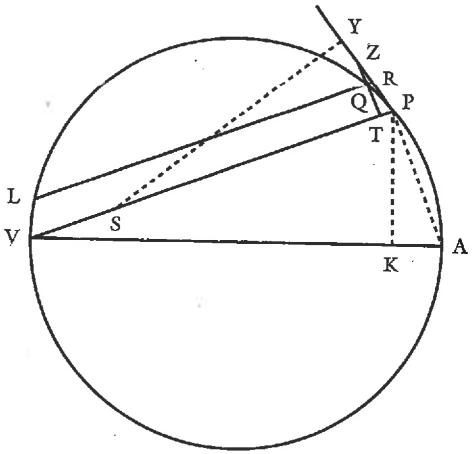


FIGURE 4
Excentric Circle Diagram from Book 1 Proposition 7, Added in the Second Edition

only *quam proxime* a Keplerian ellipse. No matter. For Newton had a much better way of showing this.

In the second edition of the *Principia* he inserted a new proposition (book 1, proposition 5), which appears in his papers from the early 1690s, that gives the rule of centripetal force for an excentric circle—that is, a circle in which a body sweeps out equal areas in equal times relative to a point S off the center. The rule is that the centripetal acceleration and force vary inversely as the product of SP^2 and PV^3. A corollary to this result then provides the basis for the CG^3/RP^2 rule I used above. Now, PV can readily be expressed in terms of SP in the case of a circle. I have instead chosen to apply the CG^3/RP^2 rule to the excentric circle. Either way, the upshot is that the centripetal force and hence acceleration vary inversely as a combination of four terms in which SP occurs respectively in powers of 5, 3, 1, and –1:

$$\left(\frac{SP}{a}\right)^5 + 3(1-\varepsilon^2)\left(\frac{SP}{a}\right)^3 + 3(1-\varepsilon^2)^2\left(\frac{SP}{a}\right) + (1-\varepsilon^2)^3\left(\frac{SP}{a}\right)^{-1}$$

Notice that SP to a power of 2 is nowhere to be found in this expression.[6]

The evidence at the time that the planetary orbits are ellipses was confined to Mercury and Mars; and even in the case of Mercury, the most elliptical of the orbits then known, the minor axis is only 2 percent shorter than the major axis. The orbits really are *nearly* circular—so much so that

an excentric circular orbit together with Kepler's area rule gives results for Venus, Jupiter, and Saturn of the same level of accuracy as Kepler's ellipses gave. In other words, to the level of precision of the data at the time, the orbits of Venus, Jupiter, and Saturn were not observationally distinguishable from excentric circles in which the planets sweep out equal areas in equal times with respect to the Sun. But then, postulating that the orbits are Keplerian ellipses and inferring a force rule exponent of –2 from these ellipses was a risky move.[7] If some of these orbits are Keplerian ellipses only *quam proxime* and excentric circles exactly, then an exponent of –2 will not hold even remotely *quam proxime*.

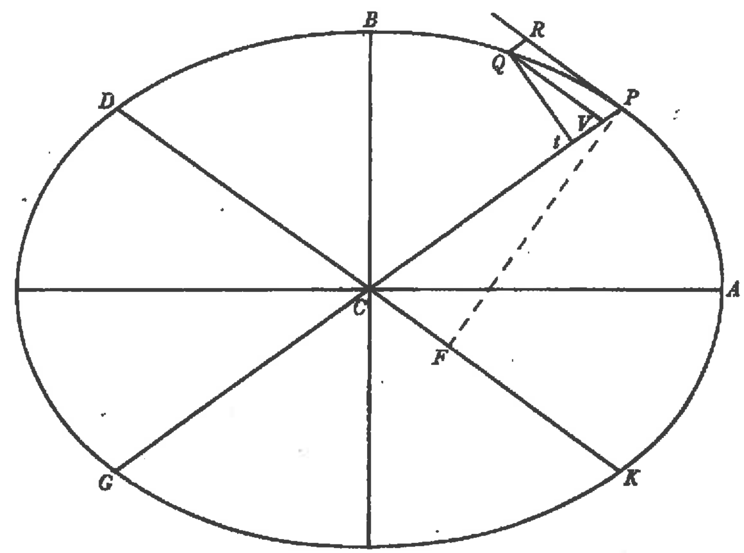
Did Newton know this? I don't know if he took the trouble to derive the full formula, but his published result for the excentric circle is enough to make clear that the inverse-square need not hold *quam proxime*. The more interesting question is not whether he knew it, but when he knew it. The excentric circle proposition first appears in Newton's surviving papers from the early 1690s. But the corollary to it—the force varies inversely with the fifth power when the center of force is on the circumference—appears in the version of *De Motu* registered by the Royal Society in December 1684. Newton scholars have long found this proposition strange, for why would Newton or anyone else have asked what the rule of force is when the center of force is on the circumference? The proof of the full excentric circle proposition is easy, and the diagram for it is virtually the same as the one for the special case that appears in *De Motu*. Perhaps Newton had the full result early on, before writing *De Motu*, and then included only the simple limiting case in the tract. If so, he had already explored the excentric circle before the first edition of the *Principia* and would have seen that the inverse-square need not hold *quam proxime* when the orbit approximating the ellipse too closely approximates an excentric circle as well.

This then is what I regard as the best answer to my "why not" question: even though the Keplerian ellipse entails the inverse-square, one cannot always infer that the inverse-square holds *quam proxime* when the Keplerian ellipse holds only *quam proxime*. In particular, observations at the time were unable to distinguish clearly between the Keplerian ellipse and the excentric circle for Venus, Jupiter, and Saturn. The centripetal acceleration rule for a body in a circular orbit sweeping out equal areas in equal times about any point off center is far removed from inverse-square. What Newton did instead was to infer the inverse-square for the planets from their nearly circular, nonprecessing excentric orbits. This licensed the further inference that the orbits must be exact ellipses in the absence of any further components of acceleration. To quote more of the entry in Huygens's notebook from which I quoted earlier:

Problem 2. Proof for "Central" Ellipse

Background (from Apollonius)

- $PV \times VG : QV^2$ as $PC^2 : CD^2$
- $QV^2 : Qt^2$ as $PC^2 : PF^2$ (from similar triangles, QVt and PFC)
- All circumscribed parallelograms equal in area: $4CD \times PF = 4CB \times CA$

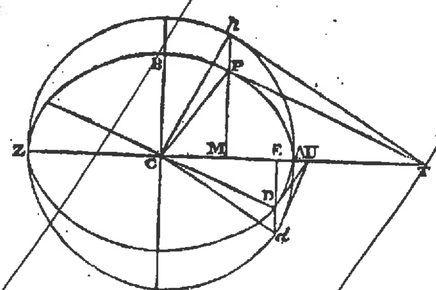


Proof

- $PV \times VG/Qt^2 = (PC^2 \times PC^2)/(CD^2 \times PF^2)$
- $QR \times VG/Qt^2 = PC^4/(BC \times CA)^2$
- But $VG \Rightarrow 2PC$
- $\mathbf{QR/PC^2 \times Qt^2 = PC/2(BC \times CA)^2}$

  $\mathbf{\propto PC}$

P, D; and let PT, pT, DU, dU be tangents to the ellipse and to the circle. (Prop. 4.)



Since Mp, Ed are perpendicular to CA, by similar triangles $CM \cdot MT = Mp^2$, hence (B. 1. Art. 17)

$$CM \cdot MT : MP^2 :: Ed^2 : ED^2,$$

and $CM \cdot MT : Ed^2 :: MP^2 : ED^2$;

whence $CM \cdot MT : Ed^2 :: MT^2 : CE^2$, by parallels PT, CD,

whence $CM \cdot MT : MT^2 :: Ed^2 : CE^2$,

or $CM^2 : CM \cdot MT :: Ed^2 : CE^2$;

therefore $CM^2 : Mp^2 :: Ed^2 : CE^2$,

and $CM : Mp :: Ed : EC$.

Hence the triangles CMp, dEC are similar, the angle dCE is the complement of MCp, and pCd is a right angle. Q. E. D.

Cor. 1. If CD be conjugate to CP, CP is conjugate to CD.

Cor. 2. $CM = Ed$, and $CE = Mp$.

Cor. 3. $CP^2 + CD^2 = AC^2 + BC^2$.

For $CP^2 + CD^2 = CM^2 + MP^2 + CN^2 + ED^2$

$$= CM^2 + MP^2 + Mp^2 + ED^2$$

$$= Cp^2 + MP^2 + ED^2.$$

But $\dfrac{MP^2 + ED^2}{BC^2} = \dfrac{MP^2}{BC^2} + \dfrac{ED^2}{BC^2} = \dfrac{Mp^2}{AC^2} + \dfrac{Ed^2}{AC^2}$

$$= \frac{Mp^2 + CM^2}{AC^2} = 1.$$

Hence $MP^2 + ED^2 = BC^2$, and $CP^2 + CD^2 = AC^2 + BC^2$.

Cor. 4. If $CM = x$, $Ed = x$, $ED^2 = \dfrac{b^2}{a^2}x^2$,

and $CE^2 = Mp^2 = a^2 - x^2$.

Hence $CD^2 = a^2 - x^2 + \dfrac{b^2}{a^2}x^2 = a^2 - \dfrac{a^2 - b^2}{a^2}x^2 = a^2 - e^2x^2$.

Cor. 5. In like manner $CP^2 = b^2 + e^2x^2$.

Cor. 6. $SP \cdot HP = CD^2$. For, as in Art. 14, Book 1, $SP = a + ex$, $HP = a - ex$:

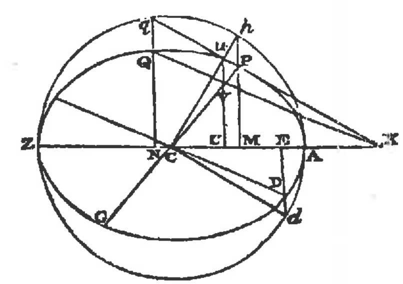
hence $SP \cdot HP = a^2 - e^2x^2 = CD^2$, by Cor. 4.

Prop. VI.

In an ellipse, if QV be an ordinate to a diameter CP, CD the semi-conjugate diameter to CP;

$$PV \cdot VG : QV^2 :: CP^2 : CD^2.$$

Let $APZD$ be an ellipse, GP a diameter, QV an ordinate to the diameter, CD the semi-diameter conjugate to CP. Let $ApZd$ be the circular projection of the ellipse; p, q, d the circular projections of points P, Q, D. Let UV, perpendicular to AZ, meet Cp in v; and join qv.



By Prop. 5, Cd is a perpendicular to Cp. Also

$$Uv : UV :: Mp : MP, \text{ which is } :: Nq : NQ.$$

Hence qv, QV will meet NU in the same point X. And since VX is parallel to CD (because QV is an ordinate), and that $UV : Uv :: ED : Ed$, it is easily seen that Xv is parallel to Cd; and therefore qv is perpendicular to Cp: and hence $Cp^2 - Cv^2 = Cq^2 - Cv^2 = qv^2$.

Now $CP^2 : CV^2 :: Cp^2 : Cv^2$;

$$\therefore\ CP^2 - CV^2 : Cp^2 - Cv^2 :: CP^2 : Cp^2;$$

$$\therefore\ CP^2 - CV^2 : \quad qv^2 \quad :: CP^2 : Cp^2.$$

$$\text{Also}\ qv^2 : QV^2 :: Cd^2\ (Cp^2) : CD^2;$$

∴ compounding the two last propositions,

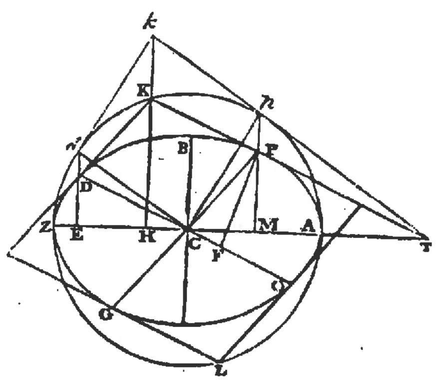
$$CP^2 - CV^2 : QV^2 :: CP^2 : CD^2,$$

$$\text{or}\quad PV.VG : QV^2 :: CP^2 : CD^2.$$

Prop. VII.

The parallelograms made by drawing tangents at the extremities of two conjugate diameters of an ellipse are all equal in area.

Let $APDZ$ be an ellipse, $ApdZ$ its circular projection,



CP, CD two semi-diameters conjugate to each other: KL a parallelogram made by drawing tangents at the extremities of the diameters PC, DC.

Draw HK perpendicular to AZ, meeting in k the tangent of the circular projection at p. Therefore since the tangent of the ellipse and of its circular projection meet AZ in the same point T, we have $HK : Hk :: MP : Mp$, that is, $HK : Hk :: BC : AC$. For the same reason the tangent at D will meet Hk in a point determined by the same proportion. Therefore the two tangents at p and d meet HK in the same point k. And Cp is at right angles to Cd and equal to it; therefore $Cpkd$ is a square.

Now the triangles THK, THk are as their bases $HK : Hk$; that is,

$$THK : THk :: BC : AC.$$

Also $TMP : TMp :: BC : AC$; hence the differences are in the same proportion; that is,

$$\text{trapezium}\ MPKH : MpkH :: BC : AC.$$

In like manner, trapezium $EDKH : EdkH :: BC : AC$.

Also triangle $CPM : CpM :: BC : AC$,

and triangle $CDE : CdE :: BC : AC$.

Add together the two former of these four sets of proportionals, and subtract the two latter, and we have

$$CPKD : Cpkd :: BC : AC.$$

$$\text{Whence,}\ CPKD : Cpkd :: AC.BC : AC^2.$$

But $Cpkd$ is equal to Cp^2 or AC^2. Therefore $CPKD$ is equal to $AC.BC$.

The parallelogram KL is four times the parallelogram $CPKD$. Therefore the parallelogram $KL = 4AC.BC$, and is constant.

Cor. If PF be drawn perpendicular on DC, the parallelogram $CPKD$ is equal to $CD.PF$. Therefore

$$CD.PF = AC.BC.$$