INVERSE PROBLEM FOR THE YARD-SALE MODEL OF WEALTH DISTRIBUTION

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Abstract

Wealth inequality has become one of the most pressing issues throughout the world today. Among the different approaches trying to understand wealth inequality, "Econophysics" applies methods from physics to establish microscopic models from which macroeconomic wealth distributions can be derived. Asset Exchange Models (AEMs) are some of the most successful models of this kind. In this thesis, we study a particular AEM called the Yard-Sale Model (YSM). In Chapter 2, we further extend the basic YSM to include redistribution and Wealth-Attained Advantage (WAA), and we derive a Fokker-Planck equation for the resulting Extended Yard-Sale Model (EYSM). In Chapter 3, we discuss the numerical method we use to solve for steady-state solutions to the EYSM. In Chapter 4, we mainly discuss the existence of a "duality" symmetry between the supercritical and the subcritical solutions to the EYSM. In Chapter 5, we further extend the EYSM so that it allows for agents with negative wealth, which are important empirically. We show how to derive and solve the Fokker-Planck equation for the resulting Affine Wealth Model (AWM) based on the tools and methods we used for EYSM. Finally in Chapter 6, we compare our results with U.S. wealth distribution data from the Survey of Consumer Finances (SCF). We first demonstrate the superiority of the AWM to the other models for its remarkable faithfulness to the empirical data . We also compare our model results from the AWM to U.S. wealth distribution data from 1989 to 2016. We argue that the time series of model parameters thus obtained provides a valuable new diagnostic tool for analyzing wealth inequality.

To my son, Spencer

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Introduction

1.1 Motivation and Background

The success of Thomas Piketty's book "Capital in the Twenty-First Century" [44], and its top placement on the *New York Times* bestseller list is evidence that economic inequality has become one of the most pressing issues throughout the world today. A substantial amount of economic study has been undertaken to explain the dynamics of economic inequality. Although economic inequality might be estimated differently within different contexts*, all the studies seem to converge to one universal conclusion: The level of economic inequality has become extremely high worldwide and it is still increasing rapidly. There exists an urgent need to understand the mechanisms behind economic inequality.

In spite of all this popular interest, the scale of inequality that exists in today's world is still staggering. For example, according to a study conducted by Oxfam International, reported at the World Economic Forum in Davos in 2018: in 2017, the richest 8 individuals in the world (led by Microsoft founder Bill Gates) have a combined total wealth of $426 billion dollars, equivalent to the combined wealth of the poorest half of the world's population [24]. According to Oxfam's past studies on the same topic, this "number of richest individuals who own as much wealth as the poorer half of world's population" was 61 in 2016 and 380 in 2009 [24, 25]. Even more unbelievable is the fact that, even among the richest people in the world, there is still substantial wealth inequality and their wealth continues to condense rapidly to the hands of even fewer people.

Another recent study that has drawn much attention was done cooperatively by

*The three most common metrics for economic inequalities are wealth, income, and consumption. [17] In our research, we focus on wealth inequality.

Michael I. Norton from Harvard Business School and Dan Ariely from Duke University, who focused on the wealth inequality in United States and tried to measure the gap between American people's perception of wealth inequality and the reality of wealth inequality [40]. To achieve their goal, they conducted a survey by asking people to describe in quintiles how much inequality they would expect in an ideal society, as well as how much inequality they believe exists in reality. The result shows that the level of inequality that people think exists in reality is much higher than what they would prefer in an ideal society, yet still far below actual levels of inequality. The result from Norton and Ariely's work means that the level of wealth inequality has been dramatically underestimated by the general population, even though people are aware that reality must be worse than what they consider ideal. There is clearly plenty of room left to improve our collective understanding about inequality.

There is a long history of human's effort to understand inequality. Dating back to the Age of Enlightenment, the famous French philosopher Jean-Jacques Rousseau published his thoughts on this topic in his work *"Discourse on the Origin and Basis of Inequality Among Men (French: Discours sur l'origine et les fondements de l'inégalité parmi les hommes)"*. According to Rousseau, inequality did not exist when humans were in the *"state of nature"* – the hypothesized condition of the lives of people before societies came into existence. Then inequality appeared and developed alongside the development of civil society and the concept of private property. Rousseau also pointed out the importance of wealth inequality compared to other forms of inequalities: Since wealth can have the most direct impact on one's well-being and is also very convenient to exchange or trade with others, all other forms of inequalities tend to finally transfer into inequality of wealth. Thus, from a strictly philosophical point of view, we can find supporting arguments as to why wealth inequality is of special interest to us and why it should be considered a dynamic process.

In the field of economics, some of the earliest and most important quantitative work to measure empirical wealth inequality was done by the Italian sociologist and economist Vilfredo Pareto in the early 20th century [42, 43]. By collecting and studying the data on land ownership in Italy, Pareto found that nearly 80% of the land was owned by only 20% of the population. Similar patterns were also found in nearby European countries. Pareto's finding is now widely known as the *"Pareto principle"* or the *"80/20 rule"*. This principle is consistent with a power-law distribution (the main asymptotic characteristic of a Pareto distribution) of wealth. In other words, the decay of a wealth distribution is proportional to a power function $w^{-1-\alpha}$ with a parameter α, known as the *"Pareto index"*. The Pareto distribution has more recently been widely used to describe different types of observable phenomena beyond wealth distributions. In the meantime, there were people proposing a different model of a wealth distributions almost at the same period of Pareto. In 1931, Robert Gibrat claimed that, rather than a power-law, wealth distribution should follow a log-normal distribution. [20] While both Pareto's and Gibrat's observations can explain part of the empirical data observed in the world, no rules were found governing which law one should apply under which circumstances.

Also in the early 20th century, the mathematical tools and metrics that can be used to analyze inequality were also developed. In 1905, the American economist Max O. Lorenz first introduced the *Lorenz curve* to describe inequality in wealth (or income) [31, 37]. Given a wealth distribution, the Lorenz curve is defined by plotting the cumulative fraction of wealth held versus the cumulative fraction of people holding it. In this way, any distribution would be represented by a concave up curve within the unit square and below the diagonal line. Based on Lorenz's work, the Italian statistician and sociologist Corrado Gini later invented the *Gini coefficient* as a measure of quantifying inequality [21–23]. The Gini coefficient is defined to be the ratio of the area between the Lorenz curve and the diagonal line of the unit square and the total area under the diagonal line of the unit square. The Gini coefficient is a real number between 0 and 1, where a higher value corresponds to a higher level of inequality, and a lower value to the contrary. Owing to its convenience and other advantages, the Gini coefficient has become one of the most widely used measures for inequality since it was first introduced.

In spite of all the work on the topic of wealth distribution and inequality, however, there were few if any attempts to find microeconomic models governing the behavior of individual agents that might underpin a particular macroeconomic theory or phenomenon such wealth distribution. In other words, Macroeconomics lacked *microfoundations* [2, 28]. The subject of economics had been divided into micro- and macro-, and both divisions of the field developed their own models and theories respectively. Most macroeconomic theories employ assumptions and models that cannot be derived or reconciled with those used in microeconomics. A unified foundation between the two is missing.

This discrepancy between the macro- and micro- theories is reminiscent of a similar situation in the field of physics, with a very different ending. Classical thermodynamics was a well-established macroscopic theory that described the states of thermodynamic systems. With the introduction of atomic theory in the early 19th century, however, there was an urgent necessity to interpret macroscopic concepts in classical thermodynamics (such as pressure, temperature, etc.) in terms of microscopic concepts (such as atoms, particles, etc.). This work was successfully carried out by Ludwig E. Boltzmann, Josiah W. Gibbs, James C. Maxwell, and others who together developed the subfield knowns as *statistical mechanics*. In statistical mechanics, microscopic models were developed to describe the movements and collisions of the atomic particles, and the rules of the macroscopic quantities in classical thermodynamics were derivable from those microscopic models. In that way, classical thermodynamics was reconciled with atomic theory.

The above-described success in physics shed light on economics. Although there was a long history of interaction between physicists and economists, the term "Econophysics" was not officially introduced until the mid-1990s by H. Eugene Stanley in a conference on statistical physics, in order to summarize the collective work done by several physicists applying tools and methods of statistical mechanics to analyze financial markets and stock prices, and to explain more general economic phenomena [18]. Since its inauguration, the field of Econophysics has become an active field of study with many exciting findings [19, 47].

Our research lies in this field of Econophysics where we explore idealized microfoundations for the dynamics of wealth distributions. Following essential ideas from previous work in the field, we apply mathematical tools from statistical physics and thereby develop microscopic models for single economic agents, and we study how these models can be used to analyze the dynamics of macroscopic wealth distributions. Among all the microscopic models, we focus on of one specific model called the Asset Exchange Model, which has been proven to be very useful. In this thesis, we will show results from various aspects of our research on this model, both theoretically and empirically. I hope my research can make a humble contribution to the long-lasting human endeavor of understanding inequality.

1.2 Asset Exchange Models

Stochastic agent-based Asset-Exchange Models (AEMs) for wealth distribution were first introduced in 1986 [1] by Angle in the social sciences literature. AEMs are highly idealized but nonetheless very useful models for understanding the statics and dynamics of wealth distributions. Though there are many variants of AEMs, most consider a closed economy involving a fixed number of economic agents, each possessing a certain amount of a resource, which we shall refer to as wealth. Pairs of agents are chosen randomly to engage in binary transactions in which a small amount of wealth w, might move from one agent to the other. In the limits of large populations and long times, the agent density function may be described by a continuous distribution, either classical or singular in nature.

In 1998, Ispolatov, Krapivsky and Redner [27] first applied the methods of kinetic theory to a specific AEM, which was later named the *Theft-and-Fraud* model by Hayes [26], and they derived a Boltzmann equation for the time evolution of the agent density function. In the Theft-and-Fraud model, agents with single property – wealth – repeatedly engage in binary transactions in which they exchange some of their wealth. In a transaction, each agent has equal chance of either winning or losing the transaction, as determined by the flip of a fair coin. Then, a fixed proportion of the wealth of the losing agent is transferred to the winning agent. One may argue this model is unrealistic because it somehow disadvantages the rich. In any transaction, the wealthier agent has to stake more wealth than the poorer agent and therefore the expected income for the wealthier agent is negative.

The model discussed in my thesis is based on another more realistic AEM which was first proposed in 2002 by Chakraborti [12], and was named the *Yard-Sale Model* (YSM) later that same year also by Hayes [26]. This model is almost identical to the Theft-and-Fraud model, but the amount of wealth transferred in each transaction is a fixed proportion of the wealth of the *poorer* of the two agents. This assumption is more realistic than that used in the Theft-and-Fraud model. If we compare it to a game between two gamblers, the bet is usually determined before the game begins, and gamblers are not allowed to stake more than they have. Therefore, a fraction of the wealth of the poorer of the two gamblers would be a reasonable bet on which both gamblers would agree. Numerical simulations indicated that the time-asymptotic state of this model was one of complete *wealth condensation* [8, 10], in which all the wealth falls into the hands of a single agent.

In 2014, inspired by the two studies mentioned above, Boghosian applied the approaches of Ispolatov and his colleagues to the Yard-Sale Model and derived a Boltzmann equation description for its agent density function [4]. Later in the same year, he showed that the Boltzmann equation for the agent density function reduces to a Fokker-Planck equation [16, 33, 45] in the limit of a large number of small transactions – i.e., the *small-transaction limit* [3]. These observations facilitated the study of the properties of these distributions, both analytically and by numerical methods. A year later it was definitively proven that the time-asymptotic limit of the evolution of either of these equations, from any initial condition at all, was the state of complete wealth condensation [7].

In an effort to avert the state of complete wealth condensation and thereby make the YSM more realistic, a model of redistribution was introduced whereby a flat "wealth tax" χ per unit time was imposed on all agents on a per-transaction basis, and redistributed uniformly to the entire population. It was shown that this has the effect of adding an Ornstein-Uhlenbeck [48] term to the Fokker-Planck equation which stabilizes the wealth distribution, so that it remains a classical distribution without wealth condensation [3].

A subsequent extension of the model introduced the idea of Wealth-Attained Advantage (WAA), in order to account for the well documented privileges that wealthier agents enjoy over poorer agents, such as higher returns on investment and lower interest rates on loans [5]. Mathematically, this was accomplished by biasing the coin in favor of the wealthier agent. The amount of the bias was taken to be proportional to the difference in wealth between the richer and poorer agent normalized to the mean wealth, and multiplied by a coefficient ζ. Because the bias is proportional to the wealth difference, it naturally reduces to zero when the transacting agents have equal wealth.

The resulting Extended Yard-Sale Model (EYSM) [5], with redistribution rate χ and WAA parameter ζ, admits much more interesting phenomenology. The agent density function is a classical distribution in the *subcritical* regime defined by $\zeta < \chi$. When $\zeta \geq \chi$, this passes to a partially wealth-condensed *supercritical* regime, characterized by coexistence between a classical distribution of agents with a fraction χ/ζ of the total wealth, and an oligarchy with the remaining $1 - \chi/\zeta$ of the total wealth [5,6]. The EYSM serves as the starting point of my research and a foundation of this thesis.

1.3 Outline of the thesis

The main goal of my research is to try to relate the EYSM with empirical wealth data and thereby solve the inverse problem for the model. Along the way, there have been both theoretical contributions and empirical studies.

In Chapter 2, we shall review the development of the EYSM with redistribution and WAA. I shall introduce all the mathematical symbols used and review the derivation of the Fokker-Planck equation for the agent density function. I shall also discuss the existence of two invariants in the model, and how the steady-state Fokker-Planck equation for the EYSM can be reduced to its *canonical-form* – an ordinary differential equation with only two free parameters, namely the *redistribution coefficient* χ and the *WAA coefficient* ζ.

In Chapter 3, I shall explain in detail the numerical method that we used to solve the canonical-form of the steady-state equation for the EYSM. I shall also discuss the numerical difficulties we encountered in using this numerical method.

In Chapter 4, I shall demonstrate the existence of a remarkable symmetry between the supercritical and subcritical regimes of the steady-state solutions to the EYSM. This symmetry has to do with the replacement $\{\chi \to \zeta,\ \zeta \to \chi\}$, which obviously has the effect of inverting the order parameter ζ/χ and thereby providing a one-to-one correspondence between subcritical and supercritical states. I will show that the wealth distribution of the subcritical state is identical to that of the corresponding supercritical state when the oligarchy is removed from the latter. I will argue that this symmetry is an example of the phenomenon of "duality", which appears in other subfields of the physics of critical phenomena.

Toward the end of relating the model to the real world, one big flaw of the EYSM is that it cannot model agents with negative wealth, which is however widely observed in empirical data. Therefore, in Chapter 5, we introduce a further modification to the EYSM to allow for agents with negative wealth. Realistic models for agent density functions possess invariance properties under scalings of the total number of agents and the total wealth; we accomplish the extension to negative wealth by additionally requiring invariance of the wealth distribution under additive shifts. Because the new model is invariant under both scalings and shifts, we refer to it as the *Affine Wealth Model* (AWM). I will derive the Fokker-Planck equation obeyed by the agent density function for the AWM, and describe its numerical solution for steady-state wealth distributions in the subcritical regime (without an oligarchy). I will also explain how numerical solutions in the supercritical regime (with an oligarchy) can be obtained from their subcritical counterparts by exploiting the above-mentioned duality [36].

Finally, in Chapter 6, we present a detailed comparison of the results of the AWM with empirical wealth data. In particular, we compare the steady-state solutions of the Fokker-Planck equation with data from the United States Survey of Consumer Finances [9] over a time period of 27 years. In doing so, we demonstrate both that (i) each of the extensions that we introduced in the basic AEM resulted in improved agreement, and (ii) of all these models the AWM is the one most faithful to the empirical data by a wide margin. Additionally, we present fitting parameters for the U.S. wealth distribution data as a function of time over a thirty-year period, under the assumption that the wealth distribution responds to changes in those parameters adiabatically. We argue that this time series of model parameters provides a new way to extract useful information about wealth inequality in an economy. As an example, we demonstrate a precisely defined way of quantifying the extent to which the U.S. wealth distribution is partially wealth-condensed.

Chapter 2

The Extended Yard-Sale Model

In this chapter, We introduce the mathematics of the Extended Yard-Sale Model. We begin by explaining the microscopic random walk process that models the transactions between the economic agents of a basic Yard-Sale Model, from which we can derive a deterministic, non-linear, non-local, integrodifferential Fokker-Planck equation. We then discuss extensions of the basic Yard-Sale Model so that it can model redistribution and Wealth-Attained Advantage (WAA). This extended model is called the Extended Yard-Sale Model (EYSM). Finally, we focus on the steady-state solution of the EYSM, introduce the so-called "canonical-form" solution, and discuss its invariance properties.

2.1 The basic Yard-Sale Model

2.1.1 Microscopic random walk process

In the simplest version of the Yard-Sale Model, we consider a closed economy with economic agents who have only a single attribute – the wealth w held by each of them. We assume that w is non-negative for all the agents. By "closed economy", we mean two things: (i) no new agents will join the system nor there will be agents exiting the system, and hence the total number of agents N is conserved; (ii) no extra wealth will be produced or flux into the system nor will wealth be consumed or flux out of the system, and hence the total wealth W is also conserved. Under these assumptions, the dynamics of the wealth distribution of the agents can be described by the evolution of the agents density function $P(w,t)$. The two conserved quantities in this closed economy can be written as the zeroth and the first moments of $P(w,t)$:

$$N = \int_0^\infty P(w,t)\,\mathrm{d}w \tag{2.1}$$

$$W = \int_0^\infty P(w,t)w\,\mathrm{d}w. \tag{2.2}$$

At the microscopic level of description, the agents engage in pairwise transactions with each other in a sequential manner: At each time t, a pair of two agents are randomly selected and some amount of wealth is transferred from one agent to the other, then another pair of two agents will be selected and the process is repeated. At each transaction, the amount of the wealth transferred is a fixed fraction β of the wealth of the *poorer* of the two agents. The direction of the wealth transfer is determined by a flip of a fair coin. Mathematically, at time t, we consider an agent with wealth w transacting with another agent with wealth x, and at time $t+\Delta t$ the wealth of the first agent will become

$$w' = w + \Delta w, \tag{2.3}$$

where the change of the wealth of the first agent is given by

$$\Delta w = \beta \min(w,x)\eta. \tag{2.4}$$

Here η is a scaled and shifted Bernoulli random variable that models the coin flip,

$$\eta = \begin{cases} +1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2}. \end{cases} \tag{2.5}$$

If $\eta = 1$, Δw is positive which means the agent with wealth w won the transaction and wealth $\beta \min(w,x)$ will be transferred to him/her. On the contrary, if $\eta = -1$, Δw becomes negative and the agent with wealth w will lose that amount of wealth. Since the coin flip is fair, the expected value of η equals 0, i.e. $E[\eta] = 0$. It is also obvious that $E[\eta^2] = 1$.

2.1.2 Fokker-Planck equation

To derive a *Fokker-Planck* equation [16, 45], alternatively called *Kolmogorov's Forward equation* [33], for the random walk process described by Eq. (2.4), we first need to take the limit of large amount of small transactions, which we call the *small transaction limit*. Mathematically, it means that let $\beta \to 0$ as $\Delta t \to 0$. In order to obtain a sensible scaling limit as will be shown later, we let $\beta = \sqrt{\gamma \Delta t}$, where Δt is the characteristic time associated with the transaction and $\gamma > 0$ is a parameter which we will refer as the *transaction coefficient*.

We now rewrite Eq. (2.4) that describes the binary transactions and put it together with the expected values of η and η^2 for a fair coin flip. These three equations serve as a summary of the microscopic random walk process that underlines the basic Yard-Sale Model,

$$\Delta w = \sqrt{\gamma \Delta t} \min(w, x)\eta \tag{2.6}$$

$$\begin{cases} E[\eta] = 0 \\ E[\eta^2] = 1. \end{cases} \tag{2.7}$$

Next, we need to go over what is called the Kramers-Moyal expansion of the Chapman-Kolmogorov equation [35, 39]. By taking the limit of $\Delta t \to 0$, the partial derivative of the agent density function $P(w, t)$ with respect to time t can be found by averaging the transition probability described by the Chapman-Kolmogorov equation over the distribution function $P(w, t)$ itself. In specific, for a function $f(\eta, x)$ of the agent with wealth x and the coin flip η, we define its expectation $\mathcal{E}[f]$ as

$$\mathcal{E}[f] = \frac{1}{N} \int_0^\infty P(x, t) E[f(\eta, x)] \, \mathrm{d}x. \tag{2.8}$$

Here, we use the scripted letter $\mathcal{E}$ to distinguished the expectation over the wealth of the transaction partner x and the coin flip η, from the expectation E over the coin flip alone.

The Fokker-Planck equation has the general form

$$\frac{\partial P(w,t)}{\partial t} = -\frac{\partial}{\partial w}\left[\sigma P(w,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial w^2}\left[D P(w,t)\right], \tag{2.9}$$

where σ is called the *drift term*, and D is called the *diffusion term*. In our basic YSM, these two terms can be obtained by calculating the following two limits:

$$\sigma = \lim_{t\to 0}\mathcal{E}\left[\frac{\Delta w}{\Delta t}\right] \tag{2.10}$$

$$D = \lim_{t\to 0}\mathcal{E}\left[\frac{(\Delta w)^2}{\Delta t}\right], \tag{2.11}$$

where $\mathcal{E}[f]$ is defined in Eq. (2.8) and Δw is described in Eqs. (2.6) and (2.7). We can plug in these two equations back into Eq. (2.10) and Eq. (2.11) to get the σ and D in the Fokker-Planck equation. The calculations are as follows:

$$\begin{aligned}\sigma &= \lim_{t\to 0}\mathcal{E}\left[\frac{\Delta w}{\Delta t}\right] \\ &= \lim_{t\to 0}\frac{\sqrt{\gamma}}{N\sqrt{\Delta t}}\int_0^\infty P(x,t)\min(w,x)E[\eta]\,\mathrm{d}x \\ &= 0 \end{aligned} \tag{2.12}$$

and

$$\begin{aligned}D &= \lim_{t\to 0}\mathcal{E}\left[\frac{(\Delta w)^2}{\Delta t}\right] \\ &= \frac{\gamma}{N}\int_0^\infty P(x,t)\min(w,x)^2 E[\eta^2]\,\mathrm{d}x \\ &= \frac{\gamma}{N}\int_0^w P(x,t)x^2\,\mathrm{d}x + \frac{\gamma}{N}\int_w^\infty P(x,t)w^2\,\mathrm{d}x \\ &= 2\gamma\left[B(w,t) + \frac{w^2}{2}A(w,t)\right], \end{aligned} \tag{2.13}$$

where we have defined $A(w,t)$ and $B(w,t)$ as partial moments of $P(w,t)$:

$$A(w,t) = \frac{1}{N}\int_w^\infty P(x,t)\,\mathrm{d}x \tag{2.14}$$

$$B(w,t) = \frac{1}{N}\int_0^w P(x,t)\frac{x^2}{2}\,\mathrm{d}x. \tag{2.15}$$

With Eqs. (2.9), (2.12) and (2.13), the resulting Fokker-Planck equation for the basic Yard-Sale Model is a non-linear, non-local, integrodifferential partial differential equation which may be written

$$\frac{\partial P(w,t)}{\partial t} = \frac{\partial^2}{\partial w^2}\left[\gamma\left(B(w,t) + \frac{w^2}{2}A(w,t)\right)P(w,t)\right]. \tag{2.16}$$

2.2 Modeling redistribution

The basic Yard-Sale Model could be regarded as a model of a complete free-market economy. However, in real economies, governments usually regulate macroeconomics via monetary policy and public finance, taxation and welfare being the most commonly known methods. We can think of the overall effect of such policies as redistributing of the wealth of the society. Therefore, it would be nice to extend the basic Yard-Sale Model to add this feature.

Another reason to include redistribution in the model is that the time-asymptotic solution to the basic Yard-Sale Model is complete wealth condensation [8, 10], which is unrealistic and not observed in real word economies. Wealth condensation will be discussed in more detail later in Chapter 4. It will be shown that adding redistribution to the model stabilizes the distribution to a non-trivial classical distribution without singularity.

We model the redistribution of wealth by applying a flat wealth tax rate χ on every agent, then evenly distributing the total tax thereby collected to every agent. This constant tax rate χ will be called the *redistribution coefficient*. At the microscopic level, at time t and within time interval Δt, an agent with wealth w will pay tax in the amount

$$T(w) = \chi w \Delta t, \tag{2.17}$$

and hence the total tax collected from the entire society will be

$$\int_0^\infty T(w)P(w)\,\mathrm{d}w = \chi\Delta t\int_0^\infty P(w)w\,\mathrm{d}w = \chi W\Delta t. \tag{2.18}$$

The total tax collected $T(w)$ is then evenly distributed to every agent, so each agent receives a share

$$I = \frac{\chi W \Delta t}{N}. \tag{2.19}$$

Therefore, the overall effect of redistribution on a single agent with wealth w in a transaction time Δt is to change their wealth by the amount

$$I - T = \chi \left(\frac{W}{N} - w \right) \Delta t. \tag{2.20}$$

If we add the above term to the random walk process described by Eq. (2.6) and retain Eq. (2.7), the microscopic random walk process that underlies the Extended Yard-Sale Model with redistribution can be summarized as

$$\Delta w = \sqrt{\gamma \Delta t} \min(w, x)\eta + \chi \left(\frac{W}{N} - w \right) \Delta t \tag{2.21}$$

$$\begin{cases} E[\eta] = 0 \\ E[\eta^2] = 1. \end{cases} \tag{2.22}$$

To obtain the new Fokker-Planck equation with redistribution, we first notice that adding redistribution to the random walk process will alter only the drift term. A computation similar to that in Eq. (2.12) results in

$$\begin{aligned} \sigma &= \lim_{\Delta t \to 0} \mathcal{E}\left[\frac{\Delta w}{\Delta t} \right] \\ &= \lim_{t \to 0} \frac{\sqrt{\gamma}}{N\sqrt{\Delta t}} \int_0^\infty P(x,t) \min(w,x) E[\eta] \, \mathrm{d}x \\ &\quad + \lim_{\Delta t \to 0} \frac{1}{N} \int_0^\infty \chi \left(\frac{W}{N} - w \right) P(x,t) \, \mathrm{d}x \\ &= \chi \left(\frac{W}{N} - w \right), \end{aligned} \tag{2.23}$$

The resulting Fokker-Planck equation involves one extra term compared to that of

Eq. (2.16),

$$\frac{\partial P(w,t)}{\partial t} = -\frac{\partial}{\partial w}\left[\chi\left(\frac{W}{N}-w\right)P(w,t)\right] + \frac{\partial^2}{\partial w^2}\left[\gamma\left(B(w,t)+\frac{w^2}{2}A(w,t)\right)P(w,t)\right]. \quad (2.24)$$

We will refer to the new term with the first partial derivative with respect to w on the right of the Eq. (2.24) as the *redistribution term*. For convenience, we also give a name – the *transaction term* – to the second-order partial derivative term that already appeared in Eq. (2.16). It is obvious that Eq. (2.24) reduces to Eq. (2.16) when the redistribution coefficient χ is set to zero.

2.3 Modeling Wealth-Attained Advantage (WAA)

In the definition of η, we have assumed that each transaction is "fair" for both of the agents, meaning each one has the same probability to either win or lose Δw in the pairwise transactions. In reality, however, the game between the rich and the poor is never fair and usually favors the rich. There is much evidence that richer people have an advantage over poorer people in economic activities. This advantage may due to access to better education, health care and legal service, etc. We give the name *Wealth-Attained Advantage (WAA)* to the overall effect of this advantage of the rich over the poor, and we would like to further extend the Yard-Sale Model to include this feature as well.

We model WAA by biasing the coin flip in each transaction. We assume that the WAA gained by the richer agent over the poorer agent in a transaction is proportional to the wealth difference between these two agents. More specifically, if an agent with wealth w transacts with an agent with wealth x, we want the expected value of the random variable η that models the coin flip, instead of being zero, to vary linearly with $w - x$.

With this idea in mind, the microscopic random walk process that underlies the

Extended Yard-Sale Model with both redistribution and WAA is updated to

$$\Delta w = \sqrt{\gamma \Delta t}\min(w,x)\eta + \Delta t \chi\left(\frac{W}{N} - w\right) \tag{2.25}$$

$$\begin{cases} E[\eta] = \zeta\sqrt{\frac{\Delta t}{\gamma}}\left(\frac{w-x}{W/N}\right) \\ E[\eta^2] = 1. \end{cases} \tag{2.26}$$

In Eq. (2.26), we have introduced a new parameter ζ which will be referred as the *WAA coefficient*. The wealth difference is divided by the mean wealth of the society W/N so that it is dimensionless. As in Subsection 2.1.2, the scale factor $\sqrt{\Delta t/\gamma}$ is introduced to achieve the desired scaling limit. Therefore, the larger the wealth difference between the two agents, the greater the advantage the richer agent would gain in the transaction. Only in the case when $w = x$, i.e., the two agents have the same wealth, will $E[\eta]$ reduce to zero so that the transaction becomes fair.

To obtain the Fokker-Planck equation, we again notice that modeling WAA would affect only the drift term of the Fokker-Planck equation, and therefore the Fokker-Planck equation can be updated by recalculating the σ as follows

$$\begin{aligned} \sigma &= \lim_{\Delta t\to 0}\mathcal{E}\left[\frac{\Delta w}{\Delta t}\right] \\ &= \lim_{\Delta t\to 0}\frac{1}{N}\int_0^\infty \chi\left(\frac{W}{N} - w\right)P(x,t)\,\mathrm{d}x \\ &\quad + \lim_{\Delta t\to 0}\zeta\frac{1}{W}\int_0^\infty \min(w,x)(w-x)P(x,t)\,\mathrm{d}x \\ &= \chi\left(\frac{W}{N} - w\right) - \zeta\Bigg[-w\frac{1}{W}\int_0^w xP(x,t)\,\mathrm{d}x + \frac{2N}{W}\frac{1}{N}\int_0^w \frac{x^2}{2}P(x,t)\,\mathrm{d}x \\ &\quad - \frac{2N}{W}\frac{w^2}{2}\frac{1}{N}\int_w^\infty P(x,t)\,\mathrm{d}x + w\frac{1}{W}\int_w^\infty xP(x,t)\,\mathrm{d}x\Bigg] \\ &= \chi\left(\frac{W}{N} - w\right) - \zeta\left[\frac{2N}{W}\left(B(w,t) - \frac{w^2}{2}A(w,t)\right) + w\Big(1 - 2L(w,t)\Big)\right], \end{aligned} \tag{2.27}$$

where we have defined another partial moment of $P(w,t)$ as:

$$L(w,t) = \frac{1}{W}\int_0^w P(x,t)x\,\mathrm{d}x. \tag{2.28}$$

Finally, the Fokker-Planck equation of the Extended Yard-Sale Model with both redistribution and WAA may be written

$$\begin{aligned}\frac{\partial P(w,t)}{\partial t} = & -\frac{\partial}{\partial w}\left[\chi\left(\frac{W}{N}-w\right)P(w,t)\right] \\ & +\frac{\partial}{\partial w}\left\{\zeta\left[\frac{2N}{W}\left(B(w,t)-\frac{w^2}{2}A(w,t)\right)+w\left(1-2L(w,t)\right)\right]P(w,t)\right\} \\ & +\frac{\partial^2}{\partial w^2}\left[\gamma\left(B(w,t)+\frac{w^2}{2}A(w,t)\right)P(w,t)\right], \qquad (2.29)\end{aligned}$$

where the partial moments $A(w,t)$, $B(w,t)$ and $L(w,t)$ were defined in Eq. (2.14), Eq. (2.15) and Eq. (2.28).

In analogy to the transaction term and the redistribution term, we call the newly introduced term the *WAA term*. We also call the partial moments $A(w,t)$, $B(w,t)$ and $L(w,t)$ the *Pareto potentials* of the density function $P(w,t)$.

In summary, Eq. (2.29) is the complete form of the Fokker-Planck equation that models the EYSM with both redistribution and WAA. We have three terms on the right of the equation that are associated with the redistribution, the WAA, and the transaction respectively. We also have three free parameters χ, ζ and γ corresponding to each of these three terms. While γ is always positive by definition, χ and ζ can be set to zero, so the EYSM can be reduced to simpler models with certain feature(s) switched off.

2.4 Steady-state equation and the canonical-form

2.4.1 Steady-state Fokker-Planck equation

The agent density function $P(w,t)$ described by Eq. (2.29) is a function of both w and t; in other words, the wealth distribution changes over time. To obtain a steady-state distribution, we examine its long-time limit. The steady-state Fokker-Planck equation can be derived by setting $\frac{\partial P(w,t)}{\partial t} = 0$ and then integrating the rest of the terms once with respect to w. Then the second-order partial differential equation reduces to a first order ordinary differential equation with only one independent variable w. We can also drop the time variable t in the density function and write $P(w)$ instead of $P(w,t)$. The same convention applies to the Pareto potentials $A(w)$, $B(w)$ and $L(w)$. Therefore the steady-state Fokker-Planck equation together with the Pareto potentials are summarized by

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}w}\left[\gamma\left(B(w)+\frac{w^2}{2}A(w)\right)P(w)\right]\\ &\quad= \chi\left(\frac{W}{N}-w\right)P(w)-\zeta\left[\frac{2N}{W}\left(B(w)-\frac{w^2}{2}A(w)\right)+w\left(1-2L(w)\right)\right]P(w),\end{aligned} \tag{2.30}$$

where

$$A(w) = \frac{1}{N}\int_w^\infty P(x)\,\mathrm{d}x \tag{2.31}$$

$$B(w) = \frac{1}{N}\int_0^w P(x)\frac{x^2}{2}\,\mathrm{d}x \tag{2.32}$$

$$L(w) = \frac{1}{W}\int_0^w P(x)x\,\mathrm{d}x. \tag{2.33}$$

Notice that in Eq. (2.30), we have three terms each lead by a scaling parameter. Since γ is never zero, we can reduce the degrees of freedom in the number of free parameters by dividing γ on both sides of the equation and defining two new parameters by:

$$\bar{\chi} := \chi/\gamma \tag{2.34}$$

$$\bar{\zeta} := \zeta/\gamma. \tag{2.35}$$

Then, Eq. (2.30) can be rewritten as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}w}\left[\left(B(w)+\frac{w^2}{2}A(w)\right)P(w)\right]\\ &\quad= \bar{\chi}\left(\frac{W}{N}-w\right)P(w)-\bar{\zeta}\left[\frac{2N}{W}\left(B(w)-\frac{w^2}{2}A(w)\right)+w\left(1-2L(w)\right)\right]P(w).\end{aligned} \tag{2.36}$$

Remember that γ has the dimension $1/\Delta t$ where Δt is the characteristic time of a transaction. Therefore, γ has the dimension "transaction per unit time". The newly defined $\bar{\chi}$ and $\bar{\zeta}$ still measure the tax rate and the level of WAA, but now in the sense of "per transaction" instead of "per unit time". In order to avoid introducing new symbols and parameters, we simply omit the bars on $\bar{\chi}$ and $\bar{\zeta}$ and still refer to them as the *redistribution coefficient* and the *WAA coefficient*, keeping in mind that their dimension has changed from "per unit time" to "per transaction". By doing so, we have reduced the number of parameters from three to two. We rewrite our steady-state equation with the reduced number of parameters as:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}w}&\left[\left(B(w)+\frac{w^2}{2}A(w)\right)P(w)\right]\\ &=\chi\left(\frac{W}{N}-w\right)P(w)-\zeta\left[\frac{2N}{W}\left(B(w)-\frac{w^2}{2}A(w)\right)+w\Big(1-2L(w)\Big)\right]P(w).\end{aligned}\tag{2.37}$$

Hereafter we will focusing on studying the solutions of Eq. (2.37), where $A(w)$, $B(w)$ and $L(w)$ are defined in Eqs. (2.31), (2.32) and (2.33). We will refer to this system as the steady-state Fokker-Planck equation of the EYSM, or the *steady-state equation* for short.

2.4.2 The canonical-form of the steady-state equation

There are two symmetries that instantaneously follow from Eq. (2.37). The first symmetry is that the steady-state solution for $P(w)$ is invariant up to a scaling factor. One may think this is obvious because $P(w)$ appear linearly on both sides of Eq. (2.37). However, this reasoning is wrong because the definition of W, N, A, B and L also depends on $P(w)$ and hence Eq. (2.37) is a non-linear equation for $P(w)$.

The right way of reasoning is that, with the transformation $\bar{P}(w)=\alpha P(w)$, we must first define $\bar{W}$, $\bar{N}$, $\bar{A}$, $\bar{B}$ and $\bar{L}$ that associate to $\bar{P}(w)$ the same way as their unbarred counterparts do to $P(w)$, and then we need to calculate the transformations for W, N, A, B and L. We have $\bar{W} = \alpha W$ and $\bar{N} = \alpha N$, and therefore $\bar{A}(w) = A(w)$, $\bar{B}(w) = B(w)$ and $\bar{L}(w) = L(w)$. We plug all these terms into Eq. (2.37) and write it as an equation for the barred terms, and thereby find it invariant under this transformation. Therefore, we have the following Lemma:

Lemma 2.4.1 *Assume $P(w)$ is a solution to Eq. (2.37). For any positive real number α, we define $\bar{P}(w) := \alpha P(w)$. Then $\bar{P}(w)$ also obeys Eq. (2.37) with $\bar{W}$, $\bar{N}$, $\bar{A}$, $\bar{B}$ and $\bar{L}$ defined the same way for $\bar{P}(w)$ as their unbarred counterparts.*

Proof: As described above. $\square$

The other symmetry requires a little bit more insight. It turns out that Eq. (2.37) is also invariant under a horizontal scaling of the independent variable w as well. So, we have our second Lemma:

Lemma 2.4.2 *Assume $P(w)$ is a solution to Eq. (2.37). For any positive real number β, we define $z := \beta w$ and $\bar{P}(z) := P(w)$. Then $\bar{P}(z)$ also obeys Eq. (2.37) with $\bar{W}$, $\bar{N}$, $\bar{A}$, $\bar{B}$ and $\bar{L}$ defined the same way for $\bar{P}(z)$ as their unbarred counterparts.*

Proof: The same reasoning that we used to prove Lemma 2.4.1 can be followed to prove this Lemma as well. We define the transformation:

$$z = \beta w,$$

and we denote the function of z as:

$$\bar{P}(z) = \bar{P}(\beta w) = P(w).$$

Then the Pareto potentials of $\bar{P}(z)$ as well as the total amount of wealth and the total number of agents are defined in the same way as those for $P(w)$, and are denoted by $\bar{A}(w)$, $\bar{B}(w)$, $\bar{L}(w)$, $\bar{N}$, $\bar{W}$. We want to derive a equation for $\bar{P}(z)$ from the steady-state equation for $P(w)$ by finding the relationships between the barred and unbarred symbols. Here are the calculations:

$$
\begin{aligned}
N &= \int_0^\infty P(w)\,\mathrm{d}w = \frac{1}{\beta}\int_0^\infty \bar{P}(\beta w)\,\mathrm{d}(\beta w) = \frac{1}{\beta}\int_0^\infty \bar{P}(z)\,\mathrm{d}z = \frac{1}{\beta}\bar{N} \\
W &= \int_0^\infty P(w)w\,\mathrm{d}w = \frac{1}{\beta^2}\int_0^\infty \bar{P}(\beta w)\beta w\,\mathrm{d}(\beta w) = \frac{1}{\beta^2}\int_0^\infty \bar{P}(z)z\,\mathrm{d}z = \frac{1}{\beta^2}\bar{W} \\
A(w) &= \frac{1}{N}\int_w^\infty P(x)\,\mathrm{d}x = \frac{1}{\bar{N}}\int_{z/\beta}^\infty \bar{P}(\beta x)\,\mathrm{d}(\beta x) = \frac{1}{\bar{N}}\int_z^\infty \bar{P}(y)\,\mathrm{d}y = \bar{A}(z) \\
B(w) &= \frac{1}{N}\int_0^w P(x)\frac{x^2}{2}\,\mathrm{d}x = \frac{1}{\beta^2}\frac{1}{\bar{N}}\int_0^{z/\beta} \bar{P}(\beta x)\frac{(\beta x)^2}{2}\,\mathrm{d}(\beta x) \\
&= \frac{1}{\beta^2}\frac{1}{\bar{N}}\int_0^z \bar{P}(y)\frac{y^2}{2}\,\mathrm{d}y = \frac{1}{\beta^2}\bar{B}(z) \\
L(w) &= \frac{1}{W}\int_0^w P(x)x\,\mathrm{d}x = \frac{\beta^2}{\bar{W}}\int_0^{z/\beta} \bar{P}(\beta x)x\,\mathrm{d}x = \frac{1}{\bar{W}}\int_0^{z/\beta} \bar{P}(\beta x)(\beta x)\,\mathrm{d}(\beta x) \\
&= \frac{1}{\bar{W}}\int_0^z \bar{P}(y)(y)\,\mathrm{d}(y) = \bar{L}(z),
\end{aligned}
$$

and we also have:

$$\frac{\mathrm{d}}{\mathrm{d}w} = \beta\frac{\mathrm{d}}{\mathrm{d}z}.$$

Now, we can plug all the above back into Eq. (2.37) and thereby rewrite that equation in terms of $\bar{P}(z)$, $\bar{N}$, $\bar{W}$, $\bar{A}$, $\bar{B}$ and $\bar{L}$:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}z}\beta\left[\left(\frac{1}{\beta^2}\bar{B} + \frac{1}{\beta^2}\frac{z^2}{2}\bar{A}\right)\bar{P}\right] \\
&\quad = \chi\left(\frac{1}{\beta}\frac{\bar{W}}{\bar{N}} - \frac{1}{\beta}z\right)\bar{P} - \zeta\left[\beta\frac{2\bar{N}}{\bar{W}}\left(\frac{1}{\beta^2}\bar{B} - \frac{1}{\beta^2}\frac{z^2}{2}\bar{A}\right) + \frac{1}{\beta}z\left(1 - 2\bar{L}\right)\right]\bar{P},
\end{aligned}
$$

or

$$
\begin{aligned}
&\frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}z}\left[\left(\bar{B} + \frac{z^2}{2}\bar{A}\right)\bar{P}\right] \\
&\quad = \frac{1}{\beta}\chi\left(\frac{\bar{W}}{\bar{N}} - z\right)\bar{P} - \frac{1}{\beta}\zeta\left[\frac{2\bar{N}}{\bar{W}}\left(\bar{B} - \frac{z^2}{2}\bar{A}\right) + z\left(1 - 2\bar{L}\right)\right]\bar{P},
\end{aligned}
$$

or

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}z}\left[\left(\bar{B} + \frac{z^2}{2}\bar{A}\right)\bar{P}\right] \\
&\quad = \chi\left(\frac{\bar{W}}{\bar{N}} - z\right)\bar{P} - \zeta\left[\frac{2\bar{N}}{\bar{W}}\left(\bar{B} - \frac{z^2}{2}\bar{A}\right) + z\left(1 - 2\bar{L}\right)\right]\bar{P}. \qquad (2.38)
\end{aligned}
$$

Notice that Eq. (2.38) is identical to Eq. (2.37), so we know that $P(\beta w) = \bar{P}(z)$ also obeys Eq. (2.37). $\square$

With the two symmetries we described above, we know that for different positive number pairs N and W the solutions to the steady state equation are actually a two-parameter family of functions. Therefore, without loss of generality, we define the *canonical-form* of the steady-state equation to be the one when both N and W are set equal to 1, i.e.,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}w}&\left[\left(B(w)+\frac{w^2}{2}A(w)\right)P(w)\right]\\ &=\chi\,(1-w)\,P(w)-\zeta\left[2\left(B(w)-\frac{w^2}{2}A(w)\right)+w\Big(1-2L(w)\Big)\right]P(w).\end{aligned} \tag{2.39}$$

To distinguish the above from Eq. (2.37), We will refer to the latter as the *general form* of the steady-state equation and we have the following theorem:

Theorem 2.4.3 *Assume $P(w)$ is a solution to the canonical-form of the steady-state equation described by Eq. (2.39). Then, for any positive numbers W and N,*

$$\bar{P}(w)=\frac{N^2}{W}P\left(\frac{w}{W/N}\right) \tag{2.40}$$

is a solution to the general form of the steady-state equation described by Eq. (2.37), with N agents and total wealth W.

Proof: It follows from Lemma 2.4.1 and Lemma 2.4.2 that Eq. (2.40) is a solution to the general form of the steady state equation Eq. (2.37). Therefore, it is enough to show that

$$\int_0^\infty \frac{N^2}{W}P\left(\frac{w}{W/N}\right)\mathrm{d}w=N$$

and

$$\int_0^\infty \frac{N^2}{W}P\left(\frac{w}{W/N}\right)w\,\mathrm{d}w=W.$$

This can be accomplished by the transformation

$$y = \frac{w}{W/N},$$

whence

$$\begin{aligned}
&\int_0^\infty \frac{N^2}{W} P\left(\frac{w}{W/N}\right) \mathrm{d}w \\
&= N \int_0^\infty P\left(\frac{w}{W/N}\right) \mathrm{d}\left(\frac{w}{W/N}\right) \\
&= N \int_0^\infty P(y)\,\mathrm{d}y = N \times 1 = N
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty \frac{N^2}{W} P\left(\frac{w}{W/N}\right) w\,\mathrm{d}w \\
&= W \int_0^\infty P\left(\frac{w}{W/N}\right) \frac{w}{W/N}\,\mathrm{d}\left(\frac{w}{W/N}\right) \\
&= W \int_0^\infty P(y) y\,\mathrm{d}y = W \times 1 = W,
\end{aligned}$$

and we are done. □

The scaling invariants and the canonical-form of the steady-state equation are very useful. In reality, when we want to compare the wealth distributions between two independent economies, it is usually the case that the total number of economic agents and the total amount of wealth are different. Then, it is desirable that we normalize N and W to 1 and compare canonical-form density functions. This idea is inherent in the Lorenz curve and the Gini coefficient that will be discussed later in Chapter 4. As we shall see, the Lorenz curve contains precisely as much information as the canonical-form density function.

The canonical-form is also very useful in finding numerical solutions to the steady-state equation. With Thoerem 2.4.3, we need only focus on finding the solution to the canonical-form of the steady-state equation; then any solution to the general form of the steady-state equation can be obtained by applying Theorem 2.4.3.

In summary, we now focus on the steady-state equation of the agent density function. By refreshing our definitions to the parameters, we have reduced the number of parameters from three to two. The remaining free parameters are still called the redistribution coefficient χ and the WAA coefficient ζ, but these now measure the redistribution and the level of WAA in a "per transaction" sense. We have also found that the solution to this steady-state equation has two invariants, corresponding to vertical and horizontal scaling of the density function. Based on these two invariants, we have defined the canonical-form of the steady-state equation to that satisfied by steady-state density functions with $N = W = 1$.

It is worth pointing out that our canonical-form steady-state equation has two free parameters and two invariants. Later in Chapter 5, we will extend our model further to incorporate negative wealth and that will introduce a third parameter, the *affine transformation coefficient* and a third invariant which is the invariance under a horizontal shift.

Chapter 3

Solving the steady-state equation with a shooting method

In this chapter, we shall discuss the numerical method we used for solving the canonical-form of the steady-state Fokker-Planck equation for the EYSM with redistribution and WAA. The key idea is that the steady-state equation together with the definitions of the Pareto potentials can be reduced to a system of ordinary differential equations with constraints, which can be treated as a boundary-value problem. Then, we can apply a shooting method to turn the boundary-value problem into an initial-value problem.

There are two main difficulties in applying this method: First, $P(w)$ is non-analytic at the origin where all its derivatives vanish; Second, the domain of $P(w)$ is $[0, \infty)$ which is not compact. We shall show how we overcome the first difficulty by using asymptotic approximations near the origin, and the second by domain compactification.

Finally, looking for the solution of the steady-state equation becomes equivalent to looking for the correct constant multiplier C_0 of the asymptotic solution in the vicinity of the origin. We will also discuss issues of sensitivity in finding C_0.

3.1 The shooting algorithm

We first focus on solving the canonical-form steady-state equation, Eq. (2.39), with Pareto potentials defined by Eqs. (2.31), (2.32) and (2.33). Our first goal is to turn these four equations into a set of simultaneous ordinary differential equations, rather than an integrodifferential equation. We can differentiate Eqs. (2.31), (2.32) and (2.33) and also expand Eq. (2.39) to obtain the system of equations

$$\frac{\mathrm{d}P(w)}{\mathrm{d}w} = \frac{1}{B(w)+\frac{w^2}{2}A(w)}\Bigg\{\chi\,(1-w)-wA(w) -\zeta\left[2\left(B(w)-\frac{w^2}{2}A(w)\right)+w\left(1-2L(w)\right)\right]\Bigg\}P(w) \tag{3.1}$$

$$\frac{\mathrm{d}A(w)}{\mathrm{d}w} = -P(w) \tag{3.2}$$

$$\frac{\mathrm{d}B(w)}{\mathrm{d}w} = \frac{w^2}{2}P(w) \tag{3.3}$$

$$\frac{\mathrm{d}L(w)}{\mathrm{d}w} = wP(w). \tag{3.4}$$

This is a system of four simultaneous ordinary differential equations for four unknown functions.

The boundary conditions that we would like to impose for $P(w)$ are $P(0) = 0$ and $\lim_{w\to\infty} P(w) = 0$ and we consider the constraints $N = 1$ and $W = 1$ to obtain the canonical form. We need to translate these conditions and constraints into a set of initial conditions that we can use to initialize the shooting, and a final condition that we need to check.

If we exploit the fact that $N = 1$ and take the origin as our starting point, we can write the initial conditions as

$$P(0) = 0 \tag{3.5}$$

$$A(0) = 1 \tag{3.6}$$

$$B(0) = 0 \tag{3.7}$$

$$L(0) = 0. \tag{3.8}$$

Eq. (3.5) comes directly from our boundary conditions for $P(w)$. Eq. (3.7) and Eq. (3.8) are obvious from the definitions of $B(w)$ and $L(W)$, and Eq. (3.6) comes from the fact that $N = 1$.

Since all four functions vary simultaneously, only one final condition is needed to solve the system. We can simply use the fact that

$$\lim_{w \to \infty} A(w) = 0, \tag{3.9}$$

which follows from the definition of $A(w)$, as the final condition.

Theoretically, we could alternatively exploit the fact that $W = 1$ and alternatively use the final condition

$$\lim_{w \to \infty} L(w) = 1. \tag{3.10}$$

As we will explain in Chapter 4, however, there are important cases when this second final condition is not satisfied. Specifically, there exists a phase transition that is determined by the two free parameters in the steady-state equation, the redistribution coefficient χ and the WAA coefficient ζ. When $\chi \geq \zeta$, the two final conditions Eq. (3.9) and Eq. (3.10) always agree simultaneously. However, in the regime $\chi < \zeta$, while $A(\infty)$ still converges to 0, $L(\infty)$ converges to χ/ζ instead of 1. This is due to the effect of wealth condensation or we call the "oligarchy" in the system.

3.2 Asymptotic approximation near the origin

In applying the shooting method, we encounter a difficulty. Looking carefully at our system of differential equations and we find that all the derivatives of $P(w)$ are zero at the origin. In fact, as we are going to show, the wealth distribution function $P(w)$ has a depleted region near the origin so that the solution at the origin is non-analytic. A constant zero function is certainly a solution for $P(w)$, however it will not satisfy the final condition. To find non-trivial solutions for $P(W)$, we can use an asymptotic analysis near the origin.

With our initial conditions Eqs. (3.6), (3.7) and (3.8), we can assume that $A(w) \approx 1$, $B(w) \approx 0$ and $L(w) \approx 0$ in the vicinity of the origin for $w \in [0, \delta w]$. Inserting these approximations back into Eq. (3.1) results in the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}w} \approx \frac{2}{w^2}\Big[\chi(1-w) - w - \zeta(-w^2+w)\Big]P. \tag{3.11}$$

Eq. (3.11) can be solved analytically, and the solution is given by

$$P(w) \approx \frac{C_0}{w^{2\chi+2\zeta+2}} \exp\left(-\frac{2\chi}{w} + 2\zeta w\right), \tag{3.12}$$

where C_0 is a constant of integration. Hence, near the origin, $P(w)$ is asymptotically equal to $\exp(-1/w)$. This function is non-analytic at the origin because all of its derivatives vanish there, hence its Taylor series vanishes at the origin. This is why $P(w)$ is depleted near the origin.

To find a non-trivial solution for $P(w)$, We can use Eq. (3.12) in the vicinity of the origin until $P(w)$ is appreciable in magnitude. Then we can revert to the system of differential equations described in Eqs. (3.1 – 3.4). Our numerical results confirm that this method works very well in practice.

3.3 Domain Compactification

Another difficulty we encountered was due to the non-compactness of the domain of $P(w)$. The support of $P(w)$ is $[0,\infty)$, and we cannot deal with ∞ numerically. One way to get around this problem is domain compactification. We define a bijection

$$w = f(\theta) \tag{3.13}$$

$$\theta = g(w) := f^{-1}(w), \tag{3.14}$$

where $w \in [0,\infty)$, and $\theta \in [0,1]$. Then we can transform all the functions in the system of differential equations as described in Eqs. (3.1), (3.2), (3.3) and (3.4) in terms of of θ instead of w. Then we can solve the system of equations on the domain $[0,1]$ and transform back to obtain $P(w)$.

Similar to the proof of Theorem 2.4.3, we define our new function of θ to be $\bar{P}(\theta)$

as a transformation of $P(w)$, and the desired relationship between the two is

$$\frac{1}{N}\int_0^w P(x)\,\mathrm{d}x = \frac{1}{N}\int_0^{f(\theta)} P(x)\,\mathrm{d}x = \frac{1}{N}\int_0^{\theta} \bar{P}(y)\,\mathrm{d}y. \tag{3.15}$$

Differentiating both sides, we obtain the desired relationship between $P(w)$ and $\bar{P}(\theta)$, which is

$$P(w) = \frac{\bar{P}(\theta)}{f'(\theta)}. \tag{3.16}$$

Parallel to $A(w)$, $B(w)$, $L(w)$, W and N defined for $P(w)$, we want to derive the equations of $\bar{A}(\theta)$, $\bar{B}(\theta)$, $\bar{L}(\theta)$ for $\bar{P}(\theta)$ under the transformation between θ and w. By the transformation Eq. (3.16), all the partial potentials are preserved, i.e.,

$$\begin{aligned} \bar{A}(\theta) &= A(w) && (3.17)\\ \bar{B}(\theta) &= B(w) && (3.18)\\ \bar{L}(\theta) &= L(w). && (3.19) \end{aligned}$$

In other words, we can write explicitly the definitions for $\bar{A}(\theta)$, $\bar{B}(\theta)$, and $\bar{L}(\theta)$ in term of $\bar{P}(\theta)$ as

$$\begin{aligned} \bar{A}(\theta) &= \int_\theta^1 \bar{P}(y)\,\mathrm{d}y && (3.20)\\ \bar{B}(\theta) &= \int_0^\theta \bar{P}(y)\frac{f(y)^2}{2}\,\mathrm{d}y && (3.21)\\ \bar{L}(\theta) &= \int_0^\theta \bar{P}(y)f(y)\,\mathrm{d}y. && (3.22) \end{aligned}$$

and we also have

$$\frac{\mathrm{d}P(w)}{\mathrm{d}w} = \frac{\mathrm{d}\bar{P}(\theta)}{\mathrm{d}w} = \frac{\mathrm{d}\bar{P}(\theta)}{\mathrm{d}\theta}\frac{1}{\frac{\mathrm{d}\theta}{\mathrm{d}w}} = \frac{\mathrm{d}\bar{P}(\theta)}{\mathrm{d}\theta}\frac{1}{f'(\theta)} \tag{3.23}$$

Therefore, by using Eqs. (3.16 – 3.19) and (3.23) in Eq. (3.1) and also differentiating Eqs. (3.20 – 3.22), the system of differential equations can be rewritten as functions of θ by

$$\begin{aligned}
\frac{\mathrm{d}\bar{P}}{\mathrm{d}\theta} &= \frac{1}{\left(\bar{B}+\frac{f(\theta)^2}{2}\bar{A}\right)}\Bigg\{\chi\left(1-f(\theta)\right)\bar{P}-f(\theta)\bar{A}\bar{P} \\
&\quad -2\zeta\left[\left(\bar{B}-\frac{f(\theta)^2}{2}\bar{A}\right)+f(\theta)\left(\frac{1}{2}-\bar{L}\right)\right]\bar{P}\Bigg\} && (3.24) \\
\frac{\mathrm{d}\bar{A}}{\mathrm{d}\theta} &= -\bar{P}(\theta) && (3.25) \\
\frac{\mathrm{d}\bar{B}}{\mathrm{d}\theta} &= \bar{P}(\theta)\frac{f(\theta)^2}{2} && (3.26) \\
\frac{\mathrm{d}\bar{L}}{\mathrm{d}\theta} &= \bar{P}(\theta)f(\theta), && (3.27)
\end{aligned}$$

with the initial conditions:

$$\begin{aligned}
\bar{P}(0) &= 0 && (3.28) \\
\bar{A}(0) &= 1 && (3.29) \\
\bar{B}(0) &= 0 && (3.30) \\
\bar{L}(0) &= 0, && (3.31)
\end{aligned}$$

We can also work out the approximation near the origin in the same way. Omitting some straightforward calculations, we write the result

$$\bar{P}(\theta) \approx \frac{c_0}{f(\theta)^{2+2\chi+2\zeta}}\exp\left(-\frac{2\chi}{f(\theta)}+2\zeta f(\theta)\right). \qquad (3.32)$$

Although there are many possible choices for the transformation function f and its inverse function g, the following natural choice has been proven to work very well in practice

$$\begin{aligned}
w &= f(\theta) = \frac{\theta}{1-\theta} && (3.33) \\
\theta &= g(w) = \frac{w}{1+w}. && (3.34)
\end{aligned}$$

From this, we can easily find the specific form of the system of equations and the approximation near the origin under this transformation by inserting Eqs. (3.33) into

Eqs. (3.24 – 3.27). We obtain the results

$$\begin{aligned}
\frac{\mathrm{d}\bar{P}}{\mathrm{d}\theta} &= \frac{1}{\bar{B}+\frac{\theta^2}{2}\bar{A}}\left\{\chi\left(1-\frac{\theta}{1-\theta}\right)\bar{P}-\frac{\theta}{1-\theta}\bar{A}\bar{P}\right. \\
&\left. -2\zeta\left[\left(\bar{B}-\frac{\theta^2}{2(1-\theta)^2}\bar{A}\right)+\frac{\theta}{1-\theta}\left(\frac{1}{2}-\bar{L}\right)\right]\bar{P}\right\} && (3.35) \\
\frac{\mathrm{d}\bar{A}}{\mathrm{d}\theta} &= -\bar{P}(\theta) && (3.36) \\
\frac{\mathrm{d}\bar{B}}{\mathrm{d}\theta} &= \frac{\theta^2}{2(1-\theta)2}\bar{P}(\theta) && (3.37) \\
\frac{\mathrm{d}\bar{L}}{\mathrm{d}\theta} &= \frac{\theta}{1-\theta}\bar{P}(\theta). && (3.38)
\end{aligned}$$

The initial conditions are still given by Eqs.(3.27 – 3.31), and we have

$$\bar{P}(\theta) \approx \frac{C_0(1-\theta)^{2+2\chi+2\zeta}}{\theta^{2+2\chi+2\zeta}}\exp\left(-\frac{2\chi\mu}{\left(\frac{\theta}{1-\theta}\right)}+\frac{2\zeta}{\mu}\frac{\theta}{1-\theta}\right) \tag{3.39}$$

near the origin.

The above is a complete description of the domain compactification and hence of the shooting method. We can apply our shooting method to $\bar{P}(\theta)$ on $\theta \in [0,1]$, instead of to $P(w)$ on $w \in [0,\infty)$. Then we can apply Eq. (3.16) to transform $\bar{P}(\theta)$ back to obtain $P(w)$.

3.4 Numerical instability in searching for C_0

For any given non-negative χ and ζ, our shooting algorithm for the canonical-form of the steady-state equation can be summarized as below

I stated the algorithm in terms of $P(w)$ for consistency assuming the application of domain compactification applied in practice. The small number δw were chosen so that the distribution function can escape the depleted region so that it has numerically non-zero derivatives. In practice, we use the approximation in Eq. (3.12) until $P(w) \geq 10^{-4}$. This choice has been proven to be convenient and successful.

Since the wealth distribution is highly skewed, to obtain sufficient accuracy, the differential equation solver used is the *6th-order Runge-Kutta-Butcher Method*, which

Algorithm 1 Shooting Algorithm for the canonical-form steady-state equation for the EYSM

1. Choose an initial guess for C_0 in Eq. (3.12)
2. In the vicinity of $w \in [0, \delta w]$, for some small value $\delta w > 0$, use Eq. (3.12) with C_0 chosen from step 1 and the initial values from Eq. (3.6 – 3.8) and solve the system of ordinary differential equations.
3. For $w \in [\delta w, \infty]$, revert to Eq. (3.1) and use the final condition obtained from step 3 as the initial conditions; then solve the system of ordinary differential equations with any accurate differential equation solver.
4. Check if the final condition is satisfied:

if true **then**
  $P(w)$ is a solution. Done.
else
  Go back to step 1 and use a different initial guess for C_0, chosen to bring the final condition closer to $A(\infty) = 0$.
end if

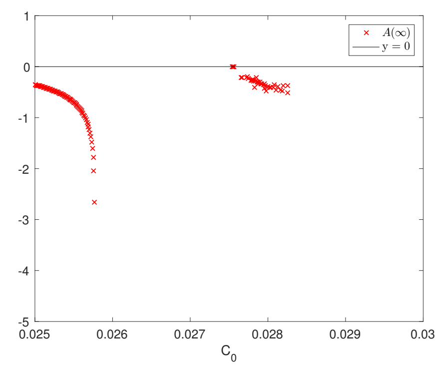
is summarized below

$$y_{i+1} = y_i + \frac{1}{90}\left(7k_1 + 32k_2 + 12k_4 + 32k_5 + 7k_6\right), \tag{3.40}$$

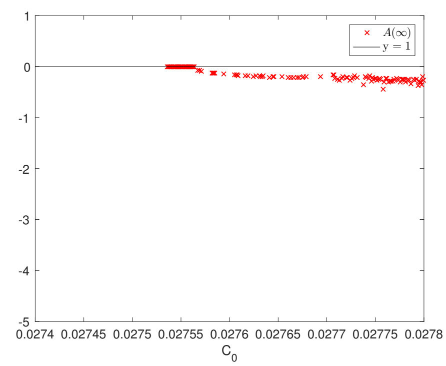
where

$$\begin{cases} k_1 = f\left(t_i, y_i\right) \\ k_2 = f\left(t_i + \frac{1}{4}h, y_i + \frac{1}{4}k_1 h\right) \\ k_3 = f\left(t_i + \frac{1}{4}h, y_i + \frac{1}{8}k_1 h + \frac{1}{8}k_2 h\right) \\ k_4 = f\left(t_i + \frac{1}{2}h, y_i - \frac{1}{2}k_2 h + k_3 h\right) \\ k_5 = f\left(t_i + \frac{3}{4}h, y_i + \frac{3}{16}k_1 h + \frac{9}{8}k_4 h\right) \\ k_5 = f\left(t_i + h, y_i - \frac{3}{7}k_1 h + \frac{2}{7}k_2 h + \frac{12}{7}k_3 h - \frac{12}{7}k_4 h + \frac{8}{7}k_5 h\right). \end{cases} \tag{3.41}$$

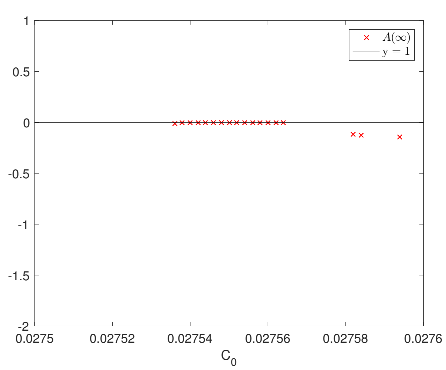
With the shooting method described above, we can see that for any given positive parameters χ and ζ, solving the steady-state solution is equivalent to searching for the correct value of C_0 so that the final condition is satisfied. In practice, we found it extremely difficult to automate the shooting method algorithm for searching C_0. This is mainly due to the high sensitivity of the final condition on the initial conditions.



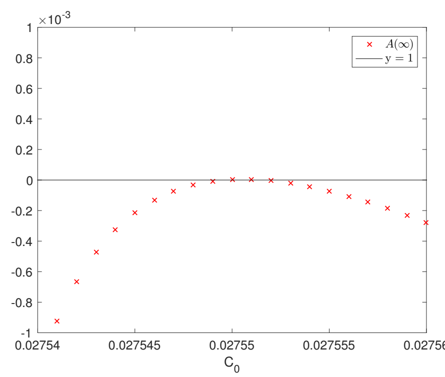
(a) $C_0 \in [0.025, 0.03]$



(b) $C_0 \in [0.0274, 0.0278]$



(c) $C_0 \in [0.0275, 0.0276]$



(d) $C_0 \in [0.02754, 0.02756]$

Figure 3.1: Final conditions $\lim_{w\to\infty} A(w)$ as a function of C_0 for different scopes of C_0. The residual becomes numerically unstable very close to the correct value of C_0, which is about 0.2755.

This sensitivity is envisioned in Figure 3.1, where we plot the final condition $\lim_{w\to\infty} A(w)$ as a function of C_0 for a specific pair of χ and ζ. From Figure 3.1, we can see that the correct value of C_0 is in the vicinity of 0.02755. However, it becomes numerically unstable closely both to the left and right side of this value. Therefore, automatically searching for C_0 with a algorithm such as bisection is no plausible. Also due to its sensitivity, C_0 needs to have at least eight significant digits to give results with sufficient numerical accuracy. In practice, the values of C_0 can be searched only by hand, and saved as a table of fine grid in the (χ, ζ) plane. This is a serious flaw for the shooting method because searching by hand is very costly in time, and it does not scale favorably when a large domain in the (χ, ζ) plane is needed. Table 3.1 shows a tiny fraction of this C_0 table.

In summary, the shooting method works well for solving for the canonical-form of the steady-state equation with satisfactory numerical accuracy. All the numerical results done in the scope of this thesis were done with the shooting method.

		ζ											
		0	0.002	0.004	0.006	0.008	0.010	0.012	0.014	0.016	0.018	0.020	...
χ	0	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	...
	0.002	0.0019	0.0018	0.0036	0.0054	0.0071	0.0088	0.0105	0.0122	0.0139	0.0156	0.0173	...
	0.004	0.0037	0.0036	0.0036	0.0053	0.0070	0.0087	0.0104	0.0121	0.0138	0.0154	0.0171	...
	0.006	0.0054	0.0054	0.0053	0.0052	0.0069	0.0086	0.0103	0.0120	0.0136	0.0153	0.0169	...
	0.008	0.0072	0.0071	0.0070	0.0069	0.0068	0.0085	0.0102	0.0119	0.0135	0.0151	0.0168	...
	0.010	0.0089	0.0088	0.0087	0.0086	0.0085	0.0084	0.0101	0.0117	0.0134	0.0150	0.0166	...
	0.012	0.0107	0.0105	0.0104	0.0103	0.0102	0.0101	0.0100	0.0116	0.0132	0.0148	0.0164	...
	0.014	0.0124	0.0122	0.0121	0.0120	0.0119	0.0117	0.0116	0.0115	0.0131	0.0147	0.0163	...
	0.016	0.0141	0.0139	0.0138	0.0136	0.0135	0.0134	0.0132	0.0131	0.0129	0.0145	0.0161	...
	0.018	0.0158	0.0156	0.0154	0.0153	0.0151	0.0150	0.0148	0.0147	0.0145	0.0144	0.0160	...
	0.020	0.0174	0.0173	0.0171	0.0169	0.0168	0.0166	0.0164	0.0163	0.0161	0.0160	0.0158	...
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 3.1: A fraction of the C_0 values found in (χ, ζ) plane with a fixed grid width of 0.002. More significant digits were retained than are shown. Hand searching for C_0 is a major disadvantage of the shooting method.

Chapter 4

Duality

In this chapter, we are going to describe a remarkable symmetry in the model, called *duality*. First, we shall explain the existence of a phase transition in the steady-state solutions of the EYSM with redistribution and WAA. This phase transition is determined by the model's two free parameters, χ and ζ. In the regime $\chi \geq \zeta$, which we call the *subcritical* regime, the density function $P(w)$ is a classical distribution. However, in the regime $\chi < \zeta$, which we call the *supercritical* regime, we observe partial *wealth condensation* resulting in what we call the *oligarchy*.

Then, digressing from the EYSM for a moment, we shall introduce the Lorenz curve and the Gini coefficient. Beyond their popularity in analyzing inequality, it turns out that these tools have a number of advantages over the classical distribution, especially when describing wealth distributions with oligarchy. Our empirical work in later chapters also employ these tools.

Finally, we shall describe duality in the model. In short, duality is a one-to-one correspondence between the supercritical and the subcritical solutions to the steady-state equation. More specifically, we have found that the non-oligarchical part of a supercritical solution for $P(w)$ is identical to that of a subcritical solution with the two free parameters in the EYSM swapped, i.e. $\{\chi \to \zeta, \zeta \to \chi\}$. We will explain this symmetry from both the microscopic and the macroscopic perspectives. We will also explain how we can visualize this duality in the Lorenz curve. One direct and practical application of duality is that it significantly reduces the workload of tabulating numerical solutions discussed in the previous chapter. Finally, we will discuss the origin of the term "duality" owing to its relationship with a similar concept in physics.

4.1 Wealth condensation and oligarchy

4.1.1 Complete wealth condensation

Recall that complete wealth condensation was first observed in the steady-state solution to the basic YSM with neither redistribution nor WAA (i.e., $\chi = 0$ and $\zeta = 0$). We can think the steady-state solution to Eq. (2.37) as a limit

$$P(w) = \lim_{t\to\infty} P(w,t). \tag{4.1}$$

In the discrete case, when we have a positive integer N agents and total wealth W, our Monte-Carlo simulation tells us that the time-asymptotic limit of the density function is when all the wealth concentrates to only one agent and the other $N-1$ agents have no wealth at all, i.e.,

$$P(w) = \lim_{t\to\infty} P(w,t) = (N-1)\delta(w) + \delta(w-W) \tag{4.2}$$

where $\delta(w)$ is the Dirac delta. More interesting things happen in the continuum case, when the distribution $P(w,t)$ will not stabilize as in Eq. (4.2), but keeps concentrating further and it becomes possible that only a fraction ϵ of an agent holds the amount of wealth W/ϵ, where $0 < \epsilon < 1$. So, what can happen when time $t \to \infty$ and hence $\epsilon \to 0$?

To properly describe the steady-state solution in the continuum case, we write $P(w)$ as

$$P(w) = N\delta(w) + W\Xi(w), \tag{4.3}$$

where we have introduced the term $\Xi(w)$, which can be thought of as the limit of a sequence of functions

$$\Xi(w) = \lim_{\epsilon\to 0} \Xi_\epsilon(w), \tag{4.4}$$

where we have introduced

$$\Xi_\epsilon(w) = \begin{cases} 1 & \text{for } \frac{1}{\epsilon} - \frac{\epsilon}{2} \leq w \leq \frac{1}{\epsilon} + \frac{\epsilon}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

The above definition is a descriptive one, it helps us to understand the intuition behind this function but it lacks mathematical rigor. A formal and rigorous definition of $\Xi(w)$ was later provided by using Sobolev-Schwarz distribution theory [29] and by non-standard analysis [14]. It is not the focus of this thesis to go deep into the mathematics behind the $\Xi(w)$ function, therefore we will just state two very important properties of $\Xi(w)$ that we are about to use later without any rigorous proof.

Theorem 4.1.1 $\Xi(w)$ *has vanishing zeroth moment and unit first moment, i.e.,*

$$\int_0^\infty \Xi(w)\,\mathrm{d}w = 0 \tag{4.6}$$

$$\int_0^\infty \Xi(w) w\,\mathrm{d}w = 1. \tag{4.7}$$

Theorem 4.1.2 *Any incomplete moment of* $\Xi(w)$ *vanishes, i.e.,*

$$\int_0^w \Xi(x) x^k\,\mathrm{d}x = 0, \text{ for } k = 0, 1, 2, \ldots \tag{4.8}$$

for any finite $w \in \mathbb{R}$.

Even without rigorous proof, the consistency between Theorem 4.1.1 and Theorem 4.1.2 and the descriptive definition of $\Xi(w)$ given by Eqs. (4.4) and (4.5) is clear. Further intuition for and a more detailed mathematical treatment of the distribution $\Xi(w)$ is provided in Appendix A of [5].

Now, we revisit Eq. (4.3) and think about what it really means. It indicates that the steady-state solution for wealth distribution in the basic Yard-Sale Model is an absolute oligarchy, where 100% of the population have no wealth at all, while an infinitesimal fraction of the population holds all the wealth of the society. In other word, it is a complete separation of the agents and the wealth. While all the agents concentrate at zero, as described by the term $N\delta(w)$ in Eq. (4.3), all the wealth concentrates to infinity for which we introduced the distribution $\Xi(w)$.

4.1.2 Classical distribution

Complete wealth condensation is unrealistic, but, as noted earlier, it can be eliminated by introducing redistribution. Our numerical solutions show that, in the EYSM with only redistribution (i.e., $\chi \neq 0$ and $\zeta = 0$), the steady-state solution becomes a classical distribution.

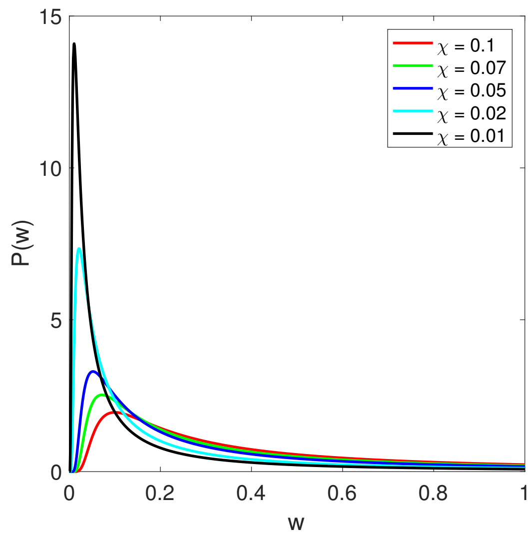


Figure 4.1: A family of steady-state solutions to the EYSM with redistribution for various values of χ.

Figure 4.1 shows a family of solutions to the EYSM with only redistribution. As can be seen, the solutions for the agent density $P(w)$ all have a depleted region near zero, and decay after a single peak. We can also notice that the density function becomes flatter with higher values of χ. When χ approaches zero, the wealth distribution approaches a Dirac delta at zero as described in Eq. (4.3); unfortunately the $\Xi(w)$ term is impossible to envision in this same manner.

4.1.3 Partial wealth condensation and oligarchy

While the solution of complete wealth condensation is obtained for the basic YSM, and a classical distribution is obtained in the ESYM with only redistribution, the more interesting case is the EYSM with both redistribution and WAA, which can give rise to coexistence between oligarchical and non-oligarchical components of the population.

In Section 3.1, solutions to Eq. (2.39) with the boundary conditions

$$P(0) = 0 \tag{4.9}$$

and

$$\lim_{w\to\infty} A(w) = 1 \tag{4.10}$$

were studied. From Eqs. (2.31) and (2.33), it is clear that Eq. (4.10) implies that

$$\lim_{w\to\infty} P(w) = 0, \tag{4.11}$$

but remarkably it does not imply that $\lim_{w\to\infty} L(w) = 1$. In fact, our numerical method has shown that there are cases when $\lim_{w\to\infty} L(w) < 1$ This is surprising because we might have expected the total wealth to be conserved as 1. Now with the discussion in Subsection 4.1.1, we know that this is actually due to a *partial wealth condensation* or a coexistence of the classical distribution and the distributional $\Xi(w)$ function. More generally, we can write the solution in the form of:

$$P(w) = p(w) + c_\infty W \Xi(w), \tag{4.12}$$

where $p(w)$ is a classical function defined on $w \in [0, \infty)$ and $c_\infty \in [0, 1]$.

Intuitively, we may think of the second term of Eq. (4.12) as corresponding to the presence of an "oligarchy" – a vanishingly small fraction of the total number of economic agents who nevertheless possess a finite fraction of the total wealth. To see this, note that the second term of Eq. (4.12) contributes nothing to N because of Eq. (4.6), whereas it contributes $W_\Xi := c_\infty W$ to the total wealth W because of Eq. (4.7). We may therefore surmise that the zeroth and first moments of the classical part of the distribution are given by

$$\int p(w)\,\mathrm{d}w = N \tag{4.13}$$

$$\int p(w)w\,\mathrm{d}w = (1 - c_\infty)\,W. \tag{4.14}$$

A distribution with the form of Eq. (4.12) with $c_\infty \neq 0$ will be called an *oligarchical distribution.*

To summarize, the contributions of the classical and oligarchical terms to N and W can be written

$$N = \int_0^\infty P(w)\,\mathrm{d}w = \underbrace{\int_0^\infty p(w)\,\mathrm{d}w}_{N_p:=N} + \underbrace{c_\infty W \int_0^\infty \Xi(w)\,\mathrm{d}w}_{N_\Xi:=0} \tag{4.15}$$

and

$$W = \int_0^\infty P(w)w\,\mathrm{d}w = \underbrace{\int_0^\infty p(w)w\,\mathrm{d}w}_{W_p:=(1-c_\infty)W} + \underbrace{c_\infty W \int_0^\infty \Xi(w)w\,\mathrm{d}w}_{W_\Xi:=c_\infty W}, \tag{4.16}$$

where we have used Eqs. (4.6) and (4.7) to evaluate the integrals over Ξ.

4.2 The Lorenz curve and Gini coefficient

It is important to note that in any graph of $P(w)$ versus w, the second, oligarchical term in Eq. (4.12) will be essentially invisible – an imperceptible, measure-zero adjustment to the extreme tail of the distribution. This is because this term contributes nothing to the total amount of economic agents present, which is, after all, the zeroth moment of the distribution. This term can not be neglected, however, because it contributes significantly to the total amount of wealth present, which is the first moment of the distribution.

If you had a way of knowing N and W in advance, you might be able to *infer* the presence of the oligarchical term from the portion of the distribution $p(w)$ that you can see as follows: If you take the zeroth moment of $p(w)$, and you confirm that the result is N, you can be sure that you have accounted for the full measure of the classical distribution $p(w)$. If you then take the first moment of $p(w)$ and find that the result is $W_p < W$, you can infer the existence of a second, oligarchical term in Eq. (4.12) with $c_\infty = 1 - \frac{W_p}{W}$.

A more direct way of recognizing the presence of an oligarchical term in Eq. (4.12) is to employ the *Lorenz curve*, first introduced by Max O. Lorenz in 1905 [37] as a way to represent inequality in wealth (or income). For a given distribution, the Lorenz curve plots the cumulative share of wealth against the cumulative share of economic agents. Hence a point (x, y) on Lorenz curve can be interpreted as the bottom x% of the population of the society possessing y% of the total wealth of the society.

The culmulative share of economic agents is given by

$$F(w) := \frac{1}{N}\int_0^w P(x)\,\mathrm{d}x = 1 - A(w), \tag{4.17}$$

while the cumulative share of wealth is just the function $L(w)$ introduced in Eq. (2.33). Hence, the Lorenz curve is a parametric plot of $L(w)$ versus $F(w)$, where the parameter w runs from zero to infinity. Going forward, we shall refer to this functional form as $\mathcal{L}(\mathcal{F})$, defined so that $\mathcal{L}(\mathcal{F}) = L(w)$ when $\mathcal{F} = F(w) \in [0,1]$, i.e., $\mathcal{L}(F(w)) = L(w)$. Three important properties of the function $\mathcal{L}(\mathcal{F})$ follow from this definition:

1. It is easy to see that the graph of $\mathcal{L}(\mathcal{F})$ must include the points $(0,0)$ and $(1,1)$, and must necessarily lie below the straight diagonal line connecting those two points. In fact, twice the area between the Lorenz curve and the diagonal is a commonly used measure of wealth (or income) inequality called the *Gini coefficient* [21], introduced by Corrado Gini in 1912 [23].

2. It is also clear that if one is given the distribution $P(w)$, one can straightforwardly compute N, W, F and L and hence the Lorenz curve $\mathcal{L}(\mathcal{F})$. It is less clear that the opposite is true: If one knows N, W and $\mathcal{L}(\mathcal{F})$, one can recover

the distribution $P(w)$. To see this, note that the slope of the Lorenz curve is

$$\mathcal{L}'(\mathcal{F}) = \frac{L'(w)}{F'(w)} = \frac{w}{W/N}, \tag{4.18}$$

which is the wealth, normalized to the average wealth. If you are given the Lorenz curve, you can suppose that you know $\mathcal{L}'(\mathcal{F})$ as a function of $\mathcal{F}$, and hence this function of $\mathcal{F}$ must be equal to $w/(W/N)$. Knowing W/N, you can invert this relation to obtain F as a function of w. Differentiating that, recalling that $F'(w) = P(w)/N$, and knowing N, you can recover $P(w)$. Hence the triplet $\{N, W, \mathcal{L}(\mathcal{F})\}$ is equivalent in information content to $P(w)$.

3. Finally, by examining the second derivative $\mathcal{L}''(\mathcal{F})$ is possible to show that the graph of $\mathcal{L}(\mathcal{F})$ must be concave up.

From the above observations, it follows that Lorenz curves corresponding to subcritical solutions are continuous and concave up, extending from point $(0,0)$ to point $(1,1)$. In the supercritical case, however, when the oligarchical term $c_\infty \Xi(w)$ is present, the graph of the Lorenz curve on the clopen domain $\mathcal{F} \in [0,1)$, though still concave up, approaches the point $(1, 1 - c_\infty)$ on the right-hand boundary of its domain, instead of $(1,1)$; it then discontinuously jumps to the point $(1,1)$ when $\mathcal{F} = 1$.

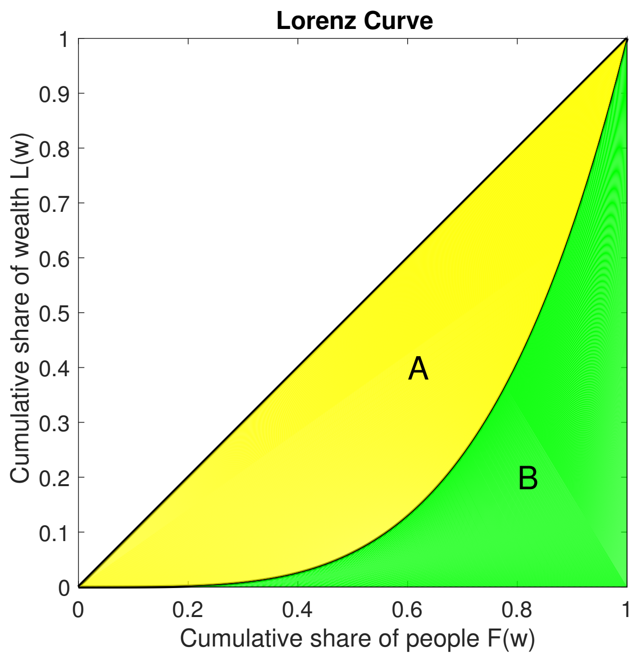
Figure 4.2 shows examples of Lorenz curves in both the supercritical and subcritical cases. The Gini coefficient can be defined either by the ratio of the shaded areas as shown in Figure 4.2 or mathematically with the Lorenz curve $\mathcal{L}(\mathcal{F})$ as

$$G := \int_0^1 \mathcal{L}(\mathcal{F})\,\mathrm{d}\mathcal{F}. \tag{4.19}$$

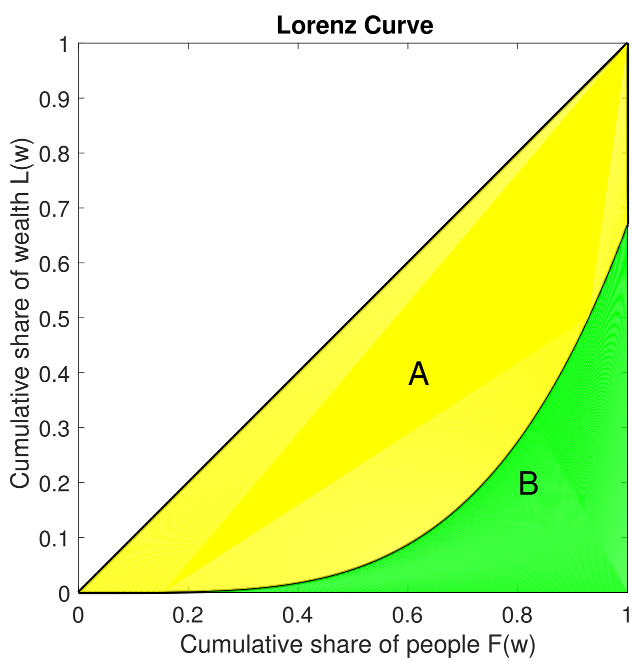
It is not difficult to show that Eq. (4.19) is mathematically equivalent to

$$G = 1 - \frac{2}{W}\int_0^\infty P(w)A(w)w\,\mathrm{d}w. \tag{4.20}$$

In practice, we can either use Eq. (4.19) or use Eq. (4.20) to calculate the Gini



(a) Lorenz Curve for subcritical state with classical wealth density function



(b) Lorenz Curve for supercritical state with partial wealth condensation

Figure 4.2: In the EYSM, the Lorenz curve cannot go below zero and always terminates at the point $(1,1)$. The Gini coefficient is defined to be $G = \frac{A}{A+B}$, where A and B are the areas of the shaded regions.

coefficient, based on what kind of information we have.

It is also worth pointing out that, the Gini coefficient G is a number between 0 and 1, no matter whether a supercritical case or a subcritical case. This is due to the fact that all the agents have non-negative wealth and hence $L(w)$ cannot dip down below 0. In Chapter 5, when we further expand our model to allow agents with negative wealth. The fact that $G \in [0,1]$ is then no longer true.

A property of Lorenz curves closely related to Property 2 above is that they are invariant under scaling of both the abscissa and the ordinate of $P(w)$. Both $F(w)$ and $L(w)$ are invariant under scaling of the ordinate of P, because their definitions in Eqs. (4.17) and (2.33) include division by N and W, respectively. Scaling of the abscissa involves a scaling of w, which changes the parametrization of the Lorenz curve, but not the functional form of the curve itself. This scale invariance is reflected in the fact that W and N are needed along with the Lorenz curve $\mathcal{L}(\mathcal{F})$ to recover the distribution $P(w)$. This means that to each Lorenz curve there corresponds an entire equivalence class of possible distributions, and that this equivalence class is isomorphic to $(\mathbb{R}^+)^2$ (since N and W are both positive). Conversely, each equivalence class can be specified by a single representative distribution, obtained by taking $W = N = 1$, which is what we referred to as the *canonical form*.

It was shown in Chapter 2 by Eq. (2.40) that the Fokker-Planck equation, Eq. (2.37), is also invariant under the above-described equivalence relation. Eq. (2.40) is therefore a two-parameter relation between every member of the equivalence class and its canonical form representative. Because P and $\overline{P}$ are in the same equivalence class in the sense described above, they will have exactly the same Lorenz curve. The Lorenz curve may thus be thought of as a property of the entire equivalence class, rather than of any single representative thereof, which makes it the most appropriate metric for comparison with empirical results.

In the following section, we shall investigate the Lorenz curves for a subcritical solution with parameters χ and ζ, with $\zeta < \chi$, and that of its dual supercritical solution with the two parameters swapped. We shall demonstrate that the two Lorenz curves are identical to within an overall scale factor. Specifically, the supercritical

Lorenz curve is $\frac{\zeta}{\chi} < 1$ times the subcritical Lorenz curve. The proof of this assertion will involve using Eqs. (4.30) and (4.32) to show that for a specific supercritical solution, $P(w)$ with total wealth W, its non-oligarchical population $p(w)$ corresponds to its dual subcritical solution with total wealth W_p, and the Lorenz curve of the previous one is just a scaling of the latter. Together with the above-described invariance property of the Lorenz curve, this will establish that it is true between all dual solution pairs, even though their total wealths, W and W_p, may differ. This establishes the one-to-one correspondence between the subcritical and supercritical solutions that is the hallmark of duality. After establishing this for the macroscopic Fokker-Planck description, we shall trace the origin of the duality back to the underlying microscopic process, where it is slightly more difficult to recognize.

4.3 Duality

The first and perhaps best known example of duality in physics is that which was discovered by Kramers and Wannier in 1941 in the context of the two-dimensional square-lattice Ising model of ferromagnetism, and which they used to make the first prediction of the critical temperature of that model [34]. They did this by comparing the high-temperature and low-temperature expansions of the partition function of the model, and supposing that the partition function has a singularity at the critical point and nowhere else. This forces a mathematical identity from which one can back out the critical temperature. When Onsager presented an exact solution for the two-dimensional Ising model in 1944 [41], Kramers' and Wannier's prediction for the critical temperature was verified, and moreover it became clear that their approach could be understood as establishing a deep one-to-one correspondence between the subcritical and supercritical states of the model [32]. If the temperature is scaled so that the critical temperature is equal to unity, the associated subcritical and supercritical temperatures are multiplicative inverses of one another, just as the case for χ/ζ in our economic model. This correspondence is not the least bit obvious, especially because it associates highly disordered states with highly ordered ones.

The notion of duality in physics exploded in importance after Maldacena's 1997 conjecture that there is a duality between the anti-de Sitter (AdS) spaces used in theories of quantum gravity, and conformal field theories (CFT) which are quantum field theoretical descriptions of elementary particles on the boundaries of those AdS spaces [38]. This conjectured association is sometimes called the AdS/CFT correspondence. Because this association relates strongly coupled field theories which can not be treated perturbatively with weakly coupled field theories which can, the methodology has the potential to enhance our understanding of strongly coupled field theories.

Given the importance attached to duality in modern physics, we find it fascinating that the very same concept appears in a simple agent-based model of the economy. In the following sections, we shall demonstrate this duality analytically, both from the microscopic agent-level model and from its macroscopic Fokker-Planck description, as well as numerically, using both Monte Carlo simulations and numerical solutions of the Fokker-Planck equation.

4.3.1 Duality in the macroscopic, Fokker-Planck description

It has been demonstrated [5] that Eq. (2.37) exhibits a second-order phase transition, in that the character of its solutions abruptly changes at the critical value $\zeta = \chi$. When $\zeta < \chi$, the solutions are subcritical, and when $\zeta > \chi$ they are supercritical. In the framework of Eq. (4.12), we can write these two types of solutions in a unified way by writing c_∞ as follows:

$$c_\infty = \begin{cases} 0 & \text{for } \chi \geq \zeta \\ \left(1 - \frac{\chi}{\zeta}\right) W & \text{for } \chi < \zeta. \end{cases} \tag{4.21}$$

This phase transition corresponding to the sudden appearance of oligarchy as ζ is increased is an example of a phenomenon sometimes known as *wealth condensation* [8].

To investigate the duality between the supercritical and subcritical solutions, we begin by considering the supercritical case (i.e. $\zeta > \chi$). Then, as shown in Eq. (4.16), the total wealth of the society W can be written

$$W = W_p + W_\Xi. \tag{4.22}$$

Then, by applying Eqs. (4.16) and (4.21) to the above equation in the supercritical case, we have the wealth of the classical part of the distribution,

$$W_p = \frac{\chi}{\zeta} W, \tag{4.23}$$

and that of the oligarchical part,

$$W_\Xi = \left(1 - \frac{\chi}{\zeta}\right) W. \tag{4.24}$$

That is, a fraction of $1 - \chi/\zeta$ of the total wealth of the entire society is held by the oligarchy, while the rest of the population holds the remaining fraction χ/ζ. In the subcritical case, by contrast, the oligarchy vanishes and the non-oligarchical population holds the entire wealth of the society, i.e., $W_p = W$.

Now, let us focus on $p(w)$, the non-oligarchical population part of a supercritical solution, and see if we can write an integrodifferential equation for it alone. Analogous to W_p, we can define variants of the Pareto potentials in Eqs. (2.31) through (2.33), for $p(w)$ alone,

$$N_p = \int_0^\infty p(x)\,\mathrm{d}x \tag{4.25}$$

$$A_p = \frac{1}{N_p} \int_w^\infty p(x)\,\mathrm{d}x \tag{4.26}$$

$$B_p = \frac{1}{N_p} \int_0^w p(x) \frac{x^2}{2}\,\mathrm{d}x \tag{4.27}$$

$$L_p = \frac{1}{W_p} \int_0^w p(x) x \kappa x. \tag{4.28}$$

Using Eqs. (4.6) through (4.8), we can insert Eq. (4.12) back into Eqs. (2.31) through (2.33) and derive the relationships between the Pareto potentials for $p(w)$ and those for $P(w)$ as follows,

$$N_p = N \tag{4.29}$$

$$A_p = A \tag{4.30}$$

$$B_p = B \tag{4.31}$$

$$L_p = \frac{\zeta}{\chi} L \tag{4.32}$$

Now, with the above equations established, we can insert Eq. (4.12) back into Eq. (2.37) and rewrite the steady-state Fokker-Planck equation as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}w}\left[\left(B_p + \frac{w^2}{2}A_p\right)p\right] + \underbrace{c_\infty \frac{\mathrm{d}}{\mathrm{d}w}\left[\left(B_p + \frac{w^2}{2}A_p\right)\Xi\right]}_{I} \\
= \;& \zeta\left(\frac{W_p}{N_p} - w\right)p + \underbrace{c_\infty \zeta\left(\frac{W_p}{N_p} - w\right)\Xi}_{II} - 2\chi\left[\frac{N_p}{W_p}\left(B_p - \frac{w^2}{2}A_p\right) + w\left(\frac{1}{2} - L_p\right)\right]p \\
&\underbrace{+\, 2c_\infty \chi\left[\frac{N_p}{W_p}\left(B_p - \frac{w^2}{2}A_p\right) + w\left(\frac{1}{2} - L_p\right)\right]\Xi}_{III}\,.
\end{aligned} \tag{4.33}$$

When considering distributional solutions to the above equation, we can choose any smooth test function $\phi_\alpha(w)$ with compact support. By multiplying both sides of Eq. (4.33) by $\phi_\alpha(w)$, and integrating with respect to w, the contribution of the terms I, II and III are all zero, leaving the equation:

$$\frac{\mathrm{d}}{\mathrm{d}w}\left[\left(B_p + \frac{w^2}{2}A_p\right)p\right] = \zeta\left(\frac{W_p}{N_p} - w\right)p - 2\chi\left[\frac{N_p}{W_p}\left(B_p - \frac{w^2}{2}A_p\right) + w\left(\frac{1}{2} - L_p\right)\right]p \tag{4.34}$$

The principal observation of this chapter can be seen in the above equation. Notice that Eq. (4.34) is of exactly the same form as the original Fokker-Planck equation in steady-state Eq. (2.37), but with the redistribution and the WAA coefficients swapped. Recalling that $p(w)$ is the non-oligarchical part of a supercritical solution $P(w)$, we know from Eq. (4.34) that $p(w)$ would correspond to a subcritical solution of Eq. (2.37). Hence we have established the promised one-to-one correspondence between subcritical and supercritical solutions, which we recognize as the key feature of duality.

4.3.2 Duality in the microscopic random process

The duality between the supercritical and the subcritical solutions can be also understood at the microscopic process level. Again, we begin by considering the supercritical case when $\chi < \zeta$, and we return to the random walk process described in Eqs. (2.25) and (2.26). We rewrite these two equations in terms of W_p instead of W by applying Eq. (4.23),

$$\Delta w = \sqrt{\gamma\Delta t}\min(w,x)\eta + \zeta\left(\frac{W_p}{N} - w\right)\Delta t + (\zeta - \chi)w\Delta t \tag{4.35}$$

$$E[\eta] = \chi\sqrt{\frac{\Delta t}{\gamma}}\frac{N}{W_p}(w - x). \tag{4.36}$$

Notice that the above two equations are of the very same form as Eqs. (2.25) and (2.26), aside from redefinitions of parameters. If we were to apply Eqs. (2.10) and (2.11) to compute the drift coefficient and diffusivity of the corresponding Fokker-Planck equation, we would again end up with Eq. (2.29). To gain some insight into the relationship between the non-oligarchical population and the oligarchy, however, instead of directly averaging over $P(w)$, we apply Eq. (4.12) to break up the average defined in Eq. (2.8) into two contributions,

$$\begin{aligned}\mathcal{E}[f(\eta,x)] &= \frac{1}{N}\int_0^\infty \mathrm{d}x\ P(x,t)E[f(\eta,x)] \\ &= \underbrace{\frac{1}{N}\int_0^\infty \mathrm{d}x\ p(x,t)E[f(\eta,x)]}_{\mathcal{E}_p[f(\eta,x)]} + \underbrace{\frac{1}{N}\int_0^\infty \kappa x\ c_\infty \Xi(x)E[f(\eta,x)]}_{\mathcal{E}_\Xi[f(\eta,x)]}.\end{aligned} \tag{4.37}$$

By breaking up the averages over the non-oligarchical population $p(w)$ and the oligarchy $\Xi(w)$ in this way, we can likewise break up the drift coefficient and the diffusivity into contributions from the non-oligarchical population and the oligarchy separately, i.e., $D = D_p + D_\Xi$ and $\sigma = \sigma_p + \sigma_\Xi$, respectively. Straightforward calculation yields

$$D_p = \lim_{\Delta t\to 0}\mathcal{E}_p\left[\frac{\Delta w^2}{\Delta t}\right] = 2\left(B_p + \frac{w^2}{2}A_p\right) \tag{4.38}$$

$$D_\Xi = \lim_{\Delta t \to 0} \mathcal{E}_\Xi \left[\frac{\Delta w^2}{\Delta t} \right] = 0 \tag{4.39}$$

$$\sigma_p = \lim_{\Delta t \to 0} \mathcal{E}_p \left[\frac{\Delta w}{\Delta t} \right] = \zeta \left(\frac{W_p}{N} - w \right) + (\zeta - \chi) w$$
$$-2\chi \left[\frac{N}{W_p} \left(B_p - \frac{w^2}{2} A_p \right) + w \left(\frac{1}{2} - L_p \right) \right] \tag{4.40}$$

$$\sigma_\Xi = \lim_{\Delta t \to 0} \mathcal{E}_\Xi \left[\frac{\Delta w}{\Delta t} \right] = (\chi - \zeta) w. \tag{4.41}$$

If we write the Fokker-Planck equation in steady state using Eqs. (4.38) through Eq. (4.41), we obtain Eq. (4.34) exactly.

From the above derivation, we can see that the random process within the non-oligarchical population $p(w)$ is equivalent to a subcritical random process with redistribution coefficient χ and WAA coefficient ζ swapped, plus an extra term, $(\zeta - \chi)w$. Remembering that we are considering the case $\zeta > \chi$, we see that this extra term is positive. This extra term is the wealth flow into the non-oligarchical population due to the tax on the oligarchy. To see this, we can compute the $\mathcal{E}_p$ average of the extra term $(\chi - \zeta)w$, obtaining $\chi(\zeta/\chi - 1)W_p$, which is exactly the amount of the tax per unit time paid by the oligarchy at tax rate χ.

Conversely, when we compute the $\mathcal{E}_\Xi$ average of the extra term, we obtain a negative term $(\chi - \zeta)w$, balancing the above-described wealth flow into the non-oligarchical population. This is due to the transaction between the non-oligarchical population and the oligarchy. Because the oligarchy is a vanishingly small fraction of an agent possessing infinite wealth in the continuum limit, the WAA model guarantees that the oligarchy wins in every such transaction. Therefore, the WAA acts like a "effective tax" on the non-oligarchical population which is balanced only by the actual redistributive tax on the oligarchy.

The above argument provides a heuristic explanation of why there exists a symmetry between the redistribution and WAA coefficients. When the wealth flows between the two systems balance each other, the steady-state is reached. The distribution of the non-oligarchical population $p(w)$ then satisfies Eq. (4.34), and corresponds to a subcritical solution of Eq. (2.37).

4.3.3 Numerical Results

Figure 4.3 confirms the presence of the duality by comparing two numerical solutions of Eq. (2.37) with different parameters, corresponding subcritical and supercritical cases. These numerical solutions are found using a shooting method, as described in Chapter 3 of this thesis. We can see that the wealth distribution of a subcritical solution when $\chi = 0.03$, $\zeta = 0.02$ and $W = 1$ (plotted in green) is identical to the wealth distribution for the non-oligarchical population of a supercritical solution when $\chi = 0.02$, $\zeta = 0.03$ and $W = 1.5$ (plotted in red).

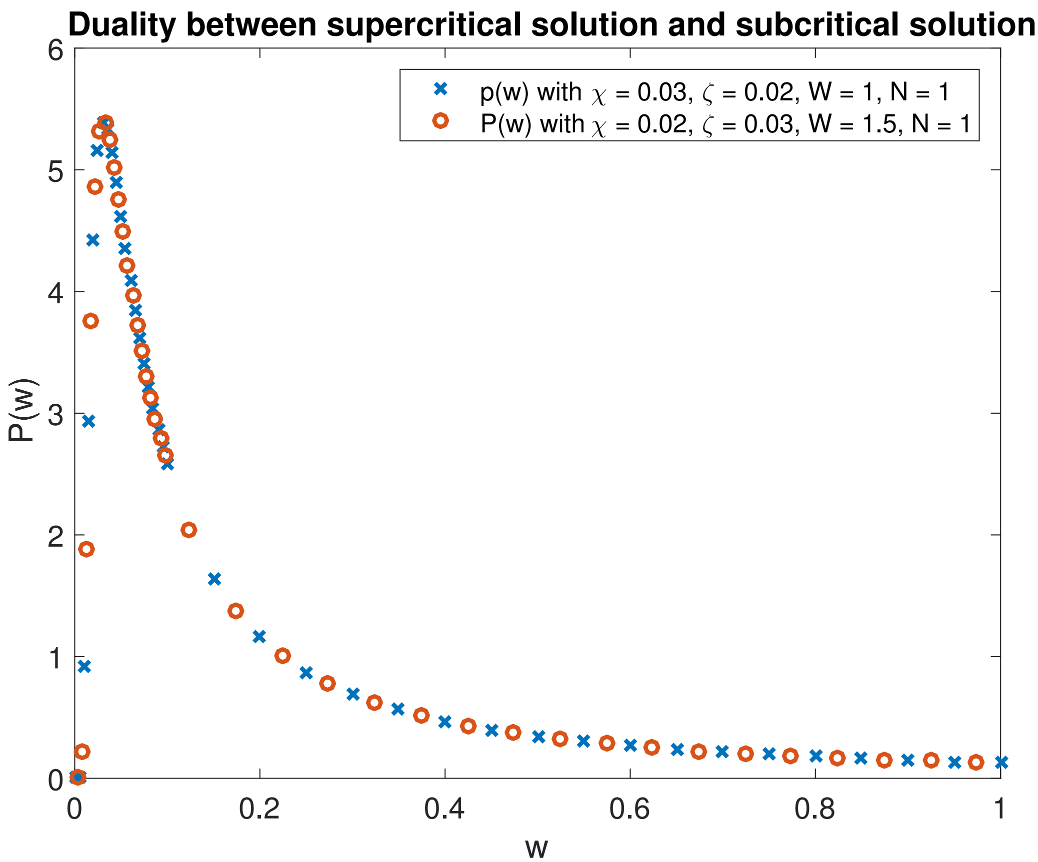


Figure 4.3: Comparison of the wealth distribution of a subcritical solution $P(w)$ and the wealth distribution of the non-oligarchical population of a supercritical solution $p(w)$, with the swapping of the redistribution coefficient χ and WAA coefficient ζ.

From a practical point of view, the duality provides an effective way to solve for a supercritical distributional solution to Eq. (2.37). If we want to solve for a supercritical distributional solution with redistribution coefficient χ and WAA coefficient ζ and total wealth W, we can instead solve for the dual subcritical classical solution with the two parameters swapped and with total wealth equal to $\frac{\chi}{\zeta}W$. Then we can simply augment this solution by the addition of an oligarchy with wealth $(1-\frac{\chi}{\zeta})W$.

Figure 4.4 shows the Lorenz curves between the subcritical solution and the supercritical solution obtained by swapping the two parameters χ and ζ. We can see that while the Lorenz curve for the subcritical solution is a curve from $(0,0)$ to $(1,1)$, the Lorenz curve for the supercritical solution is just a scaling of the previous one by a factor of $\frac{\chi}{\zeta}$, and hence it intersects the right boundary at $(1,\frac{\chi}{\zeta})$

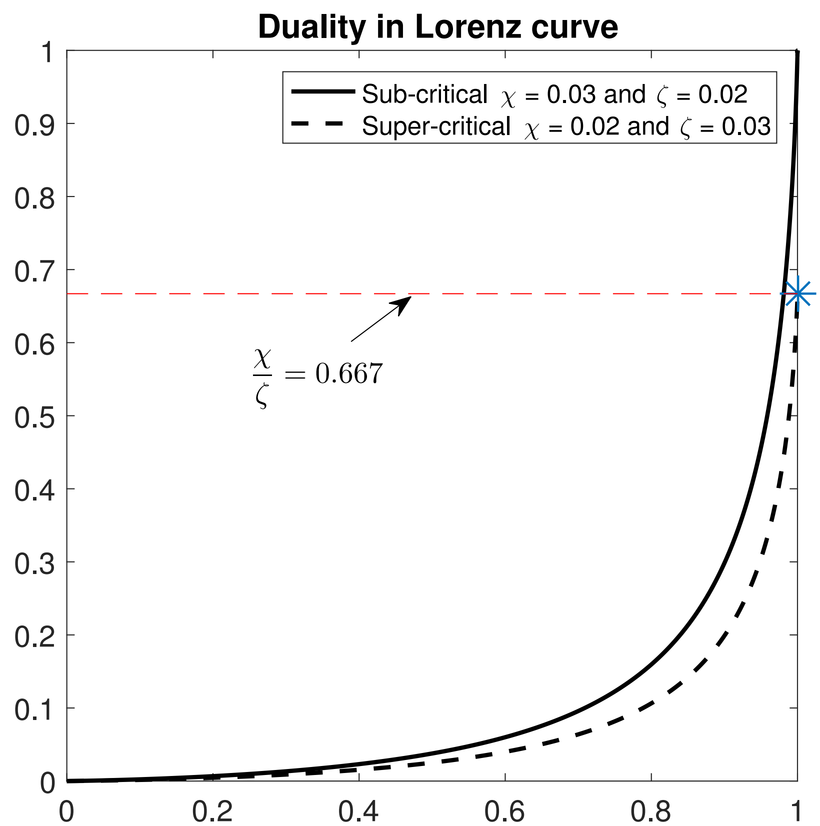


Figure 4.4: Comparison of the Lorenz curves of a subcritical solution (solid curve) and its dual supercritical solution (dotted curve). When scaled vertically, the curves perfectly coincide.

From a practical point of view, the duality provides a numerical algorithm for solving for the Lorenz curve. To compute the Lorenz curve for a supercritical distributional solution, we can just compute that for its dual subcritical classical solution and scale it appropriately.

4.4 Discussion and Conclusion

We have demonstrated a very non-trivial one-to-one correspondence between two classes of steady-state solutions of the agent-based asset-exchange model considered in [5]. The first are distributional solutions of the corresponding Fokker-Planck equation, which are characterized by wealth condensation and oligarchy, and which we refer to as *supercritical* solutions. The second are classical solutions of the corresponding Fokker-Planck equation, which exhibit neither wealth condensation nor oligarchy, and which we refer to as *subcritical solutions*. We have identified this one-to-one correspondence as an example of the phenomenon of duality.

More specifically, we have shown that the wealth distribution of the non-oligarchical part of a supercritical distribution is precisely equal to the subcritical solution obtained by swapping the redistribution and WAA coefficients. If we think of the ratio of these two coefficients as the order parameter $z = {}^{\zeta}/_{\chi}$, then the swapping of the two coefficients is equivalent to taking the inverse of z. As noted earlier, this is very similar to the Krammer-Wannier duality where the free energy of an Ising model with a high temperature is "dual" to an Ising model with the inverse of the temperature of the previous one. Hence, the order parameter $z = \zeta/\chi$ in this context plays the role of temperature in the Ising model.

We presented two mathematical arguments explaining the origin of the above-described duality, one based on the macroscopic Fokker-Planck description of the agent steady state, and the other based on the microscopic process-level description thereof. From the microscopic description, we were able to identify the crucial balance between the effects of taxation and redistribution on the oligarchy on one hand, and those of biased transaction outcomes on the non-oligarchical population on the other.

It should be noted that hints of the existence of this duality were present in earlier work. In [5], for example, it was noted that for very large values of w, the distribution exhibits an asymptotic Gaussian tail of the form:

$$\exp\left(-a|\zeta - \chi|w^2 - bw\right). \tag{4.42}$$

With hindsight, it is evident that the above equation is indicative of the symmetry described in this chapter – if you swap χ and ζ, the Gaussian tail would decay at exactly the same rate *. From the results of this chapter, however, we can see this symmetry is indicative of a much deeper exact symmetry between the subcritical and supercritical steady-state solutions to the model.

As noted above, the presence of duality has already had practical benefit in reducing the numerical work involved in finding supercritical solutions of the model. We hope that it will also have theoretical benefit in allowing us to understand and analyze the mathematical properties of this fascinating model of wealth distribution.

*The reality is slightly more complicated than this because, although not mentioned in Reference [5], the parameters a and b in Eq. (4.42) will differ above and below criticality. Presumably, this can be accounted for by the fact that the total wealth of a supercritical solution is different from that of its dual subcritical partner, as described in this chapter.

Chapter 5

The Affine Wealth Model

In this chapter, we further extend the EYSM to allow for agents with negative wealth. This will introduce a third parameter κ and a third invariant – invariance under horizontal shift. We call this new model the *Affine Wealth Model* (AWM). Although the Fokker-Planck equation for the AWM is even more complicated than that for the EYSM, luckily it can be derived by applying a straightforward function transformation to the Fokker-Planck equation for the EYSM. Better still, the numerical solution to the AWM, including the form of its Lorenz curves, can also be easily obtained by applying this transformation to the solutions to the EYSM. Therefore, solving the AWM involves very little extra numerical work.

5.1 Definition of model

A key deficiency of the EYSM is its inability to account for negative wealth. Agent wealth is initially positive, and the dynamics keep it so by design, so the agent density function P is supported on $[0, \infty)$. However, negative wealth is widely observed in real world economies. For example, according to data collected by the Federal Reserve, roughly 10.9% of the population of the United States in 2016 had negative wealth [9]. To overcome this deficiency, we generalize the EYSM by demanding a new kind of symmetry. In addition to the multiplicative scalings described by Eq. (2.40), we demand invariance under a certain additive shift of the wealth distribution, to be described below. Because the new model is invariant under both scalings and shifts, we refer to it as the *Affine Wealth Model* (AWM).

In the AWM, the agent density function $P(w)$ is supported on $[-\Delta, \infty)$, where the fixed positive quantity Δ will be described shortly. At the beginning of each transaction, the transacting agents both add Δ to their wealths. With the positive wealths that result, they perform an EYSM transaction. Finally, they both subtract

Δ from their wealths to complete the transaction. A nice feature of this approach is that it is unnecessary to modify the EYSM algorithm to deal with negative wealth.

Note that, even if both agents begin with positive wealths, w and x, the quantity

$$\Delta w = \sqrt{\gamma \Delta t}\ \min(w + \Delta, x + \Delta)\eta \tag{5.1}$$

may be larger than w and/or x, so that an agent may lose more wealth than he/she currently possesses, and thereby end up with negative wealth. The overall effect is to create an EYSM distribution in the "shifted wealth," $\bar{w} = w + \Delta \in (0, \infty)$, but then to transform that distribution, $\bar{P}(\bar{w})$, to be one in terms of w rather than $\bar{w}$, which is easily accomplished as follows,

$$P(w) = \bar{P}(\bar{w}) = \bar{P}(w + \Delta), \tag{5.2}$$

where $\bar{P}$ is an EYSM distribution, since $\mathrm{d}\bar{w}/\,\mathrm{d}w = 1$.

Now $\mathrm{d}w = \mathrm{d}\bar{w}$, so it follows that

$$N = \int_{-\Delta}^{\infty} dw\ P(w) = \int_{0}^{\infty} d\bar{w}\ \bar{P}(\bar{w}) = \bar{N} \tag{5.3}$$

and

$$\begin{aligned} W &= \int_{-\Delta}^{\infty} dw\ P(w)w \\ &= \int_{0}^{\infty} d\bar{w}\ \bar{P}(\bar{w})(\bar{w} - \Delta) = \bar{W} - \Delta\bar{N}, \end{aligned} \tag{5.4}$$

and hence the average wealth, $\mu := W/N$ is given in terms of the shifted average wealth, $\bar{\mu} := \bar{W}/\bar{N}$, by

$$\mu = \bar{\mu} - \Delta. \tag{5.5}$$

Going forward, we write Δ as a fraction of the *shifted* average wealth, $\bar{\mu}$, which is guaranteed to be positive, $\Delta = \kappa\bar{\mu}$, where $\kappa \geq 0$ is a new parameter of the model. It follows that $\mu = (1-\kappa)\bar{\mu}$, and hence

$$\Delta = \lambda\mu = \kappa\bar{\mu}, \tag{5.6}$$

where we have defined

$$\lambda := \frac{\kappa}{1-\kappa} \tag{5.7}$$

for convenience. While there is nothing preventing κ from exceeding one in principle, we shall see that empirically determined values of κ tend to be small.

In similar fashion, we can compute the Lorenz-Pareto potentials, F, A, L and B, in terms of their barred counterparts as follows:

$$\begin{aligned}
F(w) &= \bar{F}(\bar{w}) && (5.8)\\
A(w) &= \bar{A}(\bar{w}) && (5.9)\\
L(w) &= (1+\lambda)\bar{L}(\bar{w}) - \lambda\bar{F}(\bar{w}) && (5.10)\\
B(w) &= \bar{B}(\bar{w}) - \kappa\bar{\mu}^2\left[\bar{L}(\bar{w}) - \kappa\frac{\bar{F}(\bar{w})}{2}\right]. && (5.11)
\end{aligned}$$

The corresponding inverse transformations are then

$$\begin{aligned}
\bar{F}(\bar{w}) &= F(w) && (5.12)\\
\bar{A}(\bar{w}) &= A(w) && (5.13)\\
\bar{L}(\bar{w}) &= (1-\kappa)L(w) + \kappa F(w) && (5.14)\\
\bar{B}(\bar{w}) &= B(w) + \lambda\mu^2\left[L(w) + \lambda\frac{F(w)}{2}\right]. && (5.15)
\end{aligned}$$

Note that the transformation reduces to the identity when $\kappa = \lambda = 0$. For the detailed calculation process for the above relationships, please refer to Appendix A.

5.2 Fokker-Planck equation for the AWM

With the above transformations in hand, and restoring the time dependence of P for a moment, we wish to derive the Fokker-Planck equation for the AWM. We do this by supposing that the shifted wealth distribution is the result of an EYSM, so that $\bar{P}$ and its associated barred Pareto-Lorenz potentials should obey a version of Eq. (2.29).

We begin by writing Eq. (2.25) for the AWM,

$$\Delta\bar{w} = \sqrt{\gamma\Delta t}\ \min(\bar{w},\bar{x})\eta + \chi\left(\bar{\mu}-\bar{w}\right)\Delta t. \tag{5.16}$$

Note that

$$\bar{\mu}-\bar{w} = (\mu+\Delta)-(w+\Delta) = \mu - w, \tag{5.17}$$

so that the redistribution term is invariant under the shift.

Next, we wish to modify Eq. (2.26). Clearly $\bar{w}-\bar{x} = w-x$, so the numerator of the fraction in parentheses in that equation is invariant. In the denominator, we use $\bar{W}/\bar{N}$ since that is guaranteed to be positive. The modified version of Eq. (2.26) for the AWM is then

$$E[\eta] = \zeta\sqrt{\frac{\Delta t}{\gamma}}\left(\frac{\bar{w}-\bar{x}}{\bar{\mu}}\right). \tag{5.18}$$

Since Eqs. (5.16) and (5.18) are identical to Eqs. (2.25) and (2.26) but for the presence of the bars, the equation obeyed by $\bar{P}$ is

$$\begin{aligned}\frac{\partial\bar{P}}{\partial t} + \frac{\partial}{\partial\bar{w}}\Bigg\{&\chi\left(\bar{\mu}-\bar{w}\right)\bar{P}\\ &-\zeta\left[\frac{2}{\bar{\mu}}\left(\bar{B}-\frac{\bar{w}^2}{2}\bar{A}\right)+\left(1-2\bar{L}\right)\bar{w}\right]\bar{P}\Bigg\}\\ &= \frac{\partial^2}{\partial\bar{w}^2}\left[\gamma\left(\bar{B}+\frac{\bar{w}^2}{2}\bar{A}\right)\bar{P}\right].\end{aligned} \tag{5.19}$$

We now insert the transformation described in Eqs. (5.2)–(5.7), and Eqs. (5.12)–(5.15) of section 5.1 to obtain, after some calculation, the Fokker-Planck equation for the AWM,

$$\begin{aligned}\frac{\partial P}{\partial t} \quad + \quad \frac{\partial}{\partial w}&\left\{(\chi-\kappa\zeta)\left(\mu-w\right)P-(1-\kappa)\zeta\left[\frac{2}{\mu}\left(B-\frac{w^2}{2}A\right)+(1-2L)\,w\right]P\right\}\\ &= \frac{\partial^2}{\partial w^2}\left\{\gamma\left[\left(B+\frac{w^2}{2}A\right)+\lambda\mu\left(\mu L+Aw\right)+\frac{\lambda^2\mu^2}{2}\right]P\right\}.\end{aligned}$$

(5.20)

where $w \in [-\lambda\mu, \infty)$.

The equation for the steady-state agent density function can be derived by setting $\frac{\partial P}{\partial t} = 0$ and integrating once with respect to w to obtain

$$\frac{d}{dw}\left\{\gamma\left[\left(B + \frac{w^2}{2}A\right) + \lambda\mu\left(\mu L + Aw\right) + \frac{\lambda^2\mu^2}{2}\right]P\right\}$$
$$= \left(\chi - \kappa\zeta\right)\left(\mu - w\right)P - (1-\kappa)\zeta\left[\frac{2}{\mu}\left(B - \frac{w^2}{2}A\right) + \left(1 - 2L\right)w\right]P, \tag{5.21}$$

again for $w \in [-\lambda\mu, \infty)$. For a detailed calculation, please see Appendix A.

Eqs. (5.20) and (5.21) are one-parameter deformations of Eqs. (2.29) and (2.37), respectively. The former reduce to the latter when κ (and hence $\lambda = \frac{\kappa}{1-\kappa}$) is set to zero. Though the deformed equations appear more complicated, they have the same basic structure – namely a transaction term, a redistribution term and a WAA term.

Furthermore, the WAA term and the redistribution term of Eqs. (5.20) and (5.21) are of the exact same form as Eqs. (2.29) and (2.37), but with two interesting changes: First, the WAA coefficient is scaled by $1-\kappa$; Second, the redistribution rate χ is effectively reduced by $\kappa\zeta$.

For the first observation, we need $\kappa < 1$ to keep the scaled WAA coefficient positive. From Eq. (5.6), this is also equivalent to having a positive mean wealth μ. As will be shown later in Subsections 6.4 and 6.5, empirical fittings suggest that reasonable values of κ are all far less than 1. The effect of a negative WAA coefficient on the solution to the differential equation is a mathematical question out of the scope of this thesis.

For the second observation, we need $\chi > \kappa\zeta$ to keep the reduced redistribution coefficient positive. As will be shown in Eq. (5.33), given that $\kappa < 1$, this is also equivalent to the Lorenz curve hitting the right boundary at a positive value. In other words, the wealth held by the non-oligarchical population must be positive. Again, empirical evidence shows that this inequality is always satisfied by a large margin in reality, and we leave the case when it breaks down as a mathematical question for future study. Our current intuition is that this may cause an instability or even nonexistence of solutions to the differential equation.

5.3 Lorenz curve and duality for the AWM

For the convenience of our discussion, we shall briefly review the model symmetries and duality while giving them new expression in this context. We write the Lorenz curve as

$$\mathcal{L}(\mathcal{F}) := L(F^{-1}(\mathcal{F})) \tag{5.22}$$

for $\mathcal{F} \in [0,1]$. The definitions of F and L are given by Eqs.(4.17) and (2.33).

Let $P(w;\chi,\zeta;W,N)$ denote the solution to Eq. (2.37) for redistribution coefficient χ, WAA coefficient ζ, total number of agents N, and total amount of wealth W. Then Eq. (2.40) can be rewritten as

$$P(w;\chi,\zeta;N,W) = \frac{N}{W/N} P\left(\frac{w}{W/N};\chi,\zeta;1,1\right). \tag{5.23}$$

Now, let

$$\mathcal{L}^{\text{sup}}_{\langle\chi,\zeta\rangle}(\mathcal{F}) := L(F^{-1}(\mathcal{F};\chi,\zeta);\chi,\zeta) \tag{5.24}$$

denote a supercritical Lorenz curve function with redistribution coefficient χ and WAA coefficient $\zeta > \chi$. It follows that swapping χ and ζ will result in a subcritical Lorenz curve function $\mathcal{L}^{\text{sub}}_{\langle\chi,\zeta\rangle}(\mathcal{F})$. Then the duality in terms of the Lorenz curve can be written

$$\mathcal{L}^{\text{sup}}_{\langle\chi,\zeta\rangle}(\mathcal{F}) = \frac{\chi}{\zeta}\mathcal{L}^{\text{sub}}_{\langle\zeta,\chi\rangle}(\mathcal{F}), \tag{5.25}$$

for $\mathcal{F} \in [0,1)$, still assuming that $\zeta > \chi$.

To better understand the nature of the shift invariance, denote solutions to Eq. (5.21) by $P(w;\chi,\zeta,\kappa;W,N)$ for $w \in [-\lambda\mu,\infty)$. From the construction of Eq. (5.21), it is clear that the shifted density function $\bar{P}$, obeys the Fokker-Planck equation for the EYSM, Eq. (5.19), which is the same as that for the AWM when $\kappa = 0$. It follows that we have the new symmetry

$$P(w;\chi,\zeta,\kappa;W,N) = P(w+\lambda\mu;\chi,\zeta,0;(1+\lambda)W,N), \tag{5.26}$$

and so it is possible to solve the steady-state Fokker-Planck equation for the AWM by solving the much simpler version for the EYSM and shifting the result.

The above observation gives us a complete strategy for solving for the agent density function for the AWM. When asked to find $P(w;\chi,\zeta,\kappa;W,N)$ for $w \in [-\lambda\mu,\infty)$:

1. Use Eq. (5.26) to transform it to a problem for which $\kappa = 0$ and $w \in [0,\infty)$.

2. If $\zeta > \chi$ so that the problem is supercritical, use Eq. (5.25) to transform it to a subcritical one.

3. Finally, use Eq. (2.40) to transform the problem to one in so-called *canonical form*, for which $N = W = 1$.

Hence, the only numerical solutions needed for solving the AWM are subcritical, canonical-form solutions for the EYSM.

We can go one step further and directly relate the Lorenz curves of the three stages of the above strategy. We first consider the subcritical case, for which Step 2 is unnecessary. Moreover, we suppose that we start in canonical form, so that Step 3 is unnecessary. We again denote the Lorenz curve of the subcritical AWM with redistribution coefficient χ and WAA coefficient $\zeta < \chi$ by $\mathcal{L}^{\text{sub}}_{\langle\chi,\zeta\rangle}(\mathcal{F}) = L(F^{-1}(\mathcal{F}))$, and that of the corresponding EYSM solution by $\bar{\mathcal{L}}^{\text{sub}}_{\langle\chi,\zeta\rangle}(\mathcal{F}) = \bar{L}(\bar{F}^{-1}(\mathcal{F}))$. Then, using Eqs. (5.8) and (5.10), we have

$$\begin{aligned}
\mathcal{L}^{\text{sub}}_{\langle\chi,\zeta\rangle}(\mathcal{F}) &= L(F^{-1}(\mathcal{F})) \\
&= L(\bar{F}^{-1}(\mathcal{F})) \\
&= (1+\lambda)\bar{L}(\bar{F}^{-1}(\mathcal{F})) - \lambda\bar{F}(\bar{F}^{-1}(\mathcal{F})),
\end{aligned}$$

or

$$\mathcal{L}^{\text{sub}}_{\langle\chi,\zeta\rangle}(\mathcal{F}) = (1+\lambda)\overline{\mathcal{L}}^{\text{sub}}_{\langle\chi,\zeta\rangle}(\mathcal{F}) - \lambda\mathcal{F}, \tag{5.27}$$

which directly relates the Lorenz curve of the subcritical AWM to that of the corresponding EYSM. Note that

$$\mathcal{L}^{\text{sub}}_{\langle\chi,\zeta\rangle}(0) \quad = \quad (1+\lambda)0 - \lambda 0 = 0 \tag{5.28}$$

and

$$\mathcal{L}^{\text{sub}}_{\langle\chi,\zeta\rangle}(1) \quad = \quad (1+\lambda)1 - \lambda 1 = 1, \tag{5.29}$$

as expected.

We next consider the supercritical case in which $\zeta > \chi$, again using the canonical form so that Step 3 is unnecessary. A line of reasoning similar to that used above yields

$$\mathcal{L}^{\text{sup}}_{\langle\chi,\zeta\rangle}(\mathcal{F}) = (1+\lambda)\overline{\mathcal{L}}^{\text{sup}}_{\langle\chi,\zeta\rangle}(\mathcal{F}) - \lambda\mathcal{F}. \tag{5.30}$$

We can now use Eq. (5.25) to rewrite this as

$$\mathcal{L}^{\text{sup}}_{\langle\chi,\zeta\rangle}(\mathcal{F}) = (1+\lambda)\frac{\chi}{\zeta}\overline{\mathcal{L}}^{\text{sub}}_{\langle\zeta,\chi\rangle}(\mathcal{F}) - \lambda\mathcal{F}, \tag{5.31}$$

It still follows that

$$\mathcal{L}^{\text{sup}}_{\langle\chi,\zeta\rangle}(0) = 0, \tag{5.32}$$

but now we have

$$\mathcal{L}^{\text{sup}}_{\langle\chi,\zeta\rangle}(1) = (1+\lambda)\frac{\chi}{\zeta} - \lambda \tag{5.33}$$

for the fraction of wealth held by the non-oligarchical part of the population, and

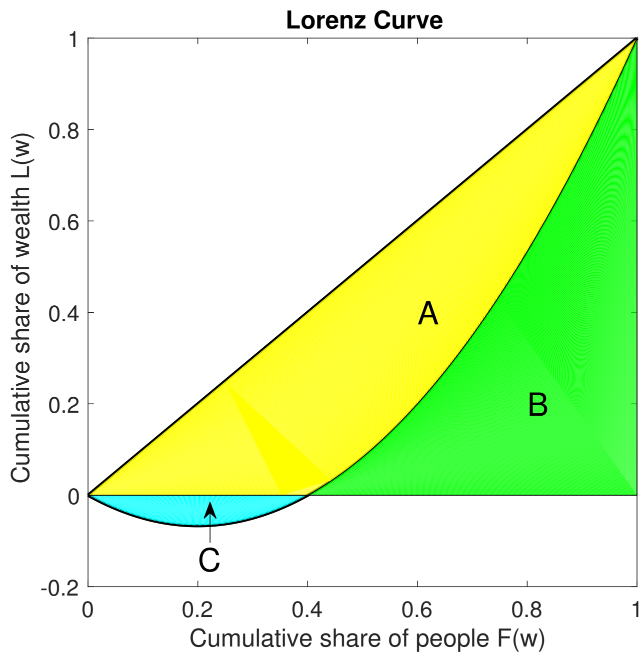
$$1 - \mathcal{L}^{\text{sup}}_{\langle\chi,\zeta\rangle}(1) = (1+\lambda)\left(1 - \frac{\chi}{\zeta}\right) \tag{5.34}$$

for the fraction of wealth held by the oligarchy.

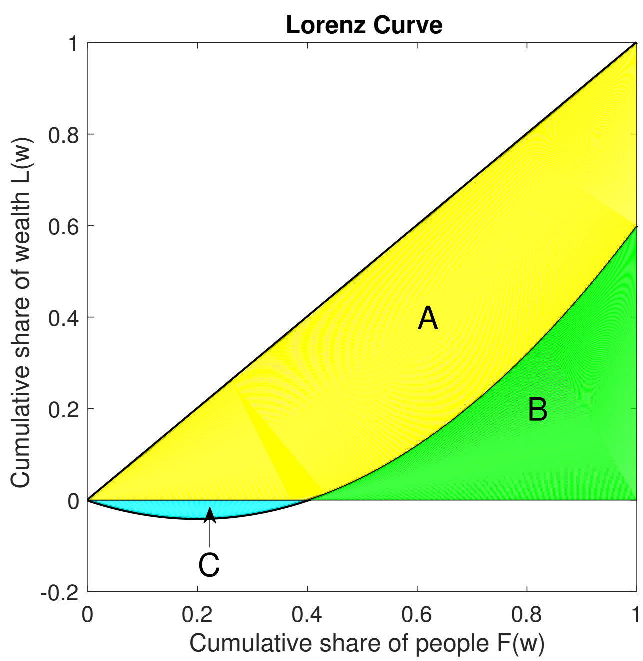
Notice that, given $\kappa < 1$, $\mathcal{L}^{\text{sup}}_{\langle\chi,\zeta\rangle}(1) > 0$ is equivalent to $\chi > \kappa\zeta$, and this is the condition we discussed in a different context in Subsection 5.2.

Using the methodology described in this subsection, we can plot the Lorenz curve for the AWM for any given parameter triplet $\langle \chi, \zeta, \kappa \rangle$ by applying transformations to Lorenz curves for subcritical solutions of the EYSM. This observation enormously facilitated obtaining the fitting results presented later in this thesis. Examples of subcritical and supercritical Lorenz curves for the AWM with negative-wealth agents are presented in Figs. 5.1a and 5.1b.

A key difference between Figure 5.1 and Figure 4.2 is that, because we now allow agents with negative wealth, the Lorenz curve can go below zero. Therefore, it is necessary to modify the geometric definition of the Gini coefficient G as the ratio between certain shaded areas, as shown in Figure 5.1. The mathematical definition of G as described in Eq. (4.19) and Eq. (4.20), however, remains unchanged. That means we can always use these two equations to calculate G, whether there are agents with negative wealth or not. In particular, G is no longer bounded above by one. In extreme cases, G could be greater than one.



(a) Lorenz Curve for AWM subcritical state with negative wealth



(b) Lorenz Curve for AWM supercritical state with negative wealth

Figure 5.1: In the subcritical case with negative wealth, the Lorenz curve dips below zero but terminates at point $(1,1)$. The Gini coefficient is defined to be $G = \frac{A}{A+B}$, where A and B are the areas of the shaded regions. With our AWM and in empirical wealth distributions, the Lorenz curve can: (i), go negative due to the existence of agents with negative total wealth; (ii), hit the right boundary somewhere below point (1,1) due to the existence of oligarchy. The Gini coefficient in this case is defined to be $G = \frac{A-C}{A+B-C}$, where A, B and C are the areas of the shaded regions.

Chapter 6

Empirical Results

6.1 Description of data used

The data we used for the U.S. wealth distribution was taken from the U.S. Survey of Consumer Finances (SCF) conducted by the Federal Reserve Board in cooperation with the U.S. Department of the Treasury [9]. It is a triennial cross-sectional survey of U.S. families, which includes information on families' balance sheets, pensions, income, and other demographic characteristics.

Among the data fields collected for the households surveyed in the SCF is one called `networth`, which represents the total wealth of a household * Because `networth` is calculated as the difference between assets and liabilities, its value can be negative. Indeed, as noted earlier, about 10.9% of the U.S. population has negative net wealth, and so the Lorenz curve for the U.S. actually does dip below zero, as described in Section 5.3.

In the remainder of this section, we will compare several different models with empirical data from the SCF. Of these models, only the AWM is capable of producing a Lorenz curve with negative values. For the other models considered, there will necessarily be a significant error at low wealth, where the Lorenz curve of the model is positive, but that of the data is negative.

For reasons of confidentiality, the published SCF data intentionally excluded people listed on the so-called "Forbes 400" list of the wealthiest people in the U.S. [30]. Because this group of people, though small in number, are so wealthy as to have a nonnegligible impact of the overall distribution especially in the upper tail, we felt it important to add them back into the SCF data. Fortunately, the journal *Forbes*

*Technically, `networth` is not contained in the original microdata of the SCF. Users of SCF data must calculate it themselves by summing over a number of other financial variables that are among the microdata. In fact, an example explaining how to do exactly this is provided with the SCF data [9], and we followed this example closely when preparing the data for this study.

publishes this list of people on an annual basis, including an estimate of their net wealth, so we simply merged the Forbes 400 dataset with the SCF dataset and used their union to conduct our analyses. We checked the resulting dataset by using Eq. (4.19) and Eq. (4.20) to calculate the wealth Gini coefficient, and comparing that to published values; for example, for the 2013 SCF data, we calculated a Gini coefficient of 85.5%, which is very close to that which was reported (85.1%) in the "Global Wealth Databook," published by Credit Suisse in 2013 [46].

The empirical wealth distribution obtained in the manner described above is discretized by groups of households. The jth such group is represented as having net wealth w_j, and weight p_j. The weights p_j are presumably proportional to the number of households in each group, and are normalized over the population so that $\sum_j p_j = 1$. The density function of the wealth can therefore be written as a sum of weighted Dirac delta distributions,

$$P(w) = \sum_{j=1}^{N} p_j \delta(w - w_j). \tag{6.1}$$

It is clear that $N = \int dw\ P(w) = \sum_j p_j = 1$, and $W = \int dw\ P(w) w = \sum_j p_j w_j$. The w_j can all be uniformly scaled so that this last quantity is also equal to one, so that the empirical data is in canonical form, with $N = W = 1$.

To plot the Lorenz ordinates, we need to compute the cumulative sum of the population with wealth less than w and their corresponding cumulative wealth. This is equivalent to plotting the points (f_j, ℓ_j), where

$$f_j \quad := \quad \sum_{i\,:\,w_i \le w_j} p_i \tag{6.2}$$

and

$$\ell_j \quad := \quad \sum_{i\,:\,w_i \le w_j} p_i w_i. \tag{6.3}$$

The empirical Lorenz curve to which we compare our models in this section is a linear interpolation of the Lorenz ordinates (f_j, ℓ_j), described above. Since SCF data is very fine, including tens of thousands of households, the interpolation appears as a smooth curve, as does the numerical solution to our theoretical model.

6.2 Fitting method

The fitting was done by minimizing the L_1 norm of the difference between the empirical Lorenz curve and the theoretical Lorenz curve obtained by numerical solution of our models. Consistent with notation we have already adopted above, if a model's parameters are the components of a *parameter vector* θ, we shall write $\mathcal{L}_\theta(\mathcal{F})$ for the theoretical (model) Lorenz curve corresponding to that parameter vector. If we then write $\mathcal{L}(\mathcal{F})$ for the empirical Lorenz curve, our fitting methodology can be described mathematically as

$$\theta_{\text{optimal}} = \arg\min_\theta J(\theta), \tag{6.4}$$

where we have defined the *discrepancy*

$$J(\theta) := \int_0^1 df\ \left|\mathcal{L}(\mathcal{F}) - \mathcal{L}_\theta(\mathcal{F})\right| \tag{6.5}$$

The choice of the L_1 norm here is inspired by the definition of Gini coefficient; just as the Gini coefficient is twice the area between the Lorenz curve and the diagonal, the discrepancy is equal to the area between the theoretical and empirical Lorenz curves.

The parameter vector θ will have different dimensions for different models. In what follows, we shall consider models with between one and three parameters, so the dimension of θ ranges from one to three. These models are described in detail in Section 6.3. In all cases, there are no guarantees for the concavity of $J(\theta)$ and therefore we employ a global numerical search for the optimal parameter(s).

For some of the results presented in Section 6.5, the model and empirical Lorenz curves are so close that it is difficult to distinguish them graphically. For this reason, we display the local error between the two curves, plotted versus F, in an inset to each of the figures presented. Since the slope of the Lorenz curve ranges from zero (or slightly less) to infinity, defining the local error as the vertical distance between the two curves would be misleading. Instead, we define the local error as the length of a line segment connecting the empirical data point (f_j, ℓ_j) to the model Lorenz curve, constructed so as to be perpendicular to the latter, as shown in Fig. 6.1. If there is more than one such line segment, the length of the shortest is used. In other words, the local error is the shortest distance from the empirical data point to the model Lorenz curve.

For model Lorenz curves in the supercritical regime, when $\mathcal{L}$ is greater than the fraction of wealth held by the non-oligarchical population and less than one, the solution will coincide with the boundary $\mathcal{F} = 1$. A line segment perpendicular to this part of the model Lorenz curve is, therefore, horizontal, so the local error is the horizontal distance from the point (f_j, ℓ_j) to the vertical boundary $\mathcal{F} = 1$, i.e., it is equal to $|f_j - 1|$. This is also shown in Fig. 6.1.

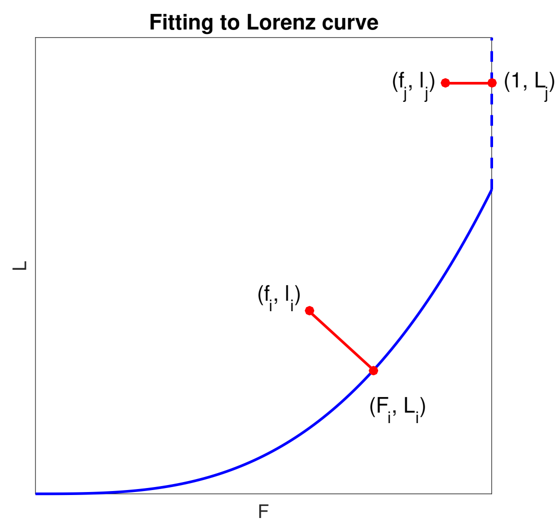


Figure 6.1: Geometry of computing the local errors for the data points (f_i, ℓ_i) and (f_j, ℓ_j) to the fitting curve $\mathcal{L}(\mathcal{F})$. For (f_i, ℓ_i), we compute its perpendicular distance to the point $(f, \mathcal{L}_\theta(\mathcal{F}))$, which is the closest point on $\mathcal{L}_\theta(\mathcal{F})$. While for (f_j, ℓ_j), we compute its horizontal distance to the boundary $\mathcal{F} = 1$.

6.3 Description of models used

6.3.1 Single-agent model

As a baseline for our comparisons, we use a linear model similar to one employed in earlier work by a number of authors. (See, e.g., [8]). Instead of randomly selecting pairs of agents to engage in binary transactions, the model selects single agents to engage in unary transactions. For this reason, we henceforth refer to it as the Single-Agent Model (SAM). In a transaction, an agent with wealth w has even odds of winning or losing a fraction $\sqrt{\gamma\Delta t}$ of his/her own wealth. If we again employ a redistribution term of Ornstein-Uhlenbeck form, as in Eq. (2.21), the analog of that equation is

$$\Delta w = \sqrt{\gamma\Delta t}\, w\eta + \chi\left(\frac{W}{N} - w\right)\Delta t. \tag{6.6}$$

where $E[\eta] = 0$ and $E[\eta^2] = 1$. This statistical process obviously conserves the total number of agents, but it conserves wealth only in a mean sense. The easily derived linear Fokker-Planck equation corresponding to this model,

$$\begin{aligned}\frac{\partial P(w,t)}{\partial t} &= -\frac{\partial}{\partial w}\left[\chi\left(\frac{W}{N} - w\right)P(w,t)\right] \\ &\quad + \frac{\partial^2}{\partial w^2}\left[\gamma\frac{w^2}{2}P(w,t)\right],\end{aligned} \tag{6.7}$$

however, conserves both N and W. In the same spirit as our derivations of Eqs. (2.37) and (5.21), we see that the steady-state solutions of Eq. (6.7) are described by

$$\frac{\mathrm{d}}{\mathrm{d}w}\left[\frac{w^2}{2}P(w)\right] = \chi\left(\frac{W}{N} - w\right)P(w), \tag{6.8}$$

which can be solved analytically. If the constant of integration is chosen to satisfy the normalization requirement of Eq. (2.1), and if we adopt transactional units by taking $\gamma = 1$ as before, the result for the agent density function is

$$P(w) = \frac{N}{\mu}\frac{(2\chi)^{2\chi}}{\Gamma(2\chi)}\left(\frac{\mu}{w}\right)^{2\chi+2} e^{-2\chi\frac{\mu}{w}}, \tag{6.9}$$

where $\mu := W/N$ is the mean wealth. Notice that near $w = 0$ this solution for $P(w)$ is very flat and depleted, while for w very large, it is asymptotically a power-law consistent with the observations of Pareto.

From Eq. (6.9), it is straightforward to calculate

$$F(w) = Q\left(2\chi + 1, \frac{2\chi\mu}{w}\right) \tag{6.10}$$

and

$$L(w) = Q\left(2\chi, \frac{2\chi\mu}{w}\right), \tag{6.11}$$

where $Q := \frac{\Gamma(a,z)}{\Gamma(a)}$ is the regularized upper incomplete gamma function. The inverse of this function is typically denoted by $Q^{-1}(a,z)$, so that $Q(a, Q^{-1}(a,z)) = z$, in terms of which the one-parameter Lorenz curve function is

$$\mathcal{L}^{\mathrm{SAM}}_{\langle\chi\rangle}(\mathcal{F}) = Q\left(2\chi, Q^{-1}\left(2\chi + 1, f\right)\right) \tag{6.12}$$

Note that the parameter vector for this model, $\theta = \langle\chi\rangle$, is one-dimensional since the Lorenz curve depends only on the redistribution coeffiient χ.

6.3.2 EYSM with redistribution

The second model that we consider is the EYSM as described in Section 2.2, with redistribution coefficient χ, but without WAA so $\zeta = 0$. Again, the parameter vector is one-dimensional, and we denote the functional form of the Lorenz curve by $\mathcal{L}^{\mathrm{EYSM}}_{\langle\chi\rangle}(\mathcal{F})$.

6.3.3 EYSM with redistribution and WAA

The third model that we consider is the EYSM as described in Section 2.3, but this time with both redistribution coefficient χ, and WAA coefficient ζ. The parameter vector is therefore two-dimensional, and we denote the functional form of the Lorenz curve by $\mathcal{L}^{\mathrm{EYSM}}_{\langle\chi,\zeta\rangle}(\mathcal{F})$.

6.3.4 AWM

The fourth model that we consider is the AWM as described in Chapter 5. The parameter vector for that model is three-dimensional, $\theta = \langle\chi,\zeta,\kappa\rangle$, and from the discussion in Section 5.3, we know that we can write

$$\mathcal{L}^{\mathrm{AWM}}_{\langle\chi,\zeta,\kappa\rangle}(\mathcal{F}) = (1+\lambda)\mathcal{L}^{\mathrm{EYSM}}_{\chi,\zeta}(\mathcal{F}) - \lambda f, \tag{6.13}$$

where it should be recalled that λ is given by Eq. (5.7).

Now a global search in a three-dimensional parameter space would be computationally expensive. Notice, however, that if we were using the L_2 norm to define the discrepancy instead of the L_1 norm, and if χ and ζ were held fixed, the optimal value of λ would be given by

$$\lambda^{L_2}_{\mathrm{opt}} = \frac{\int_0^1 df\ \left[\mathcal{L}^{\mathrm{EYSM}}_{\langle\chi,\zeta\rangle}(\mathcal{F}) - \ell(\mathcal{F})\right]\left[f - \mathcal{L}^{\mathrm{EYSM}}_{\langle\chi,\zeta\rangle}(\mathcal{F})\right]}{\int_0^1 df\ \left[f - \mathcal{L}^{\mathrm{EYSM}}_{\langle\chi,\zeta\rangle}(\mathcal{F})\right]^2}, \tag{6.14}$$

where $\ell(\mathcal{F})$ is the empirical Lorenz curve. From this we could compute

$$\kappa^{L_2}_{\mathrm{opt}} = \frac{\lambda^{L_2}_{\mathrm{opt}}}{1+\lambda^{L_2}_{\mathrm{opt}}}. \tag{6.15}$$

In our numerical simulations, we used $\kappa^{L_2}_{\mathrm{opt}}$ as the initial guess in a line search for the true optimal value $\kappa^{L_1}_{\mathrm{opt}}$, obtained by minimizing the L_1 norm of the discrepancy.

In the above-described fashion, a three-dimensional optimization problem is reduced to a two-dimensional one in $\langle\chi,\zeta\rangle$ – albeit with a quick line search at each point, for which we have an excellent initial guess. We have found this method to work reliably and significantly reduce the computational work involved.

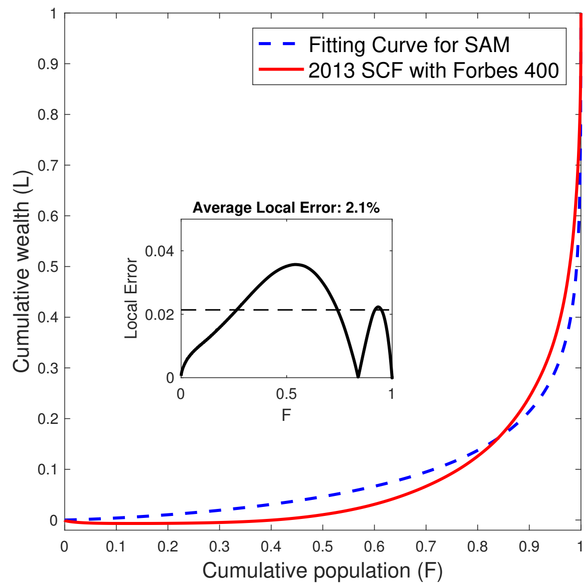
6.4 Comparison of models

In this subsection, we apply the fitting technique of Section 6.2 to all four models described in Section 6.3. For this purpose, we use the 2013 SCF data, including the Forbes 400, as described in Section 6.1. All the fitting results are shown in Fig. (6.2), and all the optimal parameters found as well as the comparisons between the fitting Ginis and the empirical Ginis are summarized in Table.(6.1)

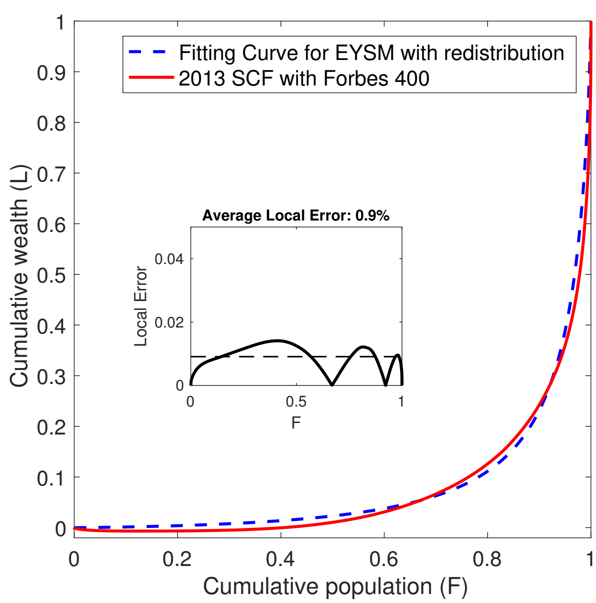
Fig. (6.2a) shows a fitting to the SCF data for the baseline SAM model. Although the Gini coefficient of the fitting curve is close to the empirical Gini, the discrepancy between the two curves clearly leaves something to be desired. The fit suggests that the SAM is unable to capture the behavior of the actual Lorenz curve both in the lower-wealth region and in the upper tail. It seriously overestimates the Lorenz curve in the lower-wealth region, far beyond what can be explained by that fact that it does not allow for negative wealth. Moreover, the SAM's asymptotically power-law tail is seen to badly underestimate the empirical upper tail. The local error incurred, plotted in the inset, has an average in the vicinity of 2%.

Fig. (6.2b) shows the fitting to the SCF data for the EYSM with redistribution only, as described in Section 2.2. This is the simplest nonlinear, binary-transaction model that we considered that yields a stable distribution. Even though there is only a single parameter in this model, namely the redistribution χ, just as there was in the SAM, the extent to which the fit has improved throughout the entire range of f is remarkable. This suggests that nonlinear models with binary transactions have large advantages over linear models. Once again, the largest local error occurs in the low-wealth region, but this time it may well be due to the fact that the model does not allow for negative wealth.

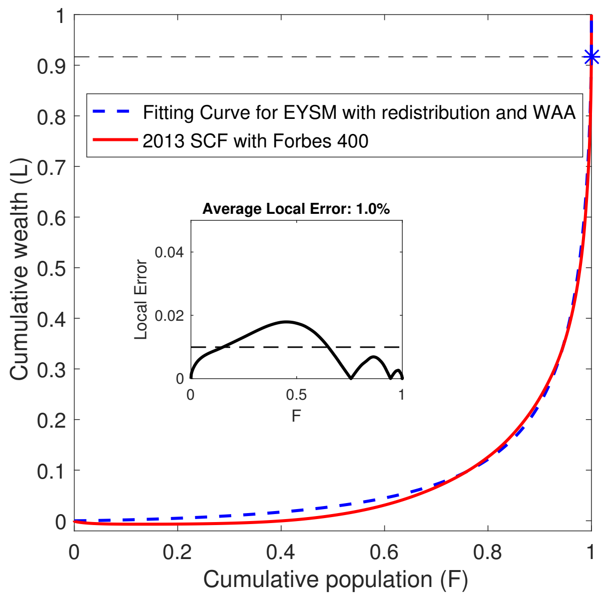
Fig. (6.2c) shows the fit to the 2013 SCF data for the EYSM with both redistribution and WAA, as described in Section 2.3. Recall that this model is capable of wealth condensation. The result clearly demonstrates that the best fit to the data lies in the supercritical regime, suggesting that the U.S. wealth distribution at that



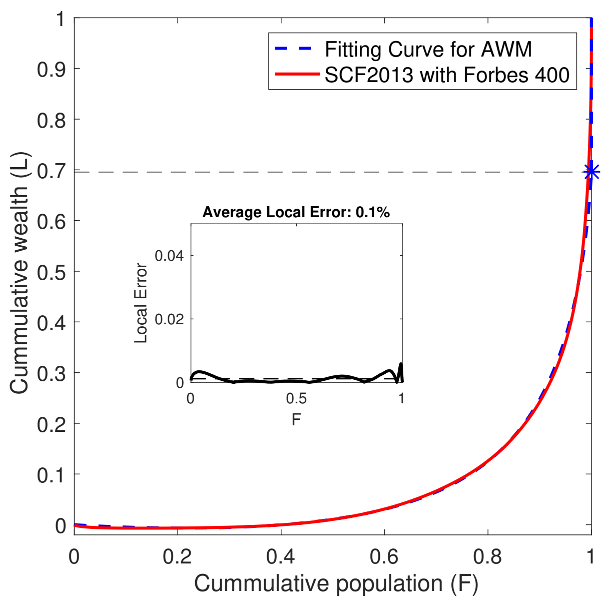
(a) Single Agent Model



(b) EYSM with redistribution



(c) EYSM with redistribution and WAA



(d) Affine Wealth Model

Figure 6.2: Fits of four different models to the empirical Lorenz Curve for the 2013 U.S. Survey of Consumer Finances data, with the addition of the Forbes 400. For each fit, we searched for the optimal parameter vector that minimizes the L_1 norm between the empirical and model Lorenz curves. The local error plotted in the inset was computed as described in Section 6.2. The four fits demonstrate increasing improvement in their ability to fit empirical data, in the order of their presentation. The fit for the AWM, introduced in chapter 5, is for superior to the other three models in its ability to capture the characteristics of the wealth distribution both in the lower-wealth region (including negative wealth) as well as in the the upper tail.

Models	χ_{opt}	ζ_{opt}	κ_{opt}	Fitting Gini	Empirical Gini
Single Agent Model	0.0066	-	-	83.29%	85.50%
EYSM w/ redist.	0.016	-	-	83.85%	
EYSM w/ redist. & WAA	0.022	0.024	-	83.76%	
Affine Wealth Model	0.046	0.064	0.076	85.59%	

Table 6.1: Optimal values of the parameters and fitted Ginis found for each model in Fig. 6.2

time was oligarchical, with 8.33% of the total wealth of the country held by a vanishingly small number of agents. Note that the upper tail of the fit is significantly improved, as compared to the two previous fits. Still, the EYSM does not allow for negative wealth, and hence there is still a large discrepancy in the lower-wealth region.

Finally, Fig. (6.2d) shows the fitting to the 2013 SCF data for the AWM. Even a glance at the figure is sufficient to tell that the AWM is better than any of the other three models considered. Of course, it has more parameters than the others, but three parameters does not seem like a high price to pay for this kind of faithfulness to empirical data. Because the AWM allows for negative wealth, it is able to capture what is happening in the low-wealth region of the Lorenz curve, yet without losing its accuracy in the upper tail. The model and empirical curves lie nearly on top of one another, the discrepancy is reduced by nearly an order of magnitude compared with the other fits, and the average local error is reduced to about only one tenth percent. In summary, we feel that the AWM has provided a reasonable way to extend the EYSM to the negative wealth regime, enabling very accurate quantitative modeling of empirical wealth distribution data.

6.5 Results for the SCF from 1989 to 2016

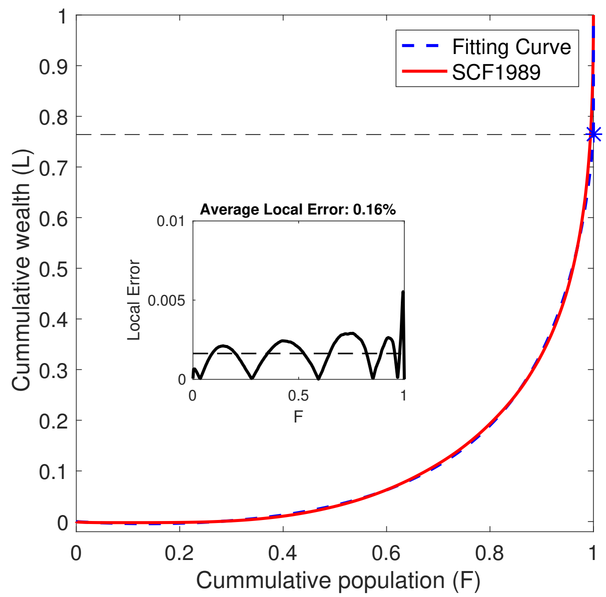
Having established the superiority of the AWM over the other three models considered in Section 6.4, we henceforth restrict our attention to the AWM and examine empirical data from the SCF over the course of time. The SCF has been conducted

Years	χ_{opt}	ζ_{opt}	κ_{opt}	Fitting Gini	Empirical Gini	fraction of wealth held by oligarch
1989	0.088	0.112	0.092	79.17%	78.96%	23.60%
1992	0.102	0.134	0.100	78.72%	78.59%	26.53%
1995	0.104	0.146	0.096	79.27%	79.06%	31.82%
1998	0.096	0.134	0.098	80.20%	79.99%	31.44%
2001	0.074	0.100	0.080	80.86%	80.54%	28.26%
2004	0.070	0.092	0.080	81.07%	80.92%	25.99%
2007	0.070	0.100	0.076	81.81%	81.61%	32.47%
2010	0.046	0.058	0.076	84.59%	84.56%	22.39%
2013	0.048	0.066	0.078	85.24%	85.05%	29.58%
2016	0.036	0.050	0.058	86.18%	85.94%	29.72%

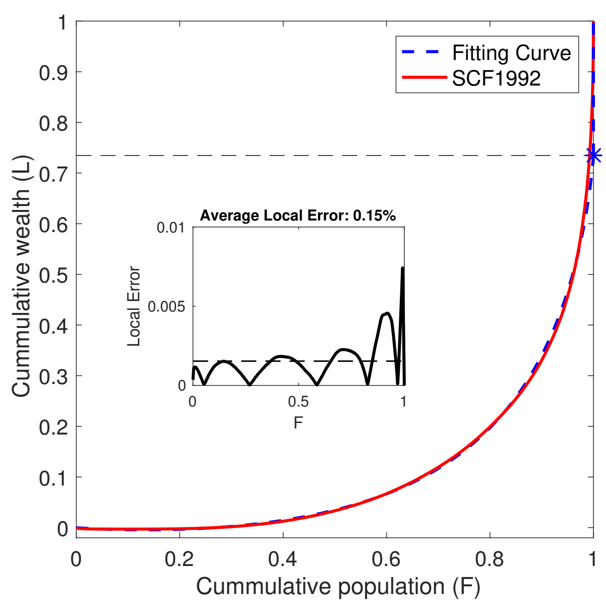
Table 6.2: Optimal values of the parameters and fitted Ginis found for each year in Fig. 6.3

in three years intervals, from 1989 to 2016, so there are ten plots in total. For each of these ten datasets, we solved the "inverse problem" of finding the AWM parameter vector $\theta = \langle \chi, \zeta, \kappa \rangle$ that minimizes the L_1 norm of the difference between the empirical and model Lorenz curves.

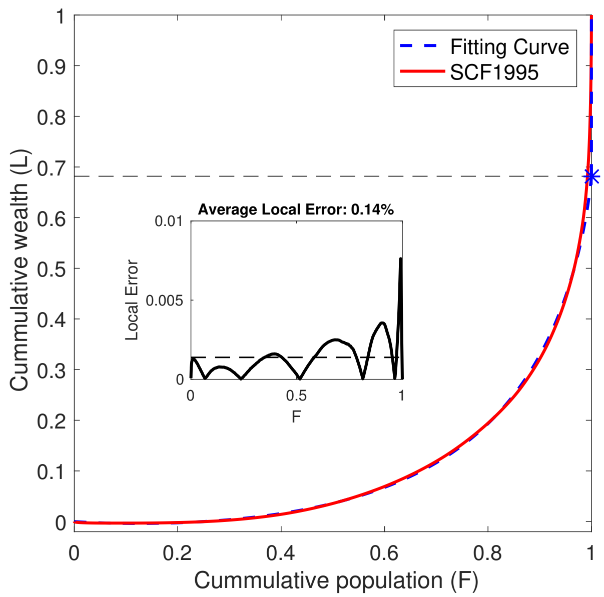
The results of the fits are shown in Fig. 6.3, and all the optimal parameters found as well as comparisons between the fitting Ginis and the empirical Ginis are summarized in Table (6.2). For each year considered, we report the best fittings as well as the optimal parameters. The plots indicate that the AWM fits to the empirical data remarkably well. For all the three-parameter fits, the empirical and model Lorenz curves are nearly indistinguishable, and average local errors are in the vicinity of 0.15%. One key observation is that all fits fall into the supercritical regime with $\chi < \zeta$. Therefore the Lorenz curves computed from the numerical solutions all hit the right-hand boundary at $((1+\lambda)\chi/\zeta - \lambda, 1)$, as described in Eq.(5.33), instead of at $(1, 1)$. The horizontal dotted lines indicate where the Lorenz curves hit the right-hand boundary, and the model Lorenz curve is a vertical line above this point. This strongly suggests that the U.S. wealth distribution has been in a state of partial wealth condensation – or partial oligarchy – for all the years of the SCF. The fraction of the total societal wealth held by the oligarchy is $(1+\lambda)(1-\chi/\zeta)$, by Eq. (5.34). From the plots, we can see that, this fraction is in the vicinity of 20% to 30%.



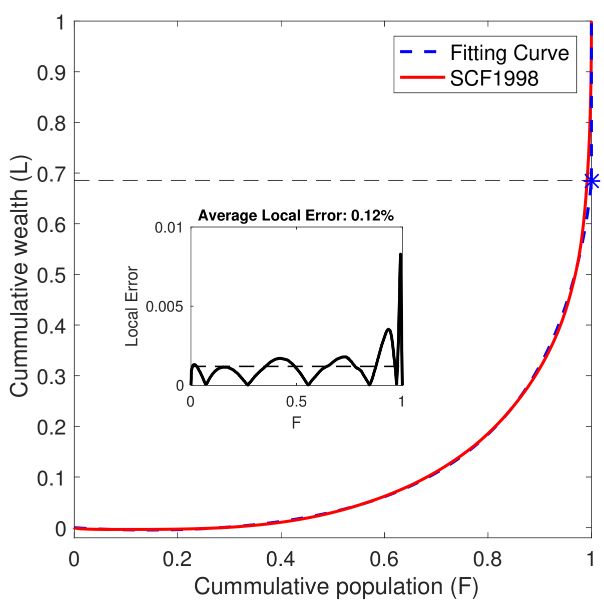
(a) Lorenz Curve Fit of SCF1989



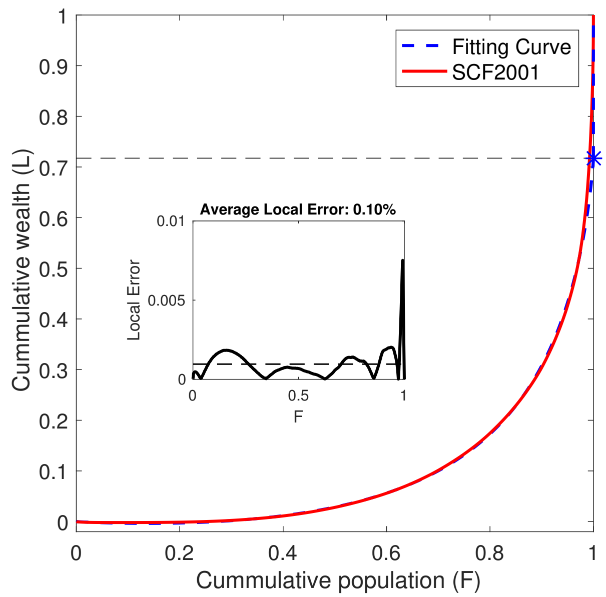
(b) Lorenz Curve Fit of SCF1992



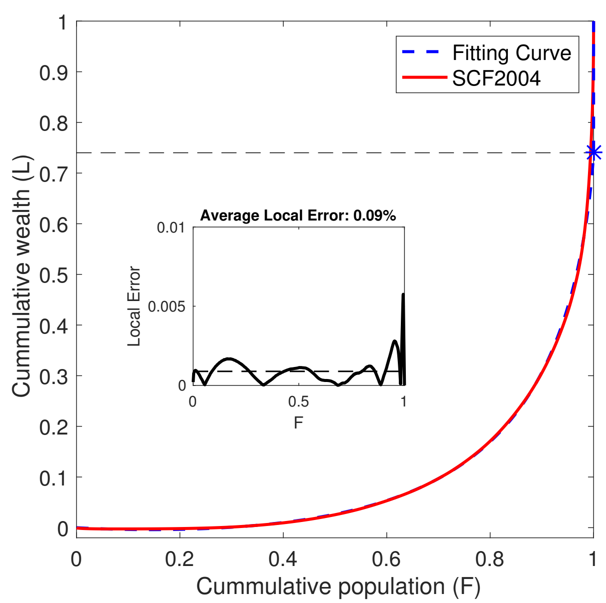
(c) Lorenz Curve Fit of SCF1995



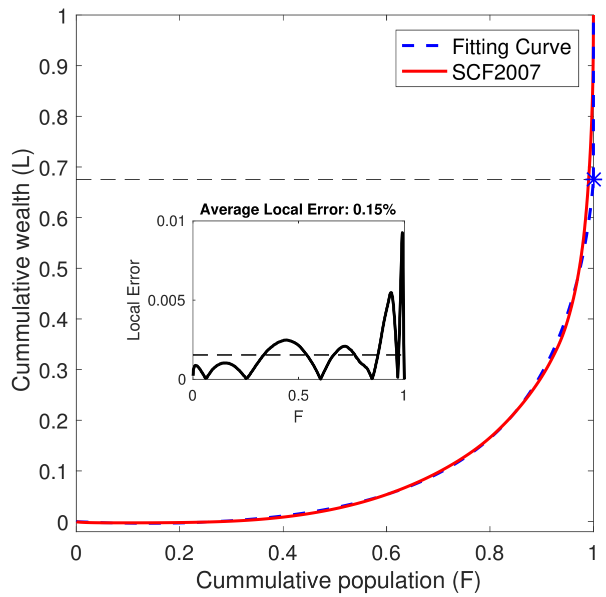
(d) Lorenz Curve Fit of SCF1998



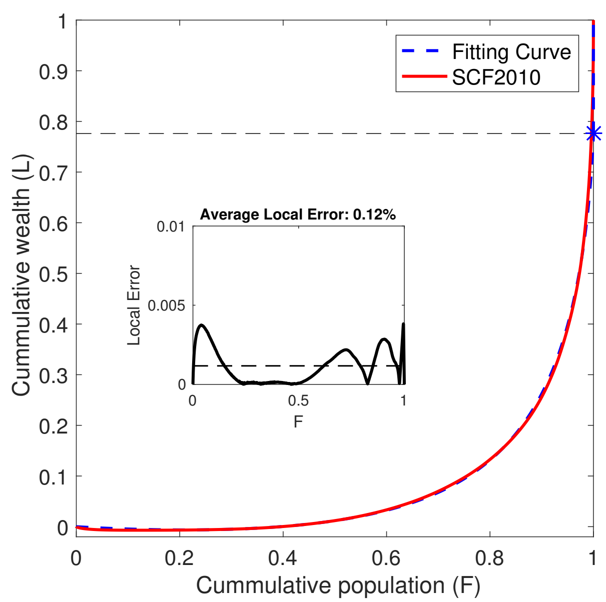
(e) Lorenz Curve Fit of SCF2001



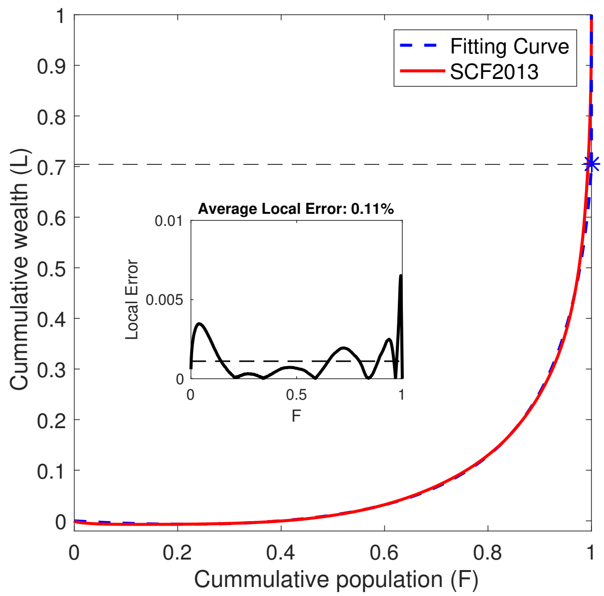
(f) Lorenz Curve Fit of SCF2004



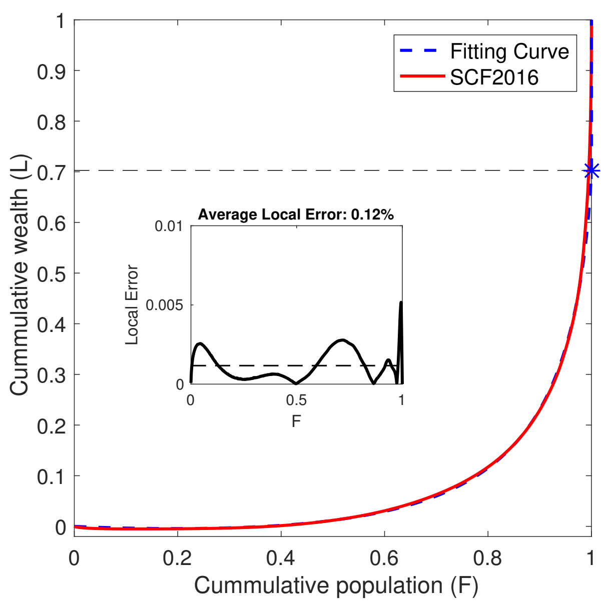
(g) Lorenz Curve Fit of SCF2007



(h) Lorenz Curve Fit of SCF2010



(i) Lorenz Curve Fit of SCF2013



(j) Lorenz Curve Fit of SCF2016

Figure 6.3: Optimal fits of the AWM to SCF data from 1989 to 2016. For each year, we determined the parameter triplet $\langle \chi, \zeta, \kappa \rangle$ that minimizes the L_1 norm of the difference between the empirical and model Lorenz curves. Our results demonstrate that the AWM fits the empirical data remarkably well for all ten datasets. All the fits are in the supercritical regime, which strongly suggests that U.S. wealth distribution is partially wealth-condensed; i.e., a finite fraction of the total wealth of the society is held by an infinitesimal fraction of the agents. This fraction can be estimated by Eq. (5.34), which was consistently in the vicinity of 30% for all the years of the study.

The ten Lorenz curves from the different years appear very similar to one another, but the fitting parameters have varied from year to year. Since each AWM fitting curve is entirely determined by the three model parameters, we can study the trend of the U.S. wealth distribution over time by plotting these optimal values of the parameters versus time.

It should be kept in mind that all of our fits are to *steady-state* Lorenz curves of the AWM. In plotting the AWM fitting parameters versus time, we are therefore supposing that their time variation is *adiabatic* in nature; in other words, we suppose that the variation of the fitting parameters is too slow to induce the $\frac{\partial P}{\partial t}$ term in the Fokker-Planck equation to make any significant contribution.

In Fig. (6.4a), we plot the optimal values of the three parameters $\langle \chi, \zeta, \kappa \rangle$ of the AWM, corresponding to the ten Lorenz curves plotted in Fig. (6.3) as a functions of time. As mentioned earlier, we can confirm that $\chi < \zeta$ throughout this entire period, so the U.S. wealth distribution was oligarchical during the entire 27-year period of the study. It should be pointed out that by "oligarchical" here, we mean a very precise thing: In the space of all valid (classical and distributional) solutions to our Fokker-Planck equation, Eq. (5.20), those solutions closest to the SCF data (in the sense that the L_1 norm of the discrepancy is smallest) are all distributional solutions exhibiting wealth condensation, i.e., for which $\lim_{f \to 1^-} \mathcal{L}(\mathcal{F}) < 1$. This gives a mathematically precise definition of the phenomenon of oligarchy.

A second feature we can observe from Fig. (6.4a) is that κ is much less variable than either χ or ζ. Hence the ratio of the lower extreme of the negative-wealth region to the average wealth is relatively constant over the years.

A third feature that is evident from the plot of the parameters versus time is that there seems to be a correlation between χ and ζ. This may be because these two parameters are, at least to some extent, redundant in their effect on the agent density function. Increasing WAA is similar (though certainly not identical) in effect to decreasing redistribution. Hence, an increase in ζ can be mitigated to some extent by a simultaneous increase in χ. It is therefore perhaps not surprising that the ratio of these two parameters, χ/ζ, is more robust than either one individually. Since both

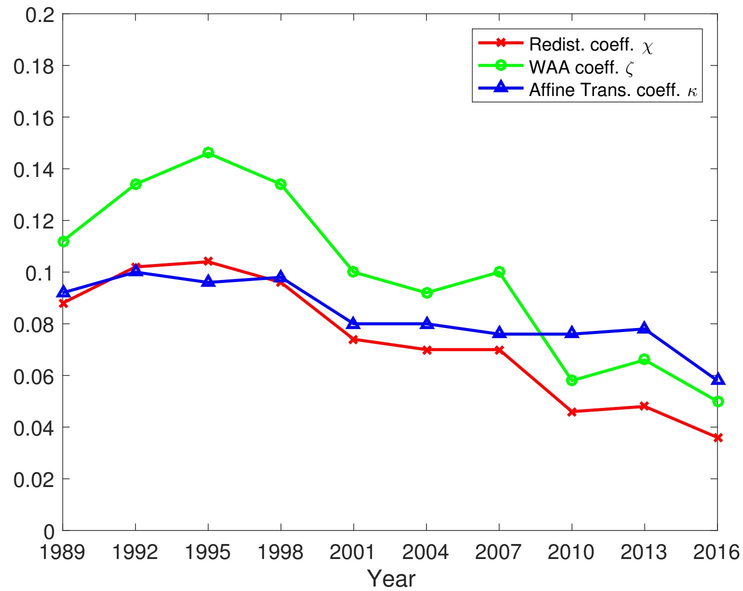
χ/ζ and κ are reasonably constant, from Eq. (5.34) one would expect the oligarchy wealth fraction to be likewise, and this is verified by the lowest curve in Fig. (6.4b), which indicates an oligarchy wealth fraction of between 20% and 30% over the course of the study.

The upper curves in Fig. (6.4b) are plots of the empirical and model Gini coefficients. It is unsurprising that these two curves are nearly identical, given the accuracy of our Lorenz curve fits. Over the 27-year time period of the study, the Gini coefficient has increased from 79% to 86%. This is consistent with figures published by leading economic institutes [13, 46].

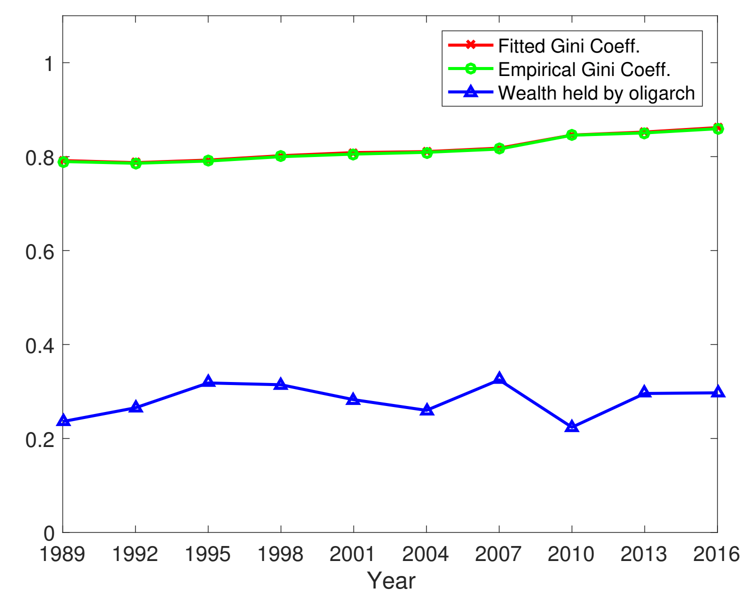
6.6 Conclusions and Future Work

The AWM, described in this work, provides a new model of wealth distribution that is able to describe empirical data with unprecedented accuracy. Because the model's parameters are related to specific features of its agent-level description, the trends of these parameters in time, as shown for example in Fig. (6.4), enable us to glimpse underlying mechanisms for wealth distribution evolution. The AWM thereby, at least to some extent, bridges the gap between microeconomics and macroeconomics. The economic turbulence of 2008, for example, is clearly reflected in a sharp downward movement of the agent-level parameters χ and ζ, accompanied by a pronounced upward movement in the Gini coefficient – the latter arguably being a macroeconomic indicator. For all of the above reasons, we feel that the approach shows great promise, though we acknowledge that a precise relationship between the AWM model parameters and more conventional economic indicators will probably require the involvement of economists, political scientists and public policy specialists to sort out.

Another observation that warrants future study is the nature of the tail of the agent density function, $P(w)$. It has been demonstrated that the non-oligarchical part of the EYSM agent density function for positive χ and nonnegative ζ has a gaussian tail [5, 11]. This seems at least somewhat at odds with – if not in outright



(a) Trend for the three parameters $\langle \chi, \zeta, \kappa \rangle$ in the AWM from 1989 to 2016



(b) Trend for the Gini coefficient (theoretical and empirical) and the fraction of the total wealth condensed to the oligarchy. from 1989 to 2016

Figure 6.4: Since the fitting curves are completely determined by the three parameters of the AWM, the change in wealth distribution over time can be summarized by plotting optimal values of these parameters found as a function of time, under the assumption that they change adiabatically. Our plots demonstrate a correlation between the redistribution parameter χ and the WAA parameter ζ. The affine transformation coefficient κ is less variable than the other two parameters which suggests that the ratio of the lower extreme of the negative-wealth region to the average wealth is relatively stable over the years. The Gini coefficient of the model data is also very close to that of the empirical data. The fraction of the wealth held by the oligarch can be computed by Eq. (5.34), and it is shown that this ratio is relatively stable within the range of 20% to 30%.

contradiction to – the conventionally held belief, dating back to Pareto [43], that wealth distributions have power-law tails. We feel that there are a number of good reasons to question this conventional belief. First, while there seems to be solid empirical evidence for a power-law tail for income distributions [15, 49], much less work has been done for wealth distributions, perhaps owing to the relative paucity of available data †. Second, while the tail of EYSM distributions is always gaussian, the midrange in the limit of small χ has been shown (albeit only for the case $\zeta = 0$) to be nearly power-law in nature [11]. Since our fitted values of χ are in the vicinity of 4% to 10%, this suggests that it is probably very easy to confuse empirical and EYSM distributions with power laws, especially if most of one's data is in the midrange – as is obviously always the case. Third, we have shown that it is mathematically important to separate the oligarchy, which is best described by distribution theory or by nonstandard analysis in the continuum limit [6], from the tail of the classical part of the agent density function, since conflating these will make the the tail seem longer than it really is. This requires delicate numerical analysis which no prior work would have had the motivation to adopt. Because the accuracy of our fits in Fig. 6.3 speak for themselves, we believe that it is time to take a fresh look at the actual empirical evidence underlying the long-held belief in power-law tails of wealth distributions.

There is plenty of room for future work in this area. As mentioned earlier, our fits of the empirical data were made to steady-state solutions of the Fokker-Planck equation for the AWM, Eq. (5.21). Yet the optimal model parameters $\langle \chi, \zeta, \kappa \rangle$ found and plotted in Fig. 6.4 are changing in time. This is no doubt because levels of redistribution, WAA and extreme poverty, respectively, change in time due to public policy and political decisions. Our approach is nonetheless valid under the assumption that the changes in these parameters are slow enough to be adiabatic in nature – i.e., not rapid enough for the time derivative, $\frac{\partial P}{\partial t}$, to become appreciable. Still,

† As of this writing, only about twenty countries in the world directly collect wealth data on their household surveys.

future work might focus on taking the time evolution into account. For this purpose, a more sophisticated fit would be necessary, including time as an independent variable, and fitting to a parameter vector that is a function of time, probably with some smoothness conditions. This would be a much more difficult fit, and we leave it to future work.

Appendix A

Derivation of the Affine Wealth Model

Recall the EYSM with redistribution and WAA as described in Section 2.3, and its Fokker-Planck equation given by Eq. (2.29). In this section, we denote the solution to Eq. (2.29) as $\bar{P}(\bar{w},t)$, while the total number of agents, the total wealth and the Pareto potentials associated with $\bar{P}(\bar{w},t)$ are denoted as $\bar{N}$, $\bar{W}$, $\bar{F}(\bar{w},t)$, $\bar{A}(\bar{w},t)$, $\bar{B}(\bar{w},t)$ and $\bar{L}(\bar{w},t)$, respectively. Correspondingly, we denote the distribution function for the newly introduced AWM by $P(w,t)$ and its associated quantities by N, W, $F(w,t)$, $A(w,t)$, $B(w,t)$ and $L(w,t)$.

The Affine Wealth Model is obtained by shifting the EYSM with redistribution and WAA horizontally by the amount $-\kappa\bar{\mu}$, where we have defined the mean wealth $\bar{\mu}$ to be

$$\bar{\mu} = \frac{\bar{W}}{\bar{N}}. \tag{A.1}$$

Therefore, we have the following transformation between the two independent variables w and $\bar{w}$

$$w = \bar{w} - \kappa\bar{\mu}, \tag{A.2}$$

It is obvious that $w \in [-\kappa\bar{\mu}, \infty)$ while $\bar{w} \in [0, \infty)$. We also have the following relationship between the two distribution functions

$$P(w,t) = \bar{P}\left(w + \kappa\bar{\mu}, t\right) = \bar{P}(\bar{w},t) \tag{A.3}$$

For convenience, we also defined a new parameter

$$\lambda = \frac{\kappa}{1-\kappa}. \tag{A.4}$$

To obtain the Fokker-Planck equation for the AWM, we need to be able to write each barred quantity in terms of the quantities without bars. We use the variable transformation described in Eq. (A.2),

$$x = \bar{x} - \kappa\bar{\mu}. \tag{A.5}$$

We carry out the following calculations for each quantity without bars. For N

$$\begin{aligned} N &= \int_{-\kappa\bar{\mu}}^{\infty} P(w,t)\,\mathrm{d}w \\ &= \int_{0}^{\infty} \bar{P}(\bar{w},t)\,\mathrm{d}\bar{w} \\ &= \boxed{\bar{N}.} \end{aligned} \tag{A.6}$$

For W

$$\begin{aligned} W &= \int_{-\kappa\bar{\mu}}^{\infty} wP(w,t)\,\mathrm{d}w \\ &= \int_{0}^{\infty} (\bar{w} - \kappa\bar{\mu})\,\bar{P}(\bar{w},t)\,\mathrm{d}\bar{w} \\ &= \int_{0}^{\infty} \bar{w}\bar{P}(\bar{w},t)\,\mathrm{d}\bar{w} - \kappa\bar{\mu}\int_{0}^{\infty} \bar{P}(\bar{w},t)\,\mathrm{d}w \\ &= \bar{W} - \kappa\frac{\bar{W}}{\bar{N}}\bar{N} \\ &= \boxed{(1-\kappa)\bar{W}.} \end{aligned} \tag{A.7}$$

For $F(w,t)$

$$\begin{aligned} F(w,t) &= \frac{1}{N}\int_{-\kappa\bar{\mu}}^{w} P(x,t)\,\mathrm{d}x \\ &= \frac{1}{\bar{N}}\int_{0}^{\bar{w}} \bar{P}(\bar{x},t)\,\mathrm{d}\bar{x} \\ &= \boxed{\bar{F}(w,t).} \end{aligned} \tag{A.8}$$

For $A(w,t)$

$$\begin{aligned} A(w,t) &= 1 - F(w,t) = 1 - \bar{F}(\bar{w},t) \\ &= \boxed{\bar{A}(\bar{w},t).} \end{aligned} \tag{A.9}$$

For $B(w,t)$

$$\begin{aligned}
B(w,t) &= \frac{1}{N}\int_{-\kappa\bar{\mu}}^{w}\frac{x^2}{2}P(x,t)\,\mathrm{d}x \\
&= \frac{1}{\bar{N}}\int_{0}^{\bar{w}}\frac{(\bar{x}-\kappa\bar{\mu})^2}{2}\bar{P}(\bar{x},t)\,\mathrm{d}\bar{x} \\
&= \frac{1}{\bar{N}}\left\{\int_{0}^{\bar{w}}\frac{\bar{x}^2}{2}\bar{P}(\bar{x},t)\,\mathrm{d}\bar{x}-\kappa\bar{\mu}\int_{0}^{\bar{w}}\bar{x}\bar{P}(\bar{x},t)\,\mathrm{d}\bar{x}+\frac{\kappa^2\bar{\mu}^2}{2}\int_{0}^{\bar{w}}\bar{P}(\bar{x},t)\,\mathrm{d}\bar{x}\right\} \\
&= \frac{1}{\bar{N}}\int_{0}^{\bar{w}}\frac{\bar{x}^2}{2}\bar{P}(\bar{x},t)\,\mathrm{d}\bar{x}-\kappa\bar{\mu}^2\frac{1}{\bar{W}}\int_{0}^{\bar{w}}\bar{x}\bar{P}(\bar{x},t)\,\mathrm{d}\bar{x}+\frac{\kappa^2\bar{\mu}^2}{2}\frac{1}{\bar{N}}\int_{0}^{\bar{w}}\bar{P}(\bar{x},t)\,\mathrm{d}\bar{x} \\
&= \boxed{\bar{B}(\bar{w},t)-\kappa\bar{\mu}^2\bar{L}(\bar{w},t)+\frac{\kappa^2\bar{\mu}^2}{2}\bar{F}(\bar{w},t).} \qquad (A.10)
\end{aligned}$$

For $L(w,t)$

$$\begin{aligned}
L(w,t) &= \frac{1}{W}\int_{-\kappa\bar{\mu}}^{w}xP(x,t)\,\mathrm{d}x \\
&= \frac{1}{(1-\kappa)\bar{W}}\int_{0}^{\bar{w}}(\bar{x}-\kappa\bar{\mu})\,\bar{P}(\bar{x},t)\,\mathrm{d}\bar{x} \\
&= \frac{1}{1-\kappa}\frac{1}{\bar{W}}\int_{0}^{\bar{w}}\bar{x}\bar{P}(\bar{x},t)\,\mathrm{d}\bar{x}-\frac{\kappa}{1-\kappa}\frac{1}{\bar{N}}\int_{0}^{\bar{w}}\bar{P}(\bar{x},t)\,\mathrm{d}\bar{x} \\
&= \frac{1}{1-\kappa}\bar{L}(\bar{w},t)-\frac{\kappa}{1-\kappa}\bar{F}(\bar{w},t) \\
&= \boxed{(1+\lambda)\bar{L}(\bar{w},t)-\lambda\bar{F}(\bar{w},t).} \qquad (A.11)
\end{aligned}$$

We can reverse the relationships to obtain

$$\bar{N} = \boxed{N} \qquad (A.12)$$

$$\bar{W} = \boxed{\frac{W}{1-\kappa}} \qquad (A.13)$$

$$\bar{F}(\bar{w},t) = \boxed{F(w,t)} \qquad (A.14)$$

$$\bar{A}(\bar{w},t) = \boxed{A(w,t)} \qquad (A.15)$$

$$\bar{L}(\bar{w},t) = \boxed{(1-\kappa)L(w,t)+\kappa F(w,t)} \qquad (A.16)$$

$$\bar{B}(\bar{w},t) = \boxed{B(w,t)+\lambda\mu^2L(w,t)+\frac{\lambda^2\mu^2}{2}F(w,t)}. \qquad (A.17)$$

We also define the mean wealth,

$$\mu = \frac{W}{N}, \tag{A.18}$$

so we have

$$\begin{aligned}
\bar{w} &= w + \kappa\bar{\mu} \\
&= w + \kappa\frac{\bar{W}}{\bar{N}} \\
&= w + \kappa\frac{1}{1-\kappa}\frac{W}{N} \\
&= \boxed{w + \lambda\mu}.
\end{aligned} \tag{A.19}$$

We can plug the above transformations back into the Fokker-Planck equation for the EYSM with redistribution and WAA to obtain the equation for the AWM, Eq. (2.29), which reads

$$\frac{\partial \bar{P}}{\partial t} + \frac{\partial}{\partial \bar{w}}\left[\chi\left(\bar{\mu} - \bar{w}\right)\bar{P}\right] - \frac{\partial^2}{\partial \bar{w}^2}\left[\gamma\left(\bar{B} + \frac{\bar{w}^2}{2}\bar{A}\right)\bar{P}\right] \tag{A.20}$$

$$= \frac{\partial}{\partial \bar{w}}\left\{2\zeta\left[\frac{1}{\bar{\mu}}\left(\bar{B} - \frac{\bar{w}^2}{2}\bar{A}\right) + \bar{w}\left(\frac{1}{2} - \bar{L}\right)\right]\bar{P}\right\}. \tag{A.21}$$

The time derivative term is unchanged,

$$\frac{\partial \bar{P}}{\partial t} = \frac{\partial P}{\partial t}.$$

The redistribution term is

$$\begin{aligned}
&\frac{\partial}{\partial \bar{w}}\left[\chi\left(\bar{\mu} - \bar{w}\right)\bar{P}\right] \\
= {} & \frac{\partial}{\partial \bar{w}}\left\{\chi\left[\frac{W}{(1-\kappa)N} - \left(w + \kappa\frac{W}{(1-\kappa)N}\right)\right]P\right\} \\
= {} & \frac{\partial}{\partial w}\left\{\chi\left[\left((\lambda+1)\mu - \lambda\mu\right) - w\right]P\right\} \\
= {} & \frac{\partial}{\partial w}\left[\chi\left(\mu - w\right)P\right].
\end{aligned} \tag{A.22}$$

The transaction term is

$$\begin{aligned}
&\frac{\partial^2}{\partial \bar{w}^2}\left[\gamma\left(\bar{B}+\frac{\bar{w}^2}{2}\bar{A}\right)\bar{P}\right] \\
= \quad & \frac{\partial^2}{\partial w^2}\left[\gamma\left(B+\lambda\mu^2 L+\frac{\lambda^2\mu^2}{2}F+\frac{(w+\lambda\mu)^2}{2}A\right)P\right] \\
= \quad & \frac{\partial^2}{\partial w^2}\left[\gamma\left(\bar{B}+\lambda\mu^2 L+\frac{\lambda^2\mu^2}{2}F+\frac{w^2}{2}\bar{A}+\lambda\mu\bar{w}\bar{A}+\frac{\lambda^2\mu^2}{2}A\right)P\right] \\
= \quad & \frac{\partial^2}{\partial w^2}\left\{\gamma\left[\left(B+\frac{w^2}{2}A\right)+\lambda\mu\left(\mu L+wA\right)+\frac{\lambda^2\mu^2}{2}\right]\bar{P}\right\}. \qquad \text{(A.23)}
\end{aligned}$$

The WAA term is

$$\frac{\partial}{\partial \bar{w}}\left\{2\zeta\left[\frac{\bar{N}}{\bar{W}}\left(\bar{B}-\frac{\bar{w}^2}{2}\bar{A}\right)+\bar{w}\left(\frac{1}{2}-\bar{L}\right)\right]\bar{P}\right\}, \qquad \text{(A.24)}$$

where we can break down and show the calculations term by term. For the first term in the bracket, we have

$$\begin{aligned}
&\bar{B}-\frac{\bar{w}^2}{2}\bar{A} \\
= \quad & B+\lambda\mu^2 L+\frac{\lambda^2\mu^2}{2}F-\frac{(w+\lambda\mu)^2}{2}A \\
= \quad & B+\lambda\mu^2 L+\frac{\lambda^2\mu^2}{2}F-\frac{w^2}{2}A-\lambda\mu wA-\frac{\lambda^2\mu^2}{2}A \\
= \quad & \left(B-\frac{w^2}{2}A\right)+\lambda\mu\left(\mu L-wA\right)+\frac{\lambda^2\mu^2}{2}\left(F-A\right), \qquad \text{(A.25)}
\end{aligned}$$

so that

$$\begin{aligned}
&\frac{\bar{N}}{\bar{W}}\left(\bar{B}-\frac{\bar{w}^2}{2}\bar{A}\right) \\
= \quad & (1-\kappa)\frac{1}{\mu}\left(B-\frac{w^2}{2}A\right)+(1-\kappa)\lambda\left(\mu L-wA\right)+\frac{(1-\kappa)\lambda^2\mu}{2}\left(F-A\right) \\
= \quad & (1-\kappa)\frac{1}{\mu}\left(B-\frac{w^2}{2}A\right)+\kappa\left(\mu L-wA\right)+\frac{\kappa\lambda\mu}{2}\left(F-A\right). \qquad \text{(A.26)}
\end{aligned}$$

For the second term in the bracket, we have

$$\bar{w}\left(\frac{1}{2}-\bar{L}\right)$$

$$
\begin{aligned}
&= (w+\lambda\mu)\left(\frac{1}{2}-(1-\kappa)L-\kappa F\right)\\
&= \frac{1}{2}w-(1-\kappa)wL-\kappa wF+\frac{1}{2}\lambda\mu-\kappa\mu L-\kappa\lambda\mu F. \qquad \text{(A.27)}
\end{aligned}
$$

Now, combine the above two terms, we can simplify and obtain the expression for the whole term in the bracket in the WAA term. The calculation follows

$$
\begin{aligned}
&\frac{\bar{N}}{\bar{W}}\left(\bar{B}-\frac{\bar{w}^2}{2}\bar{A}\right)+\bar{w}\left(\frac{1}{2}-\bar{L}\right)\\
=\ & (1-\kappa)\frac{1}{\mu}\left(B-\frac{w^2}{2}A\right)+\underline{\kappa\mu L}-\kappa wA+\frac{\kappa\lambda\mu}{2}F-\frac{\kappa\lambda\mu}{2}A\\
&+\frac{1}{2}w-(1-\kappa)wL-\kappa wF+\frac{1}{2}\lambda\mu-\underline{\kappa\mu L}-\kappa\lambda\mu F\\
=\ & (1-\kappa)\frac{1}{\mu}\left(B-\frac{w^2}{2}A\right)-\underline{\kappa wA}+\frac{\kappa\lambda\mu}{2}F-\frac{\kappa\lambda\mu}{2}A\\
&+\frac{1}{2}w-(1-\kappa)wL-\underline{\kappa wF}+\frac{1}{2}\lambda\mu-\kappa\lambda\mu F\\
=\ & (1-\kappa)\frac{1}{\mu}\left(B-\frac{w^2}{2}A\right)-\kappa w+\underline{\frac{\kappa\lambda\mu}{2}F}-\frac{\kappa\lambda\mu}{2}A\\
&+\frac{1}{2}w-(1-\kappa)wL+\frac{1}{2}\lambda\mu-\underline{\kappa\lambda\mu F}\\
=\ & (1-\kappa)\frac{1}{\mu}\left(B-\frac{w^2}{2}A\right)-\kappa w-\underline{\frac{\kappa\lambda\mu}{2}F}-\underline{\frac{\kappa\lambda\mu}{2}A}\\
&+\frac{1}{2}w-(1-\kappa)wL+\frac{1}{2}\lambda\mu\\
=\ & (1-\kappa)\frac{1}{\mu}\left(B-\frac{w^2}{2}A\right)-\kappa w-\underline{\frac{\kappa\lambda\mu}{2}}+\frac{1}{2}w-(1-\kappa)wL+\underline{\frac{1}{2}\lambda\mu}\\
=\ & (1-\kappa)\frac{1}{\mu}\left(B-\frac{w^2}{2}A\right)+(1-\kappa)\frac{1}{2}\lambda\mu-\frac{1}{2}\kappa w+\left(\frac{1}{2}w-\frac{1}{2}\kappa w\right)-(1-\kappa)wL\\
=\ & (1-\kappa)\frac{1}{\mu}\left(B-\frac{w^2}{2}A\right)+\left(\frac{1}{2}\kappa\mu-\frac{1}{2}\kappa w\right)+\frac{1}{2}(1-\kappa)w-(1-\kappa)wL\\
=\ & (1-\kappa)\frac{1}{\mu}\left(B-\frac{w^2}{2}A\right)+(1-\kappa)w\left(\frac{1}{2}-L\right)+\frac{1}{2}\kappa\left(\mu-w\right). \qquad \text{(A.28)}
\end{aligned}
$$

Therefore, the whole WAA term can be written as

$$
\frac{\partial}{\partial w}\left\{(1-\kappa)\zeta\left[\frac{2}{\mu}\left(B-\frac{w^2}{2}A\right)+w\left(\frac{1}{2}-L\right)\right]P\right\}+\frac{\partial}{\partial w}\left[\kappa\zeta\left(\mu-w\right)P\right]. \qquad \text{(A.29)}
$$

Finally, combining every term from the above calculations, we obtain the full Fokker-Planck equation for the Affine Wealth Model

$$\begin{aligned}
&\frac{\partial P}{\partial t}+\frac{\partial}{\partial w}\left[(\chi-\kappa\zeta)(\mu-w)P\right]-\frac{\partial}{\partial w}\left\{(1-\kappa)\zeta\left[\frac{2}{\mu}\left(B-\frac{w^2}{2}A\right)+w\left(\frac{1}{2}-L\right)\right]P\right\}\\
=\;&\frac{\partial^2}{\partial\bar{w}^2}\left\{\gamma\left[\left(\bar{B}+\frac{\bar{w}^2}{2}\bar{A}\right)+\lambda\mu\left(\mu L+wA\right)+\frac{\lambda^2\mu^2}{2}\right]P\right\}. \qquad \text{(A.30)}
\end{aligned}$$

Bibliography

[1] J. Angle, *The Surplus Theory of Social Stratification and the Size Distribution of Personal Wealth*, Social Forces, 65 (1986), pp. 293–326.

[2] R.J. Barro, *Macroeconomics, 4th ed.*, John Wiley & Sons Inc., 1993.

[3] B.M. Boghosian, *Fokker-Planck description of Wealth Dynamics and the Origin of Pareto's Law*, International Journal of Modern Physics C, 25 (2014), p. 1441008.

[4] ——, *Kinetics of wealth and the Pareto law*, Physical review. E, Statistical, nonlinear, and soft matter physics, 89 (2014), p. 042804.

[5] B.M. Boghosian, A. Devitt-Lee, M. Johnson, J. Li, J.A. Marcq, and H. Wang, *Oligarchy as a phase transition: The effect of wealth-attained advantage in a Fokker-Planck description of asset exchange*, Physica A: Statistical Mechanics and its Applications, 476 (2017), pp. 15–37.

[6] B.M. Boghosian, A. Devitt-Lee, and H. Wang, *The Growth of Oligarchy in a Yard-Sale Model of Asset Exchange - a Logistic Equation for Wealth Condensation*, in Proceedings of the 1st International Conference on Complex Information Systems - Volume 1: COMPLEXIS,, INSTICC, SciTePress, 2016, pp. 187–193.

[7] B.M. Boghosian, M. Johnson, and J.A. Marcq, *An H Theorem for Boltzmann's Equation for the Yard-Sale Model of Asset Exchange*, Journal of Statistical Physics, 161 (2015), pp. 1339–1350.

[8] J.-P. Bouchaud and M. Mézard, *Wealth condensation in a simple model of economy*, Physica A: Statistical Mechanics and its Applications, 282 (2000), pp. 536–545.

[9] J. Bricker, L.J. Dettling, A. Henriques, J.W. Hsu, K.B. Moore, J. Sabelhaus, J. Thompson, and R.A. Windle, *Changes in U.S. Family Finances from 2010 to 2013: Evidence from the Survey of Consumer Finances*, Federal Reserve Bulletin, 100 (2014).

[10] Z. Burda, D. Johnston, J. Jurkiewicz, M. Kamiński, M. Nowak, G. Papp, and I. Zahed, *Wealth Condensation in Pareto Macro-Economies*, Physical review. E, Statistical, nonlinear, and soft matter physics, 65 (2002), p. 02610.

[11] R. Bustos-Guajardo, *Distribuciones de fortuna en sistemas multiplicativos con intercambioe inversión*, Ph.D. thesis, Departamento de Fisica Aplicada, Unidad Merida, Mérida, Yucatán, México, (2015).

[12] A. Chakraborti, *Distributions of money in model markets of economy*, International Journal of Modern Physics C, 13 (2002), pp. 1315–1321.

[13] J.B. Davies, S. Sandstrom, A. Shorrocks, and E.N. Wolff, *The level and distribution of global household wealth*, The Economic Journal, 121 (2011), pp. 223–254.

[14] A. Devitt-Lee, H. Wang, J. Li, and B.M. Boghosian, *A nonstandard description of wealth concentrationin large-scale economies*, SIAM J. Appl. Math, (2018).

[15] A. Dragulescu and V.M. Yakovenko, *Statistical Mechanics of Money*, The European Physical Journal B - Condensed Matter and Complex Systems, 17 (2000), pp. 723–729.

[16] A.D. Fokker, *Die mittlere Energie rotierender elektrischer Dipole im Strahlungsfeld*, Annalen der Physik, 348 (1914), pp. 810–820.

[17] Z. G., *A three-headed hydra*, The Economist Group Limited, 2014.

[18] K. Gangopadhyay, *Interview of H. E. Stanley on Econophysics*, IIM Kozhikode Society & Management Review, Sage publication (USA), 2 (2013), pp. 73–78.

[19] A. Ghosh, *Econophysics Research in India in the last two Decades (1993-2013)*, IIM Kozhikode Society & Management Review, Sage publication (USA), 2 (2013), pp. 135–146.

[20] R. Gibrat, *Les Inégalités économiques*, Recueil Sirey, 1931.

[21] C. Gini, *Measurement of Inequality of Incomes*, The Economic Journal, 31 (1921), pp. 124–126.

[22] ——, *On the Measure of Concentration with Special Reference to Income and Statistics*, Colorado College Publication, General Series, 208 (1936), pp. 73–39.

[23] ——, *Variabilità e Mutabilità*, in E. Pizetti and T. Salvemini, Eds, Memo rie di Metodologica Statistica, Libreria Eredi Virgilio Veschi, Rome, 1955.

[24] D. Hardoon, *An Economy for the 99%*, Oxfam Briefing Paper, (2017).

[25] D. Hardoon, S. Ayele, and R. Fuentes-Nieva, *An Economy for the 1%*, 210 Oxfam Briefing Paper, (2016).

[26] B. Hayes, *Computing Science: Follow the money*, American Scientist, 90 (2002), pp. 400–405.

[27] S. Ispolatov, P.L. Krapivsky, and S. Redner, *Wealth Distributions in Asset Exchange Models*, The European Physical Journal B - Condensed Matter and Complex Systems, 2 (1998), pp. 267–276.

[28] M. Janssen, *Microfoundations*, The New Palgrave Dictionary of Economics, 2nd ed., (2008).

[29] R.P. Kanwal, *The Schwartz-Sobolev Theory of Distributions. In: Generalized Functions.*, Birkhäuser, Boston, MA, 2004.

[30] A.B. Kennickell, *Modeling wealth with multiple observations of income: Redesign of the sample for the 2001 Survey of Consumer Finances*, Statistical Journal of the IAOS, 33 (2017), pp. 51–58.

[31] W.I. King, *The Elements of Statistical Method*, New York: Macmillan, 1912.

[32] J.B. Kogut, *An introduction to lattice gauge theory and spin systems*, Reviews of Modern Physics, 51 (1979), pp. 659–713.

[33] A. Kolmogorov, *Über die analytischen methoden in der Wahrscheinlichkeitsrechnung*, Mathematische Annalen, 104 (1931), pp. 415–458.

[34] H.A. Kramers and G.H. Wannier, *Statistics of the two-dimensional ferromagnet*, Physical Review, 60 (1941), pp. 252–262.

[35] H. A. Kramers, *Brownian motion in a field of force and the diffusion model of chemical reactions*, Physica, 7 (1940), pp. 284–304.

[36] J. Li and B.M. Boghosian, *Duality in an asset exchange model for wealth distribution*, Physica A: Statistical Mechanics and its Applications, 497 (2018), pp. 154 – 165.

[37] M.O. Lorenz, *Methods of Measuring the Concentration of Wealth*, Publications of the American Statistical Association, 9 (1905), pp. 209–219.

[38] J. Maldacena, *The Large-N Limit of Superconformal Field Theories and Supergravity*, International Journal of Theoretical Physics, 38 (1999), pp. 1113–1133.

[39] J. E. Moyal, *Stochastic processes and statistical physics*, Journal of the Royal Statistical Society. Series B (Methodological)., 11 (1949), pp. 140–210.

[40] M.I. Norton and Ariely D., *Buildling a better America – One Wealth Quintile at a Time*, Perspectives on Psychological Science, 6 (2011), pp. 9–12.

[41] L. Onsager, *Crystal Statistics I: A two-dimensional model with an order-disorder transition*, Physical Review, 65 (1944), pp. 117–149.

[42] V. Pareto, *The Mind and Society–Trattato Di Sociologia Generale*, Harcourt, Brace, New York, 1935.

[43] ——, *La courbe de la repartition de la richesse*, in G. Busino, editor, *Oevres Completes de Vilfredo Pareto*, page 15. Geneva: Librairie Droz, 1965.

[44] T. Piketty and A. Goldhammer, *Capital in the Twenty-First Century*, Belknap Harvard, 2017.

[45] M. Planck, *Über einen satz der statistischen dynamik und seine erweiterung in der quantentheorie*, Sitzungsber. Preuss. Akad. Wiss., 24 (1917), pp. 324–341.

[46] A. Shorrocks, J. Davies, and R. Lluberas, *Global Wealth Data Book*, Credit Suisse Group AG, 2013.

[47] S. Sinha, A. Chatterjee, A. Chakraborti, and B.K. Chakrabarti, *Econophysics: An Introduction*, Wiley-VCH, Berlin, 2010.

[48] G.E. Uhlenbeck and L.S. Ornstein, *On the Theory of the Brownian Motion*, Phys. Rev., 36 (1930), pp. 823–841.

[49] V.M. Yakovenko and J.B. Rosser Jr., *Colloquium: Statistical mechanics of money, wealth, and income*, Reviews of Modern Physics, 81 (2009), pp. 1703–1725.