i,j}$ as in the previous paragraph, $0 \leq i < j \leq t$ has an ab-diagram τ_k satisfying $\pi(\tau_k) = \rho(\tau_{k+1}) =: \sigma_k$, where σ_k is an ε-diagram of a nilpotent element in $\mathfrak{g}(V_k)$ (i.e. $\pi(\mathcal{O}_{\tau_k}) = \rho(\mathcal{O}_{\tau_{k+1}}) = \mathcal{O}_{\sigma_k}$) such that $\sigma_k \leq \lambda^k$ where λ^k is the Young diagram of $D_k = D|_{V_k}$ as in section 2.3.1. In other words, $\mathcal{O}_{\sigma_k} \subseteq \overline{\mathcal{O}_{D_k}}$. This follows from the condition that $\pi(X_k) = \rho(X_{k+1}) \in \overline{\mathcal{O}_{D_k}}$.

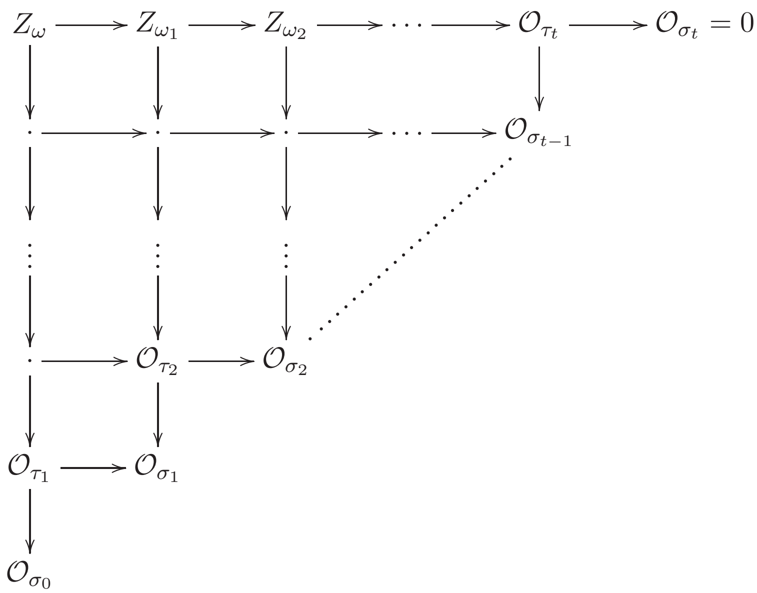
Thus we have a finite set Ω of strings $\omega = (\tau_1, \tau_2, \ldots, \tau_t)$ of orthosymplectic ab-diagrams with $\mathcal{O}_{\tau_i} \subseteq \mathrm{L}(V_{i-1}, V_i)$ satisfying

(i) $\pi(\tau_i) = \rho(\tau_{i+1}) = \sigma_i$ for $1 \leq i \leq t-1$, with σ_i as above an ε-diagram with $\sigma_i \leq \lambda^k$, and

(ii) $\sigma_t = 0$, i.e. $\mathcal{O}_{\sigma_t} = 0$.

We define $\sigma_0 := \rho(\tau_1)$. For $\omega = (\tau_1, \ldots, \tau_t) \in \Omega$, define the locally closed subset $Z_\omega \subseteq Z$ by $Z_\omega = \{(X_1, \ldots, X_t) \in Z \mid X_i \in \mathcal{O}_{\tau_i}\}$.

The conditions on the $\omega \in \Omega$ listed above imply that we have a fiber product diagram subordinate to the previous diagram defining Z:



where ω_i denotes the string $(\tau_{i+1}, \ldots, \tau_t)$ and $\omega_0 = \omega$. All of the maps in this diagram are smooth, being either projections or the maps ρ or π at the bottom of the columns and right-hand ends of the rows. Thus each Z_ω is a smooth variety, and we get from Proposition 3.10

$$\begin{aligned}
\dim Z_\omega &= \dim \mathcal{O}_{\tau_1} - \dim \mathcal{O}_{\sigma_1} + \dim Z_{\omega_1} \\
&= \frac{1}{2}(\dim \mathcal{O}_{\sigma_0} + \dim \mathcal{O}_{\sigma_1} + n_0 n_1 - \Delta_{\tau_1}) - \dim \mathcal{O}_{\sigma_1} + \dim Z_{\omega_1} \\
&= \frac{1}{2}(\dim \mathcal{O}_{\sigma_0} + n_0 n_1 - \Delta_{\tau_1}) - \frac{1}{2} \dim \mathcal{O}_{\sigma_1} + \dim Z_{\omega_1}
\end{aligned}$$

and for any Z_{ω_i}, $0 \leq i \leq t-3$, we have the analagous

$$\begin{aligned}
\dim Z_{\omega_i} &= \dim \mathcal{O}_{\tau_{i+1}} - \dim \mathcal{O}_{\sigma_{i+1}} + \dim Z_{\omega_{i+1}} \\
&= \frac{1}{2}(\dim \mathcal{O}_{\sigma_i} + \dim \mathcal{O}_{\sigma_{i+1}} + n_i n_{i+1} - \Delta_{\tau_{i+1}}) - \dim \mathcal{O}_{\sigma_{i+1}} + \dim Z_{\omega_{i+1}} \\
&= \frac{1}{2} \dim \mathcal{O}_{\sigma_i} + \frac{1}{2}(n_i n_{i+1} - \Delta_{\tau_{i+1}}) - \frac{1}{2} \dim \mathcal{O}_{\sigma_{i+1}} + \dim Z_{\omega_{i+1}}
\end{aligned}$$

For $t = t-2$, we have the fiber product diagram

$$\begin{CD} Z_{\omega_{t-2}} @>>> \mathcal{O}_{\tau_t} \\ @VVV @VVV \\ \mathcal{O}_{\tau_{t-1}} @>>> \mathcal{O}_{\sigma_{t-1}} \end{CD}$$

which implies

$$\begin{aligned} \dim Z_{\omega_{t-2}} &= \dim \mathcal{O}_{\tau_{t-1}} - \dim \mathcal{O}_{\sigma_{t-1}} + \dim \mathcal{O}_{\tau_t} \\ &= \frac{1}{2}(\dim \mathcal{O}_{\sigma_{t-2}} + \dim \mathcal{O}_{\sigma_{t-1}} + n_{t-2}n_{t-1} - \Delta_{\tau_{t-1}}) - \dim \mathcal{O}_{\sigma_{t-1}} + \dim \mathcal{O}_{\tau_t} \\ &= \frac{1}{2}\dim \mathcal{O}_{\sigma_{t-2}} + \frac{1}{2}(n_{t-2}n_{t-1} - \Delta_{\tau_{t-1}}) - \frac{1}{2}\dim \mathcal{O}_{\sigma_{t-1}} + \dim \mathcal{O}_{\tau_t} \end{aligned}$$

Proposition 3.10 implies $\dim \mathcal{O}_{\tau_t} = \frac{1}{2}(\dim \mathcal{O}_{\sigma_{t-1}} + \dim \mathcal{O}_{\sigma_t} + n_{t-1}n_t - \Delta_{\tau_t})$. Recall that $\mathcal{O}_{\sigma_t} = 0$, so we have

$$\dim Z_{\omega_{t-2}} = \frac{1}{2}\dim \mathcal{O}_{\sigma_{t-2}} + \frac{1}{2}(n_{t-2}n_{t-1} + n_{t-1}n_t - \Delta_{\tau_{t-1}} - \Delta_{\tau_t}).$$

Thus inductively,

$$\dim Z_{\omega} = \frac{1}{2}\dim \mathcal{O}_{\sigma_0} + \frac{1}{2}\sum_{i=1}^{t}(n_{i-1}n_i - \Delta_{\tau_i}).$$

3.4.2 From Orbits to Z

Recall from section 2.3.3 the map $\vartheta : Z \to \overline{\mathcal{O}_D}$ given by $\vartheta(X_1, \dots, X_t) = X_1^* X_1$.

Lemma 3.12. *For every conjugacy class $\mathcal{O} \subseteq \overline{\mathcal{O}_D}$ we have*

$$\operatorname{codim}_Z \vartheta^{-1}(\mathcal{O}) \geq \frac{1}{2}\operatorname{codim}_{\overline{\mathcal{O}_D}} \mathcal{O}.$$

Proof. Recall that the orbit $\mathcal{O}_D$ is open in its closure $\overline{\mathcal{O}_D}$. As a consequence of Lemma 2.8 and the discussion in section 2.3.3, there is a unique open stratum Z_{λ^0} that is the iterated fiber product of the open orbits $\mathcal{O}_{\lambda^i} = \mathcal{O}_{D_i} \subseteq \overline{\mathcal{O}_{D_i}}$, i.e. sitting "above" $\mathcal{O}_D$. We have , $\lambda^0 = (\tau_1^0, \dots, \tau_t^0) \in \Omega$ where τ_i^0 is the ab-diagram of $D|_{V_{i-1}} : V_{i-1} \to V_i$.

Define $M' := \{(X_1, \dots, X_t) \in M \mid \text{all } X_i \text{ surjective}\} = \prod_{i=1}^t \mathrm{L}'(V_{i-1}, V_i)$ and let $Z' = Z \cap M'$. We have by Lemma 2.8 that $Z_{\lambda^0} \subseteq Z'$, and $\Delta_{\tau_i^0} = 0$ by the remark after Proposition 3.10 since the $D|_{V_i}$ are by definition surjective. Hence by the dimension result of the previous section

$$\dim Z_{\lambda^0} = \frac{1}{2}\dim \mathcal{O}_D + \frac{1}{2}\sum_{i=1}^{t} n_{i-1}n_i.$$

The classes $\mathcal{O} \in \mathfrak{g}(V)$ have even dimension, so for $\mathcal{O} \subset \overline{\mathcal{O}_D}$, $\dim \mathcal{O} \leq \dim \mathcal{O}_D - 2$. Thus for all other $\omega \in \Omega$, $\dim Z_\omega \leq \dim Z_{\lambda^0} - 1$. Z is a finite union of strata Z_ω, and so we have that $\dim Z = \dim Z_{\lambda^0} = \dim Z'$. So for $\omega = (\tau_1, \ldots, \tau_t) \neq \lambda^0$ above $\mathcal{O} \neq \mathcal{O}_D$, we have

$$\begin{aligned}\operatorname{codim}_Z Z_\omega &= \dim Z_{\lambda^0} - \dim Z_\omega \\ &= \frac{1}{2}(\dim \mathcal{O}_D + \sum_{i=1}^{t} n_{i-1}n_i - \dim \mathcal{O} - \sum_{i=1}^{t}(n_{i-1}n_i - \Delta_{\tau_i})) \\ &= \frac{1}{2}(\operatorname{codim}_{\mathcal{O}_D} \mathcal{O} + \sum_{i=1}^{t} \Delta_{\tau_i}).\end{aligned}$$

Since $\vartheta^{-1}(\mathcal{O})$ is a finite union of strata Z_ω satisfying the above[1], we have that

$$\operatorname{codim}_Z \vartheta^{-1}(\mathcal{O}) \geq \frac{1}{2}\operatorname{codim}_{\overline{\mathcal{O}_D}} \mathcal{O}.$$

□

Remark 3.13. As a result of the above proof, those strata Z_ω, $\omega = (\tau_1, \ldots, \tau_t)$, of codimension 1 are iterated fiber products above those conjugacy classes $\mathcal{O} \subset \overline{\mathcal{O}_D}$ of codimension 2 satisfying $\Delta_{\tau_i} = 0$.

3.4.3 Proof of Theorem 2.13(a)

Theorem (2.13a)**.** *Let* $\mathrm{G}(V)$ *be a symplectic or orthogonal group for a finite dimensional vector space* V *over a field* K *of prime characteristic* $p \neq 2$*. For a nilpotent element* $D \in \mathfrak{g}(V)$*, construct* $M = \prod_{i=1}^{t} \mathrm{L}(V_{i-1}, V_i)$ *and* $Z \subseteq M$ *the set of tuples* $(X_1, \ldots, X_t)$ *satisfying equations* $(*)$ *below, the* V_i *as in section 2.3.1. Then the*

[1]See Appendix B for a detailed discussion.

affine scheme Z is reduced, so that Z isan affine variety. Moreover, Z is a complete intersection in M with respect to the equations $(*)$*.*

$$(*) \qquad \begin{aligned} X_1X_1^* &= X_2^*X_2 \\ X_2X_2^* &= X_3^*X_3 \\ &\vdots \\ X_{t-1}X_{t-1}^* &= X_t^*X_t \\ X_tX_t^* &= 0 \end{aligned}$$

Proof. Consider the map

$$\zeta : M \to \prod_{i=1}^{t} \mathfrak{g}(V_i) =: N$$

given by $(X_1, X_2, \ldots, X_t) \mapsto (X_1X_1^* - X_2^*X_2, X_2X_2^* - X_3^*X_3, \ldots, X_tX_t^*)$ as in Remark 2.11. As a scheme, Z is the fiber $\zeta^{-1}(0)$. In order to see that Z is an affine variety, we show that it is a complete intersection and nonsingular in codimension zero, which implies by the discussion in section 1.4 that the ideal defining Z is reduced.

First, we show that ζ is smooth in

$$M' := \{(X_1, \ldots, X_t) \mid \text{all } X_i \text{ surjective}\} = \prod_{i=1}^{t} \mathrm{L}'(V_{i-1}, V_i).$$

To see this we compute the differential $d\zeta$ at a point $\alpha = (X_1, \ldots, X_t) \in M'$. Taking a tangent vector $(T_1, \ldots, T_t) \in M$ we get:

$$(d\zeta)_\alpha(T_1, \ldots, T_t) = (T_1X_1^* + X_1T_1^* - T_2^*X_2 - X_2^*T_2 \quad , \ldots ,$$

$$T_{t-1}X_{t-1}^* + X_{t-1}T_{t-1}^* - T_t^*X_t - X_t^*T_t, \quad T_tX_t^* + X_tT_t^*).$$

Since each X_i is surjective, we can solve the equation

$$(d\zeta)_\alpha(T_1, \ldots, T_t) = (S_1, \ldots, S_t)$$

inductively: If $T_t, T_{t-1}, \ldots, T_{j+1}$ have been determined, one has to solve an equation

$$T_j X_j^* + X_j T_j^* = S_j'$$

for some S_j' satisfying $(S_j')^* = -S_j'$. This can be done setting $S_j' = R_j - R_j^*$ and then solving $X_j T_j^* = R_j$ using the fact that X_j is surjective. Thus $(d\zeta)_\alpha$ is surjective for $\alpha \in M'$, and ζ is smooth in M'.

In particular, Z, as a scheme, is smooth in $Z' := Z \cap M'$. Applying Lemma 2.8 (b) and (c) inductively, we see that $\vartheta^{-1}(\mathcal{O}_D) \subseteq Z'$, where $\vartheta : Z \to \overline{\mathcal{O}_D}$ is the map $(X_1, \ldots, X_t) = X_1^* X_1$. Hence $Z' \neq \emptyset$.

ζ is surjective. Indeed, since $\pi : \mathrm{L}(V_{t-1}, V_t) \to \mathfrak{g}(V_t)$ given by $\pi(X) = XX^*$ is surjective, ζ is surjective in the last coordinate. Proceeding inductively, assume that we have chosen $(X_i, \ldots, X_t)$ such that $X_i X_i^* = X_{i+1}^* X_{i+1} + S_i, \ldots, X_t X_t^* = S_t$ for some $(S_1, \ldots, S_t) \in N$. Then, again, the surjectivity of $\pi : \mathrm{L}(V_{i-2}, V_{i-1}) \to \mathfrak{g}(V_{i-1})$ implies that we can find $X_{i-1} \in \mathrm{L}(V_{i-2}, V_{i-1})$ such that $\pi(X_{i-1}) = X_{i-1} X_{i-1}^* = X_i^* X_i + S_{i-1}$.

Thus we have that the dimension of the generic fiber Z' is $\dim M - \dim N$. Hence $\operatorname{codim}_M Z = \operatorname{codim}_M Z' = \dim N$, since we showed $\dim Z = \dim Z'$ during the proof of Lemma 3.12. As noted in Remark 2.12, equations $(*)$ give exactly $\dim \mathfrak{g}(V_1) \times \ldots \times \dim \mathfrak{g}(V_t) = \dim N$ equations in $\prod_{i=1}^{t} \mathbb{A}^{n_i n_{i-1}} \cong M$. Hence Z is a complete intersection.

$\overline{\mathcal{O}_D} \setminus \mathcal{O}_D$ consists of finitely many conjugacy classes $\mathcal{O}_i$ and $\overline{\mathcal{O}_D}$ is a finite union $\mathcal{O}_D \cup (\bigcup \mathcal{O}_i)$. $\vartheta^{-1}(\mathcal{O}_D) \subseteq Z'$ so we must have that $Z_{sing} \subset \vartheta^{-1}(\bigcup \mathcal{O}_i)$. For each we have $\operatorname{codim}_Z \vartheta^{-1}(\mathcal{O}_i) \geq \frac{1}{2} \operatorname{codim}_{\overline{\mathcal{O}_D}} \mathcal{O}_i \geq 1$ by Lemma 3.12, which shows that Z is smooth in codimension 0.

Thus Z is an affine variety which is a complete intersection in M, proving (a).

□

Chapter 4

Part B

In this chapter we introduce the concept of a good pair of varieties as well as some specific results in the symplectic and orthogonal groups. These will be used to prove part (b) of Theorem 2.13, showing that there is a quotient map $Z \to \overline{\mathcal{O}_D}$ for a suitable algebraic group.

4.1 General Results on Good Pairs

We begin by stating some definitions and results needed from [Don90] and [AJ84]. In the following section, let G be an arbitrary reductive linear algebraic group over K. A G-module connotes a rational, possibly infinite dimensional, G-module. Denote by B a Borel subgroup, $T \subset B$ a maximal torus, and $\mathrm{X}(T)$ the character group of T.

We choose a system of positive roots such that the weights of T in $\mathrm{Lie}(B)$ are zero and the *negative* roots. This determines a partial order on $\mathrm{X}(T)$, denoted by $\leq$. Let $\mathrm{X}^+(T)$ be the set of dominant weights. For $\xi \in \mathrm{X}(T)$, let K_ξ be the one-dimensional B-module on which T acts via ξ and let $\mathrm{Y}(\xi) = \mathrm{Ind}_B^G K_\xi$ be the induced module as in [Jan03].

4.1.1 Good Filtrations

By a *good filtration* of a G-module M we mean an ascending filtration of M

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots$$

with $M = \bigcup_i M_i$ such that for each $i > 0$, M_i/M_{i-1} is either 0 or isomorphic to $\mathrm{Y}(\xi_i)$ for some $\xi_i \in \mathrm{X}^+(T)$.

The following results, which we will need later, are proven for general reductive G in [Don90] where H^* is Hochschild cohomology.

Proposition 4.1. *Let M be a G-module.*

(a) If M has a good filtration then $H^i(G, M \otimes \mathrm{Y}(\xi)) = 0$ for all $i > 0$ and $\xi \in \mathrm{X}^+(T)$.

(b) If M has countable dimension and $H^1(G, M \otimes \mathrm{Y}(\xi)) = 0$ for all $\xi \in \mathrm{X}^+(T)$ then M has a good filtration.

(c) If M has a good filtration and $\xi \in \mathrm{X}^+(T)$, then the filtration multiplicity $(M : \mathrm{Y}(\xi))$ of $\mathrm{Y}(\xi)$ in M is independent of the choice of good filtration and equal to $\dim(M \otimes \mathrm{Y}(\xi^*))^G$.

Proof. This is Proposition 1.2a(i-iii) in [Don90]. Part (a) is Corollary 3.3 of [CPSvdK77] and part (b) is due to Friedlander in [Fri85], which generalizes the finite dimensional case in [Don81]. Part (c) follows from [Don85, 12.1.1]. □

Define as in [FP86] the *good filtration dimension* $d = d(M)$ of a G-module M by the two conditions

1. $H^i(G, M \otimes Y) = 0$ for all $i > d$ and G-modules Y admitting a good filtration, and
2. $H^d(G, M \otimes Y) \neq 0$ for some G-module Y admitting a good filtration.

If $M \neq 0$ and there is no non-negative integer d satisfying 1 and 2, we put $d(M) = +\infty$. We follow the convention $d(0) = -\infty$.

Lemma 4.2. *(a) If* $0 \to M \to N_0 \to N_1 \to \ldots \to N_n \to 0$ *is an exact sequence of* G*-modules then* $d(M) \leq \max\{d(N_i) + i \mid 0 \leq i \leq n\}$.

(b) If $0 \to N_n \to N_{n-1} \to \ldots \to N_0 \to M \to 0$ *is an exact sequence of* G*-modules then* $d(M) \leq \max\{d(N_i) - i \mid 0 \leq i \leq n\}$.

Proof. This is Lemma 1.2c in [Don90]. The result follows from the long exact sequence of cohomology for $n = 1$, and by induction, splicing exact sequences and using the case $n = 1$. When $n = 0$, there is nothing to prove. □

Corollary 4.3. *Let* M *be a finite dimensional* G*-module such that every symmetric power of* M *admits a good filtration. Then* $d(\bigwedge^r M) \leq r-1$ *for every positive integer* r.

Proof. This is Corollary 1.2d in [Don90]. □

We will often be concerned with a product $G = G_1 \times G_2$. In this case, we take $B = B_1 \times B_2$ and $T = T_1 \times T_2$, where T_i is a maximal torus in the Borel subgroup $B_i \subset G_i$, $i = 1, 2$. Then we have $\mathrm{X}^+(T) = \mathrm{X}^+(T_1) \times \mathrm{X}^+(T_2)$ and $\mathrm{Y}(\xi, \eta) = \mathrm{Ind}_B^G K_{(\xi,\eta)} = \mathrm{Y}_1(\xi) \otimes \mathrm{Y}_2(\eta)$ for $\xi \in \mathrm{X}^+(T_1)$ and $\eta \in \mathrm{X}^+(T_2)$ where $\mathrm{Y}_1(\xi) = \mathrm{Ind}_{B_1}^{G_1} K_\xi$ and $\mathrm{Y}_2(\eta) = \mathrm{Ind}_{B_2}^{G_2} K_\eta$ [Don90, 1.1].

Proposition 4.4. *Suppose that* $G = \mathrm{G}_1 \times G_2$ *and* M *is a* G*-module.*

(a) If M *has a good* G*-module filtration then* M *has a good* G_1*-module filtration.*

(b) If G_2 *acts trivially on* M *then* M *has a good* G_1*-module filtration if and only if* M *has a good* G*-module filtration.*

(c) If M *has a good* G*-module filtration then* M^{G_1} *has a good* G_2*-module filtration and* $(M^{G_1} : \mathrm{Y}_2(\eta)) = (M : \mathrm{Y}(0, \eta))$ *for* $\eta \in \mathrm{X}^+(T_2)$.

Proof. This is Proposition 1.2e in [Don90]. □

4.1.2 Good Pairs

Let $\mathcal{V}$ be a reduced affine algebraic variety over K with coordinate ring $K[\mathcal{V}]$. If $\phi : \mathcal{U} \to \mathcal{V}$ is a morphism of varieties, then we also have the comorphism $\phi^* : K[\mathcal{V}] \to K[\mathcal{U}]$, given by $\phi^* f = f \circ \phi \in K[\mathcal{U}]$ for $f \in K[\mathcal{V}]$. Let $\mathcal{V}$ be a G-variety, i.e. a variety $\mathcal{V}$ equipped with a left action $G \times \mathcal{V} \to \mathcal{V}$, which is also a morphism of varieties. Then $K[\mathcal{V}]$ is a left G-module with action given by $(gf)(v) = f(g^{-1}v)$ for $g \in G$, $f \in K[\mathcal{V}]$, and $v \in \mathcal{V}$.

A *good G-variety* is a G-variety $\mathcal{V}$ such that $K[\mathcal{V}]$ admits a good G-module filtration. A *good pair of G-varieties* is a pair $(\mathcal{V}, \mathcal{A})$ where $\mathcal{V}$ is a good G-variety and $\mathcal{A} \subset \mathcal{V}$ is a G-stable closed subset such that the defining ideal $I_{\mathcal{A}} \subset K[\mathcal{V}]$ of $\mathcal{A}$ admits a good G-module filtration. If $(\mathcal{V}, \mathcal{A})$ is a good pair of G-varieties, then $\mathcal{A}$ is a good G-variety since $K[\mathcal{A}] = K[\mathcal{V}]/I_{\mathcal{A}}$ has a good filtration by Lemma 4.1(a) and (b) and the exact sequence $0 \to I_{\mathcal{A}} \to K[\mathcal{V}] \to K[\mathcal{A}] \to 0$.

The following are a few useful results about good pairs.

Lemma 4.5. *Let $\mathcal{V}$ be a G-variety and let $\mathcal{A} \subset \mathcal{A}'$ be G-stable closed subsets of $\mathcal{V}$.*

(a) If $(\mathcal{V}, \mathcal{A}')$ and $(\mathcal{A}', \mathcal{A})$ are good pairs then $(\mathcal{V}, \mathcal{A})$ is a good pair.

(b) If $(\mathcal{V}, \mathcal{A})$ and $(\mathcal{V}, \mathcal{A}')$ are good pairs then $(\mathcal{A}', \mathcal{A})$ is a good pair.

Proof. This is Lemma 1.3a in [Don90]. □

Proposition 4.6. *(a) If $(\mathcal{V}_1, \mathcal{A}_1)$ and $(\mathcal{V}_2, \mathcal{A}_2)$ are good pairs of G-varieties, then $(\mathcal{V}_1 \times \mathcal{V}_2, \mathcal{A}_1 \times \mathcal{A}_2)$ is a good pair of G-varieties for the action $g(v_1, v_2) = (gv_1, gv_2)$ $(g \in G, v_i \in \mathcal{V}_i)$ of G on $\mathcal{V}_1 \times \mathcal{V}_2$.*

(b) If $G = G_1 \times G_2$, $(\mathcal{V}_1, \mathcal{A}_1)$ is a good pair of G_1-varieties, and $(\mathcal{V}_2, \mathcal{A}_2)$ is a good pair of G_2-varieties, then $(\mathcal{V}_1 \times \mathcal{V}_2, \mathcal{A}_1 \times \mathcal{A}_2)$ is a good pair of G-varieties for the action $(g_1, g_2)(v_1, v_2) = (g_1 v_1, g_2 v_2)$ $(g_i \in G_i, v_i \in \mathcal{V}_i)$ of G on $\mathcal{V}_1 \times \mathcal{V}_2$.

Proof. This is Proposition 1.3e in [Don90]. □

The concept of a *good complete intersection* is a useful tool for finding good pairs of varieties. Let $\mathcal{V}$ be a good, irreducible and smooth G-variety with a G-stable closed subvariety $\mathcal{A}$ of codimension r, and suppose that E is a G-stable subspace of $K[\mathcal{V}]$ of dimension r that generates the ideal of $\mathcal{A}$. Then $\mathcal{A}$ is a *good complete intersection in $\mathcal{V}$ with respect to E* if the symmetric algebra $\mathrm{S}(E)$ has a good G-module filtration.

Proposition 4.7. *Suppose $\mathcal{V}$ is a good, irreducible, smooth G-variety and $\mathcal{A}$ is a good complete intersection with respect to some G-stable subspace E of $K[\mathcal{V}]$. Then $(\mathcal{V}, \mathcal{A})$ is a good pair of G-varieties.*

Proof. This is Proposition 1.3b(i) in [Don90]. We include the proof for flavor.

Consider the Koszul resolution

$$0 \longrightarrow K[\mathcal{V}] \otimes \bigwedge^r(E) \longrightarrow K[\mathcal{V}] \otimes \bigwedge^{r-1}(E) \longrightarrow \cdots \longrightarrow K[\mathcal{V}] \otimes E \longrightarrow I_{\mathcal{A}} \longrightarrow 0$$

as in [Mat90, page 135]. The maps are G-equivariant. If Y is a G-module with a good filtration, then so is $Y' = Y \otimes K[\mathcal{V}]$ by [Mat90] and [Don85, 3.1.1]. Therefore, we have $d(K[\mathcal{V}] \otimes \bigwedge^i(E)) \le d(\bigwedge^i(E)) \le i - 1$ for $i > 0$ by Corollary 4.3. Hence we have from the Koszul resolution and Lemma 4.2(b),

$$\begin{aligned} d(I_{\mathcal{A}}) &\le \max\{d(K[\mathcal{V}] \otimes \textstyle\bigwedge^{i+1}(E)) - i) \mid 0 \le i \le r - 1\} \\ &\le \max\{d(\textstyle\bigwedge^{i+1}(E)) - i) \mid 0 \le i \le r - 1\} \\ &\le 0. \end{aligned}$$

Thus $I_{\mathcal{A}}$ has a good filtration and $(\mathcal{V}, \mathcal{A})$ is a good pair. ☐

Given a finite dimensional G-module, $\mathcal{M}$, over K, one can view $\mathcal{M}$ as the affine G-variety $\mathbb{A}^{\dim \mathcal{M}}$. Then $\mathcal{M}$ is a good G-variety if and only if all symmetric powers $\mathrm{S}^n(\mathcal{M}^\vee)$ of the dual of $\mathcal{M}$ have good filtrations as G-modules, hence $\mathrm{S}(\mathcal{M}^\vee)$ has a good filtration. The following Lemma implies that one must only examine the exterior powers of $\mathcal{M}$, of which there are finitely many, in order to know about the symmetric powers.

Lemma 4.8. *If $\mathcal{M}$ is a finite dimensional G-module such that $\bigwedge(\mathcal{M}^\vee)$ has a good filtration, then* $\mathrm{S}(\mathcal{M}^\vee)$ *has a good filtration.*

Proof. This is [AJ84, 4.3(1)]. □

Lemma 4.9. *If $\mathcal{M}$ and $\mathcal{N}$ are finite dimensional G-modules such that $\bigwedge(\mathcal{M}^\vee)$ and $\bigwedge(\mathcal{N}^\vee)$ have good filtrations, then $\mathcal{M} \otimes \mathcal{N}^\vee$ is a good G-variety.*

Proof. If $\bigwedge(\mathcal{M}^\vee)$ and $\bigwedge(\mathcal{N}^\vee)$ have good filtrations, [AJ84, 4.3(2)] states that $\mathrm{S}(\mathcal{M}^\vee \otimes \mathcal{N}^\vee)$ has a good filtration. This then implies that $\mathcal{M} \otimes \mathcal{N}^\vee$ is a good G-variety, identifying $\mathcal{N}$ with its dual $\mathcal{N}^\vee$. □

The following Corollary gives another means of determining good pairs of varieties.

Corollary 4.10. *Let $\mathcal{M}$ be a finite dimensional G-module with submodule $\mathcal{A}$. If the symmetric algebras* $\mathrm{S}(\mathcal{M}^\vee)$ *and* $\mathrm{S}((\mathcal{M}/\mathcal{A})^\vee)$ *admit good filtrations, then $(\mathcal{M}, \mathcal{A})$ is a good pair when regarded as G-varieties.*

Proof. This is Corollary 1.3c in [Don90]. □

4.1.3 Quotient Maps

As advertised, good pairs of varieties are the key to generalizing the proof of Theorem 5.3 in [KP82]. Over a field of characteristic 0, quotient maps restrict to quotient maps on stable subvarieties. This is not the case in positive characteristic, where the restriction of a quotient map to a stable subvariety may or may not be a quotient map. However, the following proposition shows that a quotient map of a good variety $\mathcal{V}$ does restrict to a quotient map on a subvariety $\mathcal{A}$ if $(\mathcal{V}, \mathcal{A})$ is a good pair of varieties.

Recall that if $\mathcal{V}$ is a G-variety and $q : \mathcal{V} \to \mathcal{U}$ a morphism of varieties. We call q, or less precisely $\mathcal{U}$, a *G-quotient* if q is closed, surjective, and $\mathrm{q}^* K[\mathcal{U}] = K[\mathcal{V}]^G$.

Proposition 4.11. *Suppose $G = G_1 \times G_2$ and $(\mathcal{V}, \mathcal{A})$ is a good pair of G-varieties. If* $\mathrm{q} : \mathcal{V} \to \mathcal{U}$ *is a G-equivariant morphism of varieties which is a G_1-quotient map,* *then* $(\mathcal{U}, \mathrm{q}(\mathcal{A}))$ *is a good pair of* G_2*-varieties and the restriction* $\mathcal{A} \to \mathrm{q}(\mathcal{A})$ *is a* G_1*-quotient.*

Proof. This is Proposition 1.4a in [Don90]. We include the proof as reference for the remark following.

$\mathrm{q}(\mathcal{A})$ is closed in $\mathcal{U}$ by [Fog69] Lemma 5.4 and [Hab75]. Now, let I be the ideal of $\mathcal{A}$ in $\mathcal{V}$ and let J be the ideal of $\mathrm{q}(\mathcal{A})$ in $\mathcal{U}$. I has good G-module filtration by definition, and hence has good G_1-module filtration by Proposition 4.4(a). Thus $H^1(G_1, I) = 0$ by Proposition 4.1(a) (with $\xi = 0$). Thus we have a commutative diagram of G_2-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & K[\mathcal{U}] & \longrightarrow & K[\mathrm{q}(\mathcal{A})] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^{G_1} & \longrightarrow & K[\mathcal{V}]^{G_1} & \longrightarrow & K[\mathcal{A}]^{G_1} & \longrightarrow & 0 \end{array}$$

The rows are exact, and the middle vertical map, which is the restriction of the comorphism q^*, is an isomorphism. We have that $\mathrm{q} : \mathcal{A} \to \mathrm{q}(\mathcal{A})$ is a surjection, so the right hand map is injective, and therefore an isomorphism. Therefore q restricts to a G_1-quotient $\mathcal{A} \to \mathrm{q}(\mathcal{A})$. We also have that J is isomorphic to I^{G_1}, which has good G_2-filtration by Proposition 4.4(c). Hence $(\mathcal{U}, \mathrm{q}(\mathcal{A}))$ is a good pair of G_2-varieties. □

Remark 4.12. If $(\mathcal{V}, \mathcal{A})$ is a good pair of $\mathrm{SO}(V)$-varieties such that $\mathcal{A}$ is a $\mathrm{O}(V)$-stable subvariety, then a $\mathrm{O}(V)$-quotient map $\mathcal{V} \to \mathcal{U}$ restricts to a $\mathrm{O}(V)$-quotient $\mathcal{A} \to q(\mathcal{A})$.

Indeed, we have $H^1(\mathrm{SO}(V), I) = 0$ when $G_1 = \mathrm{SO}(V)$. Since $H^i(\mathrm{O}(V), I) = H^i(\mathrm{SO}(V), I)^{\mathbb{Z}/2\mathbb{Z}}$ by I 6.9(3) of [Jan03], we have that $H^1(\mathrm{O}(V), I) = 0$ when $\mathcal{V}$ is a good $\mathrm{SO}(V)$-variety. The argument above in the proof of Proposition 4.11 (from the commutative diagram on) then shows that a $\mathrm{O}(V)$-quotient $\mathrm{q} : \mathcal{V} \to \mathcal{U}$ restricts to a $\mathrm{O}(V)$-quotient $\mathcal{A} \to \mathrm{q}(\mathcal{A})$ letting $G_1 = \mathrm{O}(V)$ and $G = G_1^0 \times G_2$.

4.2 Orthosymplectic Good Pairs

The following are results on good filtrations and good pairs in the orthosymplectic setting. One notes that it does not make sense to ask if a variety is a good O_n-variety, since O_n is not connected. Instead, we identify certain varieties that are good SO_n-varieties. In other words, those that are good $\mathrm{G}^0(V)$-varieties, where V is a space of type $\varepsilon = +1$. In the symplectic setting, i.e. when V is a space of type $\varepsilon = -1$, we have $\mathrm{G}^0(V) = \mathrm{G}(V)$ in the results below.

Lemma 4.13. *V is a good $\mathrm{G}^0(V)$-variety.*

Proof. [AJ84] proves in section 4.9 that each $\bigwedge^i(V^\vee)$ has a good filtration when $\mathrm{G}(V) = \mathrm{Sp}(V)$, proving by Lemma 4.8 that V is a good $\mathrm{Sp}(V)$-variety. When $\mathrm{G}^0(V) = \mathrm{SO}(V)$, [AJ84] remark that when $\operatorname{char} K = p \neq 2$, the $\bigwedge^i(V^\vee)$ are of the form $H^0(G/B, \lambda_i)$ for some $\lambda_i \in \mathrm{X}(T)^+$. This is sufficient for our purposes, but it is worth remarking that they go on to prove directly that all $\mathrm{S}^i(V^\vee)$ have good filtrations when $p = 2$. ☐

Proposition 4.14. *$\mathrm{L}(V,U)$ is a good $\mathrm{G}^0(V)\times\mathrm{G}^0(U)$-variety for the action $(g,h)X = hXg^{-1}$, V an ε-space and U a $-\varepsilon$-space.*

Proof. We first observe that we may identify $\mathrm{L}(V,U) = \mathrm{Hom}(V,U)$ with $V^\vee \otimes U$ where $\mathrm{G}^0(V)$ acts on $V^\vee$ and $\mathrm{G}^0(U)$ acts on U. We have already seen that the exterior powers $\bigwedge^i(V)$ and $\bigwedge^j(U^\vee)$ have good $\mathrm{G}^0(V)$- and $\mathrm{G}^0(U)$-filtrations, respectively. Hence by considering the trivial action of $\mathrm{G}^0(V)$ on U and of $\mathrm{G}^0(U)$ on V, we have by Proposition 4.4(b) and Lemma 4.9 that $V^\vee \otimes U$ is a good $\mathrm{G}^0(V)\times\mathrm{G}^0(U)$-variety. ☐

Remark 4.15. We have by the same argument that $\mathrm{L}(V,U)$ is a good $\mathrm{G}^0(V)\times\mathrm{GL}(U)$-variety, or a good $\mathrm{GL}(V)\times\mathrm{G}^0(U)$-variety, since by [AJ84] sections 4.3 and 4.9, a finite dimensional vector space V is also a good $\mathrm{GL}(V)$-variety and the exterior powers $\bigwedge^i(V)$ have good $\mathrm{GL}(V)$-filtrations.

Proposition 4.16. *$\mathfrak{g}(V)$ is a good $\mathrm{G}^0(V)$-variety, with action given by conjugation: $g.X = gXg^{-1}$.*

Proof. This is implied by [AJ84] Proposition 4.4 since we are assuming $p \neq 2$. $\square$

Proposition 4.17. *Let V be an ε-space and let $e \leq \dim(V)$. $(\mathfrak{g}(V), \mathfrak{g}(V)_e)$ is a good pair of $\mathrm{G}^0(V)$-varieties, where $\mathfrak{g}(V)_e$ is the determinantal subvariety of matrices of rank less than or equal to e.*

First, we prove two intermediate lemmas.

Lemma 4.18. $(\mathrm{L}(V,W), \mathrm{L}(V,W)_e)$ *is a good pair of* $\mathrm{G}^0(V) \times \mathrm{GL}(W)$*-varieties, where W is a finite dimensional vector space over K with $e \leq \dim(W) \leq \dim(V)$ and action given by $(g,h)X = hXg^{-1}$ $g \in \mathrm{G}^0(V), h \in \mathrm{GL}(W), X \in \mathrm{L}(V,W)$.*

Proof. Let W' be a K-vector space with $\dim W' = \dim W$. The map

$$\mathrm{L}(V,W') \times \mathrm{L}(W',W) \xrightarrow{\psi} \mathrm{L}(V,W)$$

given by matrix multiplication $(X,Y) \mapsto YX$ is a $\mathrm{GL}(W')$-quotient by [Don90, 1.4c(ii)]. We also have that $\mathrm{L}(V,W)$ is a good $\mathrm{G}^0(V) \times \mathrm{GL}(W')$-variety by the remark following Proposition 4.14 and that $(\mathrm{L}(W',W), \mathrm{L}(W',W)_e)$ is a good pair of $\mathrm{GL}(W') \times \mathrm{GL}(W)$-varieties by [Don90, 1.4c(i)]. Thus $(\mathrm{L}(V,W') \times \mathrm{L}(W',W), \mathrm{L}(V,W') \times \mathrm{L}(W',W)_e)$ is a good pair of $\mathrm{G}^0(V) \times \mathrm{GL}(W') \times \mathrm{GL}(W)$-varieties by 4.6(b).

Hence ψ restricts to a $\mathrm{GL}(W')$-quotient $\mathrm{L}(V,W') \times \mathrm{L}(W',W)_e \to \mathrm{L}(V,W)_e$ and $(\mathrm{L}(V,W), \mathrm{L}(V,W)_e)$ is a good pair of $\mathrm{G}^0(V) \times \mathrm{GL}(W)$-varieties by 4.11.

$\square$

Lemma 4.19. $(\mathrm{L}(V,U), \mathrm{L}(V,U)_e)$ *is a good pair of* $\mathrm{G}^0(V) \times \mathrm{G}^0(U)$*-varieties, where $e < \dim(V) \leq \dim(U) \leq \dim(V)+1$ and U is a quadratic space of opposite type from V.*

Proof. We begin by downward induction on e.

First, let $d = \dim(V) = \dim(U)$ and let $e = d-1$. Then $\mathrm{L}(V,U)_e$ is a good complete intersection in $\mathrm{L}(V,U)$. To see this, note that the ideal defining $\mathrm{L}(V,U)_e$ is generated by the determinant function, *det*. The rank 1 K-module $E = K \cdot det$ is the trivial representation for the action of $\mathrm{G}^0(V) \times \mathrm{G}^0(U)$, and so is isomorphic to the induced module $Y(0)$. Thus E has a good $\mathrm{G}^0(V) \times \mathrm{G}^0(U)$-module filtration, as do the symmetric powers $S^p(E)$.

Now, let $d = \dim(V)$ and $\dim(U) = d + 1$ with $e = d - 1$. Here, the ideal of $\mathrm{L}(V, U)_e$ is generated by *det* as well as the polynomials determined by the vanishing of all $d \times d$-minors, which can be indexed $u_1, \ldots, u_{d+1}$ by choosing a basis of U. Letting E be the K-module generated by these, $E = K \otimes U$, which also has a good $\mathrm{G}^0(V) \times \mathrm{G}^0(U)$-filtration by Lemma 4.9 (we showed already that the $\bigwedge^i(U)$ have good $\mathrm{G}^0(V) \times \mathrm{G}^0(U)$-filtrations).

Finally, let $d = \dim(V) \leq \dim(U) \leq \dim(V) + 1$ and let $e < d - 1$. Assume by induction that for $m > e$, $(\mathrm{L}(V, U), \mathrm{L}(V, U)_m)$ is a good pair of $\mathrm{G}^0(V) \times \mathrm{G}^0(U)$-varieties. Let W be a vector space of dimension e. Then $\mathrm{L}(V, W) \times \mathrm{L}(W, U)$ is a good $\mathrm{G}^0(V) \times \mathrm{GL}(W) \times \mathrm{G}^0(U)$-variety with action $(g, h, f)(X, Y) = (hXg^{-1}, fYh^{-1})$. By [Don90, 1.4c(ii)], the map

$$\mathrm{L}(V, W) \times \mathrm{L}(W, U) \xrightarrow{\psi} \mathrm{L}(V, U)_e$$

given by matrix multiplication is a $\mathrm{GL}(W)$-quotient.

Consider the factorization of ψ

$$\mathrm{L}(V, W) \times \mathrm{L}(W, W') \times \mathrm{L}(W', U) \longrightarrow \mathrm{L}(V, W')_e \times \mathrm{L}(W', U) \longrightarrow \mathrm{L}(V, U)_e$$

where W' is a vector space of dimension $e + 1$, both maps given by matrix multiplication. The first map is a $\mathrm{GL}(W)$-quotient. We know that the multiplication in the second map, $\mathrm{L}(V, W') \times \mathrm{L}(W', U) \to \mathrm{L}(V, U)_{e+1}$ is a $\mathrm{GL}(W')$-quotient ([Don90, 1.4c(ii)] again). By the previous lemma, $(\mathrm{L}(V, W'), \mathrm{L}(V, W')_e)$ is a good pair of $\mathrm{G}^0(V) \times \mathrm{GL}(W')$-varieties, so by 4.11, $(\mathrm{L}(V, U)_{e+1}, \mathrm{L}(V, U)_e)$ is a good pair of $\mathrm{G}^0(V) \times \mathrm{G}(U)$-varieties, and $\mathrm{L}(V, W')_e \times \mathrm{L}(W', U) \to \mathrm{L}(V, U)_e$ is again a $\mathrm{GL}(W')$-quotient (here, $G_1 = \mathrm{GL}(W'), G_2 = \mathrm{G}^0(V) \times \mathrm{G}^0(U)$).

By induction, $(\mathrm{L}(V, U), \mathrm{L}(V, U)_{e+1})$ is a good pair of $\mathrm{G}^0(V) \times \mathrm{G}^0(U)$-varieties, hence $(\mathrm{L}(V, U), \mathrm{L}(V, U)_e)$ is a good pair of $\mathrm{G}^0(V) \times \mathrm{G}^0(U)$-varieties by 4.5(a).

□

We are now able to prove Proposition 4.17.

Proof of Proposition 4.17. The case $e = \dim(V)$ is trivial, since by Proposition 4.16 $\mathfrak{g}(V)$ is a good $\mathrm{G}^0(V)$-variety.

Now let $e < \dim(V)$ and let U be a $-\varepsilon$-space with either $\dim(U) = \dim(V)$ or $\dim(U) = \dim(V) + 1$. Then the map

$$\mathrm{L}(V, U) \xrightarrow{\rho} \mathfrak{g}(V)$$

given by $X \mapsto X^*X$ is onto and is a $\mathrm{G}(U)$-quotient by Theorem 2.7. By the previous lemma, $(\mathrm{L}(V, U), \mathrm{L}(V, U)_e)$ is a good pair of $\mathrm{G}^0(V) \times \mathrm{G}^0(U)$-varieties, hence ρ restricts to a $\mathrm{G}(U)$-quotient $\mathrm{L}(V, U)_e \to \mathfrak{g}(V)_e$ and $(\mathfrak{g}(V), \mathfrak{g}(V)_e)$ is a good pair of $\mathrm{G}^0(V)$-varieites (4.11 with $G_1 = \mathrm{G}^0(U)$ and $G_2 = \mathrm{G}^0(V)$, and the remark following).

□

4.3 Proof of Theorem 2.13(b)

Recall the definition of $M := \mathrm{L}(V_0, V_1) \times \mathrm{L}(V_1, V_2) \times \ldots \times \mathrm{L}(V_{t-1}, V_t)$ from section 2.3 and the subset $Z \subseteq M$ defined by

$$(*) \qquad \begin{aligned} X_1X_1^* &= X_2^*X_2 \\ X_2X_2^* &= X_3^*X_3 \\ &\vdots \\ X_{t-1}X_{t-1}^* &= X_t^*X_t \\ X_tX_t^* &= 0 \end{aligned}$$

with $X_i \in \mathrm{L}(V_{i-1}, V_i)$. Recall as well that $\pi(X_i) = X_iX_i^* \in \mathfrak{g}(V_i)$ and $\rho(X_i) = X_i^*X_i \in \mathfrak{g}(V_{i-1})$. We had $n_0 = \dim(V_0) \geq n_1 = \dim(V_1) \geq \ldots \geq n_t = \dim(V_t) > 0 = \dim(V_{t+1})$.

For $1 \leq i \leq t$ define

$$M_i = \mathrm{L}(V_0, V_1) \times \mathrm{L}(V_1, V_2) \times \ldots \times \mathrm{L}(V_{i-1}, V_i)$$

and define $G_i := \mathrm{G}(V_0) \times \mathrm{G}(V_1) \times \ldots \times \mathrm{G}(V_i)$ which acts on M_i by

$$(g_0, g_1, \ldots, g_i)(X_1, \ldots, X_i) = (g_1 X_1 g_0^{-1}, g_2 X_2 g_1^{-1}, \ldots, g_i X_i g_{i-1}^{-1}).$$

In M_i, define Z_i by the first i equations of $(*)$, except replacing the last $X_i X_i^* = X_{i+1}^* X_{i+1}$ by $X_i X_i^* \in \overline{C_{\lambda^i}} \subset \mathfrak{g}(V_i)$, i.e. is contained in the closure of the conjugacy class of the nilpotent element D^{i+1}. Note that for all of the above, setting $i = t$ gives the original definitions of M and Z since in that case, $D^{t+1} = 0$ and so we have $X_t X_t^* = 0$ as in $(*)$. These Z_i are in fact the same as the Z_i in the fiber product diagram of section 3.4.1. The action of each G_i is the same as the original action of $\mathrm{G}(V_0) \times \mathrm{G}(V_1) \times \ldots \times \mathrm{G}(V_t)$ restricted to the first $i+1$ coordinates, hence we can view M_i and Z_i as $G_t^0 := \mathrm{G}^0(V_0) \times \mathrm{G}^0(V_1) \times \ldots \times \mathrm{G}^0(V_t)$-varieties via the projection $G_t^0 = \mathrm{G}^0(V_0) \times \mathrm{G}^0(V_1) \times \ldots \times \mathrm{G}^0(V_t) \to G_i^0 := \mathrm{G}^0(V_0) \times \mathrm{G}^0(V_1) \times \ldots \times \mathrm{G}^0(V_i)$, meaning that

$$(g_0, g_1, \ldots, g_t)(X_1, \ldots, X_i) = (g_1 X_1 g_0^{-1}, g_2 X_2 g_1^{-1}, \ldots, g_i X_i g_{i-1}^{-1}).$$

As before, Z_i is a G_i-stable subvariety of M_i, thus also a G_t-stable subvariety.

Lemma 4.20. *(M, Z) is a good pair of G_t^0-varieties for the action described above.*

Proof. By Proposition 4.14, each $\mathrm{L}(V_{i-1}, V_i)$ is a good $\mathrm{G}^0(V_{i-1}) \times \mathrm{G}^0(V_i)$-variety, $1 \leq i \leq t$. Regarding $\mathrm{L}(V_{i-1}, V_i)$ as a $G_t = \mathrm{G}(V_0) \times \mathrm{G}(V_1) \times \ldots \times \mathrm{G}(V_t)$-variety via the action

$$(g_0, g_1, \ldots, g_t) X_i = g_i X_i g_{i-1}^{-1},$$

i.e. letting $\mathrm{G}(V_j), j \neq i-1, i$, act trivially, we get that $\mathrm{L}(V_{i-1}, V_i)$ is a good G_t^0-variety for each $1 \leq i \leq t$ by Proposition 4.4(b). Hence each M_i is a good G_t^0-variety by Proposition 4.6(a).

Let $N = \mathfrak{g}(V_1) \times \ldots \times \mathfrak{g}(V_t)$ and view N as a G_t-variety via the action

$$(g_0, g_1, \ldots, g_t)(Z_1, \ldots, Z_t) = (g_1 Z_1 g_1^{-1}, \ldots, g_t Z_t g_t^{-1}).$$

N is a good G_t^0-variety by Propositions 4.16, 4.6(b) and 4.4(b).

Recall the map $\zeta : M \to N$ given by

$$\zeta(X_1, \ldots, X_t) = (X_1 X_1^* - X_2^* X_2, X_2 X_2^* - X_3^* X_3, \ldots, X_t X_t^*).$$

ζ is G_t-equivariant and is surjective, as shown in section 3.4.3, and Z is the fiber $\zeta^{-1}(0)$. Thus the comorphism $\zeta^* : K[N] \to K[M]$ is injective, and identifying $K[N]$ with $\mathrm{S}(N^\vee)$, the symmetric algebra of the dual (as a G_t-module) of N, we have that $\zeta^*(N^\vee)$ is the span of the defining relations $(*)$ above for Z in M. We proved in section 3.4.3 that Z is a complete intersection in M given by $(*)$. Since N is a good G_t^0-variety, $\mathrm{S}(N^\vee) \cong \mathrm{S}(\zeta^*(N^\vee))$ has a good G_t^0-module filtration. Thus Z is a good complete intersection and (M, Z) is a good pair of G_t^0-varieties by Proposition 4.7(a).

□

Lemma 4.21. *For $2 \le i \le t$, the projection $\varphi : M_i \to M_{i-1}$ given by $\varphi(X_1, \ldots, X_i) = (X_1, \ldots, X_{i-1})$ induces a surjection $Z_i \to Z_{i-1}$.*

Proof. If $X = (X_1, X_2, \ldots, X_i) \in Z_i$, then $X_i X_i^* \in \overline{\mathcal{O}_{\lambda^i}}$ and therefore $X_{i-1} X_{i-1}^* = X_i^* X_i \in \overline{\mathcal{O}_{\lambda^{i-1}}}$ by Proposition 2.8(a). Thus $\varphi(X) \in Z_{i-1}$.

Conversely let $Y = (X_1, X_2, \ldots, X_{i-1}) \in Z_{i-1}$. Then $X_{i-1} X_{i-1}^* \in \overline{\mathcal{O}_{\lambda^{i-1}}}$ so that $X_{i-1} X_{i-1}^* = X_i^* X_i$ for some $X_i \in \mathrm{L}(V_{i-1}, V_i)$ (Proposition 2.8(a)) with $X_i X_i^* \in \overline{\mathcal{O}_{\lambda^i}}$ (Proposition 2.8(c)). Hence $Y = \varphi(X_1, X_2, \ldots, X_{i-1}, X_i)$ where $(X_1, X_2, \ldots, X_{i-1}, X_i) \in Z_i$. □

Theorem 4.22. *(a) (M_i, Z_i) is a good pair of G_i^0-varieties for $1 \le i \le t$ and furthermore, $(\mathfrak{g}(V), \overline{\mathcal{O}_D})$ is a good pair of $\mathrm{G}^0(V)$-varieties.*

(b) There is a map $Z_i \to Z_{i-1}$ which is $\mathrm{G}(V_0) \times \mathrm{G}(V_1) \times \ldots \times \mathrm{G}(V_t)$-equivarient and

a $\mathrm{G}(V_i)$*-quotient for* $2 \leq i \leq t$*. Furthermore, there is a map* $\vartheta : Z \to \overline{\mathcal{O}_D}$ *which is* $\mathrm{G}(V_0) \times \mathrm{G}(V_1) \times \ldots \times \mathrm{G}(V_t)$*-equivarient and a* $\mathrm{G}(V_1) \times \ldots \times \mathrm{G}(V_t)$*-quotient.*

Proof. For part (a), we proceed by downward induction on i. Lemma 4.20 is the case $i = t$. Suppose now that $1 < i \leq t$ and that (M_i, Z_i) is a good pair of G_i^0-varieties. Consider the map $\rho_i : M_i \to M_{i-1} \times \mathfrak{g}(V_{i-1})_{n_i}$ given by $\rho_i(X_1, X_2, \ldots, X_i) = (X_1, X_2, \ldots, X_{i-1}, X_i^* X_i)$. ρ_i is a $\mathrm{G}(V_i)$-quotient map by Theorem 2.7. Since (M_i, Z_i) is a good pair of G_i^0-varieties, by Proposition 4.11, we have that $(M_{i-1} \times \mathfrak{g}(V_{i-1})_{n_i}, \rho_i(Z_i))$ is a good pair of G_{i-1}^0-varieties and the restriction $Z_i \to \rho_i(Z_i)$ is a $\mathrm{G}(V_i)$-quotient. We have that $(\mathfrak{g}(V_{i-1}), \mathfrak{g}(V_{i-1})_{n_i})$ is a good pair of $\mathrm{G}^0(V_{i-1})$-varieties by Proposition 4.17, and hence by Lemma 4.4(b) a good pair of G_{i-1}^0-varieties letting $\mathrm{G}^0(V_0) \times \ldots \times \mathrm{G}^0(V_{i-2})$ act trivially (so that $(g_0, g_1, \ldots, g_{i-1})X = g_{i-1} X g_{i-1}^{-1}$). Then by Proposition 4.6(a), $(M_{i-1} \times \mathfrak{g}(V_{i-1}), M_{i-1} \times \mathfrak{g}(V_{i-1})_{n_i})$ is a good pair of G_{i-1}^0-varieties, and hence $(M_{i-1} \times \mathfrak{g}(V_{i-1}), \rho_i(Z_i))$ is a good pair of G_{i-1}^0-varieties by Lemma 4.5(a) (and also $\mathrm{G}^0(V_{i-1})$-varieties by Proposition 4.4(b)).

Consider now the G_{i-1}-equivarient map $\pi_i : M_{i-1} \to M_{i-1} \times \mathfrak{g}(V_{i-1})$ given by $\pi_i(X_1, \ldots, X_{i-1}) = (X_1, \ldots, X_{i-1}, X_{i-1} X_{i-1}^*)$. π_i is injective, and so gives an isomorphism of varieties $M_{i-1} \cong \pi_i(M_{i-1}) = \{(X_1, \ldots, X_{i-1}, X_{i-1} X_{i-1}^*) \in M_{i-1} \times \mathfrak{g}(V_{i-1})\}$, as well as an isomorphism $Z_{i-1} \cong \pi_i(Z_{i-1})$. Indeed, the map π_i is the identity in all but the last coordinate. We note that $X_{i-1} \to X_{i-1} X_{i-1}^*$ is the quotient map $\pi : \mathrm{L}(V_{i-2}, V_{i-1}) \to \mathfrak{g}(V_{i-1})$, and so $d\pi_i$ is a surjection in the last coordinate, hence in all coordinates. Since Z_{i-1} is a closed subvariety of M_{i-1}, $\pi_i : Z_{i-1} \to \pi_i(Z_{i-1})$ is also an isomorphism of varieties.

Note that $\rho_i(Z_i) \subseteq \pi_i(Z_{i-1}) \subseteq \pi_i(M_{i-1})$. Indeed, given $(X_1, X_2, \ldots, X_{i-1}, X_i^* X_i) \in \rho_i(Z_i)$, we have that $X_i^* X_i = X_{i-1} X_{i-1}^*$ since $(X_1, X_2, \ldots, X_{i-1}, X_i) \in Z_i$, hence $(X_1, X_2, \ldots, X_{i-1}, X_i^* X_i) = (X_1, X_2, \ldots, X_{i-1}, X_{i-1} X_{i-1}^*) \in \pi_i(Z_{i-1})$. Thus we have a commutative diagram

$$Z_i \twoheadrightarrow \rho_i(Z_i) \overset{\iota}{\hookrightarrow} \pi_i(Z_{i-1}) \cong Z_{i-1}, \qquad \varphi : Z_i \twoheadrightarrow Z_{i-1}$$

where the projection φ from Z_i to Z_{i-1} is a surjection by Lemma 4.21. Hence the inclusion from $\rho_i(Z_i)$ to $\pi_i(Z_{i-1})$ is surjective. Indeed, given $(X_1, \ldots, X_{i-1}, X_{i-1}X_{i-1}^*) \in \pi_i(Z_{i-1})$, $(X_1, \ldots, X_{i-1}) \in Z_{i-1}$ is the image of some $(X_1, \ldots, X_{i-1}, Y) \in Z_i$. Then $\rho_i(X_1, \ldots, X_{i-1}, Y) = (X_1, \ldots, X_{i-1}, Y^*Y) = (X_1, \ldots, X_{i-1}, X_{i-1}X_{i-1}^*)$ since by the definition of Z_i, $Y^*Y = X_{i-1}X_{i-1}^*$. Thus $(\pi_i \circ \iota)^{-1} : Z_{i-1} \to \rho_i(Z_i)$ is an isomorphism, having proven earlier that π_i is an isomorphism and ι is evidently just the identity.

We have from Proposition 4.16 that $\mathfrak{g}(V_{i-1})$ is a good $\mathrm{G}^0(V_{i-1})$-variety, hence $(\mathfrak{g}(V_{i-1}), \{0\})$ is a good pair of $\mathrm{G}^0(V_{i-1})$-varieties, and so $(M_{i-1} \times \mathfrak{g}(V_{i-1}), M_{i-1} \times \{0\})$ is a good pair of G_{i-1}^0-varieties by Proposition 4.6. The map $\theta : M_{i-1} \times \mathfrak{g}(V_{i-1}) \to M_{i-1} \times \mathfrak{g}(V_{i-1})$ given by $\theta(X_1, \ldots, X_{i-1}, Y) = (X_1, \ldots, X_{i-1}, X_{i-1}X_{i-1}^* - Y)$ is G_{i-1}-equivariant and satisfies $\theta^2 = 1$, hence is an isomorphism. We have that $\theta(M_{i-1} \times \{0\}) = \pi_i(M_{i-1})$, hence $M_{i-1} \cong \pi_i(M_{i-1})$. Thus $(M_{i-1} \times \mathfrak{g}(V_{i-1}), \pi_i(M_{i-1}))$ is a good pair of $\mathrm{G}^0(V_{i-1})$-varieties by above (hence also a good pair of G_{i-1}^0-varieties by Proposition 4.4(b)). We already showed that $(M_{i-1} \times \mathfrak{g}(V_{i-1}), \rho_i(Z_i))$ is a good pair of G_{i-1}^0-varieties, so applying Lemma 4.5(b) to $\rho_i(Z_i) \subseteq \pi_i(M_{i-1}) \subseteq M_{i-1} \times \mathfrak{g}(V_{i-1})$, $(\pi_i(M_{i-1}), \rho_i(Z_i))$ is a good pair of G_{i-1}^0-varieties, and by the preceding paragraph, $Z_{i-1} \cong \rho_i(Z_i)$, so (M_{i-1}, Z_{i-1}) is a good pair of G_{i-1}^0-varieties.

For the last assertion of part (a), we let $i = 1$. We have by induction that $(M_1 = \mathrm{L}(V_0, V_1), Z_1)$ is a good pair of $G_1^0 = \mathrm{G}^0(V_0) \times \mathrm{G}^0(V_1)$-varieties. The map $\pi_1 = \pi : \mathrm{L}(V_0, V_1) \to \mathfrak{g}(V_0)_{n_1} = \mathfrak{g}(V)_{n_1}$ given by $\pi(X_1) = X_1X_1^*$ is a $\mathrm{G}(V_1)$-quotient by Theorem 2.7. Recall from the discussion in section 2.3 that $\vartheta(Z) = \overline{\mathcal{O}_D}$ where $\vartheta(X_1, X_2, \ldots, X_t) = X_1^*X_1$. Thus $Z_1 \to \pi(Z_1) = \overline{\mathcal{O}_D}$ is a $\mathrm{G}(V_1)$-quotient and $(\mathfrak{g}(V)_{n_1}, \overline{\mathcal{O}_D})$ is a good pair of $\mathrm{G}^0(V_0) = \mathrm{G}^0(V)$-varieties by Proposition 4.11. $(\mathfrak{g}(V), \mathfrak{g}(V)_{n_1})$ is a good pair of $\mathrm{G}^0(V)$-varieties by Proposition 4.17, implying that $(\mathfrak{g}(V), \overline{\mathcal{O}_D})$ is a good pair of $\mathrm{G}^0(V)$-varieties by Lemma 4.5(a).

For part (b), we note from above that for $1 < i \leq t$, ρ_i restricts to a $\mathrm{G}(V_i)$-quotient $Z_i \to \rho_i(Z_i)$. Thus $(\pi_i \circ \iota)^{-1} \circ \rho_i : Z_i \to Z_{i-1}$ (given by the restriction of the projection $M_i \to M_{i-1}$) is a $\mathrm{G}(V_i)$-quotient, since $\pi_i \circ \iota$ is an isomorphism of varieties.

Composing these $\mathrm{G}(V_i)$-quotient maps with the $\mathrm{G}(V_1)$-quotient map $\pi : Z_i \to$

$\overline{\mathcal{O}_D}$,

$$Z = Z_t \longrightarrow Z_{t-1} \longrightarrow \cdots \longrightarrow Z_1 \xrightarrow{\pi} \overline{\mathcal{O}_D}$$

is a $\mathrm{G}(V_0) \times \ldots \times \mathrm{G}(V_t)$-equivariant, $\mathrm{G}(V_1) \times \ldots \times \mathrm{G}(V_t)$-quotient map $Z \to \overline{\mathcal{O}_D}$ given by $(X_1, \ldots, X_t) \mapsto (X_1^* X_1)$. This is the map ϑ originally introduced in section 2.3.3.

$\square$

Chapter 5

Conclusion

5.1 Final Results

The astute (determined? punctilious?) reader will notice that we have not proven the promised result that an orbit closure is normal if and only if it is normal in classes of codimension 2. We will prove this below by applying our generalization of Kraft and Procesi's result (Theorem 2.13) to Theorem 9.2 of [KP82] which shows that $\overline{\mathcal{O}_D}$ is normal if and only if it is normal in $\widetilde{\mathcal{O}}_D$ where

$$\widetilde{\mathcal{O}}_D = \mathcal{O}_D \cup \bigcup_i \mathcal{O}_i, \qquad \operatorname{codim}_{\overline{\mathcal{O}_D}} \mathcal{O}_i = 2.$$

First we state the following lemma, proven directly in [KP82, 9.1] for lack of a suitable source. The proof given therein is independent of the characteristic of K.

Lemma 5.1. *Let $\mathcal{V}$ be an affine Cohen-Macaulay variety, $\mathcal{W} \subset \mathcal{V}$ a closed subset of codimension ≥ 2. Then every regular function on $\mathcal{V} \setminus \mathcal{W}$ extends to a regular function on $\mathcal{V}$, i.e. $K[\mathcal{V} \setminus \mathcal{W}] = K[\mathcal{V}]$ where $K[\mathcal{V}]$ denotes the ring of global regular functions on a variety $\mathcal{V}$.*

As above, let $\widetilde{\mathcal{O}}_D$ denote the union of $\mathcal{O}_D$ with those conjugacy classes $\mathcal{O}_i$ of codimension 2. The variety $\widetilde{\mathcal{O}}_D$ is then the complement of the union of those classes of codimension at least 4.

Proposition 5.2. *(a) Every regular function on $\widetilde{\mathcal{O}}_D$ extends to $\overline{\mathcal{O}_D}$, i.e. $K[\widetilde{\mathcal{O}}_D] = K[\overline{\mathcal{O}_D}]$.*

(b) $\overline{\mathcal{O}_D}$ is normal if and only if $\widetilde{\mathcal{O}}_D$ is normal, i.e. if $\overline{\mathcal{O}_D}$ is normal at all points contained in orbits $\mathcal{O}_i$ of codimension 2.

Proof. For (a), let $f \in K[\widetilde{\mathcal{O}}_D]$ and consider the quotient map $\vartheta : Z \to \overline{\mathcal{O}_D}$. We have from Proposition 3.10 that

$$\operatorname{codim}_Z \vartheta^{-1}(\overline{\mathcal{O}_D} \setminus \widetilde{\mathcal{O}}_D) \geq 2$$

since $\overline{\mathcal{O}_D} \setminus \widetilde{\mathcal{O}}_D = \bigcup_{\operatorname{codim}_{\overline{\mathcal{O}_D}} \mathcal{O} \geq 4} \mathcal{O}$. We can then define the regular function $F := f \circ \vartheta$ on $\vartheta^{-1}(\widetilde{\mathcal{O}}_D)$.

Because Z is a complete intersection in M, it is Cohen-Macaulay and thus by Lemma 5.1, F extends to a regular function on Z, since $\vartheta^{-1}(\overline{\mathcal{O}_D} \setminus \widetilde{\mathcal{O}}_D)$ is a closed subset of codimension ≥ 2. Since ϑ is $G := \mathrm{G}(V_0) \times \ldots \times \mathrm{G}(V_t)$-equivariant, $g.F(x) = F(g^{-1}x) = f \circ \vartheta(g^{-1}x) = f \circ \vartheta(x) = F(x)$. Hence F is invariant on $\vartheta^{-1}(\widetilde{\mathcal{O}}_D)$ and thus also on Z, so we have $F \in K[Z]^G = \vartheta^*(K[\overline{\mathcal{O}_D}])$. Since ϑ^* is injective, F identifies with an extension of f to $\overline{\mathcal{O}_D}$. Thus every regular function on $\widetilde{\mathcal{O}}_D$ extends to $\overline{\mathcal{O}_D}$.

For (b), we note that $\operatorname{codim}_{\widetilde{\mathcal{O}}_D}(\widetilde{\mathcal{O}}_D \setminus \mathcal{O}_D) \geq 2$, so if $\widetilde{\mathcal{O}}_D$ is normal, then every regular function on $\mathcal{O}_D$ must extend to a regular function on $\widetilde{\mathcal{O}}_D$ ([Eis95] Corollary 11.4). By (a), they extend to $\overline{\mathcal{O}_D}$ as well. We note that $\overline{\mathcal{O}_D}$ is normal if and only if every regular function on $\mathcal{O}_D$ extends to $\overline{\mathcal{O}_D}$, since $\operatorname{codim}_{\overline{\mathcal{O}_D}}(\overline{\mathcal{O}_D} \setminus \mathcal{O}_D) \geq 2$.

□

5.2 Summary of Results

In sum, we have shown that an affine variety Z exists for each nilpotent $D \in \mathfrak{g}(V)$ such that $\overline{\mathcal{O}_D}$ is a G-quotient of Z for some algebraic group G (Theorem 2.13). Z is not necessarily normal, but it is Cohen-Macaulay. We use this fact in the proof of Proposition 5.2 to show that $\overline{\mathcal{O}_D}$ is normal if and only if it is normal at all points contained in codimension two orbits $\mathcal{O} \subset \overline{\mathcal{O}_D}$.

We summarize some additional intermediate results that may be of general interest:

Theorem (3.7)**.** *If U is an orthogonal space, V is a symplectic space and $X \in \mathrm{L}(V,U)$, let $\widetilde{G} := \mathrm{GL}(U \oplus V)$ and let $\widetilde{X}$ be the element $\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \in \mathrm{End}(U \oplus V)$. Then there is a cocharacter $\phi : K^\times \to \widetilde{G}$ associated to nilpotent $\widetilde{X}$ in $\widetilde{G} = \mathrm{GL}(U \oplus V)$ with $\phi(t) \in \mathrm{O}(U) \times \mathrm{Sp}(V)$ for all $t \in K^\times$.*

Proposition (3.10, generalized from $\operatorname{char} K = 0$)**.** *Let U and V be quadratic spaces of opposite types and $X \in \mathrm{L}(V,U)$. If (X^*, X) is the orthosymplectic nilpotent pair with associated ab-diagram τ, let $\mathcal{O}_X$ denote the $\mathrm{O}(U) \times \mathrm{Sp}(V)$-orbit of $X \in \mathrm{L}(V,U)$. Let $\pi(\mathcal{O}_X)$ denote the $\mathrm{G}(U)$-orbit of $XX^* \in \mathfrak{g}(U)$ and let $\rho(\mathcal{O}_X)$ denote the $\mathrm{G}(V)$-orbit of $X^*X \in \mathfrak{g}(V)$. Then we have that*

$$\dim \mathcal{O}_X = \frac{1}{2}(\dim \pi(\mathcal{O}_X) + \dim \rho(\mathcal{O}_X) + \dim U \cdot \dim V - \Delta_\tau)$$

where $\Delta_\tau = \sum_{i odd} a_i b_i$, a_i the number of rows of τ of length i starting with a and b_i the number of rows of τ of length i starting with b.

Proposition (4.17)**.** *Let V be an ε-space and let $e \leq \dim(V)$. $(\mathfrak{g}(V), \mathfrak{g}(V)_e)$ is a good pair of $\mathrm{G}^0(V)$-varieties, where $\mathfrak{g}(V)_e$ is the determinantal subvariety of matrices of rank less than or equal to e.*

5.3 More Known Results

Kraft and Procesi go on to describe in [KP82] exactly which nilpotent orbits under the action of the symplectic and orthogonal groups have normal closure when $\operatorname{char} K = 0$. This is done in terms of Young diagrams and *irriducible minimal ε-degenerations*, of which there are finitely many. They also explicitly describe the singularities occurring in these cases.

Table 5.1: Irreducible Minimal ε-degenerations

Type	a	b	c	d
Lie algebra	$\mathfrak{sp}_2$	$\mathfrak{sp}_{2n}$ $n > 1$	$\mathfrak{so}_{2n+1}$ $n > 0$	$\mathfrak{sp}_{4n+2}$ $n > 0$
ε	-1	-1	1	-1
η	(2)	$(2n)$	$(2n+1)$	$(2n+1, 2n+1)$
σ	$(1,1)$	$(2n-1, 2)$	$(2n-1, 1, 1)$	$(2n, 2n, 2)$
$\operatorname{codim}_{\overline{\mathcal{O}_{\varepsilon,\eta}}} \mathcal{O}_{\varepsilon,\sigma}$	2	2	2	2

Type	e	f	g	h
Lie algebra	$\mathfrak{so}_{4n}$ $n > 0$	$\mathfrak{so}_{2n+1}$ $n > 1$	$\mathfrak{sp}_{2n}$ $n > 1$	$\mathfrak{so}_{2n}$ $n > 2$
ε	1	1	-1	1
η	$(2n, 2n)$	$(2, 2, 1^{2n-1})$	$(2, 1^{2n-2})$	$(2, 2, 1^{2n-4})$
σ	$(2n-1, 2n-1, 1, 1)$	(1^{2n+1})	(1^{2n})	(1^{2n})
$\operatorname{codim}_{\overline{\mathcal{O}_{\varepsilon,\eta}}} \mathcal{O}_{\varepsilon,\sigma}$	2	$4n-4$	$2n$	$4n-6$

Recall that an ε-degeneration $\sigma \leq \eta$ is *minimal* if $\sigma \neq \eta$ and there is no ε-diagram ν such that $\sigma < \nu < \eta$.

Proposition ([KP82] Proposition 3.2)**.** *Let $\sigma \leq \eta$ be an ε-degeneration. Assume that for two integers r and s the first r rows and the first s columns of η and σ coincide and that $(\eta_1, \ldots, \eta_r)$ is an ε-diagram. Denote by η' and σ' the diagrams obtained by erasing these rows and columns of η and σ respectively, and put $\varepsilon' := (-1)^s\varepsilon$. Then $\sigma' \leq \eta'$ is an ε'-degeneration and*

$$\operatorname{codim}_{\overline{\mathcal{O}_{\varepsilon',\eta'}}} \mathcal{O}_{\varepsilon',\sigma'} = \operatorname{codim}_{\overline{\mathcal{O}_{\varepsilon,\eta}}} \mathcal{O}_{\varepsilon,\sigma}.$$

Though we do not give the proof of this here, it relies on the combinatorics of ε-diagrams in sections 2.1.2 through 2.1.4 and does not depend on the character of K.

In the setting of the proposition, an ε-degeneration $\sigma \leq \eta$ is *obtained* from the ε'-degeneration $\sigma' \leq \eta'$ by adding rows and columns. An ε-degeneration $\sigma \leq \eta$ is then *irreducible* if it cannot be obtained from another ε'-degeneration $\sigma' \leq \eta'$ by adding rows and columns in a non-trivial way. We also have that any ε-degeneration is obtained in a unique way from an irreducible ε'-degeneration by adding rows and columns, and that $\sigma' \leq \eta'$ is minimal if and only if $\sigma \leq \eta$ is minimal.

Table 5.1 classifies the irreducible minimal ε-degenerations. For t equal to one of a, b, c, d, e, f, g, or h, an ε-degeneration is said to be of type t if it is obtained from the corresponding minimal irreducible degeneration by adding rows and columns. Types a, b, c, d, and e are of particular interest since they correspond to those classes of codimension 2.

Kraft and Procesi show that $\operatorname{Sing}(\overline{\mathcal{O}_{\varepsilon,\eta}}, \mathcal{O}_{\varepsilon,\sigma})$, the equivalence class of singularities of $\overline{\mathcal{O}_{\varepsilon,\eta}}$ contained in $\mathcal{O}_{\varepsilon,\sigma}$ (see [KP82, 12.1] for more detail), is equal to the equivalence class of singularities of $\overline{\mathcal{O}_{\varepsilon',\eta'}}$ in $\mathcal{O}_{\varepsilon',\sigma'}$:

Theorem ([KP82] Theorem 12.3)**.** *Let the ε-degeneration $\sigma \leq \eta$ be obtained from the ε'-degeneration $\sigma' \leq \eta'$ by adding rows and columns. Then*

$$\operatorname{Sing}(\overline{\mathcal{O}_{\varepsilon,\eta}}, \mathcal{O}_{\varepsilon,\sigma}) = \operatorname{Sing}(\overline{\mathcal{O}_{\varepsilon',\eta'}}, \mathcal{O}_{\varepsilon',\sigma'}).$$

Kraft and Procesi conclude in Theorem 16.2 that an orthogonal or symplectic conjugacy class has normal closure if and only if its partition has no degenerations of type e by computing $\operatorname{Sing}(\overline{\mathcal{O}_{\varepsilon,\eta}}, \mathcal{O}_{\varepsilon,\sigma})$ for the irreducible minimal ε-degenerations. These results do not extend to the connected components of the disconnected orbits in the orthogonal case, the so-called “very even” orbits. These are addressed by Sommers in [Som05] where he concludes that orbits of nilpotent elements in $\mathfrak{so}(V)$ under the action of the special orthogonal group have normal closure, again for $\operatorname{char} K = 0$. Whether these results generalize to positive characteristic without alteration has yet to be seen. We hope to make progress towards this in the future.

Appendix A

Let U and V be finite dimensional K-vector spaces, K algebraically closed of characteristic not 2, and let $A : U \to V$ and $B : V \to U$ be linear maps such that the compositions BA and AB are nilpotent endomorphisms on U and V, respectively.

We describe here K-algebras $\mathcal{A}$ such that pairs $\mathcal{P} = \{(U, V), (A, B)\}$ of vector spaces and maps between them form a category of $\mathcal{A}$-modules.

Let W be a K-vector space with basis A, B, and T. Then let S be the free algebra

$$S := K\langle A, B, T\rangle = K \oplus W \oplus (W \otimes W) \oplus \ldots \oplus W^{\otimes^i} \oplus \ldots$$

and let $I = \langle T^2 - 1, A(1-T), B(1+T), AT + TA, BT + TB\rangle$ be the (two-sided) ideal. Denote by $\mathcal{A}$ the quotient S/I.

Consider an $\mathcal{A}$-module $\mathcal{M}$ which is also finite dimensional as a K-vector space. Multiplication by T defines an order two linear automorphism of $\mathcal{M}$. Let

$$\mathcal{M}^+ = (1+T)\mathcal{M} \quad \text{and} \quad \mathcal{M}^- = (1-T)M$$

be the $+1$ and -1 T-eigenspaces. Then $\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^-$ is their direct sum.

Since $A(1-T) = 0$ in $\mathcal{A}$, we have $A\mathcal{M}^- = 0$. Likewise $B\mathcal{M}^+ = 0$ since $B(1+T) = 0$ in $\mathcal{A}$. Additionally, we have $A\mathcal{M}^+ \subseteq \mathcal{M}^-$ and $B\mathcal{M}^- \subseteq \mathcal{M}^+$ since $AT = -TA$ and $BT = -TB$. Indeed, we have for $m \in \mathcal{M}^+$, $ATm = -TXm = Xm$, so Xm is in the -1 T-eigenspace, namely $\mathcal{M}^-$. Similarly for $m \in \mathcal{M}^-$, $BTm = -TBm = -Bm$.

Hence one can view A as a linear mapping $\mathcal{M}^+ \to \mathcal{M}^-$ and B as a linear mapping $\mathcal{M}^- \to \mathcal{M}^+$.

If $\phi : \mathcal{M} \to \mathcal{N}$ is an isomorphism of $\mathcal{A}$-modules, then $\phi(\mathcal{M}^+) = \mathcal{N}^+$ and $\phi(\mathcal{M}^-) = \mathcal{N}^-$. Thus such an isomorphism is classified by the action of $\mathrm{GL}(\mathcal{M}^+) \times \mathrm{GL}(\mathcal{M}^-)$ on the pair (A, B) in $\mathrm{Hom}(\mathcal{M}^+, \mathcal{M}^-) \times \mathrm{Hom}(\mathcal{M}^-, \mathcal{M}^+)$.

Now, for $n \geq 1$ let $\mathcal{A}_n = \mathcal{A}/\langle (AB)^n \rangle$. Since $(AB)^n = 0$ in $\mathcal{A}_n$, we must have that $(BA)^{n+1} = 0$ as well. Thus for any $\mathcal{A}_n$ module $\mathcal{M}$, both AB and BA act nilpotently. Moreover, any A-module on which AB and BA act nilpotently can be seen as a module for $\mathcal{A}_n$ for some n.

Thus given vector spaces U and V and a pair of linear maps $A : U \to V$ and $B : V \to U$ such that AB and BA are nilpotent, $U \oplus V$ can be viewed as a $\mathcal{A}_n$-module for some n with U the $+1$ T-eigenspace and V the -1 T-eigenspace. This module is then a direct sum of indecomposable modules, uniquely determined up to isomorphism.

Given $(AB)^n = 0$ with n minimal, we may have $(AB)^n A = 0$ or $(AB)^n A \neq 0$, but as stated above, we must have $B(AB)^n A = (BA)^{n+1} = 0$. Thus up to interchanging U and V (and thus A and B), there are two indecomposable $\mathcal{A}_n$-modules which are not $\mathcal{A}_{n-1}$-modules, and which we can describe by $abab \ldots abab$ and $abab \ldots ababa$, using the notation in section 3.1.1.

Appendix B

Here we describe the preimage $\vartheta^{-1}(\mathcal{O})$ of an orbit $\mathcal{O} \in \overline{\mathcal{O}_D}$ where $\vartheta : Z \to \overline{\mathcal{O}_D}$ is the map $(X_1, \ldots, X_t) = X_1^* X_1$. Let λ be the ε-diagram of D.

Let $(X_1, \ldots, X_t)$ be an element of Z. If τ_i is the ab-diagram of the pair (X_i^*, X_i), we associate to $(X_1, \ldots, X_t)$ the string of ab-diagrams $(\tau_1, \ldots, \tau_t)$. Because of the fiber product diagram describing Z in section 3.4.1, the τ_i must satisfy the following conditions:

(i) $\pi(\tau_i) = \rho(\tau_{i+1}) = \sigma_i$ for $1 \leq i \leq t-1$, with $\sigma_k \leq \lambda^k$ where λ^k is the Young diagram of $D_k = D|_{V_k}$ as in section 2.3.1, and

(ii) $\sigma_t = 0$, i.e. $\mathcal{O}_{\sigma_t} = 0$.

We define $\sigma_0 := \rho(\tau_1)$.

$\vartheta^{-1}(\mathcal{O})$ is the set of tuples $(X_1, \ldots, X_t)$ of Z such that $\sigma_0 = \mu$ where μ is the ε-diagram associated with the orbit $\mathcal{O}$. The task then is to describe those strings of ab-diagrams $(\tau_1, \ldots, \tau_t)$ such that $\rho(\tau_1) = \mu$ for ε-diagrams $\mu \leq \lambda$.

First, we consider the case $\mathcal{O} = \mathcal{O}_D$, i.e. $\lambda = \mu$. By definition, we have $|\lambda| = \dim(V_0)$, $|\lambda'| = \dim(V_1)$, and in general $|\lambda^i| = \dim(V_i)$ where $\lambda^i = (\lambda^{i-1})'$. If we draw the Young diagram of λ using b's rather than boxes, we obtain an ab-diagram τ_1 with $\rho(\tau_1) = \lambda$ by inserting a's before, between and after the b's, or by adding a row with a single a. The number of a's needed to "fill" the b-diagram of λ (to put a's between every two b's) is exactly $|\lambda'|$. This is the minimum number of a's needed to make an ab-diagram from the b-diagram. As long as the b-diagram we begin with is indeed an ε-diagram, the ab-diagram obtained from filling the b-diagram is indeed an orthosymplectic ab-diagram. Since we must have $\pi(\tau_1) \leq \lambda'$, there must be *exactly* $|\lambda'|$ a's in the ab-diagram τ_1, since the partial order defined on Young diagrams compares partitions of the same number. Thus there is only one ab-diagram τ_1 such that $\rho(\tau_1) = \lambda$ and $\pi(\tau_1) \leq \lambda'$. In fact, we have $\pi(\tau_1) = \lambda'$, since $\lambda'_i = \lambda_i - 1 = \#$ of a's needed to fill a row of λ_i b's.

For example, if $\lambda = (4, 3, 3, 2)$ then

$$\lambda = \begin{matrix} bbbb \\ bbb \\ bbb \\ bb \end{matrix} \qquad \tau_1 = \begin{matrix} bababab \\ babab \\ babab \\ bab \end{matrix} \qquad \pi(\tau_1) = \begin{matrix} aaa \\ aa \\ aa \\ a \end{matrix} = \lambda'$$

Now we need τ_2, and iteratively τ_i, such that $\rho(\tau_i) = \lambda^{i-1}$ and $\pi(\tau_i) \leq \lambda^i$. Once again, since we have exactly $|(\lambda^{i-1})'|$ a's with which to fill the b-diagram of λ^{i-1}, there is a unique τ_i.

In the above example,

$$\lambda' = \begin{matrix} bbb \\ bb \\ bb \\ b \end{matrix} \qquad \tau_2 = \begin{matrix} babab \\ bab \\ bab \\ b \end{matrix} \qquad \pi(\tau_2) = \begin{matrix} aa \\ a \\ a \end{matrix} = \lambda^2$$

and finally

$$\lambda^2 = \begin{matrix} bb \\ b \\ b \end{matrix} \qquad \tau_3 = \begin{matrix} bab \\ b \\ b \end{matrix} \qquad \pi(\tau_3) = a = \lambda^3$$

Hence we have shown

Lemma. *There is a unique stratum Z_{λ^0} with $\lambda^0 = (\tau_1^0, \ldots, \tau_t^0)$ such that $\rho(\tau_1) = \lambda$. In particular, τ_i^0 is the ab-diagram of $D_{i-1} = D|_{V_{i-1}}$.*

Proof. By definition, the ab-diagrams of the D_i form such a string. Its unicity follows from the above discussion and the τ_i^0 are built by "filling" with a's the ε-diagram of $\pi(\tau_{i-1}^0)$, written as a b-diagram, and τ_1^0 is the filling of λ. This is also Lemma 5.4 of [KP79], though their discussion refers to ab-diagrams of general nilpotent pairs, rather than orthosymplectic pairs. $\square$

Now we consider the case $\mu < \lambda$. The key observation is that $|\mu^i| \leq |\lambda^i|$ for all $i \in \mathbf{Z}^+$ and for some (at least one) $j \in \mathbf{Z}^+$, we must have $|\mu^j| < |\lambda^j|$. At this stage, when we go to construct τ_j from $\pi(\tau_{j-1})$, we first fill the b-diagram of $\pi(\tau_{j-1})$ with $|\mu^j|$ a's (in a unique way), but we must have exactly $|\lambda^j|$ a's in the ab-diagram τ_j since $\pi(\tau_j) = \sigma_j \leq \lambda^j$ (remember, comparable partitions are of the same number). So now we have an additional $|\lambda^j| - |\mu^j|$ a's to add to our b-diagram, which can be placed in many different positions, as long as the resulting ab-diagram is orthosymplectic (some union of the indecomposable types of Table 3.1). This restricts our choice for where to place the a's, but does not guarantee a unique ab-diagram.

In particular, we will have $\sigma_j \neq \mu^j$ (they are not even partitions of the same number). While the stratum Z_{λ^0} is the iterated fiber product above the $\mathcal{O}_{D_i}$, the other strata Z_ω with $\omega = (\tau_1, \ldots, \tau_t)$ and $\rho(\tau_1) = \mu \neq \lambda$ are iterated fiber products above the $\mathcal{O}_{\sigma_i}$ and *not* the $\mathcal{O}_{\mu^i}$. There are, however, finitely many ab-strings $(\tau_1, \ldots, \tau_t)$ with $\rho(\tau_1) = \mu$, each corresponding to a choice of where to place a's in the diagrams τ_j for j such that $|\mu^j| \neq |\lambda^j|$. Hence there are finitely many Z_ω sitting above each orbit $\mathcal{O}$ contained in the boundary $\overline{\mathcal{O}_D} \setminus \mathcal{O}_D$.

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