# Quantification of Trading Advantage in a Kinetic Model of Asset Exchange

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May 15, 2015

**Abstract**

Asset-exchange models (AEMs) are mathematical representations of the movement of wealth within a system of economic agents. In this work, we focused on one particular AEM, the Yard-Sale Model (YSM), in which transactions are pairwise between agents and exchanges are always fractions of the poorer agent's wealth.

In recent work, Boghosian derived a Boltzmann equation for the YSM and showed that with a kinetic approach to wealth distributions, the YSM exhibits qualitative agreement with Pareto's Law and may indeed be the first explanation of this macroeconomic observation since it was first proposed nearly a century ago. One of the shortcomings of the YSM is its symmetric trading assumption wherein both agents in a transaction have an equal likelihood of winning. In this work, we reconsidered the YSM in the presence of a trading bias based on the idea that economic agents may have an advantage or disadvantage due to their intrinsic characteristics such as race, gender, etc. We assigned agents within the YSM to have a single valued parameter representing their *trading advantage* and we derived a pair of coupled Boltzmann equations and the resulting Fokker-Planck equations. We considered the effects of this Bias due to Agent Attributes (BAA) on the kinetics of wealth and the implications this may have on the dynamics of inequality and the relationships between redistribution and trade.

## Contents

Special thanks to Bruce Boghosian and Christoph Börgers for their encouragement and support in the eleventh hour.

# 1 Introduction

Reconciling microeconomic processes with macroeconomic trends remains one of the most important challenges in economic theory. In the particular case of the economics of wealth distributions, efforts have spread beyond the social sciences in order to describe the way that transactions between two individuals act as "microfoundations" to generate realistic distributions of wealth.

In recent work, inspired by the methodologies of statistical physics, Boghosian [1] devised a Boltzmann equation for wealth exchange, offering what may be the first microfoundational explanation of Pareto's Law in nearly a century of study on the subject. In this paper I will rederive Boghosian's kinetic model in the presence of a trading bias to account for the way an economic agent's intrinsic characteristics may help or hurt them financially, based philosophically on the idea that *who* someone is affects their ability to trade.

## 1.1 Wealth distributions

One of the first serious considerations of the way in which wealth distributes came from the Italian economist and sociologist Vilfredo Pareto, in 1916. In his seminal work *"Trattato di sociologia generale"* [2], an expansive survey of wealth and income in Europe spanning several centuries of data, Pareto found that wealth accumulates in a distinct way almost invariably across economies and cultures. The shape of the distribution, shown in Fig. 1, has since come to be known as the Pareto Distribution.

The Pareto Distribution is primarily characterized by its tail: as wealth $w$ increases, the number of agents with wealth $w$ decreases like a power law. Pareto found this to be a reasonable approximation to the data he collected, though it is important to note that the decay is not *exactly* like a power law [3]. Nonetheless, this approximation was the impetus for further analysis: This power law rule gave way to a simple new comparative method for wealth distributions.

## 1.2 Measuring wealth inequality

The way in which Pareto visualized his famous distribution was by plotting wealth $w$ versus the fraction of economic agents $A(w)$ with wealth greater than $w$. He found that for a lower bound $w_{\text{min}}$ on wealth, $A(w)$ could be approximated by

$$A(w) \approx \begin{cases} 1 & \text{if } w < w_{\text{min}} \\ \left(\frac{w_{\text{min}}}{w}\right)^{\alpha} & \text{otherwise}\,, \end{cases} \tag{1}$$

where $\alpha$ is referred to as the *Pareto index.* After some thought, one can see that $\alpha \to \infty$ corresponds to a perfectly uniform distribution of wealth where each agent has the same net wealth given by $w_{\text{min}}$. On the other hand, $\alpha \to 1_+$ corresponds to an increasingly concentrated distribution, where almost all wealth has accumulated in the hands of a smaller and smaller fraction of the $N$ agents. To see that this is the lower bound of $\alpha$, we consider a distribution

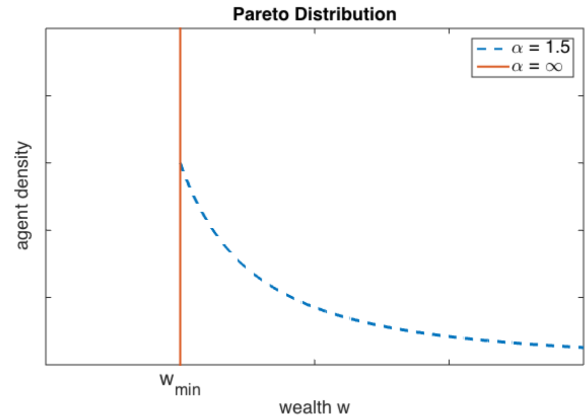



**Figure 1:** Examples of the Pareto distribution, with $\alpha = 1$, 1.5, 2, and $\infty$. When $\alpha = \infty$, the Pareto distribution becomes to Dirac $\delta$-function, corresponding to an even distribution of wealth.

$P(w)$ which describes the spread of wealth across a given population, with $P(w)$ normalized to $N$, so that the quantity $A(w)$ may be defined in the following integral form,

$$A(w) := \frac{1}{N} \int_w^\infty dx \; P(x) \,. \tag{2}$$

$P$ may be more appropriately thought of as an agent density function over wealth space. Now, by differentiating Eq. (2) and reconsidering Eq. (1) we find that

$$P(w) \approx \begin{cases} 0 & \text{if } w < w_{\min} \\ \frac{\alpha N}{w_{\min}} \left(\frac{w_{\min}}{w}\right)^{\alpha+1} & \text{otherwise} \,. \end{cases} \tag{3}$$

If we assume the total wealth of our system, given by the first moment of $P$, to be finite, then we may conclude that indeed $\alpha > 1$.

One reasonable objection to the Pareto index as a metric for wealth inequality is that it is a good approximation of the distribution of wealth only on the tail, the wealthiest in a given system, and is less accurate for the poorer bulk of the population. To introduce an alternative measure, first consider the Lorenz curve shown in Fig. 2. The American economist Max O. Lorenz, a contemporary of Pareto, devised this representation of the distribution of wealth in 1905 [4], which plots the fraction of the wealth of economic agents against the fraction of agents in the system. Mathematically, this can be thought of as a parametric relationship between two cumulative distribution functions $F$ of agents and $L$ of wealth, defined as

$$F(w) \quad := \quad \frac{1}{N} \int_0^w dx \; P(x) \tag{4}$$

$$L(w) \quad := \quad \frac{1}{W} \int_0^w dx \; P(x) \, x \,, \tag{5}$$

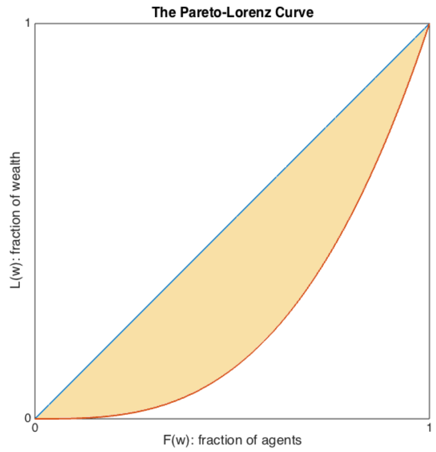



**Figure 2:** The Pareto-Lorenz curve is plotted with the cumulative distribution of agents $F(w)$, versus the cumulative distribution of wealth $L(w)$. The wealth parameter $w$ varies from zero (the lower left corner) to $\infty$ (the upper right corner). The blue diagonal then represents an even distribution of wealth, and the orange curve below the diagonal represents a system in which the distribution is largely inequal. The Gini coefficient is equal to one half of the area of the shaded region.

where $P$ is as previously defined, and $W$ is the total wealth of the system. In the above, we have defined the quantities

$$N := \int_0^\infty dw\; P(w) \tag{6}$$

$$W := \int_0^\infty dw\; P(w) w\,, \tag{7}$$

where $w$ varies from 0 to $\infty$. Note that $F$ is the lower incomplete zeroth moment that complements Pareto's distribution term $A$, in the sense that $F = 1 - A$.

The diagonal of Fig. 2 then corresponds to an even distribution of wealth, where each of the $N$ agents possesses the same fraction $W/N$ of wealth. In real economies, the Lorenz curve lies provably below the diagonal, and the area of the shaded region, as a fraction of the area of the lower triangle, thus represents the deviation of a given wealth distribution from equality. This measure of inequality is referred to as the *Gini coefficient*, and is formally defined as

$$G := \frac{\int_0^1 dF\;(F - L)}{\int_0^1 dF\; F} = 1 - 2\int_0^1 dF\; L\,. \tag{8}$$

Accordingly, when wealth is uniformly distributed, $G = 0$, and when a single agent owns all the wealth in the system, $G = 1$. Since the Gini coefficient is defined across the entire wealth spectrum, it has become a popular measure of wealth inequality as an alternative to the Pareto index.

It is important to understand what is mathematically described by the term *inequality*. Consider, for example, the two different economies described by the Lorenz curves as shown in Fig. 3. It is easy to see that both of these economies have identical Gini coefficients, and yet it would not be unreasonable to consider Population 1, which has a very poor lower quantile - as less *fair* than

Population 2, in which there is a larger middle quantile and the upper quantile is less economically dominant. The Gini coefficient is thus a measurement of net inequality, but should *not* be interpreted as a measure of *unfairness*.

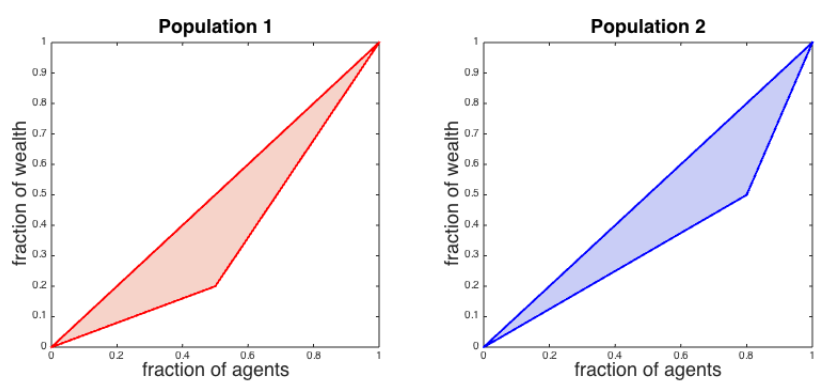
Thus far both the Pareto index and Gini coefficient have been considered as snapshots of inequality, but a more interesting measurement is surely the rate at which inequality changes. Consider the fact that in 2011, the wealthiest 388 people in the world had the same net worth as the poorest 3.5 billion, and that by earlier this year that number had dropped to 80 [5]. Indeed, the distribution of wealth is anything but static, and understanding the forces that shift wealth has rarely been more pertinent.

## 1.3 A kinetic theory of wealth

Despite Pareto's efforts and the paradigmatic shift he caused in the way economics was studied, the theory of wealth distributions did not have any significant progress in terms of microfoundational understanding for some time. In the mid-twentieth century, mathematicians Champernowne [6] and Mandelbrot [7] separately published works on stochastic models of wealth, but it was not until 1986 that a kinetic approach began to unfold, with a publication by Angle in the sociological literature [8].

Angle conceptualized asset exchange models (AEMs) as simple systems of pairwise transactions, based on fundamental assumptions about the way in which two random economic agents might trade. Mathematically, AEMs are representations of wealth exchanges between $N$ agents that yield wealth distributions of a given population, where each agent shares some fraction of the total wealth $W$ of the system. For simplicity, $N$ and $W$ are generally considered to be conserved quantities and for now we will consider them as such. This makes for

**Figure 3:** The two graphs below are hypothetical Pareto-Lorenz curves for two different populations. Despite identical Gini coefficients for the two economies and thus identical *inequality*, one might reasonably claim that the inequality of Population 1 is less *fair* because its lower class is far poorer than the lower class of Population 2.



an easily understood model - for conserved wealth, one agent's gain is another agent's loss. It should be noted that this is not entirely realistic; in the presence of consumption, production, immigration and emigration, $N$ and $W$ would fail to be conserved, and these are naturally areas of future interest. In the continuum limit of wealth $w$, the wealth distribution generated by the model may be described by the agent density function $P(w, t)$ as previously formulated in Section 1.2, though now with an added dependence on time.

In this work I consider one particular AEM, the Yard-Sale Model (YSM), proposed by Chakraborti in 2002 [9]. In the YSM, the wealth exchanged in a given pairwise transaction is a fraction of the *poorer* agent's wealth. This is a reasonable assumption, as it is exceedingly rare that we ever gain more than our net worth in a single transaction. Additionally, in the simplest version of the YSM, two agents have equal chances of "winning" in any transaction. Henceforth, I consider the value in a given transaction not to be determined by the amount the buyer is willing to spend (as is economics convention), but rather determined objectively. In this way, it becomes clear that an economic agent can make "mistakes" and that the wealth exchanged in a given transaction is not the total value of the good, but the magnitude of the mistake.

One interpretation of this model, in order to visualize its kinetic behavior, is of economic agents as particles, wealth as energy, and transactions as collisions. In this sense, a system of economic agents 'bounce' off one another exchanging wealth (though our model does not in fact depend on any spatial locality). This conceptualization then lends itself immediately to the field of statistical physics and its methodologies, especially those developed by the Austrian physicist Ludwig Boltzmann.

# 2 Prior Work

## 2.1 The Boltzmann equation and the Fokker-Planck equation

Given a few general principles of pairwise transactions, can we generate an entire dynamic distribution of wealth? An analogous problem was solved in the second-half of the nineteenth century when Boltzmann, Maxwell and Gibbs developed statistical physics in order to explain macroscopic phenomena due to kinetic models of the nascent atomic theory. In particular, in 1872, Boltzmann [10] devised an eponymous equation that describes non-equilibrium thermodynamics in dilute gases, and in the appropriate limit captures features of the atomic behavior on which it is founded. Today, in the physics literature, a Boltzmann equation may be any in which a macroscopic quantity changes due to underlying kinetic principles of binary interactions.

Nearly sixty years after Boltzmann first devised his equation, Andrey Kolmogorov developed what is known as the Fokker-Planck equation, which describes the effects of various physical forces on the evolution of a probability density function over time. In fact, if considered in the appropriate limit, the integrodifferential Boltzmann equation may reduce to the simpler differential form of the Fokker-Planck equation [11].

Inspired by both Boltzmann and Kolmogorov, in 2013 Boghosian [12] derived a Boltzmann equation for the YSM, and found a corresponding Fokker-Planck equation describing the time evolution of the YSM where the appropriate limit is the small transaction limit. This is not an unreasonable assumption: in the majority of transactions that we engage in as economic agents, the amount of wealth exchanged is a small fraction of our net wealth. Boghosian's next step was to introduce redistribution into the model, and, after some numerical analysis, he was able to demonstrate substantial agreement with Pareto's distribution, especially in the limit of small but non-zero redistribution. This is nothing short of remarkable, and it could very well be the first microfoundational explanation of Pareto's findings since they were first published in 1916.

Before formally deriving these equations, let us take a deeper look at what these models might tell us. What can be said about the forces in the Fokker-Planck equation that establish the shape of the wealth distribution? What role does redistribution play in determining this shape?

## 2.2 An $H$-theorem for the Boltzmann equation

One of the beautiful successes of the Boltzmann equation was its resulting agreement with certain features of the second law of thermodynamics. Known as the $H$-theorem, this consequence describes an entropically-behaving quantity $H$ in a dilute gas. More generally, just as a Boltzmann equation may be an intermediate kinetic equation between microscopic and macroscopic phenomena, an $H$-theorem may describe the tendency for some quantity of the kinetic system to increase or decrease in a statistically irreversible way.

In recent work, Boghosian, Marcq and I [13] proved an $H$-theorem for the YSM in the absence of redistribution, with the Gini coefficient as the corresponding $H$-functional. Specifically, we noted that the rate of change of the Gini coefficient is given by an infinite-dimensional version of the chain rule,

$$\frac{dG}{dt} = \int_0^\infty dw \, \frac{\delta G[P]}{\delta P(w)} \frac{\partial P}{\partial w} \geq 0 \, , \tag{9}$$

where $\delta G[P]/\delta P(w)$ is the Fréchet derivative of $G[P]$, and may be thought of as an infinite-dimensional analog of the gradient in function space. G is then a Liapunov functional tending inexorably toward the value 1, corresponding to complete oligarchy.

This is a fascinating consequence of the YSM. Modern economic thinking contends that market-forces maintain the steady-state of the wealth distribution - the YSM alternatively suggests that not only is the supposed "steady-state" in fact *unsteady*, but that without redistribution it would tend invariably toward oligarchy. This is even more remarkable if we recall that this simple version of the YSM assumes a symmetric probability of winning for either agent in any transaction. This is almost difficult to believe - with symmetric chances of winning, one agent still comes out on top.

In fact, what is happening here is that those agents lucky enough to win their first few transactions will never again be forfeiting a large portion of their wealth, because the stakes in their following transactions will be a fraction of the wealth of the poorer agents with whom they are now statistically likely to trade. These agents may then withstand a longer string successive losses. The unlucky agents who happen to lose their first few transactions will be unable to re-accumulate enough wealth to break out of poverty.

This is, of course, still dependent on the symmetric trading probability. What happens if we break this symmetry, and in what ways can it be broken?

## 2.3 The advantage of the wealthy

Despite the implicit advantage of the rich emergent from the YSM with symmetric probability, the model does not account *entirely* for their advantages. Consider that the wealthy can hire better tax lawyers, can more readily hide money offshore and can influence public policy in their favor. These are not happenstance effects of having more money and not spending it proportionally to those of less wealth, these are *explicit* advantages that the rich exploit to get richer.

Earlier this year, Boghosian, Marcq, Wang and I [14] devised a Boltzmann equation using an asymmetric trading probability in order to account for what we termed a Wealth-Attained Advantage (WAA), and found the corresponding Fokker-Planck equation in the small-transaction limit. One can quickly imagine the effects of the WAA on the wealth dynamics of the Fokker-Planck equation. Already prone toward oligarchy, the WAA only encourages this tendency, speeding up the rate at which the distribution unbalances. As redistribution plays the role of stabilizing the wealth distribution, it's influence must be increased in the presence of the WAA in order to maintain the same Gini coefficient.

However, this is by no means the only bias present in trade. Consider, for example, that women earn about 88 cents for every dollar that men earn [15] [1], or that there is a distinct racial bias in the mortgage lending market [16]. These are biases based on the inherent characteristics of a given trader; these are Biases due to Agent Attributes (BAA). There are, of course, an inordinate number of characteristics that may help or hurt one's ability to trade. In order to statistically encapsulate these features we may consider them as a single parameter $z$ corresponding to an agent's *trading advantage*. To introduce the BAA to the YSM, each agent may be imbued with some random value of $z$, referred to as a 'quenched attribute', representing a given characteristic that each agent maintains across transactions.

The BAA is indeed the focus of this work. The following sections will involve a more formal look at the Boltzmann equation and Fokker-Planck equation, first in the absence of trading biases and then in the presence of the BAA, before a final consideration of the role that this bias plays in the dynamics of wealth.

[1] There remains some dispute on the exact difference between the wages of women and men, and on the forces involved in forming it. Nonetheless, the existence of this disparity suggests a bias.

# 3 Deriving the Boltzmann and Fokker-Planck Equations for the Yard-Sale Model in the Absence of Biases

## 3.1 The Boltzmann equation

In a given AEM, some amount $\Delta w$ of wealth is exchanged in a pairwise transaction between two randomly selected agents. In the specific case of the YSM, we assume that $\Delta w$ must be some fraction $\beta$ of the wealth of the poorer agent. Observably this is almost always true; it is exceptionally rare that we ever gain more than our net worth in one transaction. Thus, if the wealths of two agents are given by $\overline{w}$ and $\overline{w}'$, then we may write

$$\Delta w = \beta \min(\overline{w}, \overline{w}') \tag{10}$$

where $\beta$ is some fraction sampled from a symmetric distribution $\eta(\beta)$, where $\int d\beta\, \eta(\beta) = 1$, its first moment equal to zero, and its second moment finite. It is important to note that this symmetry of $\eta$ is what guarantees equal chances for each agent to walk away from a transaction with more wealth, and is exactly what we will alter in adding a bias in the following section.

If we consider $w$ and $w'$ to be the respective post-transaction wealths of the agents starting with $\overline{w}$ and $\overline{w}'$ then we have a transformation from $\mathbb{R} \times \mathbb{R}$ to itself given by

$$\begin{aligned} w &= \overline{w} + \beta \min(\overline{w}, \overline{w}') \qquad (11) \\ w' &= \overline{w}' - \beta \min(\overline{w}, \overline{w}') \qquad (12) \end{aligned}$$

with inverse

$$\begin{aligned} \overline{w} &= w - \frac{\beta}{1-\beta} \min\left(\frac{1-\beta}{1+\beta} w, w'\right) \qquad (13) \\ \overline{w}' &= w' + \frac{\beta}{1-\beta} \min\left(\frac{1-\beta}{1+\beta} w, w'\right), \qquad (14) \end{aligned}$$

and with Jacobian

$$J(w, w', \beta) := \frac{\partial(\overline{w}, \overline{w}')}{\partial(w, w')} = \tfrac{1}{1+\beta}\theta\left(w' - \tfrac{1-\beta}{1+\beta} w\right) + \tfrac{1}{1-\beta}\theta\left(\tfrac{1-\beta}{1+\beta} w - w'\right). \tag{15}$$

To derive the Boltzmann equation for the YSM, we want to find the rate of change of the agent distribution $P(w, t)$. To do this we first consider the small interval $[w, w + dw]$ in the wealth spectrum. The flow of agents into this interval will originate from transactions between agents of wealth in $[\overline{w}, \overline{w} + d\overline{w}]$ and those with wealth in $[\overline{w}', \overline{w}' + d\overline{w}']$. The flow of agents out of this interval will of course be due to transactions between agents of wealth in $[w, w + dw]$ and agents of wealth in $[w', w' + dw']$. Thus, we find the rate of change of the agent distribution $P(w,t)$ by integrating the transaction terms corresponding to inflow and outflow with respect to $w'$ and $\overline{w}'$ over their domains,

$$\frac{\partial P(w,t)}{\partial t} = \frac{1}{N^2}\int_0^\infty dw' \int_0^\infty d\overline{w}' \int d\beta \; \eta(\overline{w},\overline{w}',\beta)\big[P(\overline{w},t) - P(w,t)\big]P(w',t)P(\overline{w}',t)\,. \tag{16}$$

Recall that we demanded for the complete zeroth moment of $P(w,t)$ to be the total number of agents in the system, that is, $N = \int_0^\infty dw\; P(w,t)$. Applying this to (16) and then employing the Jacobian we may rewrite it in the following form

$$\frac{\partial P(w,t)}{\partial t} = \frac{1}{N}\int_0^\infty dw' \int d\beta \; \eta(\overline{w},\overline{w}',\beta)\big[P(\overline{w},t)P(\overline{w}',t)J(w,w',t) - P(w,t)P(w',t)\big]\,. \tag{17}$$

Combining Eqns. (13), (14), (15) and (17) yields

$$\begin{aligned}\frac{\partial P(w)}{\partial t} \;\; &= \;\; \int_{-1}^{+1} d\beta \; \eta(\beta)\Bigg\{ -\left[P(w) - \frac{1}{1+\beta}P\left(\frac{w}{1+\beta}\right)\right] \\ &\quad + \frac{1}{N}\int_0^{\frac{w}{1+\beta}} dx\; P(x)\left[P(w-\beta x) - \frac{1}{1+\beta}P\left(\frac{w}{1+\beta}\right)\right]\Bigg\}. \end{aligned} \tag{18}$$

Eq. (18) is the integrodifferential Boltzmann equation for the fundamental form of the YSM.

## 3.2 Deriving the Fokker-Planck equation in the small transaction limit

To derive the Fokker-Planck equation, we begin by considering Eq. (18) in the small transaction (or small $\beta$) limit. First, note that when $\beta = 0$, the right-hand side of Eq. (18) vanishes. Additionally, recall our demand that the first moment of $\eta$ equal zero. Thus, if we expand the inner terms of the right-hand side of Eq. (18) in a Maclaurin series in $\beta$, the first contributing term will be of order $\beta^2$ and the higher order terms may be considered negligible. After some calculations, our Boltzmann equation then reduces to the simpler differential form given by

$$\frac{\partial P(w)}{\partial t} = \frac{\partial^2}{\partial w^2}\left[\gamma\left(B(w) + \frac{w^2}{2}A(w)\right)P(w)\right], \tag{19}$$

where $A$ is defined as before and $B$ is the following incomplete wealth moment,

$$B := \frac{1}{N}\int_0^w dx\; P(x,t)\frac{x^2}{2}\,, \tag{20}$$

and $\gamma$ is defined as the second moment of the $\eta$ distribution,

$$\gamma := \int d\beta\; \eta(\beta)\beta^2\,. \tag{21}$$

### 3.3 An expression for redistribution

The prior conclusion of the Gini coefficient as an $H$-functional of the Boltzmann equation for the YSM extends to the Fokker-Planck equation as well [13]. Thus, the precarious balancing act of maintaining the distribution in a non-oligarchical state must come in the form of redistribution. As a simple model of redistribution, we introduce a taxation $\tau$ on all economic agents. A given agent will gain or lose proportional to the difference between their wealth $w$ and the average agent wealth $W/N$ of the system. Accordingly, redistribution manifests itself as an additional term in Eq. (19) in the following way,

$$\frac{\partial P(w)}{\partial t}+\frac{\partial}{\partial w}\left[\tau\left(\frac{W}{N}-w\right)P(w)\right]=\frac{\partial^2}{\partial w^2}\left[\gamma\left(B(w)+\frac{w^2}{2}A(w)\right)P(w)\right]. \quad (22)$$

Indeed, the introduction of this redistribution term yields an agent distribution convincingly akin to the Pareto distribution. A deeper consideration of this can be found in [1].

Henceforth, we omit the redistribution term, though we return to it in later sections to reintroduce it to the Fokker-Planck equation with the BAA. Additionally, the derivation of the WAA term in the Fokker-Planck equation is in many ways similar to the following derivation of the BAA and thus we also omit it until Section 6. For further reading and a full derivation of the WAA, refer to [14].

## 4 A Boltzmann Equation for the Yard-Sale Model with a Bias Due to Agent Attributes

In order to derive a Boltzmann equation for the YSM including the BAA term, two critical changes must be made to the prior derivation of (18). As articulated in Section 2.3, the BAA manifests itself through a new quenched attribute ascribed at random to each agent. Thus, we allow our agent density function $P$ to now depend on a third variable $z \in \mathbb{R}$ that represents the *trading advantage* of an agent. If $z$ is a quantification of a unchanging characteristic of an agent, then $z = \overline{z}$. Our second alteration is to demand some asymmetry of the distribution $\eta(\beta)$. As aforementioned, the symmetry of $\eta$ in the absence of bias terms is what allowed for even trading odds between agents; thus, to introduce bias we introduce asymmetry. Indeed, we generalize Eq. (17) as follows,

$$\frac{\partial P(w,z,t)}{\partial t}=\frac{1}{N}\int_0^\infty dw'\int dz'\int d\beta\ \eta(\overline{w},\overline{w}',z,z',\beta)\big[P(\overline{w},z,t)P(\overline{w}',z',t)J(w,w',t) - P(w,z,t)P(w',z',t)\big] \quad (23)$$

where the distribution $\eta$ now also depends on the trading advantage of the two agents, $z$ and $z'$, with wealths $\overline{w}$ and $\overline{w}'$, respectively. We make the following assumptions about $\eta$: For any $\overline{w},\overline{w}', \in \mathbb{R}^+$ and for any $z, z' \in \mathbb{R}$, we demand that $\eta$ be normalized,

$$\int d\beta\ \eta(\overline{w},\overline{w}',z,z',\beta)=1\,,$$

that its first moment be odd under simultaneous interchange of $\overline{w}$ and $\overline{w}'$ and of $z$ and $z'$,

$$\int d\beta\ \eta(\overline{w},\overline{w}',z,z',\beta)\beta = R(z,z') = -R(z,z'),$$

and that its second moment be finite,

$$\int d\beta\ \eta(\overline{w},\overline{w}',z,z',\beta)\beta^2 < \infty\,.$$

Of the first moment we should note that the simultaneous interchange of $\overline{w}$ and $\overline{w}'$ and of $z$ and $z'$ is natural; this can more appropriately be thought of as an interchange of agents rather than an interchange of only the wealth or only the trading advantage of agents which would make little physical sense. Conservation of agents, wealth, and trading advantage are demonstrated in the Appendices A, B, and C, respectively.

Substituting Eqs. (11), (12) and (15) into Eq. (23) and reversing the order of integration once yields

$$\begin{aligned}\frac{\partial P(w,z,t)}{\partial t} &= -P(w,z,t) \\ &+ \frac{1}{N}\int d\beta\ \frac{1}{1+\beta}\int_{\frac{1-\beta}{1+\beta}w}^{\infty} dw'\ \eta(\overline{w},\overline{w}',z,z',\beta)P(\overline{w},z,t)P(\overline{w}',z',t) \\ &+ \frac{1}{N}\int d\beta\ \frac{1}{1-\beta}\int_0^{\frac{1-\beta}{1+\beta}w} dw'\ \eta(\overline{w},\overline{w}',z,z',\beta)P(\overline{w},z,t)P(\overline{w}',z',t)\end{aligned} \tag{24}$$

Now $R(z,z')$ must be asymmetric and must vanish when agents have equal trading advantage. To maintain a certain degree of simplicity we choose to use a linear $R$. Though this is by no means the only possible choice of $R$, the linear term is the only one that will matter in the small-transaction limit. Thus, the distribution $\eta$ becomes

$$\eta(\overline{w},\overline{w}',z,z',\beta) = \frac{1+r(z-z')}{2}\delta(\beta-\alpha) + \frac{1-r(z-z')}{2}\delta(\beta+\alpha) \tag{25}$$

where $\alpha \in [0,1)$. With this choice of distribution and the first $z$ moment of $P$ given by

$$Z(w,t) := \int dz\ P(w,z,t)z\,, \tag{26}$$

and we derive the following pair of coupled Boltzmann equations

$$\frac{\partial P(w,t)}{\partial t} = \int dz\ \frac{\partial P(w,z,t)}{\partial t} = S_0(w,z,t;\alpha,r) \tag{27}$$

$$\frac{\partial Z(w,t)}{\partial t} = \int dz\ z\frac{\partial P(w,z,t)}{\partial t} = S_1(w,z,t;\alpha,r)\,, \tag{28}$$

where we define the right-hand side of (27) as

$$
\begin{aligned}
S_0(w,z,t;\alpha,q,r) \;&=\; -P(w,t) \qquad (29)\\
&+ \tfrac{1}{2N(1+\alpha)}\int dz\int dz'\int_{\frac{1-\alpha}{1+\alpha}w}^{\infty} dw'\Big[1+r(z-z')\Big]P\Big(\tfrac{w}{1+\alpha},z,t\Big)P\Big(w'+\tfrac{\alpha}{1+\alpha}w,z',t\Big)\\
&+ \tfrac{1}{2N(1-\alpha)}\int dz\int dz'\int_{\frac{1+\alpha}{1-\alpha}w}^{\infty} dw'\Big[1-r(z-z')\Big]P\Big(\tfrac{w}{1-\alpha},z,t\Big)P\Big(w'-\tfrac{\alpha}{1-\alpha}w,z',t\Big)\\
&+ \tfrac{1}{2N(1-\alpha)}\int dz\int dz'\int_{0}^{\frac{1-\alpha}{1+\alpha}w} dw'\Big[1+r(z-z')\Big]P\Big(w-\tfrac{\alpha}{1-\alpha}w',z,t\Big)P\Big(\tfrac{w'}{1-\alpha},z',t\Big)\\
&+ \tfrac{1}{2N(1+\alpha)}\int dz\int dz'\int_{0}^{\frac{1+\alpha}{1-\alpha}w} dw'\Big[1-r(z-z')\Big]P\Big(w+\tfrac{\alpha}{1+\alpha}w',z,t\Big)P\Big(\tfrac{w'}{1+\alpha},z',t\Big)
\end{aligned}
$$

and the right-hand side of (28) as

$$
\begin{aligned}
S_1(w,z,t;\alpha,q,r) = \int dz\, z\Bigg\{& - P(w,z,t) \qquad (30)\\
&+ \tfrac{1}{2N(1+\alpha)}\int dz'\int_{\frac{1-\alpha}{1+\alpha}w}^{\infty} dw'\Big[1+r(z-z')\Big]P\Big(\tfrac{w}{1+\alpha},z,t\Big)P\Big(w'+\tfrac{\alpha}{1+\alpha}w,z',t\Big)\\
&+ \tfrac{1}{2N(1-\alpha)}\int dz'\int_{\frac{1+\alpha}{1-\alpha}w}^{\infty} dw'\Big[1-r(z-z')\Big]P\Big(\tfrac{w}{1-\alpha},z,t\Big)P\Big(w'-\tfrac{\alpha}{1-\alpha}w,z',t\Big)\\
&+ \tfrac{1}{2N(1-\alpha)}\int dz'\int_{0}^{\frac{1-\alpha}{1+\alpha}w} dw'\Big[1+r(z-z')\Big]P\Big(w-\tfrac{\alpha}{1-\alpha}w',z,t\Big)P\Big(\tfrac{w'}{1-\alpha},z',t\Big)\\
&+ \tfrac{1}{2N(1+\alpha)}\int dz'\int_{0}^{\frac{1+\alpha}{1-\alpha}w} dw'\Big[1-r(z-z')\Big]P\Big(w+\tfrac{\alpha}{1+\alpha}w',z,t\Big)P\Big(\tfrac{w'}{1+\alpha},z',t\Big)\Bigg\}.
\end{aligned}
$$

# 5 Deriving the Fokker-Planck Equation in the Presence of the BAA

Similar to the derivation of the Fokker-Planck equation for the YSM in Section 3.2, we now Taylor expand $S_0(w,z,t;\alpha,r)$ and $S_1(w,z,t;\alpha,r)$ in $r$ and $\alpha$, treating those two quantities as of the same order. For simplicity, we suppress notation of $t$. The zeroth and first-order derivatives can be verified to vanish. At second order, we find

$$
\frac{\partial^2 S_0}{\partial r^2}(w,z;0,0) = 0 \qquad (31)
$$

$$
\frac{\partial^2 S_0}{\partial r\partial\alpha}(w,z;0,0) = \frac{1}{2}\frac{\partial}{\partial w}\left[\frac{\Xi}{N}\Big(\Lambda + w\Theta\Big)P - \left(wA + \frac{W}{N}L\right)Z\right] \qquad (32)
$$

$$
\frac{\partial^2 S_0}{\partial\alpha^2}(w,z;0,0) = \frac{1}{2}\frac{\partial^2}{\partial w^2}\left[\Big(w^2A + 2B\Big)P\right] \qquad (33)
$$

and

$$\frac{\partial^2 S_1}{\partial r^2}(w,z;0,0) = 0 \tag{34}$$

$$\frac{\partial^2 S_1}{\partial r \partial \alpha}(w,z;0,0) = \frac{1}{2}\frac{\partial}{\partial w}\left[\frac{\Xi}{N}\Big(\Lambda + w\Theta\Big)Z - \left(wA + \frac{W}{N}L\right)\int dz\; z^2 P(w,z)\right] \tag{35}$$

$$\frac{\partial^2 S_1}{\partial \alpha^2}(w,z;0,0) = \frac{1}{2}\frac{\partial^2}{\partial w^2}\left[\Big(w^2 A + 2B\Big)Z\right] \tag{36}$$

where we have defined the auxiliary quantities

$$\begin{aligned} L(w) &:= \frac{1}{W}\int_0^w dx\; P(x)x \\ \Xi &:= \int_0^\infty dx\; Z(x) \\ \Theta(w) &:= \frac{1}{\Xi}\int_w^\infty dx\; Z(x) \\ \Lambda(w) &:= \frac{1}{\Xi}\int_0^w dx\; Z(x)x\,. \end{aligned}$$

Now, consider the variance $\sigma^2$ of $P$ where

$$\sigma^2 = \int dz\; P(z)(z-Z)^2\,.$$

It follows that

$$\begin{aligned} \int dz\; P(z)z^2 &= \int dz\; P(z)\big[Z + (z-Z)\big]^2 \\ &= \int dz\; P(z)Z^2 + 2\int dz\; P(z)Z(z-Z) + \int dz\; P(z)(z-Z)^2 \\ &= 2Z\int dz\; P(z)z - Z^2\int dz\; P(z) + \sigma^2 \\ &= Z^2 + \sigma^2\,. \end{aligned}$$

We make the assumption that $P$ is a narrow distribution, based on the idea that the differences in trading behavior of agents are slight. Thus, $P$ has a low variance, and the above accordingly yields the approximation

$$\int dz\; P(w,z)z^2 \cong [Z(w)]^2\,. \tag{37}$$

We now reassemble the expansions of $S_0$ and $S_1$ into a Fokker-Planck equation, employing Eq. (37), giving us

$$\frac{\partial P}{\partial t} = \frac{\partial^2}{\partial w^2}\left[\gamma\left(\frac{w^2}{2}A + B\right)P\right] + \frac{\partial}{\partial w}\left\{\sigma\left[\frac{\Xi}{N}\Big(\Lambda + w\Theta\Big)P - \left(wA + \frac{W}{N}L\right)Z\right]\right\} \tag{38}$$

$$\frac{\partial Z}{\partial t} = \frac{\partial^2}{\partial w^2}\left[\gamma\left(\frac{w^2}{2}A + B\right)Z\right] + \frac{\partial}{\partial w}\left\{\sigma\left[\frac{\Xi}{N}\Big(\Lambda + w\Theta\Big)Z - \left(wA + \frac{W}{N}L\right)Z^2\right]\right\} \tag{39}$$

where we have defined the constants $\gamma$ and $\sigma$ to quantify the overall transaction rate and the BAA, respectively.

## 6 Reintroducing Redistribution and the WAA

The WAA term as derived by Boghosian [14] in earlier work is given by

$$\frac{\partial}{\partial w}\left\{\zeta\left[\left(B(w)-\frac{w^2}{2}A(w)\right)+\frac{W}{N}w\left(\frac{1}{2}-L(w)\right)\right]P(w)\right\}. \tag{40}$$

This is straightforward to derive from (18) using the distribution

$$\begin{aligned}\eta(\overline{w},\overline{w}',z,z',\beta) &= \frac{1+q(\overline{w}-\overline{w}')+r(z-z')}{2}\delta(\beta-\alpha)\\ &\quad+\frac{1-q(\overline{w}-\overline{w}')-r(z-z')}{2}\delta(\beta+\alpha)\end{aligned} \tag{41}$$

and following the procedures of Sections 2 and 3. Master versions of the Boltzmann Eqs. (30) and (31) with both the WAA and BAA terms are given in full in Appendix C.

Now, in order to reintroduce the redistribution term and the WAA term into the set of coupled Fokker-Planck equations, we must take their zeroth and first $z$ moments as follows. The zeroth moments are straightforward to compute (they are identical to their representations in the absence of the BAA). The first $z$ moment of the redistribution term is

$$\begin{aligned}\int dz\,\frac{\partial}{\partial w}\left[\tau\left(\frac{W}{N}-w\right)P(w,z)\right]z &= \frac{\partial}{\partial w}\int dz\,\left[\tau\left(\frac{W}{N}-w\right)P(w,z)\right]z\\ &= \frac{\partial}{\partial w}\left[\tau\left(\frac{W}{N}-w\right)Z(w)\right]\end{aligned} \tag{42}$$

Taking similar steps, we find the first $z$ moment for the WAA to be

$$\frac{\partial}{\partial w}\left\{\zeta\left[\left(B(w)-\frac{w^2}{2}A(w)\right)+\frac{W}{N}w\left(\frac{1}{2}-L(w)\right)\right]Z(w)\right\}. \tag{43}$$

Thus, our full coupled Fokker-Planck equations, with $\tau$, $\gamma$, $\zeta$ and $\sigma$ quantifying the rate of redistribution, rate of transactions, the WAA and the BAA, respectively, are

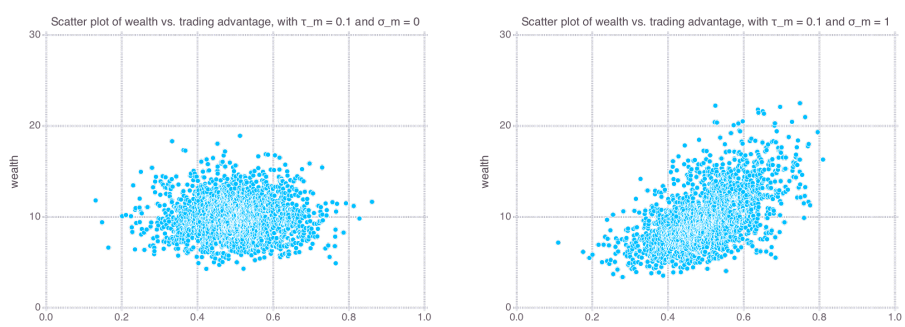
$$\frac{\partial P(w)}{\partial t} = \frac{\partial^2}{\partial w^2}\left[\gamma\left(\frac{w^2}{2}A(w)+B(w)\right)P(w)\right] - \frac{\partial}{\partial w}\left[\tau\left(\frac{W}{N}-w\right)P(w)\right] \tag{44}$$
$$+\frac{\partial}{\partial w}\left\{\zeta\left[\left(B(w)-\frac{w^2}{2}A(w)\right)+\frac{W}{N}w\left(\frac{1}{2}-L(w)\right)\right]P(w)\right\}$$
$$+\frac{\partial}{\partial w}\left\{\sigma\left[\frac{\Xi}{N}\Big(\Lambda(w)+w\Theta(w)\Big)P(w)-\left(wA(w)+\frac{W}{N}L(w)\right)Z(w)\right]\right\}$$

$$\frac{\partial Z(w)}{\partial t} = \frac{\partial^2}{\partial w^2}\left[\gamma\left(\frac{w^2}{2}A(w)+B(w)\right)Z\right] - \frac{\partial}{\partial w}\left[\tau\left(\frac{W}{N}-w\right)Z(w)\right] \tag{45}$$
$$+\frac{\partial}{\partial w}\left\{\zeta\left[\left(B(w)-\frac{w^2}{2}A(w)\right)+\frac{W}{N}w\left(\frac{1}{2}-L(w)\right)\right]Z(w)\right\}$$
$$+\frac{\partial}{\partial w}\left\{\sigma\left[\frac{\Xi}{N}\Big(\Lambda(w)+w\Theta(w)\Big)Z(w)-\left(wA(w)+\frac{W}{N}L(w)\right)Z(w)^2\right]\right\}.$$

# 7 Results

To simulate the effects of the BAA on the dynamics of wealth, I used a Monte-Carlo method to generate wealth distributions over a system of 2000 agents, where each agent is given an initial wealth $W/N = 10$. I assume trading advantage to be distributed like a Gaussian, and thus imbue each agent with a random value of $z$ pulled from a normal distribution under an affine transformation such that it is centered at 0.5 and has support on the unit interval.

**Figure 4: Scatter plots of trading advantage vs. wealth** in the absence (left plot with $\sigma_m = 0$) and presence (right plot with $\sigma_m = 1$) of the $z$ bias. Data was generated using Monte-Carlo simulations of 100,000 transactions in a system of 2000 agents with $W/N = 10$ and $\tau_m = 0.1$.

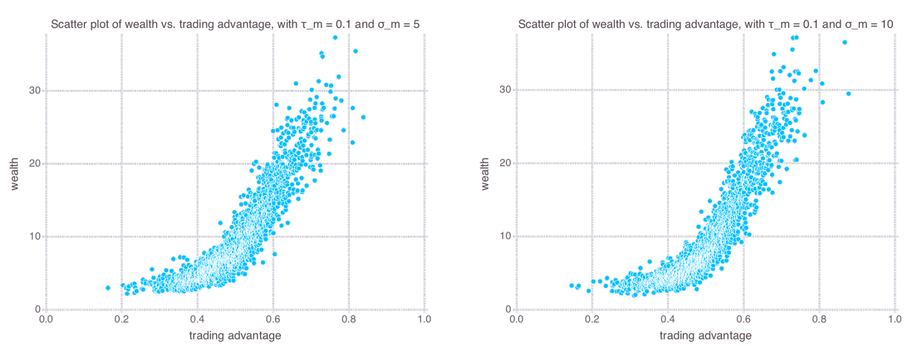


The first and perhaps most obvious way to explore the effects of the BAA is in the form of a scatter plot of the system, where the wealth of an agent is plotted against its trading advantage. Fig. 4 then offers an immediate qualitative realization of the BAA, where the left plot is a control population without any trading bias, and the right plot is generated in the presence of a moderate BAA normalized to the width of the $z$-distribution. Both plots were created by running the Monte-Carlo script for 100,000 transactions, with the amount traded in a given transaction bounded above by 10% of the poorer agent's wealth (in accord with the YSM). The parameters $\tau_m$ and $\sigma_m$ represent redistribution and the strength of the BAA, though the $m$ subscripts specify them to be parameters specifically of the Monte-Carlo script and they are indeed uncalibrated with the $\tau$ and $\sigma$ of the Fokker-Planck equations in Section 6.

In order to explore the effects of the bias and to observe its behavior, we amplify the effect of the bias. Fig. 5 demonstrates more severe results, where the left plot is for a strong bias at $\sigma_m = 5$, and the right plot is for a *very* strong bias with $\sigma_m = 10$ where agents are all but guaranteed to win a transaction in which they have the dominant $z$. The results are stark and demonstrate the extremes of the BAA. The height of the cluster of poorer agents at the bottom of both plots is maintained by the magnitude of $\tau_m$, set to 0.1, corresponding to a 10% transaction tax.

For a more qualitative inspection of the BAA, consider the cumulative distribution of wealth against the cumulative distribution of agents with respect to trading advantage. The shape of the resulting curve may accordingly be titled a *z-Lorenz curve*, where the fraction of agents with trading advantage in the interval $[z_{\min}, z]$ is plotted on the abscissa, against the fraction of wealth of said agents plotted on the ordinate. A corresponding measure of inequality is then the *z-Gini coefficient*, defined naturally as the fraction of the triangular

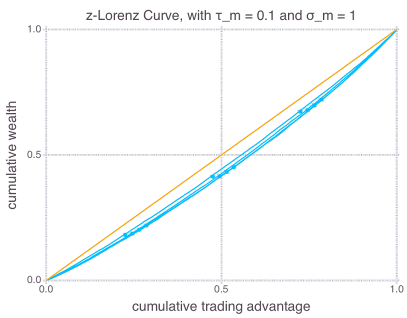
**Figure 5: Scatter plots of trading advantage vs. wealth** in the presence of a strong BAA (left plot with $\sigma_m = 5$) and a BAA in which the agent with higher trading advantage is all but guaranteed to win (right plot with $\sigma_m = 10$). Data was generated using Monte-Carlo simulations of 100,000 transactions in a system of 2000 agents with $W/N = 10$ and $\tau_m = 0.1$.



area beneath the diagonal represented by the region between the diagonal and $z$-Lorenz curve.

Plotted in Fig. 6 is a $z$-Lorenz curve for $\tau_m = 0.1$, in the presence of the BAA; $\sigma_m = 1$. The Monte-Carlo code was truncated after 80,000 transactions at which point the $z$-Gini had reached its asymptotic limit. Fig. 6 corresponds to the right scatter plot of Fig. 4.

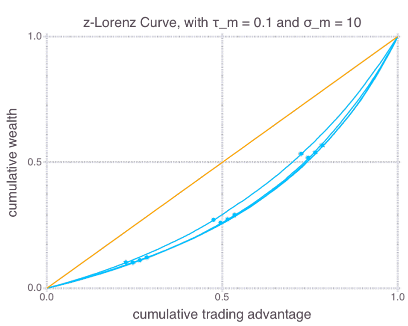
**Figure 6: $z$-Lorenz curve and Table of $z$-Gini** plotted for four different levels of transaction, with $\tau_m = 0.1$, $\sigma_m = 1$ and 95% confidence intervals. The distribution reaches a steady-state in terms of $z$-Gini in 80,000 transactions, for 2000 agents.




| Transactions | $z$-Gini |
|---|---|
| 20,000 | 0.0786 |
| 40,000 | 0.107 |
| 60,000 | 0.111 |
| 80,000 | 0.118 |

Once again, in order to understand the limits and potential of the BAA, we extremize it, setting $\sigma_m = 10$ so that in any given transaction, the agent with higher trading advantage is all but guaranteed to win. The results are plotted in Fig. 7. The obvious and predictable result is the higher degree of wealth inequality, given by a larger $z$-Gini coefficient. The subtler but no less interesting result is the acceleration of the $z$-Gini toward its asymptotic limit.

**Figure 7: $z$-Lorenz curve and Table of $z$-Gini** plotted for four different levels of transaction, with 95% confidence intervals. $\tau_m = 0.1$ and a strong bias where $\sigma_m = 10$. The distribution reaches a steady-state in terms of $z$-Gini in 60,000 transactions, for a system of 2000 agents.




| Transactions | $z$-Gini |
|---|---|
| 20,000 | 0.279 |
| 40,000 | 0.325 |
| 60,000 | 0.337 |
| 80,000 | 0.337 |

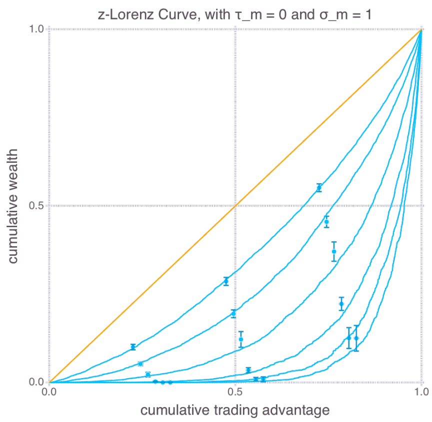
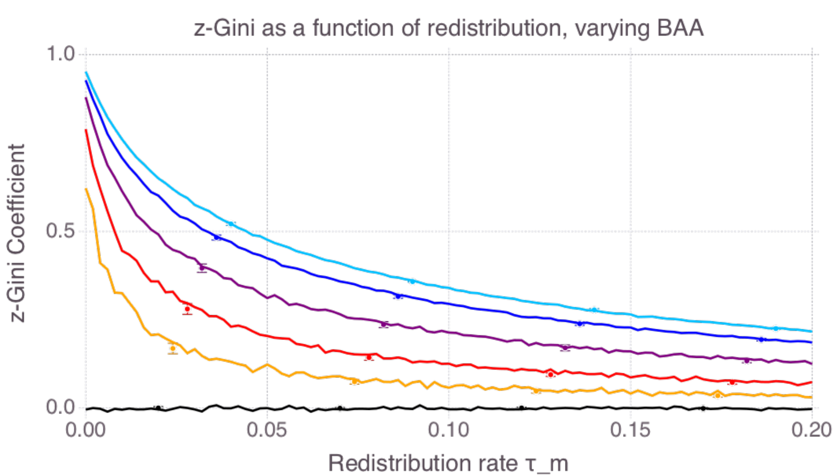



**Figure 8: $z$-Lorenz curve in the absence of tax** and the presence of BAA, with 95% confidence intervals for a system of 2000 agents. The orange diagonal is the initial distribution of wealth. The leftmost blue curve required 50,000 transactions. Each successive curve to the right required twice the transactions of the former; the rightmost curve used 1.6 million transactions. This numerically supports the claim made below that the $z$-Gini tends towards 1 in the absence of redistribution.

To reiterate one of the important conclusions of the $H$-theorem for the Boltzmann and Fokker-Planck equations, redistribution is critical in establishing a steady-state of the wealth distribution, and it thus dictates the upper bound on the asymptotic Gini coefficient. This must also be true of the $z$-Gini, suggested numerically by Fig. 8, in which the curves diverging from the diagonal correspond to higher and higher numbers of transactions. I leave this open as a conjecture for future analysis; in the absence of redistribution and the presence of the BAA, the asymptotic limit of the $z$-Gini must be 1.

**Figure 9: Asymptotic $z$-Gini as a function of $\tau_m$**, with a parametric dependence on the BAA for a system of 2000 agents. The $\sigma_m$ values corresponding with the various plots are as follows: teal = 10, blue = 4, purple = 2, red = 1, orange = 0.5, and black = 0, with 95% confidence intervals. Note that the asymptotics are approximate, and for $\tau_m \approx 0$, the upper 5 curves should more accurately tend toward 1.



Now, to further investigate the relationship of redistribution and the asymptotic $z$-Gini, consider Fig. 9 in which $z$-Gini is plotted as a function of $\tau_m$ with a parametric dependence on the BAA. All curves were generated by the Monte-Carlo script, run with a tolerance to detect the asymptotic limit, or run for an upper limit of 500,000 transactions if this tolerance was not met. For this reason, all curves are not the true asymptotic limit of $z$-Gini, but rather an approximation. Indeed, by the above prediction, all curves should intersect in the upper left corner as $z$-Gini tends toward 1 without redistribution. The $\sigma_m$ values corresponding with the various curves are as follows: teal = 10, blue = 4, purple = 2, red = 1, orange = 0.5, and black = 0.

# 8 Conclusions

The goal of this work was to quantify the BAA and to carefully consider its effects on the dynamics of wealth within the kinetic framework laid out by Boghosian. More generally, this work was intended to serve the larger goal of developing this model toward a more realistic representation of distributions of wealth.

To these ends, I was able to rederive a pair of coupled Boltzmann equations and the corresponding Fokker-Planck equation in the presence of the BAA. I numerically simulated its effects on wealth dynamics through a Monte-Carlo algorithm. I showed the influence of the BAA by demonstrating its resulting deviation from a uniform distribution of wealth. I considered these results in the proposed $z$-Lorenz curve and conjectured that in the absence of redistribution, the corresponding $z$-Gini coefficient must tend toward 1. Lastly, I explored the parametric relationship between redistribution and $z$-Gini coefficient, and showed that the BAA is a concentrating force on the distribution of wealth.

The kinetic model of the YSM expounded in this work is still in its nascent stage. The prior introduction of the WAA, and the introduction of the BAA presented above represent important steps toward a quantitative model of wealth distribution, and are in support of real macroeconomic observations. I believe that this model and, more generally, that this approach borrowed from statistical physics founded on underlying principles of the way in which people trade, will play an influential role in our future understanding of wealth. Indeed, it may only be through this lens that Pareto's hopes of understanding the forces that shape the distribution of wealth may be met, and that his macroeconomic observations may be reconciled with the microfoundations that support them.

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# A Conservation of Agents

To confirm that the YSM with BAA conserves agents, first we recall our definition of the $\eta$ distribution where

$$\int d\beta\ \eta(\overline{w}, \overline{w}', z, z', \beta) = 1\,.$$

We then integrate Eq. (23) with respect to both $w$ and $z$,

$$\begin{aligned}
\frac{dN}{dt} &= \frac{1}{N}\int dz \int_0^\infty dw \int dz' \int_0^\infty dw' \int d\beta\ \eta(\overline{w}, \overline{w}', z, z', \beta)[P(\overline{w}, z, t)P(\overline{w}', z', t)J(w, w', t) \\
&\quad - P(w, z, t)P(w', z', t)] \\
&= \frac{1}{N}\int dz \int_0^\infty d\overline{w} \int dz' \int_0^\infty d\overline{w}' \int d\beta\ \eta(\overline{w}, \overline{w}', z, z', \beta)P(\overline{w}, z, t)P(\overline{w}', z', t) \\
&\quad - \frac{1}{N}\int dz \int_0^\infty dw \int dz' \int_0^\infty dw' \int d\beta\ \eta(\overline{w}, \overline{w}', z, z', \beta)P(w, z, t)P(w', z', t) \\
&= \frac{1}{N}\int dz \int_0^\infty d\overline{w} \int dz' \int_0^\infty d\overline{w}'\ P(\overline{w}, z, t)P(\overline{w}', z', t) \\
&\quad - \frac{1}{N}\int dz \int_0^\infty dw \int dz' \int_0^\infty dw'\ P(w, z, t)P(w', z', t) \\
&= \frac{1}{N}\left(\int_0^\infty d\overline{w} \int dz\ P(\overline{w}, z, t)\right)\left(\int_0^\infty d\overline{w}' \int dz'\ P(\overline{w}', z', t)\right) \\
&\quad - \frac{1}{N}\left(\int_0^\infty dw \int dz\ P(w, z, t)\right)\left(\int_0^\infty dw' \int dz'\ P(w', z', t)\right) \\
&= \frac{1}{N}\left(\int_0^\infty d\overline{w}\ P(\overline{w}, t)\right)\left(\int_0^\infty d\overline{w}'\ P(\overline{w}', t)\right) \\
&\quad - \frac{1}{N}\left(\int_0^\infty dw\ P(w, t)\right)\left(\int_0^\infty dw'\ P(w', t)\right) \\
&= \frac{N}{N}\int_0^\infty d\overline{w}'\ P(\overline{w}', t) - \frac{N}{N}\int_0^\infty dw'\ P(w', t) \\
&= N - N \\
&= 0\,,
\end{aligned}$$

demonstrating that agents are conserved by the Boltzmann equation.

# B Conservation of Wealth

To show that the Boltzmann equation conserves wealth, first recall that for the distribution $\eta$ we demanded that

$$\int d\beta\ \eta(\overline{w}, \overline{w}', z, z', \beta) = 1$$

and that

$$\int d\beta\ \eta(\overline{w}, \overline{w}', z, z', \beta)\, \beta = R(z, z')\,.$$

We multiply Eq. (23) by $w$ and then integrate with respect to both $w$ and $z$,

$$\begin{aligned}
\frac{dW}{dt} &= \frac{1}{N}\int dz \int_0^\infty dw \int dz' \int_0^\infty dw' \int d\beta\; \eta(\overline{w},\overline{w}',z,z',\beta)[P(\overline{w},z,t)P(\overline{w}',z',t)J(w,w',t) \\
&\quad - P(w,z,t)P(w',z',t)]w \\
&= \frac{1}{N}\int dz \int_0^\infty dw \int dz' \int_0^\infty dw' \int d\beta\; \eta(\overline{w},\overline{w}',z,z',\beta)P(\overline{w},z,t)P(\overline{w}',z',t)J(w,w',t)w \\
&\quad - \frac{1}{N}\int dz \int_0^\infty dw \int dz' \int_0^\infty dw' \int d\beta\; \eta(\overline{w},\overline{w}',z,z',\beta)P(w,z,t)P(w',z',t)w \\
&= \frac{1}{N}\int dz \int_0^\infty d\overline{w} \int dz' \int_0^\infty d\overline{w}' \int d\beta\; \eta(\overline{w},\overline{w}',z,z',\beta)P(\overline{w},z,t)P(\overline{w}',z',t)(\overline{w}+\beta\min(\overline{w},\overline{w}')) \\
&\quad - \frac{1}{N}\int dz \int_0^\infty dw \int dz' \int_0^\infty dw' \int d\beta\; \eta(\overline{w},\overline{w}',z,z',\beta)P(w,z,t)P(w',z',t)w \\
&= \frac{1}{N}\int dz \int_0^\infty d\overline{w} \int dz' \int_0^\infty d\overline{w}'\; P(\overline{w},z,t)P(\overline{w}',z',t)\overline{w} \\
&\quad + \frac{1}{N}\int dz \int_0^\infty d\overline{w} \int dz' \int_0^\infty d\overline{w}'\; P(\overline{w},z,t)P(\overline{w}',z',t)\min(\overline{w},\overline{w}')R(z,z') \\
&\quad - \frac{1}{N}\int dz \int_0^\infty dw \int dz' \int_0^\infty dw'\; P(w,z,t)P(w',z',t)w \\
&= \frac{1}{N}\left(\int_0^\infty d\overline{w} \int dz\; P(\overline{w},z,t)\overline{w}\right)\left(\int_0^\infty d\overline{w}' \int dz'\; P(\overline{w}',z',t)\right) \\
&\quad - \frac{1}{N}\left(\int_0^\infty dw \int dz\; P(w,z,t)w\right)\left(\int_0^\infty dw' \int dz'\; P(w',z',t)\right) \\
&\quad + \frac{1}{N}\int dz \int_0^\infty d\overline{w} \int dz' \int_0^\infty d\overline{w}'\; P(\overline{w},z,t)P(\overline{w}',z',t)\min(\overline{w},\overline{w}')R(z,z') \\
&= W\frac{N}{N} - W\frac{N}{N} \\
&\quad + \frac{1}{N}\int dz \int_0^\infty d\overline{w} \int dz' \int_0^\infty d\overline{w}'\; P(\overline{w},z,t)P(\overline{w}',z',t)\min(\overline{w},\overline{w}')R(z,z') \\
&= \frac{1}{N}\int dz \int_0^\infty d\overline{w} \int dz' \int_0^\infty d\overline{w}'\; P(\overline{w},z,t)P(\overline{w}',z',t)\min(\overline{w},\overline{w}')R(z,z') \\
&= 0\,.
\end{aligned}$$

where the last steps follows from the assumption we made on $R$ to be odd under simultaneous interchange of $\overline{w}$ with $\overline{w}'$ and of $z$ with $z'$, while the rest of the integrals remains symmetric under these interchanges, thus demonstrating conservation of wealth.

## C Conservation of Trading Advantage

To show that the YSM conserves trading advantage, we first multiply Eq. (23) by $z$ and then integrate with respect to both $w$ and $z$,

$$
\begin{aligned}
\frac{d\Xi}{dt} &= \frac{1}{N}\int_0^\infty dw \int dz \int_0^\infty dw' \int dz' \int d\beta\; \eta(\overline{w},\overline{w}',z,z',\beta)\big[P(\overline{w},z,t)P(\overline{w}',z',t)J(w,w',t) \\
&\quad - P(w,z,t)P(w',z',t)\big]\, z \\
&= \frac{1}{N}\int_0^\infty d\overline{w} \int dz \int_0^\infty d\overline{w}' \int dz' \int d\beta\; \eta(\overline{w},\overline{w}',z,z',\beta)P(\overline{w},z,t)P(\overline{w}',z',t)\, z \\
&\quad - \frac{1}{N}\int_0^\infty dw \int dz \int_0^\infty dw' \int dz' \int d\beta\; \eta(\overline{w},\overline{w}',z,z',\beta)P(w,z,t)P(w',z',t)\, z \\
&= \frac{1}{N}\int_0^\infty d\overline{w} \int dz \int_0^\infty d\overline{w}' \int dz'\; P(\overline{w},z,t)P(\overline{w}',z',t)\, z \left(\int d\beta\; \eta(\overline{w},\overline{w}',z,z',\beta)\right) \\
&\quad - \frac{1}{N}\int_0^\infty dw \int dz \int_0^\infty dw' \int dz'\; P(w,z,t)P(w',z',t)\, z \left(\int d\beta\; \eta(\overline{w},\overline{w}',z,z',\beta)\right) \\
&= \frac{1}{N}\left(\int_0^\infty d\overline{w} \int dz\; P(\overline{w},z,t)\, z\right)\left(\int_0^\infty d\overline{w}' \int dz'\; P(\overline{w}',z',t\right) \\
&\quad - \frac{1}{N}\left(\int_0^\infty dw \int dz\; P(w,z,t)\, z\right)\left(\int_0^\infty dw' \int dz' P(w',z',t)\right) \\
&= \frac{N}{N}\left(\int_0^\infty d\overline{w} \int dz\; P(\overline{w},z,t)\, z\right) - \frac{N}{N}\left(\int_0^\infty dw \int dz\; P(w,z,t)\, z\right) \\
&= \int_0^\infty d\overline{w}\; Z(\overline{w},t) - \int_0^\infty dw\; Z(w,t) \\
&= \Xi - \Xi \\
&= \;= 0\,,
\end{aligned}
$$

demonstrating that trading advantage is conserved by the Boltzmann equation.

## D Master Boltzmann Equations Including the WAA and BAA Terms

For $\eta$ defined in (25), we derive the following pair of coupled Boltzmann equations

$$
\frac{\partial P(w,t)}{\partial t} = \int dz\; \frac{\partial P(w,z,t)}{\partial t} = S_0(w,z,t;\alpha,q,r) \tag{46}
$$

$$
\frac{\partial Z(w,t)}{\partial t} = \int dz\; z\frac{\partial P(w,z,t)}{\partial t} = S_1(w,z,t;\alpha,q,r) \tag{47}
$$

where we define the right-hand side of (46) as

$$
\begin{aligned}
&S_0(w,z,t;\alpha,q,r) = -P(w,t) \qquad (48)\\
&+ \tfrac{1}{2N(1+\alpha)} \int dz \int dz' \int_{\frac{1-\alpha}{1+\alpha}w}^{\infty} dw' \Big[1 + r(z-z') + q\left(\tfrac{1-\alpha}{1+\alpha}w - w'\right)\Big] P\Big(\tfrac{w}{1+\alpha}, z, t\Big) P\Big(w' + \tfrac{\alpha}{1+\alpha}w, z', t\Big)\\
&+ \tfrac{1}{2N(1-\alpha)} \int dz \int dz' \int_{\frac{1+\alpha}{1-\alpha}w}^{\infty} dw' \Big[1 - r(z-z') - q\left(\tfrac{1+\alpha}{1-\alpha}w - w'\right)\Big] P\Big(\tfrac{w}{1-\alpha}, z, t\Big) P\Big(w' - \tfrac{\alpha}{1-\alpha}w, z', t\Big)\\
&+ \tfrac{1}{2N(1-\alpha)} \int dz \int dz' \int_0^{\frac{1-\alpha}{1+\alpha}w} dw' \Big[1 + r(z-z') + q\left(w - \tfrac{1+\alpha}{1-\alpha}w'\right)\Big] P\Big(w - \tfrac{\alpha}{1-\alpha}w', z, t\Big) P\Big(\tfrac{w'}{1-\alpha}, z', t\Big)\\
&+ \tfrac{1}{2N(1+\alpha)} \int dz \int dz' \int_0^{\frac{1+\alpha}{1-\alpha}w} dw' \Big[1 - r(z-z') - q\left(w - \tfrac{1-\alpha}{1+\alpha}w'\right)\Big] P\Big(w + \tfrac{\alpha}{1+\alpha}w', z, t\Big) P\Big(\tfrac{w'}{1+\alpha}, z', t\Big)
\end{aligned}
$$

and the right-hand side of (47) as

$$
\begin{aligned}
&S_1(w,z,t;\alpha,q,r) = \int dz\, z\Big[ - P(w,z,t) \qquad (49)\\
&+ \tfrac{1}{2N(1+\alpha)} \int dz' \int_{\frac{1-\alpha}{1+\alpha}w}^{\infty} dw' \Big[1 + r(z-z') + q\left(\tfrac{1-\alpha}{1+\alpha}w - w'\right)\Big] P\Big(\tfrac{w}{1+\alpha}, z, t\Big) P\Big(w' + \tfrac{\alpha}{1+\alpha}w, z', t\Big)\\
&+ \tfrac{1}{2N(1-\alpha)} \int dz' \int_{\frac{1+\alpha}{1-\alpha}w}^{\infty} dw' \Big[1 - r(z-z') - q\left(\tfrac{1+\alpha}{1-\alpha}w - w'\right)\Big] P\Big(\tfrac{w}{1-\alpha}, z, t\Big) P\Big(w' - \tfrac{\alpha}{1-\alpha}w, z', t\Big)\\
&+ \tfrac{1}{2N(1-\alpha)} \int dz' \int_0^{\frac{1-\alpha}{1+\alpha}w} dw' \Big[1 + r(z-z') + q\left(w - \tfrac{1+\alpha}{1-\alpha}w'\right)\Big] P\Big(w - \tfrac{\alpha}{1-\alpha}w', z, t\Big) P\Big(\tfrac{w'}{1-\alpha}, z', t\Big)\\
&+ \tfrac{1}{2N(1+\alpha)} \int dz' \int_0^{\frac{1+\alpha}{1-\alpha}w} dw' \Big[1 - r(z-z') - q\left(w - \tfrac{1-\alpha}{1+\alpha}w'\right)\Big] P\Big(w + \tfrac{\alpha}{1+\alpha}w', z, t\Big) P\Big(\tfrac{w'}{1+\alpha}, z', t\Big)\Big].
\end{aligned}
$$

Both $S_0$ and $S_1$ verifiably conserve agents and wealth, though the proof of this is omitted as it is very similar to those found in Appendices A and B.