II. "The Third Day": Ramified Mathematical Development

A. Propositions VII-XXXVIII: The Mathematical Theory

1. Remainder of the "Third Day" devoted to 32 further propositions, with geometric proofs, establishing relationships among variables that we would now establish algebraically
   a. Propositions VIII-IX continue comparisons started in III-VI
   b. Propositions X-XXVI concern initial speeds and diverting from one initial motion to another; culminates in key Scholium following Prop. XXIII
   c. Propositions XXVII-XXXI concern minimal time trajectories
   d. Propositions XXXII-XXXVI concern time comparisons along different paths; culminates in second key Scholium
2. 16 of the Propositions called Theorems and 16, Problems, where latter give constructions (with ruler and compass) that determine unknown quantities -- e.g. height of equivalent plane
   a. From our point of view, all 32 propositions assert relations among variables of the sort expressed in algebraic equations
   b. E.g. Prop. XXI: let object fall vertically through AC, then divert along plane; space along plane in time through AC is between double and triple AC
3. Symbolic proof of Proposition XXX will help one appreciate task Galileo faced working within compass-and-ruler geometry
   a. Problem: angle of plane to reach given point at a distance along the horizontal fastest
   b. Now $s=1/2*a*\sin(\theta)*t^2$, and $d=s*\cos(\theta)$, where d is a fixed distance along the horizontal
   c. Hence $d=1/2*a*\sin(\theta)*\cos(\theta)*t^2$, so that minimum t to cover d when $\sin(\theta)*\cos(\theta)$ a maximum
   d. Calculus shows this to be when $\theta=45$ deg (as does simple trigonometry) just as Galileo says
4. Many of the results involve acquiring an initial speed, say through vertical fall, then proceeding with this initial speed along some inclined plane -- diverting the motion
   a. Physically not really possible, for do not simply or automatically preserve speed when thus changing direction, though with smooth transition can come close in rolling
   b. Thus primarily expressing mathematical relationships, whether he was aware of this or not, given his scalar notion of speed
   c. Initial vertical fall a mathematical device to specify an initial velocity, something he could not just do numerically; looked on this way, legitimate and clever, as we shall see when we get to projectile motion
5. Also, many of the quoted results are physically wrong because of comparisons of rolling and falling motions, though results hold within each distinct regime of motion
6. These caveats aside, an impressive mathematical development from one definition and one postulate, just as Sagredo acknowledges (p. [266f]

a. Since nothing on local motion remotely comparable to it before, must have been very impressive at the time, illustrating the power of Archimedean approach to science

b. Lots of interesting mathematics and history of mathematics in the results we will be bypassing

B. A Key Result: "Conversion" into Horizontal Motion

1. Some of the most interesting results from our point of view are to be found in the Scholium on diverted motion following Prop. XXIII

   a. Section primarily concerned with consequences of "diverting" motion from vertical into planes, including horizontal

   b. But discussion includes Galileo's clearest remarks on what we now call the principle of inertia

2. A mathematical result: if accelerated motion converted to horizontal, then if s the distance traversed in the accelerated motion in time t, 2*s the distance traversed in the horizontal motion in t

   a. Since measurement of speed more tractable in horizontal, offers a way of measuring maximum speed obtained during acceleration

   b. Of course, cannot divert in case of falling or sliding, but can do so, with care, in case of rolling

   c. Thus, an experimental program in the offing

3. As Galileo makes clear, reasoning here presupposes that no acceleration or deceleration at all along horizontal in the absence of air resistance etc.

   a. Reasoning presupposes that motion along horizontal "equable", which in turn entails "eternal"

   b. Behind this reasoning is claim that all acceleration or deceleration from nature alone involves the vertical

   c. Thus at least suggests a limited form of what we call the principle of inertia

4. Augment line of reasoning with claim that this natural acceleration associated with the vertical is always the same magnitude in ascent and descent, and obtain result that speed acquired in fall always just the amount needed to return to initial height

   a. Paired inclined planes of whatever slope will yield restored height (in absence of resistance etc.)

   b. Take reasoning to limit and have another argument for what one might call a horizontal principle of inertia

   c. But be careful here, for in the limit the surface of the earth is spherical, so that the continuing motion in question may be circular (in keeping with Galileo's conception of eternal motion)

   d. Another possible experimental program with rolling spheres

5. Galileo here brings out most clearly the central physics claim in his theory of local motion: only one natural mechanism of acceleration or deceleration -- i.e. of change of speed -- and it acts along the vertical in a completely uniform way

   a. Not just a mathematical theory giving theorems about uniformly accelerated motion, but a physical theory of (ideal) motion governed by natural processes near the surface of the earth

   b. Can extend to motion constrained by an inclined plane (ignoring difference between rolling and falling); but note that he is unable to extend theory to pendular motion – curvilinear motion constrained to remain at some specified distance from a point
   c. Only result for pendulums experimental: $T \propto \sqrt{\text{length}}$

C. Some Other Interesting (and Testable) Results
   1. Many of the other mathematical results provide at least a basis for a test of the theory, if not a basis for an experimental program
      a. E.g. can test claim that 45 deg plane fastest for covering a given horizontal distance (though problems in maintaining rolling likely to yield exactly the wrong result)
      b. Many of the claims permit (potential) qualitative tests of the theory, and qualitative tests in general more tractable -- something Galileo appreciated
   2. Probably the most interesting of these is that an inclined plane, though the shortest distance between two points, does not yield the shortest time between the points in natural motion (Prop. XXXVI)
      a. So long as slope of plane no greater than 45 deg, less time via two planes forming chords of same circle (assuming complete transfer of speed at point of intersection)
      b. Since proof depends on a large number of earlier results, this theorem in effect pulls a lot of them together
      c. Main idea: speed gained from D to B shortens time from B to C more than time lost in going from D to B
   3. Scholium that follows extrapolates this result into an even more striking claim: circular arc faster than inclined plane even though distance traveled greater than along any sequence of planes
      a. Argument simply by taking reasoning on planes to limit
      b. Galileo wrongly asserts that circular arc the fastest, but correct in asserting that circular arc faster
      c. Problem of determining the fastest path becomes famous in 1691 with Jacob Bernoulli challenge -- answer, the cycloid
   4. This consequence of the theory is quite unexpected and counterintuitive, and hence provides basis for a challenging test of theory
      a. At 90 deg arc the effects of rolling along plane and falling at beginning of circle will confirm the theory for the wrong reason -- reduced acceleration in rolling
      b. But at 45 deg, or maybe 30, with proper surfaces, maybe can maintain rolling of sphere in both cases, providing a true test
      c. But test very difficult to execute -- more likely to get confounded results than meaningful ones, raising the problem of how to assess actual results
   5. Note logic of evidence here: an anomalous consequence of the theory which, if confirmed, shows that theory yielding new knowledge and hence not just recapitulating what is already known