FROM FUNCTIONS AS PROCESS TO FUNCTIONS AS OBJECT:

A REVIEW OF *REIFICATION* AND *ENCAPSULATION*

A qualifying paper

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Abstract

This qualifying paper explores the processes of *reification* (Sfard, 1987, 1989, 1991, 1992) and *encapsulation* (Dubinsky, 1991a, 1991b; Dubinsky & Harel, 1992a), as they apply to the concept of mathematical functions, and attempts to determine how both processes contribute to explaining the formation of the concept of function. The review of empirical studies on students' understanding of functions leads to the conclusion that neither reification nor encapsulation is without flaw; however, they have contributed significantly to partially explaining the formation of the function concept. This implies that more theoretical and empirical work needs to be conducted within this area in order for researchers and educators to have a complete explanation of the formation of the function concept.

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Introduction

Purpose of Paper

The purpose of this qualifying paper is to explore and understand the processes of *reification* and *encapsulation*, as they apply to the concept of mathematical functions, and to determine how both processes contribute to explaining the formation of the concept of function. *Reification*, which is discussed by Sfard (1987, 1989, 1991, 1992), and *encapsulation*, which is discussed by Ayers, Davis, Dubinsky, and Lewin (1988), Breidenbach, Dubinsky, Hawks, and Nichols (1992), Dubinsky (1991a, 1991b), and Dubinsky and Harel (1992a), are two of the major theoretical models[1] used to explain the formation of conceptual entities in mathematics (Harel & Kaput, 1991). The formation of conceptual entities was first discussed by Piaget (1975/1977), when he differentiated between *form* and *content*, where "content is the whole set of sensorimotor processes which produce action, and the form is the system of concepts used by the subject to become aware of this action; therefore to conceptualize this activating content" (Piaget, 1975/1977, p. 147). The analogous terms used in describing concept formation in mathematics are *process* and *object*, where process is "a form of understanding of a concept that involves imagining a transformation of mental or physical objects that the subject perceives as relatively internal and totally under her or his control" (Dubinsky & Harel, 1992b, p. 20), and object is "a form of understanding of a concept that sees it as something to which actions and processes may be applied" (Dubinsky & Harel, 1992b, p.

[1] It should be noted that *reification* and *encapsulation* are not the *only* theoretical models used to explain concept formation. There are other theoretical models, such as *integration operation* (Harel & Kaput, 1991), as well as many others.

19). Thus, *content* can be associated with a *process*, and the *form* can be associated with an *object*.

Rationale for Paper

There are two main reasons for choosing to explore and understand the processes of *reification* and *encapsulation*, as they apply to mathematical functions. First, function is one of the fundamental objects of mathematics (Schwartz, 1999), and furthermore, it is under-recognized as the main mathematical object that should be taught in high school. Second, since function is an object of mathematics, then there must be a form-content distinction in its conceptualization, and hence an explanation for the formation of the concept of function related to this distinction. These two reasons will be discussed in further detail.

There are also two main reasons for choosing to determine how the processes of *reification* and *encapsulation* contribute to explaining the formation of the concept of function. First, if *researchers* understand the conceptualization of the mathematical object of function, they then can conduct more relevant research on students' understanding of function, and in mathematics in general. Second, if *educators* understand the conceptualization of the mathematical object of function, then they will be armed with the necessary knowledge to develop a cognitively appropriate curriculum for students, based on the premise that function is one of the fundamental objects of mathematics, and will also be able to teach this cognitively appropriate curriculum effectively.

Function as an Object of Mathematics

The fundamental objects of mathematics are number and quantity, shape and space, pattern and *function*, data, and arrangements to which the following actions can be applied: representing, formulating, and modeling; manipulating and transforming; inferring and drawing conclusions; and communicating (Schwartz, 1999).[2]

Function as the object of algebra.

Function, as one of the fundamental objects of mathematics, is under-recognized as the main mathematical object that should be taught in the high school curriculum. This is true if we consider the algebra curriculum, which for many students entails at least half the mathematics they are required to learn throughout their high school careers. Other reasons for considering the case of algebra is because it can be viewed as the gatekeeper to higher mathematics achievement (Moses, 2001), as well as because it is one of the main tools used to grant access to almost any form of post-secondary education.

The main object of algebra can be either equations or functions: algebra is generally taught from the perspective of equations, and the alternative approach offered by Schwartz (1999), Schwartz and Yerushalmy (1992), and others explicitly states that function is one of the fundamental objects of algebra. Algebra, *with equations as the main object*, implies that there is a focus on various symbolic manipulations such as factoring, multiplying, and simplifying of different polynomial expressions in order to solve various equations (in one and two variables) and inequalities (Chazan, 1993; Chazan & Yerushalmy, 2003; Yerushalmy & Chazan, 2002).

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[2] This framework and approach has been adopted by the Harvard Group of the Balanced Assessment in Mathematics Project, which is now managed by the Concord Consortium (see http://balancedassessment.concord.org/).

Algebra, *with functions as the main object*, implies that (a) there is a focus on allowing for letters to be interpreted as variables rather than unknowns, (b) expressions to be interpreted as the correspondence rules for functions, (c) the Cartesian coordinate system to be interpreted as a space for displaying the results of calculation procedures rather than the points in a solution set, (d) the equals sign to be interpreted as the assignment of a name to a particular computational process, i.e., *f(x)* =, and to also be interpreted as the indication of identity between two computational processes (Chazan & Yerushalmy, 2003).[3]

In the United States of America (USA), the typical approach to algebra has been from the perspective of equations, rather from that of functions (Yerushalmy & Chazan, 2002). This seems to be strongly related to the fact that functions are a late addition to the subject of mathematics, specifically algebra, appearing only over the past 300 years, as documented in works presented by Kleiner (1989), Malik (1980), Markovits, Eylon, and Bruckheimer (1986), O'Connor and Robertson (2005), Sfard (1992), and Sierpinska (1992). The *equational* approach is not deemed to be sufficient by many mathematics educators, and many researchers in mathematics education have developed theory and research that focus on a *functional* approach to algebra (Dubinsky & Harel, 1992b) instead of an *equational* approach to algebra. For these, the underlying premise is that function is one of the fundamental objects of mathematics. Furthermore, equations could be considered a subset of the broader topic of functions, such that an equation would be defined as the comparison of two functions.

[3] This framework and approach, with functions as the main object of algebra, has been adopted by the Israel Center for Educational Technology and the Haifa University School of Education (see http://cet.ac.il/math-international/first.htm).

In fact, functions contain equations, tables, graphs, etc., and not the other way around (Chazan, 1993). In addition, Dreyfus and Eisenberg (1982) remind us that functions tie algebra, trigonometry, and geometry together in the high school curriculum. Thus, functions are a central aspect of the high school mathematics curriculum, and as Schwartz and Yerushalmy (1992) highlight, function is the fundamental object of algebra. Hence, it would seem that a *functional* approach to algebra, and not an *equational* approach to algebra, such as that currently used overwhelmingly in the USA, would be the most suitable choice with regards to mathematics education. The concept of functions, therefore, deserves the attention I intend to give to it in this paper.

Process and Object Conceptions of Functions

If function is an object of mathematics (i.e., a conceptual entity), then there must be a form-content distinction in its conceptualization. The form-content distinction in the conceptualization of function is often referred to as a process-object distinction (Harel & Kaput, 1991), as alluded to earlier. This process-object distinction applies in general to mathematics, and also specifically to the concept of function, which is a fundamental object of algebra, and hence a fundamental object of mathematics.

Recall that a process conception of a mathematical concept is "a form of understanding of a concept that involves imagining a transformation of mental or physical objects that the subject perceives as relatively internal and totally under her or his control" (Dubinsky & Harel, 1992b, p. 20). That is, a process conception of a mathematical concept is understanding a concept as a procedure or a recipe, which can be thought of as being analogous to Piaget's (1975/1977) definition of content. An object conception of a mathematical concept is "a form of understanding of a concept that sees it as something to which actions and processes may be applied" (Dubinsky & Harel, 1992b, p. 19). Thus, an object conception of a mathematical concept is understanding a concept as a whole or as an entity that can be represented, formulated, modeled, manipulated, transformed, communicated, and upon which conclusions can be inferred and drawn (recall that these are the actions proposed by Schwartz, 1999). This definition can be thought of as being analogous to Piaget's (1975/1977) definition of form.

This process-object distinction, which is recognized implicitly and explicitly by many researchers, such as Ayers, Davis, Dubinsky, and Lewin (1988), Breidenbach, Dubinsky, Hawks and Nichols (1992), Dreyfus and Eisenberg (1982), Dreyfus and Vinner (1982), Dubinsky (1991a, 1991b), Dubinsky and Harel (1992b), Harel and Kaput (1991), Henrici (1974), Pegg (2002), Sfard (1987, 1989, 1991), and Vinner (1983), but sometimes expressed through the use of different terms (see Table 1).

Table 1

Process/Object Terms Used By Different Researchers

Researcher	Term for	
	Process	Object
Ayers, Davis, Dubinsky, and Lewin (1988)	Process	Object
Breidenbach, Dubinsky, Hawks, and Nichols (1992)	Process	Object
Dubinsky (1991a, 1991b)	Process	Object
Dubinsky and Harel (1992b)	Process	Object
Harel and Kaput (1991)	Process	Object
Henrici (1974)	Algorithmic	Dialectic
Piaget (1975/1977)	Content	Form
Sfard (1987, 1989, 1991, 1992)	Operational	Structural

For instance, Henrici (1974) uses the terms algorithmic and dialectic, but his terms refer to *types of mathematics* rather than to *types of conceptions of mathematics*. Algorithmic mathematics is a problem-solving tool that is based on the problem, requires action, and brings about results; dialectic mathematics is based on objects and on rules, and requires contemplation and insight. Here algorithmic mathematics aligns with process conceptions, and dialectic mathematics aligns with object conceptions. Henrici (1974) also points out that these are not the only two polarities in mathematics; other polarities include pure versus applied, abstract versus concrete, theory-oriented versus problem-oriented, and static versus dynamic.

For Henrici (1974), the equilibrium of these polarities in mathematics has been disturbed such that mathematicians give preference to one type of mathematics (usually dialectic mathematics), and neglect the other type of mathematics (usually algorithmic mathematics) within the classroom. This has, therefore, caused a crisis within mathematics today, especially within the teaching and learning of mathematics. The crisis within the teaching and learning of mathematics, specifically functions, is, of course, one of the major concerns of this paper, and the dichotomy to which Henrici (1974) refers is actually present within the conception of functions, i.e., the algorithmic-dialectic distinction within mathematics can be likened to the process-object distinction within the conception of functions.

More process-based mathematics.

Given today's crisis in mathematics, Henrici (1974) argues that the current mathematics curriculum should entail more algorithmically based mathematics (i.e., more process-oriented mathematics), as "the traditional curriculum as offered today is deficient in algorithmic content" (p. 81). This argument may seem to be dated, but it is still relevant today because the polarities within the mathematics curriculum are similar to the polarities that exist within the conceptualizations of mathematical objects, and these polarities will always be present in cognitive development based on Piaget's (1975/1977) theory of the formation of conceptual entities. Henrici (1974) suggests this idea even though dialectic mathematicians believe that learning dialectic mathematics, i.e., object-oriented mathematics, is sufficient, as it will eventually give rise to algorithmic mathematics. He strongly believes that algorithmic mathematics is not a subset of dialectic mathematics, but needs to be taught and nurtured independently of dialectic mathematics.

Henrici (1974), however, does not believe that one should increase the number of algorithmic mathematics courses, but rather incorporate algorithms into the current dialectic mathematics courses. In order to support this point, he makes reference to topics within complex analysis which can be taught from an algorithmic perspective such as the theory of power series, the theory of residues, the theory of conformal mapping, continued fractions, the principal of the maximum, winding numbers, iteration as a tool for finding fixed points, and the connection between the Taylor coefficients of an analytic function and its singularities. He concludes by pointing out that students will be actively engaged in mathematics if it is of an algorithmic form where they can be involved in computations. Even students who are not so strong or who do not understand the abstractness of the process will also be engaged in the mathematics. He further points out that other areas of mathematics that can be addressed in a similar fashion include linear algebra and number theory.

Henrici (1974) makes a clear and concise argument from a theoretical perspective, and even offers ideas on how to address the algorithmic versus dialectic, or if you prefer, the process versus object (or even content versus form), distinction in mathematics within the mathematics classroom. The only problem with his argument is that he presents no empirical research to justify the claim that all students will be actively engaged in mathematics if it is of an algorithmic (process) nature. Neither does he discuss the types of conceptions students will have in mathematics as they pertain to process and object conceptions. These conceptions, including the distinctions in these conceptions, as well as the formation of these conceptions, are discussed in detail by Ayers, Davis, Dubinsky, and Lewin (1988), Breidenbach, Dubinsky, Hawks and Nichols (1992), Dubinsky (1991a, 1991b), and Sfard (1987, 1989, 1991, 1992), and will be thoroughly examined in this paper.

Outline of Paper

This qualifying paper will, therefore, focus on (1) exploring and understanding the process of *reification*, which is proposed by Sfard (1987, 1989, 1991, 1992), as well as the process of *encapsulation*, which is proposed by Ayers, Davis, Dubinsky, and Lewin (1988), Breidenbach, Dubinsky, Hawks, and Nichols (1992), Dubinsky (1991a, 1991b), and Dubinsky and Harel (1992), as they apply to functions, and (2) determining how the processes of *reification* and *encapsulation* contribute to explaining the formation of the concept of function. As stated earlier, this is deemed necessary because function can be viewed as one of the fundamental objects of mathematics (Schwartz, 1999), and as an object of mathematics, it can be conceived of as both a process and as an object. Also, if researchers and educators understand the conceptualization of mathematical functions, then they will be able to conduct useful research, develop appropriate mathematics curricula, and effectively teach the concept of functions.

In order to explore and understand the processes of *reification* and *encapsulation*, as they apply to functions, as well as to determine how the two processes contribute to explaining the formation of the function concept, the following will be done:

- the terminology used in explaining *reification* will be defined,
- the cognitive developmental theoretical framework of *reification* will be presented,
- the history of the development of functions will be traced,
- the process of *reification* will be explained,
- some empirical research based on *reification* will be presented, and
- finally an analysis of *reification* will be offered.

Likewise, a similar procedure, except for the historical development of functions, will be followed for the process of *encapsulation*.

Reification

Terminology

The first major theoretical model on students' conceptions of functions to be examined is found in theory and research by Sfard (1987, 1989, 1991, 1992). She, however, does not use the terms *process* and *object*, but instead uses *operational* and *structural* to highlight the dual nature of mathematical conceptions, such as the function conception (Sfard, 1987, 1989, 1991, 1992), as well as the algebra conception (Sfard & Linchevski, 1994). For Sfard (1992), operational can be likened to process, and structural can be likened to object. This is because the operational nature of a mathematical conception is the notion that it can be perceived as a computational process that is dynamic, while the structural nature of a mathematical conception is the notion that it can behave like an object-like entity that is static. Both conceptions complement each other, and are necessary for effective problem solving to occur. Sfard (1991) uses the term conception in order to indicate understanding or knowledge of a particular concept, which in turn refers to a notion or an idea. In terms of the notion of function, an operational conception would include computational procedures, while a structural conception would include aggregates of ordered pairs (Sfard, 1989). It is in this sense that the terms defined here are used in reviewing theories and research studies on the process of reification.

Cognitive Developmental Theoretical Framework

Let us now turn to the cognitive developmental theoretical framework in which the process of reification is grounded. The idea of reification is grounded in the work of Piaget (1970/1972), as well as Piaget and Garcia (1983/1989). Piaget (1970/1972) examines the origins of knowledge (i.e., genetic epistemology) through the fields of child-psychology and biology. In examining the former, psychogenetic data is analyzed, and in examining the latter, their biological antecedents are analyzed. Piaget (1970/1972) also examines the epistemological problems present in the area of science, specifically logic, mathematics, and physics.

For the purposes of understanding Sfard's (1992) process of reification, only the first of the epistemological problems of mathematics highlighted by Piaget (1970/1972), i.e., its "fruitfulness" despite its poor foundations, is relevant; however, all are mentioned here:

> The epistemology of mathematics has three principal and classic problems: why is it so fruitful though based on very few relatively poor concepts or axioms; why has it a necessary character, thus remaining constantly rigorous despite its constructive character which could be a source of irrationality; and why does it agree with experience or physical reality in spite of its completely deductive nature? (Piaget, 1970/1972, p. 69)

In response to the first problem, Piaget (1970/1972) uses the genetic approach to understand the convergence between what mathematicians say about the construction of mathematical knowledge and what the early stages of child development reveal about the construction of mathematical knowledge. For mathematicians, mathematics is based on the construction of structures, which is a continuous process, and which leads to the formation of mathematical entities:

> Mathematical entities are no longer ideal kinds of objects, given once and for all either from within or from outside us: they no longer have an ontological meaning; their functions alter continuously as they move from one level to another; an operation on such 'entities' becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by 'stronger' structures. (Piaget, 1970/1972, p. 70)

For Piaget (1970/1972), there is a relationship between the formation of mathematical knowledge and a child's process of reflective abstraction. (Reflective abstraction is discussed later in this paper.)

Let us now turn to Piaget and Garcia (1983/1989), in which it is argued that the mechanisms of cognitive development of a concept are comparable to the mechanisms of the historical development of the said concept. This is evident in the following statement:

> Since we are not interested in the contents of the developmental levels, but in their *modes of construction* [italics added], it does not seem to be more far fetched to compare the *mechanisms involved in the sequences of stages* [italics added] in history to that found in psychological development than it is to look for common evolutionary mechanisms at vastly different levels of zoological phyla. (Piaget & Garcia, 1983/1989, p. 139)

They present their theory by tracing the historical development of various concepts in mathematics and science, such as motion, geometry, algebra, and mechanics. They then attempt to identify the obstacles in the historical development of these topics that are similar to the obstacles in the cognitive development of these same topics in the minds of young children in order to prove their point.

Piaget and Garcia (1983/1989) then conclude that the general mechanisms common to both psychogenesis and the history of sciences are: (1) the transition from an intra to an inter to a trans phase, and (2) equilibration. The former mechanism can be described as,

> The intra phase leads to the discovery of a set of properties in objects and events finding only local and particular explanations. The “reasons” to be established can thus be found only in the relations between objects, which means that they can be found only in “transformations”. These, by their nature, are characteristic of the inter level. Once discovered, these transformations require the establishment of

> relations between each other, which leads to the construction of "structures", characteristic of the trans level. (Piaget & Garcia, 1983/1989, pp. 273-274)

This means that psychogenesis, which is defined as the development of a concept from psychological origins, as well as from the history of the sciences, occurs through a three-phase mechanism of composition – intra-object, inter-object, and trans-object. In the first phase, composition is intra-object, i.e., it is exogenous and induced within states or figures. In the second phase, composition is inter-object, i.e., it occurs when transformations between states are mastered. In the third phase, composition is trans-object, i.e., it occurs when endogenous structures are organized by integrating various transformations applied to external objects.

The latter mechanism, equilibration, which is mentioned by Piaget and Garcia (1983/1989), is interpreted by Gruber and Vonèche (1977) to mean the following: "endogenous restructuration is a continuous process … the basic nature of this process is to seek out equilibrium states … *equilibration* is an active process tending toward the growth of intelligence, more and more complex, flexible, and inclusive structures" (p. 783). Thus, it is these general mechanisms (i.e., the transition between phases and equilibration) that are common to psychogenesis and to the history of the sciences.

Next, in order to apply the theoretical framework outlined by Piaget and Garcia (1983/1989) to Sfard's (1992) process of reification, which will be presented later, it is pertinent to have some knowledge about the history of the development of functions.

History of the Development of Functions

In exploring the history of the development of functions, we first see that the concept of function as understood through modern mathematics is actually not present in the ancient mathematics of the Babylonians and the Greeks. Second, we see the possibility of the emergence of the concept of function as a relationship of dependence and then as a relationship between varying quantities. Third, we see the emergence of the concept of function through symbolic and graphical representations. Fourth, we see the struggle in defining the term function from the perspective of algebra to defining it from a perspective based on set theory. Finally, we see that mathematicians settle on a definition of the general form: "A function $f : S \to T$ consists of two sets S and T together with a 'rule' that assigns to each $x \in S$ a specific element of T, denoted $f(x)$" (Marsden & Hoffman, 1993, p. 3), which is known as the Dirichlet-Bourbaki definition.

2000 B.C.E. – 1299 C.E.: Tabular Representations and Trigonometric Functions

The function concept dates back 4000 years to about 2000 B.C.E., however, not all 4000 years are of note. On the one hand, the first 3700 years consisted of "anticipations" of the function concept (Kleiner, 1989, p. 282). On the other hand, the first 3300 years consisted of ideas, which when viewed through the lens of modern mathematics, seem to resemble current ideas of the function concept (O'Conner & Robertson, 2005). This indicates that there is some discrepancy as to how much of those 4000 years are of note, and whether a given historical moment was a part of the development of the function concept as O'Conner and Robertson (2005) might suggest, or whether the historical moment gave rise to the development of function as Kleiner (1989) might suggest.

O'Connor and Robertson (2005) explore the mathematical contributions of the ancient Babylonians and the ancient Greeks to the development of the concept of function, and highlight two major points. First, the mathematics of the Babylonians

(~1700 B.C.E.) was rich in tables, such as “tables of squares of the natural numbers, cubes of the natural numbers, and reciprocals of the natural numbers” (O’Connor & Robertson, 2005). For mathematicians today, functions can be represented through tables; however, as Bell states, “It may be too generous to credit the ancient Babylonians with the instinct for functionality; for a function has been successively defined as a table or a correspondence” (as cited in O’Connor & Robertson, 2005).

Second, Ptolemy (~150 C.E.), who contributed to the mathematics of the Greeks, examined the chords of a circle, which today can be viewed as related to the notion of trigonometric functions. This, however, does not mean that Ptolemy had any real knowledge of trigonometric functions, or even of the concept of function (O’Connor & Robertson, 2005). Thus, it seems that the mathematical ideas of the first 3300 years merely resemble our current notion of the function concept when viewed through the lens of modern mathematics.

1300 C. E. – 1499 C. E.: Relationship of Dependence

This then brings the next 400 years into question. In 1350, Oresme described “the laws of nature as laws giving a dependence of one quantity on another” (O’Connor & Robertson, 2005), which can be viewed as the possibility of the emergence of the concept of function because of the relationship of dependence.

1500 C. E. – 1599 C. E.: Relationship Between Varying Quantities and Variables

Oresme’s work is followed by Galileo’s (1564-1642) work, in which he studied the concept of motion. His study of motion allowed him to investigate the relationship between two varying quantities (Malik, 1980), which can also be viewed as being related to the concept of function. There was also the introduction of a syncopated algebra into mathematics by Viète (1540-1603) in this century. In this syncopated algebra, a vowel was used to represent an unknown quantity and a consonant was used to represent a known quantity or a parameter (Boyer, 1985). This syncopated algebra is indirectly related to the concept of function because algebraic notation, or rather symbolic notation[4], is one of the many ways of representing functions.

1600 C.E. – 1699 C.E.: Symbolic and Graphical Representations

This same syncopated algebra developed into a symbolic algebra through the work of persons such as Descartes (1596-1650), in which he used letters at the beginning of the alphabet to represent a parameter, and letters at the end of the alphabet to represent an unknown (Boyer, 1985). There were also other more significant highlights in the 17th century. First, the 17th century gave birth to analytic geometry, which involved the idea of variables and equations (Kleiner, 1989). In Descartes' (1637) analytic geometry, "curves described by motion or formula referring to motion rather than by construction, were included in investigations and a relation representable in expression and its graph were now accepted as mathematical objects" (Malik, 1980, p. 490). The only problem with analytic geometry, at this time, was that it was devoid of the notion of dependent and independent variables (Kleiner, 1989). Second, the 17th century saw the beginning of the calculus by Newton and Leibniz, however, this calculus focused on geometric curves and not on functions (Kleiner, 1989), and it gave rise to operations applied to formulas. These two big ideas – analytic geometry and the calculus – together, created an environment ripe for the development of the concept of function. Finally, the 17th century witnessed

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[4] The term *symbolic notation* is preferable to *algebraic notation* because this paper is taking the stance that a *functional* approach is better than an *equational* approach to algebra, thus implying that there is more to *algebra* than equations. In addition, the types of representations used in algebra are *not* limited to only one type of representation, as the term *algebraic notation* would suggest. Algebra involves other types of representations such as tabular and graphical representations.

the first mathematical use of the term "function" by Johann Bernoulli in 1694 when he described a function as "a quantity somehow formed from indeterminate and constant quantities" (as cited in O'Connor & Robertson, 2005).

1700 C.E. – 1799 C.E.: Euler's Definition of Function from an Algebraic Perspective

The 18th century was even more eventful than the 17th century as it witnessed the separation of analysis from geometry, where the concept of function as an algebraic expression was used instead of the concept of variable, as it applies to geometric objects (Kleiner, 1989). This is evidenced in the work of Euler, who attempts to define the term function from an entirely algebraic perspective in 1748: "A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities" (as cited in O'Connor & Robertson, 2005). For Euler, the notion of function is "an analytic expression representing the relation between two variables with its graph having no corners" (Malik, 1980, p. 490). Euler, however, never defines the term "analytic expression", but implies that it includes algebraic functions (consisting of the four binary operations, powers, and roots) and transcendental functions (consisting of exponential, logarithmic, and trigonometric functions, as well as derivatives and integrals) (Kleiner, 1989; O'Connor & Robertson, 2005). In addition to defining the term function, it is Euler who first treats the calculus as a theory of functions (Kleiner, 1989). Furthermore, this calculus did not need a more sophisticated definition than the one Euler proposed, and no one seemed to question Euler's definition since it was sufficient for the continuing development of the calculus (Malik, 1980).

The 18th century witnessed yet another milestone in the development of the concept of function – the vibrating string controversy (see Figure 1) – to which persons such as Euler, d'Alembert, Daniel Bernoulli, and Lagrange offered solutions (Kleiner, 1989). The problem of the vibrating string did not have a clear solution, and was different from any problem previously posed. It did, however, have a tremendous impact on the development of the concept of functions because of the nature of the problem.

An elastic string having fixed ends (0 and ℓ, say) is deformed into some initial shape and then released to vibrate. The problem is to determine the function that describes the shape of the string at time t.

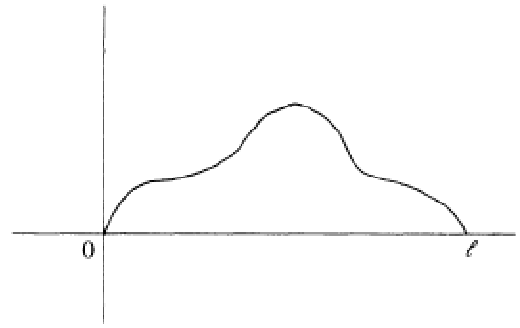

Figure 1. Motion of the vibrating string.

Note. From "Evolution of the Function Concept: A Brief Survey," by I. Kleiner, 1989, *The College Mathematics Journal, 20*(4), p. 285. Copyright 1989 by the Mathematical Association of America. Reprinted with permission.

Kleiner (1989) points out that the result of this controversy on the concept of function was to expand Euler's as well as Bernoulli's definition of function. Functions could now be defined piecewise by more than one analytic expression dependent on the interval, and could also include freehand representations that did not necessarily have an analytic expression.

1800 C.E. – 1899 C.E.: Dirichlet's Definition of Function Based on an Arbitrary Correspondence

The definition of function continued to expand throughout the 19^{th} century, where, in 1805, Fourier introduced what is now known as the Fourier series, which is used to represent some discontinuous functions (O'Connor & Robertson, 2005). The introduction of Fourier series had several effects on the concept of function (Kleiner, 1989). First, it caused the analytic and geometric representations of functions to be equally valued. It also meant that two analytically different functions could be the same in a specified interval, but not necessarily outside this interval. In addition, it challenged mathematicians to re-visit the meaning of a discontinuous function. These changes allowed for the formation and formalism of the subject of analysis, and the re-examination of the concept of function.

The introduction of the Fourier series also invited criticisms from Dirichlet, and from others such as Gauss, Abel, and Cauchy (Kleiner, 1989). Through their criticisms, Cauchy addressed issues concerning continuity, convergence, and the definite integral, and Dirichlet addressed the definition of function. Dirichlet, in studying the convergence of the Fourier series in 1829, found the need to re-define a function as "y is a function of x if for any value of x there is a rule which gives a unique value of y corresponding to x"

(Malik, 1980, p. 491). Thus, for Dirichlet, functions referred to an arbitrary correspondence defined over an interval, whilst for mathematicians before him, functions referred to analytic expressions or curves (Kleiner, 1989). Dirichlet's definition served to clarify the difference between the *definition* of a function and the *representation* of a function (O'Connor & Robertson, 2005). It also gave birth to the notions of *domain* and *range* through the influence of metric space and topology, causing the elevation of the concept of function to a more complex abstract concept (Malik, 1980).

As with previous definitions of a function, Dirichlet's definition was also not readily accepted by persons such as Chebychev, Baire, Borel, and Levesque. Dirichlet's definition opened up a "Pandora's Box", as several other definitions of functions began to emerge throughout the remainder of the century (Kleiner, 1989). Despite this fact, Dirichlet's definition did give birth to analysis, a subject derived from the calculus (Malik, 1980). In this new analysis, because of Dirichlet's definition, it was found that the processes of analysis did not apply to all functions, but rather to some classes of functions (Kleiner, 1989). Mathematicians such as Riemann and Weierstrass began to revel in the exceptions rather than the regularities in analysis. In the middle of the 19^{th} century, Riemann expanded on Cauchy's concept of integral, and thus extended the class of functions that could be represented by the Fourier series; this is now known as the Riemann integral. His work opened up the doors to looking at the mathematically discontinuous. By 1872, Weierstrass introduced his example of a continuous nowhere differentiable function, which led to the separation of the continuous from the differentiable in analysis, and overall to the "arithmetization of analysis"[5], and the use of counterexamples.

The 19th century had one last milestone: Baire's attempt to answer the question as to whether all functions could be represented analytically (Kleiner, 1989). For Baire, a function can be represented analytically if "it can be built up from a variable and constants by a finite or denumerable set of additions, multiplications, and passages to pointwise limits" (Kleiner, 1989, p. 295). Thus, any function consisting of a variable and constants, which was created by the use of the four binary operations of algebra and the operations of analysis, could be represented analytically. From this, Baire created the Baire classification scheme, and the functions that belonged to this scheme were known as Baire functions. He then stated that a function could be represented analytically if it belonged to one of the Baire classes.

1900 C.E. – present: Dirichlet-Bourbaki's Definition of Function

By the start of the 20th century, function theory was the area of mathematics that focused on counterexamples, and in 1905, Lebesgue showed that not all functions could be represented analytically using Baire's definitions through the use of a counterexample (Kleiner, 1989). Furthermore, Dirichlet's definition of function remained under the microscope, as it "was found to be too broad by some (e.g., Lebesgue) and devoid of meaning by others (e.g., Baire and Borel), but was acceptable to yet others (e.g., Hadamard)" (Kleiner, 1989, p. 296).

The 20th century also saw the development of L^2 functions, generalized functions and category theory. L^2 functions form a Hilbert space and allow one to work on

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[5] "It is customary to say that Weierstrass "arithmetized analysis" (a phrase Felix Klein coined in 1895), by which is meant that he freed analysis from the geometrical reasonings and intuitive understandings so prevalent at the time" (Burton, 2003, p 578).

equivalence classes of functions rather than on individual functions. Additionally, operators on a Hilbert space are functions whose argument is a function. Generalized functions or distributions provide a context for differentiating mathematical objects which are not functions, but expand their use in the theory of differential equations. Category theory allowed for the concept of function as a mapping between sets to become acceptable in the 20^{th} century. Algebra and analysis were the main contributors to this development in mathematics.

These 20^{th} century developments allowed for modern definitions of functions based on set theory to emerge, such as Caratheodory's definition of functions in 1917: a function is "a rule of correspondence from a set A to real numbers" (Malik, 1980, p. 491), and Bourbaki's definition of functions in 1939: A function is "a rule of correspondence between two sets" (Malik, 1980, p. 491). By the middle of the 20^{th} century "the Dirichlet-Bourbaki definition of function had become established as textbook terminology" (Malik, 1980, p. 491). An example of such terminology from a currently used college textbook is: "A function $f : S \to T$ consists of two sets S and T together with a 'rule' that assigns to each $x \in S$ a specific element of T, denoted $f(x)$" (Marsden & Hoffman, 1993, p. 3). Definitions equivalent to this one are still being used today in both high school and college; however, the emphasis placed on this type of definition may be dependent on the teacher, the curriculum, and the textbook. For instance, a high school textbook's definition may be of the form: "A function is a relationship between input and output. In a function, the output depends on the input. There is exactly one output for each input" (Holliday, et al, 2003), which is much less complex than the former definition.

Process of Reification

Having discussed the foundations of reification and traced the history of the development of the concept of function, we now turn towards the process of reification as outlined by Sfard (1987, 1989, 1991, 1992). For Sfard (1992), the historical development of functions reveals that they were reified over a period of three centuries, where they were initially viewed operationally, and then finally structurally. For her, the operational view of functions started with the newly invented symbolic algebra at the end of the seventeenth century, where functions were connected to the notion of a computational process, and the structural view of functions emerged with Dirichlet's ideas around set theory and arbitrary correspondence in the early nineteenth century, as well as Bourbaki's ideas concerning ordered pairs in the early twentieth century. Sfard (1992) proposes that this process of reification occurs not only in the historical development of functions, but likewise in an individual's development of the concept of function, where one's conception of functions is first operational and later structural. This aligns with the work of Piaget and Garcia (1983/1989), who investigated the process of psychogenesis as previously discussed, in which they emphasized that the mechanisms of cognitive development of a concept and the mechanisms of the historical development of the said concept are comparable.

The process of reification of a mathematical concept in the history of its development occurs in three hierarchical levels – interiorization, condensation, and finally reification (Sfard, 1992). For Sfard (1992), this notion of a hierarchical model is related to, but not identical to, Piaget's (1970/1972), which was previously discussed, in which, "mathematical entities move from one level to another; an operation on such 'entities' becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by 'stronger' structures" (Sfard, 1992, p. 64). She relies on this model because she believes that mathematics can be viewed "as a hierarchy in which what appears to be a process at one level must be transformed into a full-fledged abstract object at a higher level to become a building block of more advanced mathematical constructs" (Sfard, 1992, p. 64).

The three levels (interiorization, condensation, and reification) form a constant three-step pattern in moving from operational to structural conceptions. Sfard (1992) points out that "first there must be a process performed on the already familiar objects, then the idea of turning this process into a more compact, self-contained whole should emerge, and finally an ability to view this new entity as a permanent object in its own right must be acquired" (p. 64). She then continues by explaining that the latter two levels are quite easy to confuse, however, condensation should be viewed as thinking of functions as a process (with inputs and outputs) minus its in-between steps, and reification should be viewed as thinking of condensed functions as an object-like entity. Furthermore, condensation occurs gradually, whilst reification occurs rather suddenly. It should be noted that unless reification has occurred, a person continues to have an operational conception of function.

There are two potential problems with the process of reification as explained by Sfard (1992). First, in order to understand a concept structurally one may have to give up his/her understanding of it operationally. This may, however, prove to be difficult for a student who relies on the operational nature of the concept as the main source of its meaning. Second, reification does not occur unless there is an initial operational understanding, and a lower-level reification is necessary in order for a higher-level interiorization to occur. This means that if there are two concepts in which the understanding of the latter is dependent on the understanding of the former, then the former needs to be reified before the latter can be interiorized. Moreover, Sfard (1992) further claims that since the operational conception develops before the structural conception, it makes little sense to introduce and teach a new concept from a structural perspective, and, in fact, there is no need to introduce topics from a structural perspective until it is actually needed. These latter claims are used to ground her empirical research, to which I will now turn with the aim of evaluating the justification of her proposal for the process of reification.

Empirical Research

Two of the empirical studies conducted by Sfard will be discussed in this section. The goals, the sample and methods used, as well as the findings of each study will be summarized in turn. I have chosen to discuss the studies in chronological order, for us to get some sense of the growth in research, as well as how one study may or may not have impacted the other.

Study 1: Order of Operational and Structural Conceptions of Functions

In Sfard's (1987) first study, the goal was to determine the order in which the operational and structural conceptions of functions emerged, as well as to examine the theoretical and practical educational implications of the different types of conceptions of functions. In order to do the above, she collected two sets of data. In the first, the sample consisted of 60 secondary school students (31 of whom were 16 years old, and 29 of whom were 18 years old), who had already been exposed to the formal structural definition of function, as well as the notion of function. Data was collected through a questionnaire on the definition of a function and the characteristics of functions. As hypothesized, the results indicated that most students had an operational rather than a structural conception of functions. This was confirmed by the fact that many students believed that all functions possessed an algorithmic process.

In the second set, the sample consisted of 96 secondary school students (44 of whom were between the ages of 14 and 15 years, and 52 of whom were between the ages of 16 and 17 years). Data was collected through the use of two questionnaires. In the first questionnaire, students were asked to translate four word problems into equations, and in the second questionnaire students were asked to find the verbal analogy for calculating the solutions of similar problems. Each age group was divided into two sub-groups in order to administer the reverse order of questionnaires to each sub-group. The results indicated that both sets of students were significantly more successful at creating verbal prescriptions (operational tasks) than they were at creating equations (structural tasks).

From the results obtained from both data sets, Sfard (1987) concludes that the operational conception of function develops in students before the structural conception of function. This point is further supported by the fact that the operational aspect of functions was never taught to the students in either data set. In light of this finding, Sfard proposes that the curriculum should be changed to accommodate this, and should therefore teach the operational aspects of function before the structural aspects of function. She further suggests ways to do this, and also acknowledges that her data is not presented in very much detail.

Study 2: The Phenomenon of Reification

In Sfard's (1989) second study, the goal was to closely examine the phenomenon of reification. A control group (22 to 25 years of age) was given an elementary mathematics course (secondary-level) where the function concept was taught in a structural way. Data about students' conceptions of functions was collected through classroom observations and a questionnaire about functions. The experimental group (22 to 25 years of age) was given a functions-based intervention on algorithms and computability, based on (1) "it would be of little or no avail to introduce a new mathematical notion by means of its structural description" (Sfard, 1989, p. 152), and (2) "the "structural approach should not be assumed until an actual step was made toward a higher-level theory, for which this approach is indispensable" (Sfard, 1989, p. 152). Instruments similar to those used with the control group were used to collect data from the experimental group.

The questionnaire revealed several interesting aspects within the control group. First, few students had fully developed a structural conception of functions. Second, most of the students' conceptions of functions were closer to operational than to structural. Third, students developed a quasi-structural conception of functions. The data from the experimental group, however, revealed other findings. Classroom observations indicated that students formed operational conceptions of functions, and were initially resistant to understanding functions from a structural perspective. The questionnaire, which was administered at the end of the course, revealed more students than in the control group had progressed towards the structural conception of functions, and it is believed that there was less evidence of quasi-structural[6] conceptions of functions.

From these observations, Sfard (1989) concludes that using an operationally based curriculum does allow for the beginning of the reification of functions, even though the experimental group never developed a fully-fledged structural conception. In fact, some students may never reach a structural conception of functions. In addition, from a theoretical perspective, in order for reification to take place, one needs to be introduced to higher-level operations that you would perform on an object. Simultaneously, in order to handle higher-level operations, one needs to have structural conceptions in place. This can only occur if one can balance a lower-level reification with higher-level operations, which is a difficult task, even for mathematicians.

Analysis

Having now defined the terminology used to explain reification, presented the cognitive developmental theoretical framework, traced the history of the development of the function concept, explained the process of reification, and presented some of Sfard's empirical research based on reification, it is time to step back and ask ourselves the following questions:

- How is the process of reification grounded in the cognitive developmental theoretical literature?
- Are there flaws in the explanation of the process of reification?
- Does the empirical evidence presented support the explanation of reification?

[6] For Sfard (1989), a quasi-structural conception of a function is a "pupil's tendency to associate functions with algebraic formulae …. this tendency can be indicative of operational conception (the student may perceive a formula as a short description of a computational algorithm) as well as of a structural (the formula may be interpreted as a static relation between ordered pairs)" (p. 155).

- Are the data collection methods for the empirical research sufficient?

It is through these questions that we may be able to truly determine the contribution of the process of reification in explaining the formation of the concept of function.

How Is The Model Grounded?

Sfard (1992) presents the process of reification as a hierarchical three-step pattern (interiorization, condensation, and reification) grounded in the work of Piaget (1970/1972) and in the work of Piaget and Garcia (1983/1989). In choosing to ground her work in Piaget (1970/1972), Sfard (1992) remarks that her idea regarding the process of reification is similar to, but not the same as, Piaget's (1970/1972) idea regarding the formation of mathematical knowledge, which is actually the case based on her explanation of the process of reification.

However, in choosing to ground her work in Piaget and Garcia (1983/1989), Sfard (1992) is not entirely clear in her explanation. She does explicitly say that the historical development of the concept of function applies to the cognitive development of the concept of function, which truly matches the fundamental argument presented by Piaget and Garcia (1983/1989), but she is unclear in the relationship between her three-step pattern and Piaget and Garcia's (1983/1989) three-phase mechanism of composition. The reader is left to determine if the two are really related. It seems that interiorization (a process on an object) is analogous to the intra phase (object analysis); condensation (compacting the interiorized process) is analogous to the inter phase (analyzing relations or transformations between objects); and reification (seeing the condensed process as an object) is analogous to the trans phase (the building of structures). Thus, Sfard (1992) grounds her work clearly in that of Piaget (1970/1972), but not as clearly in Piaget and Garcia (1983/1989).

Are There Flaws in the Model?

If one assumes that Sfard's (1992) three-step process is related to Piaget and Garcia's (1983/1989) three-phase mechanism of composition, then it is fair to say that there is a flaw in the explanation of the process of reification, i.e., the process of reification as explained in Sfard (1992) lacks the explicitness that is required for the untrained reader to fully understand the model. Even more important for us to note is the fact that the definitions of the stages of condensation and reification are blurry, as Sfard (1992) rightly points out when she states that, "Condensation and reification may seem confusingly similar, but there is a subtle difference between them" (p. 64).

The next flaw in the explanation of the process of reification is Sfard's (1987, 1989, 1991, 1992) prescription for teaching. She makes the claim that since the operational conception develops before the structural conception, then one should not introduce and teach a new concept from a structural perspective, and moreover one should not introduce topics from a structural perspective until it is absolutely necessary and the operational conception is fully developed. For Sfard (1991), this claim is based on operational conceptions being an invariant in learning. This claim is also based on the fact that her data reveal that the operational conception develops before the structural conception.

Even though this might be the case, the claim seems to counter the ideas put forward by Piaget (Duckworth, 1973, 1996; Piaget, 1975/1985) and Vygotsky (1978), who both believed that students should be given the opportunity to explore concepts above their current level of understanding in order to get to the next level of understanding. For Piaget (1975/85),

> It appears to us that in explaining cognitive development, whether accounting for the history of science or psychogenesis, the concept of improving or optimizing equilibration imposes itself as fundamental. … On the one hand, it involves compensation of perturbations responsible for the disequilibria that motivate seeking; on the other, it involves the construction of the new factors producing improvement. (p. 139)

Thus, disequilibrium is necessary for cognitive development, and disequilibrium is more likely to occur if a student is taught above their current level of understanding, i.e., from a structural perspective rather than an operational perspective in the case of functions, which contradicts Sfard's (1991) claim.

A similar argument against Sfard can be presented through the use of Vygotsky's (1978) zone of proximal development (ZPD), which is "*the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers*" (p. 86). This implies that learning occurs when a child interacts with someone who knows more than he/she does and works on problems that are above his/her "actual level" of development. Moreover, development occurs when a child is able to move from his/her "actual level" of development to his/her "potential level" of development. "Thus, the notion of a zone of proximal development enables us to propound a new formula, namely that the only 'good learning' is that which is in advance of development" (p. 89). Therefore, based on Vygotsky (1978), in order for learning to occur in the case of functions, it may be necessary to teach at the structural level instead of at the operational level. This statement is again contradictory to Sfard's (1991) claim, as well as to her data.

Hence, it seems that there are two flaws in the explanation of the process of reification. First, it is not explicitly clear in its definitions and connections to the cognitive development literature, and second, Sfard's (1987, 1989, 1991, 1992) prescription for teaching opposes the two most influential figures in the field of cognitive development, Piaget and Vygotsky. The latter flaw is cause for concern given the consequences this may have for mathematics education. To determine which order is the most appropriate one needs to look at the results of empirical studies that evaluate Sfard's claim as opposed to those suggested by Piaget's and by Vygotsky's views. The study by Sfard (1989) reported above provides some, but not full support of her views.

Does the Empirical Evidence Support the Model?

Turning to the empirical research presented by Sfard (1987, 1989), her studies do offer evidence of operational and structural conceptions of the concept of function. The research does not, however, offer any real evidence of the three-step process of interiorization, condensation, and reification. One can only assume that interiorization maps to operational conceptions, and reification maps to structural conceptions, based on the definitions. Therefore, the evidence of operational conceptions is indicative of interiorization, and the evidence of structural conceptions is indicative of reification. This assumption is problematic because there is still no evidence of condensation because condensation maps to nothing.

The other problem with the data presented by Sfard (1987, 1989) is that the three-step process of interiorization, condensation, and reification is a cyclical process, so to speak, but the empirical research gives no evidence of this cycle. The empirical evidence implies only one level of operational conceptions and one level of structural conceptions. Thus, the empirical evidence only supports the claim of there being operational conceptions of function and structural conceptions of functions, and it offers no real evidence of interiorization, condensation, and reification. Hence, this lack of evidence of interiorization, condensation, and reification, as well as of the cyclical nature of reification, is indicative of the need for more empirical research to further support the model for the process of reification.

Are Data Collection Methods Sufficient?

Even though this paper does not ultimately seek to criticize research methods used by Sfard (1987, 1989), it is pertinent to address the sufficiency of these methods because their insufficiency may account for the lack of empirical evidence to support the process of reification. In Sfard (1987), questionnaires are used to collect data for both experiments. This may be the reason that we only see evidence of the different conceptions of function, and no evidence of the three-step pattern. Is it possible that a clinical interview (see Ginsburg, 1997; Piaget, 1929/1976), for instance, would evoke such evidence? The same question applies to Sfard (1989), even though classroom observations were also a source of data. Thus, would a change in data collection methods impact the type of data obtained?

Summary Of Responses To Questions Posed

The above questions challenge several aspects of the contribution of the process of reification in explaining the formation of the concept of function. First, Sfard (1992) grounds her work clearly in that of Piaget (1970/1972), but not as clearly in Piaget and Garcia (1983/1989). Second, there are at least two flaws in the explanation of the process of reification: (a) it lacks explicitness and clarity, and (b), Sfard's (1987, 1989, 1991, 1992) prescription for teaching opposes the two most influential figures in the field of cognitive development – Piaget and Vygotsky. Third, there is a lack of actual evidence of interiorization, condensation, and reification, as well as of the cyclical nature of reification. Fourth, the data collection methods may be insufficient, and hence a potential reason for the lack of evidence. Thus, even though Sfard's (1992) explanation of the process of reification has contributed significantly to the explanation of the formation of the concept of function, it is lacking in some areas. This gives us reason to turn now to the explanation of the process of encapsulation, and to embark on a similar process in order to explore and understand the formation of the concept of function.

Encapsulation

Terminology

The next major theoretical model on students' conceptions of functions to be discussed is found in theory and research by Ayers, Davis, Dubinsky, and Lewin (1988), Breidenbach, Dubinsky, Hawks, and Nichols (1992), Dubinsky (1991a, 1991b) and Dubinsky and Harel (1992a). All of these researchers use the terms *process* and *object* as previously defined, but, in addition, Breidenbach, Dubinsky, Hawks, and Nichols (1992), and Dubinsky and Harel (1992a) use the terms *pre-function* and *action* to describe other types of conceptions of function. All four terms are defined below:

- Pre-function conception indicates that the student does not have enough knowledge of functions to perform mathematical tasks that are related to functions.
- Action conception indicates that the student is able to compute the value of a function one step at a time, but is unable to think of it in general terms. This makes the concept static.
- Process conception indicates that the student is able to understand that the transformation of quantities by a function is dynamic, i.e., the transformation is a complete activity where you start with objects, act on the objects, and end with new objects. In addition, the student is able to understand composite and inverse functions in general terms, as well as the notion of one-to-one[7] and onto[8].
- Object conception indicates that the student is able to understand functions such that they can perform actions on them that will transform them.

The progress from one conception to the next is never in a single direction. A student will have a different conception depending on the context, situation, or representation. It is in this sense that the terms defined here are used in reviewing theories and research studies on the process of encapsulation.

Cognitive Developmental Theoretical Framework

Let us now turn to the cognitive developmental theoretical framework in which the process of encapsulation is grounded. Encapsulation is considered to be a type of

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[7] "A function $f : S \rightarrow T$ is called ***one-to-one*** or an ***injection*** if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$" (Marsdenn & Hoffman, 1993, p. 4).

[8] "We say that $f : S \rightarrow T$ is ***onto***, or is a ***surjection***, when for every $y \in T$, there is an $x \in S$ such that $f(x) = y$, in other words, when the range equals the target" (Marsdenn & Hoffman, 1993, p. 4).

reflective abstraction by Ayers, Davis, Dubinsky, and Lewin (1988), Breidenbach, Dubinsky, Hawks, and Nichols (1992), and Dubinsky (1991a, 1991b), and the idea of encapsulation is grounded in the work of Beth and Piaget (1966), which is based on reflective abstraction. In the first part of Beth and Piaget (1966), Beth traces the history of philosophical thinking about mathematics in detail, i.e., he examines the historical development of logico-mathematical thought. In the second part of Beth and Piaget (1966), Piaget examines logico-mathematical thought as it develops in the individual, i.e., he examines the cognitive development of logico-mathematical thought. Thus, Piaget's main goal is to investigate the relationship between logic and psychology. From his investigations, he proposes that reflective abstraction is the process by which logico-mathematical concepts are constructed within the individual. This means reflective abstraction is a constructive process, i.e.,

> Reflective abstraction consists in deriving from a system of actions or operations at a lower level, certain characteristics whose reflection (in the quasi-physical sense of the term) upon actions or operations of a higher level it guarantees; for it is only possible to be conscious of the processes of an earlier construction through a reconstruction on a new plane. (Beth & Piaget, 1966, p. 189)

In addition, reflective abstraction is an on-going process which consists of "constructing and characterizing new mathematical entities" (Beth & Piaget, 1966, p. 205), and which is very different from invention and from discovery.

Process of Encapsulation

Having discussed the foundations of encapsulation, we now turn towards the process of encapsulation as outlined by Ayers, Davis, Dubinsky, and Lewin (1988), Breidenbach, Dubinsky, Hawks, and Nichols (1992), and Dubinsky (1991a, 1991b). Each set of researchers builds on the previous ones. Consequently, a discussion in chronological order of publication leads to the most comprehensible summary. So, for Ayers, Davis, Dubinsky, and Lewin (1988), encapsulation is a form of reflective abstraction, where "Reflective abstraction refers to the cognitive process by which a physical or mental action is reconstructed and reorganized on a higher plane of thought and so comes to be understood by the knower" (p. 247), and there are four types of reflective abstractions – generalization, interiorization, encapsulation, and coordination.

These reflective abstractions are necessary to construct the concept of function, and are defined below in terms of the specific case of function:

- If a student is able to conceive of functions beyond an algebraic formula, i.e., as a process that transforms the domain into the range, then the student is able to *generalize*.
- If a student is able to form a mental representation of the (possibly mental) action of the function, i.e., of the above process, then the student is able to *interiorize*.
- If a student is able to transform the process for a function into a single, total entity, i.e., to consciously understand the function, then the student is able to *encapsulate*.
- If a student is able to merge the process of two or more functions into one process, i.e., perform a composition, then the student is able to *coordinate*. This new process can be further *interiorized* and *encapsulated*.

Ayers, Davis, Dubinsky, and Lewin (1988) point out that once the concept of function has been encapsulated, then a student will be able to access many ideas in mathematics and science. Unfortunately, only students who have reached the developmental stage of formal operations will be able to perform these reflective abstractions, and even then little is known about how to foster these reflective abstractions.

For Breidenbach, Dubinsky, Hawks, and Nichols (1992) and Dubinsky (1991a, 1991b) encapsulation is also a form of reflective abstraction. This relates to the ideas presented by Ayers, Davis, Dubinsky, and Lewin (1988). Dubinsky (1991b) explains that Piaget uses the term reflective abstraction to describe the process of constructing logico-mathematical structures, and that reflective abstraction is present in the coordination of sensorimotor structures and continues to be present throughout higher mathematics. Dubinsky (1991b) continues by stating that there are four forms of reflective abstraction for Piaget – interiorization, coordination, encapsulation, and generalization. He then takes the liberty of adding a fifth form of reflective abstraction – reversal – which he claims is discussed by Piaget, but never recognized as a form of reflective abstraction. He supports this claim through reference to Piaget's INRC group. This is very different from the ideas presented by Ayers, Davis, Dubinsky, and Lewin (1988), who only consider four types of reflective abstraction, and in far less detail than does Dubinsky (1991a, 1991b).

Dubinsky (1991b) defines each form of reflective abstraction used in the construction of mathematics in general, rather than for the specific case of functions. These definitions are as follows:

- Interiorization – "to construct internal processes as a way of making sense out of perceived phenomena" (pp. 101-102).
- Coordination – "the composition … of two or more processes to construct a new one" (pp. 101-102).

- Encapsulation – "conversion of a (dynamic) process into a (static) object" (pp. 101-102).
- Generalization – "to apply an existing schema to a wider collection of phenomena" (pp. 101-102).
- Reversal – "constructing a new process which consists of *reversing* the original process" (pp. 101-102), which exists internally.

Now Breidenbach, Dubinsky, Hawks, and Nichols (1992) rely on Dubinsky's (1991a, 1991b) explanation of reflective abstraction to explain the construction of mathematical processes and objects. For Breidenbach, Dubinsky, Hawks, and Nichols (1992), a subject begins with a mathematical object to which an action, a repeatable physical or mental manipulation that changes the object, is applied. Once this action occurs mentally without necessarily going through all the steps, then the action is said to be *interiorized*, i.e., it has become a *process*. This process can be used to create other processes by *reversing* it or *coordinating* it with other existing processes. Finally, when an action can be then applied to this said process to change it, the process is said to be *encapsulated*, i.e., it has become an *object*. Note that *de-encapsulation* can also occur, in which the subject is able to go from the object to the process from which it was initially created. In addition, if a subject is confronted with a new context or situation, then the subject does not necessarily construct new objects in order to deal with said situation or context, but rather *generalizes* existing processes and objects to fit said situation or context. These explanations of the different forms of reflective abstraction very much match the work of Dubinsky (1991b); however, Breidenbach, Dubinsky, Hawks, and

Nichols (1992) imply that it is an ongoing process and that each form of reflective abstraction is reliant on another form of reflective abstraction.

In fact, the construction of mathematical processes and objects is easily represented in Figure 2 for Breidenbach, Dubinsky, Hawks, and Nichols (1992), as well as for Dubinsky (1991a, 1991b):

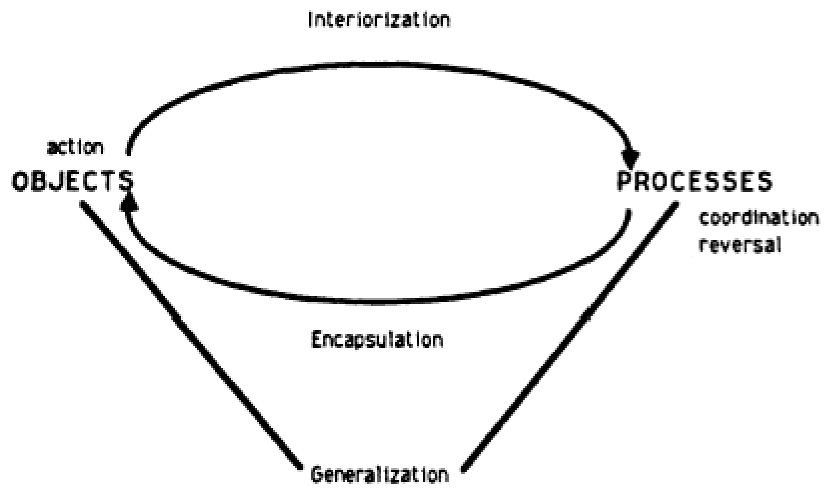

Figure 2. Construction of processes and objects.

Note. From “Development of the Process Conception of Function,” by D. Breidenbach, E. Dubinsky, J. Hawks, and D. Nichols, 1992, *Educational Studies in Mathematics, 23*, p. 250. Copyright 1992 by Kluwer Academic Publishers. Reprinted with kind permission of Springer Science and Business Media.

Breidenbach, Dubinsky, Hawks, and Nichols (1992) acknowledge two flaws with their theory. First, the distinction between an action and a process is not very clear and it will require additional research. Second, they do not believe that the only way to go from a process to an object and back is through encapsulation and de-encapsulation, since one can construct a process through at least three methods – interiorization, reversal of a process, or coordination of existing processes. Therefore, their theory should allow for more than one method through which to create an object from a process.

Empirical Research

Let us now turn to some of the empirical research that relies on the process of encapsulation as its theoretical framework. Three empirical studies will be discussed in turn. The goals, the sample and methods used, as well as the findings of each study will be highlighted in turn. I have chosen to discuss the studies in chronological order, for us to get some sense of the growth in research, as well as how one study may or may not have impacted the other.

Study 1: Fostering Reflective Abstractions

In the study conducted by Ayers, Davis, Dubinsky, and Lewin (1988), the goal was to show that the computer operating system Unix could be used to foster the reflective abstractions necessary for understanding the concept of function and compositions of functions. The initial sample consisted of 44 students in an optional first-year college mathematics lab. The students were also enrolled in an introductory calculus course. The students were placed in three sections by the registrar’s office, and the researchers determined the instructional treatment to be received by each group. The first group, Computer 1, consisted of ten students and was taught by Instructor B. The second group, Computer 2, consisted of fourteen students, and was taught by Instructor A. The third group, Paper-and-Pencil, consisted of twenty students, and was taught by Instructor A. Only 30 students were used in the final data analysis. The students all attended six weekly two-hour sessions, of which the first and the last session were used for testing. The students all received a joint traditional undergraduate lecture from Instructor A on functions and composition of functions in the fourth session. The students were, however, separated for the second, third, and fifth sessions during which they received the designated intervention. These three sessions were devoted to written exercises on functions and composition of functions for the Paper-and-Pencil group, and computer exercises on the Unix operating system, specifically the application of shell scripts to data and the creation of shell scripts[9] with or without piping[10], for the Computer groups.

Data was collected through a pre-test in the first session and a post-test in the sixth session. The pre-test focused primarily on questions about functions and composition of functions, which would normally be given in a typical advanced high school or elementary college course. The post-test focused on questions that could determine if reflective abstractions occurred. For instance, if reference was made to the composition of two functions as a new function, then encapsulation had occurred. The results indicated that the Computer students did better generally on the post-test than did the Paper-and-Pencil group. The researchers concluded from this that the computer exercises given to the Computer students were more likely to induce reflective abstractions on functions and composition of functions than did the written exercises

[9] A shell script is a Unix file containing a meaningful sequence of any number of commands which can include both standard arithmetic operations on numbers and text-processing functions (Ayers, Davis, Dubinsky, and Lewin, 1988, p. 248.)
[10] Piping is the combination of two or more shell scripts to form the equivalent of a composite function in mathematics (Ayers, Davis, Dubinsky, and Lewin, 1988, p. 249.)

given to the Paper-and-Pencil students. This matches the researchers' initial hypothesis that the shell scripts and pipes from the UNIX operating system was likely to initiate the types of reflective abstractions needed in understanding functions and composition of functions.

Study 2: Improving Students' Conceptions of Functions

In the study conducted by Breidenbach, Dubinsky, Hawks, and Nichols (1992), the goal was first, to show that college students who have been exposed to much mathematics do not necessarily understand the concept of function, and second, to show that an intervention based on computers can improve students' conceptions of function. The sample consisted of 62 pre-service high, middle, and elementary school math teachers who were sophomore and junior math majors. The participants had already been exposed to quite a few undergraduate mathematics courses including the calculus sequence. As part of the experiment, the participants were given a one-semester intervention in Discrete Mathematics, which was based on the theoretical framework outlined above. The first part of the course focused on work with ISETL (a computer environment), and the second part of the course focused on an instructional treatment on functions.

Data was collected before the intervention, after working with ISETL, and after the instructional treatment. The data collected before the intervention required the participants to answer, in writing, two questions: first, what is the definition of a function, and second, give examples of functions. After working with ISETL, data was collected through three methods. The students were asked first, to define a function, second, to give three examples of a function, and third, to determine if a given situation was a function and to explain why. After the instructional treatment, data was collected using three instruments. First, the students were given an optional un-timed, un-graded, open-book exam on functions. Second, the students were interviewed individually on the definition of a function and determining if a given situation was a function. Third, the students were given a final exam for the overall course, which involved no reference to computers, and which tested the students on composition of functions and the properties of being one-to-one and onto.

Overall observations indicate that the students' definition of function changed from pre-function to action to process after exposure to ISETL. Students were also successful on the optional, un-timed, un-graded, open-book exam on functions and the final interviews where a process conception of function was observed among the students. In addition, the students did well on the final exam, which did not ask any questions connected to ISETL. For Breidenbach, Dubinsky, Hawks, and Nichols (1992), the students' success on the final exam indicates that the students' abilities to construct a process conception of functions may have lead to improved performance in mathematics. In fact, the researchers believe that the students' construction of the process conception of function can be explained by the intervention given, i.e., both the exposure to ISETL and to the instructional treatment.

Breidenbach, Dubinsky, Hawks, and Nichols (1992) conclude the report on their study by re-visiting the theoretical framework used to explain the development of the function concept. The results of their study indicate that it is not only actions that transform objects as previously stated, but rather that both actions and processes transform objects. In addition, actions and processes are distinguished by the fact that actions occur externally to the thinking of the subject, and require the use of an explicit formula or recipe, while a process occurs internally to the thinking of the subject, and is not necessarily explicit.

Study 3: Examining the Action and Process Conceptions of Function

In Dubinsky and Harel's (1992a) study, the goal was to examine the action and process conception of function. The sample (22 undergraduate students) received an instructional treatment in Discrete Mathematics, which was based on a constructivist theory of learning and used computer activities. The treatment was designed to promote the students' conception of function. Some of the students initially had pre-function conceptions, whilst others had action conceptions, and even fewer had process conceptions. Those who did not have a process conception of function progressed to that level, and those who were already there were able to strengthen their conception. The details of this study are described in Breidenbach, Dubinsky, Hawks, and Nichols (1992), which has been previously discussed in this paper. Data was collected before, during, and after the instructional treatment. Dubinsky and Harel (1992a) reported on the post-interviews of 13 of the 22 students through a case study analysis of four students who, to the researchers, exemplified the overall findings of the interviews of the 13 students. In the post-interviews, students were asked to define a function and were also questioned on several situations. A conceptual analysis of each situation was given by the researchers so as to determine possible kinds of responses each situation would evoke. Each situation could evoke an answer in one or more of the possible conceptions of functions previously stated.

Initial observations revealed that the students go back and forth between conceptions because of self-imposed restrictions created through misconceptions of the concept of function. It was found that students had manipulation restrictions[11], quantity restrictions[12], and continuity restrictions[13]. The level and severity of restriction also varied. Students also struggled with determining the function when the information was implicit in the situation, and tended to confuse the one-to-one property and uniqueness to the right condition[14]. After analyzing the post-interviews, it was found that students believed that in order for a situation to be a function, then one had to be able to perform a manipulation on the function. Second, students relied on the presence of an algorithm as a criterion for constructing a process, as well as whether they believed they were allowed to construct the process. Third, confusion of uniqueness to the right diminished as the process conception of function strengthened. Finally, students were able to overcome the continuity restriction after the intervention.

Analysis

Having now defined the terminology used to explain encapsulation, presented the cognitive developmental theoretical framework, explained the process of encapsulation, and presented some of the empirical research based on encapsulation, it is time to step back and ask ourselves the following questions:

- How is the process of encapsulation grounded in the cognitive developmental theoretical literature?

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[11] Manipulation restriction is defined by Dubinsky and Harel (1992a) as: "you must be able to perform explicit manipulations or you do not have a function" (p. 86).
[12] Quantity restriction is defined by Dubinsky and Harel (1992a) as: "inputs and outputs must be numbers" (p. 86-87).
[13] Continuity restriction is defined by Dubinsky and Harel (1992a) as: "a graph representing a function must be continuous" (p.87).
[14] Uniqueness to the right condition "entails a unique finishing point" (Dubinsky & Harel, 1992a, p. 87), and is related to the process conception.

- Are there flaws in the explanation of the process of encapsulation?
- Does the empirical evidence presented support the explanation of encapsulation?
- Are the data collection methods for the empirical research sufficient?

It is through these questions that we may be able to truly determine the contribution of the process of encapsulation in explaining the formation of the concept of function.

How Is The Model Grounded?

The process of encapsulation is grounded in the work of Beth and Piaget (1966) on reflective abstractions. This makes perfect sense since Piaget seeks to examine the construction of logico-mathematical thought, and encapsulation is used to explain the construction of process and object conceptions of function. In terms of the notions of interiorization, coordination, encapsulation, generalization, and reversal as forms of reflective abstraction (see Ayers, Davis, Dubinsky, & Lewin, 1988; Breidenbach, Dubinsky, Hawks, & Nichols, 1992; and Dubinsky, 1991a, 1991b), Dubinsky (1991b) is the most helpful as a guide in re-visiting Piaget's work and exploring the filiations in the work:

- "reflective abstraction consists of translating a succession of material actions into a system of *interiorised* [italics added] operations, the laws of which are simultaneously implied in act" (Beth & Piaget, 1966, p. 206).
- "Reflective abstraction always consists in the introduction of new *co-ordinations* [italics added] into what is derived from earlier forms – which is already a variety of operations upon operations" (Piaget, 1970/1972, p. 71).

- "reflective abstractions that draw from more elementary forms the elements used to construct new forms" (Piaget, 1975/1985, p. 140), which can be thought of as *encapsulation*, which is to convert a process into an object (Dubinsky, 1991b).
- "To achieve constructive *generalizations* [italics added] and to overcome these limitations ("not only … but also"), however, a complete change in direction is needed and the use of reflective abstractions in required" (Piaget and Garcia, 1983/1989, p. 229).

These quotations actually imply that Ayers, Davis, Dubinsky, and Lewin (1988), Breidenbach, Dubinsky, Hawks, and Nichols (1992), and Dubinsky (1991a, 1991b) could be accurate in highlighting interiorization, coordination, encapsulation, and generalization as the possible forms of reflective abstractions.

The question that remains is whether *reversal* of a process can be considered a form of reflective abstraction. The only part of Piaget's work that implies that *reversal* could be considered a form of reflective abstraction is the fact that reversibility is discussed in terms of the INRC group as inversion or negation, and as reciprocity or symmetry (Gruber & Vonèche, 1977). The presence of the INRC group is characteristic of formal thought as is the ability to do reflective abstractions. Thus, it is possible that reversal could be considered a form of reflective abstraction (Dubinsky, 1991b). Hence, it seems that the types of reflective abstraction highlighted by Ayers, Davis, Dubinsky, and Lewin (1988), Breidenbach, Dubinsky, Hawks, and Nichols (1992), and Dubinsky (1991a, 1991b) do exist within Piaget's work, and that the explanation of the process of encapsulation is thoroughly grounded in the work of Piaget.

Are There Flaws in the Model?

The process of encapsulation seems fairly clear and properly grounded, as do the combined reflective abstractions. There is a problem, however, with matching the theory of encapsulation, or if you prefer reflective abstractions, with the definitions of each type of conception. The problem is that encapsulation is used only to explain the construction of mathematical processes and objects, and it does not explain the construction of pre-function or action conceptions. How then do Breidenbach, Dubinsky, Hawks, and Nichols (1992) and Dubinsky and Harel (1992a) explain the construction of a pre-function conception or of an action conception of function through the process of encapsulation? How are these two conceptions formed or are they innate? This lack of a connection between the types of reflective abstraction and the types of conception of function implies a flaw in the explanation of the process of encapsulation.

Does the Empirical Evidence Support the Model?

Turning to the empirical research presented by Ayers, Davis, Dubinsky, and Lewin (1988), Breidenbach, Dubinsky, Hawks, and Nichols (1992), and Dubinsky and Harel (1992a), it is seen that there is little or no evidence of the process of encapsulation or of each type of reflective abstraction. Ayers, Davis, Dubinsky, and Lewin (1988) state that their experimental group did better than their control group, and hence, it is likely that the computer exercises on the Unix operating system fostered the reflective abstractions necessary in constructing the concept of function and composition of functions, which could be considered evidence of the process of encapsulation. Also, there is no real evidence of each individual type of reflective abstraction. The same is true for the empirical data presented by Breidenbach, Dubinsky, Hawks, and Nichols (1992), and Dubinsky and Harel (1992), in which there is no explicit mention of each type of reflective abstraction, and hence no correlation of the data with the actual process of encapsulation. Does this all mean that there is no evidence of interiorization, coordination, encapsulation, generalization, and reversal in the empirical research, and thus no evidence to support the entire process of encapsulation?

In addition to the lack of evidence to support the process of encapsulation, there seems to be no evidence of the object conception being present in the data discussed by all three sets of researchers. Also, even though the function conceptions (pre-function, action, process, and object) which were proposed by Breidenbach, Dubinsky, Hawks, and Nichols (1992) and Dubinsky and Harel (1992a), are fairly clear in their theoretical definition, the conceptions are very blurry in the data they present.

The next critique is more of a question. All three sets of researchers use a computer-based intervention in their empirical studies. The computer-based interventions are of a programming nature. Does this imply that the process of encapsulation, as well as the other reflective abstractions, will not occur without this kind of context? Is such an intervention necessary to induce and accelerate the pace of the reflective abstractions?

Based on the above comments, it seems that the empirical data offers no real evidence of the actual process of encapsulation, nor of the presence of the object conception. Moreover, computer-based stimulations may be necessary for inducing reflective abstractions.

Are Data Collection Methods Sufficient?

Again, even though this paper does not ultimately seek to criticize research methods used by Ayers, Davis, Dubinsky, and Lewin (1988), Breidenbach, Dubinsky, Hawks, and Nichols (1992), and Dubinsky and Harel (1992a), it is pertinent to address the sufficiency of these methods because their insufficiency may account for the lack of empirical evidence to support the process of encapsulation. Breidenbach, Dubinsky, Hawks, and Nichols (1992), and Dubinsky and Harel (1992a) rely on both qualitative and quantitative methods to collect data. Therefore, it is a surprise that the data does not offer the kind of evidence one would wish to encounter. Ayers, Davis, Dubinsky, and Lewin (1988) only rely on a pre-test and a post-test, which I imagine should be sufficient for their research question. I still wonder, however, whether their data could be further corroborated with the use of clinical interviews (see Ginsburg, 1997; Piaget, 1929/1976), which may elicit the evidence needed to support the model for the process of encapsulation. Thus, as was brought up for the case of Sfard's research (1987, 1989), would a change in data collection methods impact the type of data obtained?

Summary Of Responses To Questions Posed

The above questions focus attention on several pertinent aspects of the contribution of the process of encapsulation in explaining the formation of the concept of function. First, the proposal for five types of reflective abstraction is supported through the work of Piaget. Second, there is a lack of a connection between the types of reflective abstraction and the types of conception of function. Third, there is no evidence of the presence of the actual process of encapsulation, nor of the object conception. Furthermore, computer-based stimulations may be necessary for inducing reflective abstractions. Fourth, the data collection methods may need modification, in order to provide more evidence. Thus, the explanation of the process of encapsulation by the three sets of researchers has been significant, but like the explanation of the process of reification provided by Sfard (1992), it is lacking in some areas.

Conclusions

This paper has focused on (1) exploring and trying to understand the processes of reification and encapsulation as they apply to changing one's understanding of functions from that of process to that of object, and (2) trying to determine how the processes of reification and encapsulation contribute to explaining the formation of the concept of function. Here a process conception is procedure oriented (related to Piaget's definition of content – see Piaget, 1975/1977), whilst an object conception is entity oriented (related to Piaget's definition of form – see Piaget, 1975/1977).

In reviewing the processes of reification and encapsulation several ideas have come to light. On the one hand, it has been found that reification is essentially a three-step process of interiorization, condensation, and reification. In this three-step process, "first there must be a process performed on the already familiar objects, then the idea of turning this process into a more compact, self-contained whole should emerge, and finally an ability to view this new entity as a permanent object in its own right must be acquired" (Sfard, 1992, p. 64). Encapsulation, on the other hand, is essentially one type of reflective abstraction (interiorization, coordination, encapsulation, generalization, and reversal) in which an action is applied to a process in order to change it into an object, and hence be encapsulated.

The explanation of both the process of reification and the process of encapsulation is a starting point for understanding the formation of the concept of function. They are both strongly grounded in the work of Piaget, and are heavily reliant on his ideas. Like

Piaget's ideas, the theories of reification and encapsulation focus on actions and on objects. Despite the fact that the explanation of both processes has contributed significantly to explaining the formation of the concept of function and are recognized as some of the major theoretical models used to explain conception formation (Harel & Kaput, 1991), neither explanation is without flaw.

It has been found that the theoretical explanation, as well as the empirical evidence, offered for both reification and encapsulation, indicate a mismatch between the actual process of concept formation and the different types of conceptions one can have of the concept of function. The empirical research framed within either process does not provide sufficient evidence to support the process in which either theory is based, which may be partly due to the research methods employed. This may also be the reason for the lack of evidence of structural/object conceptions within the empirical research.

Given the flaws in the theory and in the research for reification and encapsulation, several other areas should be explored. First, more theoretical models should be reviewed, such as *integration operation*, which is another major theoretical model used to explain concept formation (Harel & Kaput, 1991). Second, other cognitive developmental theoretical frameworks other than Piaget's should be considered as the basis for the theoretical models.

Last, but not least, more empirical research needs to be carried out; however, different research methods could be employed. For instance, research including more clinical interviews (see Ginsburg, 1997; Piaget, 1929/1976) could be carried out. In addition, a broader sample in terms of age and/or skill level should be considered. Both strategies may allow us to obtain (a) evidence of structural/object conceptions and (b) evidence of the actual process of reification or encapsulation. The strategies may also allow us to better match the actual process of concept formation and the different types of conceptions one can have of the concept of function.

References

Ayers, T., Davis, G., Dubinsky, E., & Lewin, P. (1988). Computer experiences in learning composition of functions. *Journal for Research in Mathematics Education, 19*(3), 246-259.

Beth, E. W., & Piaget, J. (1966). *Mathematical epistemology and psychology* (W. Mays, Trans.). Dordrecht, Holland / Boston, USA: D. Reidel Publishing Company.

Boyer, C. B. (1985). *A history of mathematics*. Princeton, NJ: Princeton University Press.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*, 247-285.

Burton, D. M. (2003). *The history of mathematics: An introduction* (5^{th} ed.). New York: McGraw-Hill.

Chazan, D. (1993). F(x) = G(x)?: An approach to modeling with algebra. *For the Learning of Mathematics, 13*(3), 22-26.

Chazan, D. & Yerushalmy, M. (2003). On appreciating the cognitive complexity of school algebra: Research on algebra learning and directions of curricular change. In J. Kilpatrick, G. W. Martin, & D. Schifter (Eds.), *A Research Companion to the Principles and Standards for School Mathematics* (pp. 123-135). Reston, NCTM.

Dreyfus, T. & Eisenberg, T. (1982). Intuitive function concepts: A baseline study on intuitions. *Journal for Research in Mathematics Education, 13*(5), 360-380.

Dreyfus, T. & Vinner, S. (1982). Some aspects of the function concept in college students and junior high school teachers. *Proceedings of the Sixth International Conference for the Psychology of Mathematics Education*, Antwerp, Belgium.

Dubinsky, E. (1991a). Constructive aspects of reflective abstraction in advanced mathematics. In L. P. Steffe (Ed.), *Epistemological Foundations of Mathematical Experience* (pp. 160-202). New York: Springer-Verlag.

Dubinsky, E. (1991b). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 95-123). Netherlands: Kluwer Academic Publishers.

Dubinsky, E., & Harel, G. (1992a). The nature of the process conception of function. In E. Dubinsky & G. Harel (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp. 85-106). Washington, Mathematical Association of America.

Dubinsky, E. & Harel, G. (Eds.). (1992b). *The concept of function: Aspects of epistemology and pedagogy*. Washington, Mathematical Association of America.

Duckworth, E. (1973, October). Piaget takes a teacher's look. *Learning*, 22-27.

Duckworth, E. (1996). Either we're too early and they can't learn it, or we're too late and they know it already: The dilemma of "applying Piaget." In E. Duckworth, *The having of wonderful ideas and other essays on teaching and learning* (2nd ed., pp. 31-49). NY: Teachers College Press.

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice.* Cambridge; New York: Cambridge University Press.

Gruber, H. E., & Vonèche, J. J. (Eds.). (1977). *The essential Piaget*. New York: Basic Books, Inc.

Harel, G. & Kaput, J. (1991) The role of conceptual entities and their symbols in building advanced mathematical concepts. In D. Tall (Ed.), *Advanced Mathematical Thinking*, London: Reidel.

Henrici, P. (1974). Computational Complex Analysis. *Proceedings of Symposia in Applied Mathematics, 20*, 79-86.

Holliday, Marks, Cuevas, Casey, Moore-Harris, Day, et al. (2003). *Glencoe: Algebra 1*. New York, Illinois, California: Glencoe/McGraw Hill.

Kaput, J. (in press). Transforming algebra from an engine of inequity to an engine of mathematical power by “algebrafying” the K-12 curriculum. To appear in S. Fennel (Ed.) *A Framework for Integrating Algebra Across K-12 Mathematics.* Reston, VA: NCTM.

Kleiner, I. (1989). Evolution of the function concept: A brief survey. *The College Mathematics Journal, 20*(4), 282-300.

Malik, M. A. (1980). Historical and pedagogical aspects of the definition of function. *International Journal of Mathematics Education in Science and Technology, 11*(4), 489-492.

Markovits, Z., Eylon, B., & Bruckheimer, M. (1986). Functions today and yesterday. *For the Learning of Mathematics, 6*(2), 18-24.

Marsden, J. E. & Hoffman, M. J. (1993). *Elementary classical analysis* (2nd ed.). New York: W. H. Freeman and Company.

Moses, B. (2001). *Radical equations: Math literacy and civil rights*. Boston: Beacon Press.

O'Connor, J. J. & Robertson, E. F. (2005). History topic: The function concept. In *MacTutuor History of Mathematics.* Retrieved April 15, 2006, from http://www-history.mcs.st-andrews.ac.uk/history/index.html

Pegg, J. (2002). Fundamental cycles of cognitive growth. *Proceedings of the 26th International Conference for the Psychology of Mathematics Education.* Norwich, England.

Piaget, J. (1972). *The principles of genetic epistemology* (W. Mays, Trans.). New York: Basic Books, Inc. (Original work published in 1970.)

Piaget, J. (1976). *A child's conception of the world* (J. and A. Tomlinson, Trans.). Maryland: Littlefield Adams Quality Paperbacks. (Original work published in 1929)

Piaget, J. (1977). *The development of thought* (A. Rosin, Trans.). New York: The Viking Press. (Original work published in 1975)

Piaget, J. (1985). *The equilibration of cognitive structures* (T. Brown and K. J. Thampy, Trans.). Chicago: The University of Chicago Press. (Original work published in 1975).

Piaget, J. & Garcia, R. (1989) *Psychogenesis and the history of science* (H. Feider, Trans.). New York: Columbia University Press. (Original work published in 1983).

Schwartz, J. L. (1999). Can technology help us make the mathematics curriculum intellectually stimulating and socially responsible? *International Journal of Computers for Mathematical Learning, 4*(2-3), 99-119.

Schwartz, J. & Yerushalmy, M. (1992). Getting students to function on and with algebra. In E. Dubinsky & G. Harel (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp. 261-289). Washington, Mathematical Association of America.

Sfard, A. (1987). Two conceptions of mathematical notions: Operational and structural. *Proceedings of the 11th International Conference for the Psychology of Mathematics Education*. Montreal: Canada.

Sfard, A. (1989). Transition from operational to structural conception: The notion of function revisited. *Proceedings of the 13th International Conference for the Psychology of Mathematics Education*. Paris, France.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on process and objects on different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification – the case of function. In E. Dubinsky & G. Harel (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp. 59-84). Washington, Mathematical Association of America.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification – The case of algebra. *Educational Studies in Mathematics, 26*, 191-228.

Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky & G. Harel (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp. 25-58). Washington, Mathematical Association of America.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematics Education in Science and Technology, 14*(3), 293-305.

Vygotsky, L. (1978). *Mind In Society*. Harvard University Press.

Yerushalmy, M. & Chazan, D. (2002). Flux in school algebra: Curricular change, graphing technology, and research on student learning and teacher knowledge. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 725-755). Hillsdale, NJ: Erlbaum.