SURVIVAL NUMBERS OF GROUPS AND GRAPHS WITH EMPHASIS ON $\mathbb{Z}^d$ AND DIESTEL-LEADER GRAPHS

A dissertation

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Abstract

The survival number of a metric space is a newly created statistic. For a given space, it equals the minimum amount of convex regions necessary to bound another convex region. Survival number is initially defined in terms of tessellations but can also be defined as a covering property. In this dissertation, we examine simple, connected graphs equipped with the path metric and finitely generated groups equipped with word metrics. We show the survival number of $\mathbb{Z}^2$, depending on the generating set, equals 3 or 4; and we then equate finding the survival number of $\mathbb{Z}^d$ for any $d \geq 2$ with an open problem in combinatorial geometry. Last, we compute survival numbers of Diestel-Leader graphs and lamplighter groups.

Dedicated to my two brothers Brandon and Jason Buckles. We share in each other's struggles and successes. I got my brothers' backs, and they have mine.

Acknowlegements

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Chapter 1

Introduction

Let T be a set of points in a metric space (X, d). The *Voronoi decomposition of X (relative to T)* is the division of X into sets, called *Voronoi cells*, where for each $t \in T$ the Voronoi cell $C_T(t)$ is defined by the nearest neighbor rule: $C_T(t) = \{x \in X : d(x,t) \leq d(x,t') \text{ for all } t' \in T\}$. From this definition, we observe that $C_T(t)$ is non-empty (in particular, $t \in C_T(t)$) and the union $\bigcup_{t \in T} C_T(t) = X$.

The idea of survival numbers began with Itai Benjamini in his 2001 article "Survival of the Weak in Hyperbolic Spaces, A Remark on Competition and Geometry". [1] Benjamini examines a competition model on graphs which begins with two vertices x_0 and y_0 and a positive integer m. The vertices determine two disjoint clusters (path-connected subsets) X and Y, which grow from the singleton sets $\{x_0\}$ and $\{y_0\}$, respectively. The cluster X grows m times faster Y. Assuming $m > 1$, such a competition in $\mathbb{Z}^d$ for any $d > 1$ will result in X eventually surrounding Y. When $m = 1$, this results in both X and Y being unbounded. Benjamini's article discusses why both X and Y will be unbounded for all m if the competition is in a Gromov hyperbolic space rather than a Euclidean space.

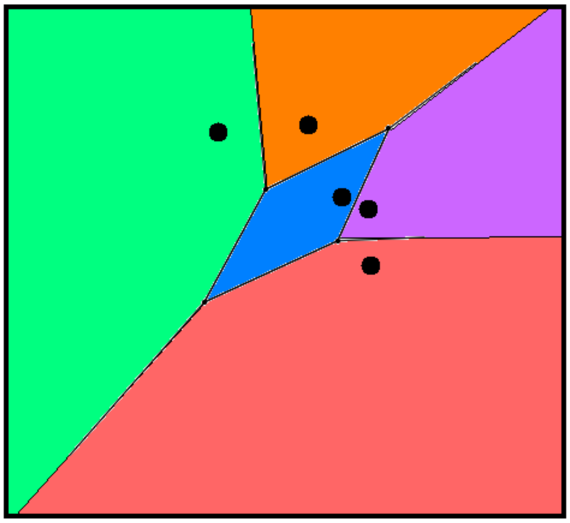
Hilary Finucane in 2013 modified Benjamini's competition model. She fixed $m = 1$ and, rather than having only two competing clusters, asked what is the maximum number of unbounded clusters provided the initial singleton sets are sufficiently spaced apart. Because $m = 1$, Finucane's question is equivalent to asking what is the maximum number of infinite Voronoi cells. An infinite Voronoi cell will "survive" the competition, and the survival number measures how many cells will survive if we are given only the geometry of the underlying metric space. [10]

Finucane's article defines the survival number of a metric space and focuses on vertex transitive graphs equipped with path metrics. She proves some important properties of the survival numbers of such graphs. These properties include the

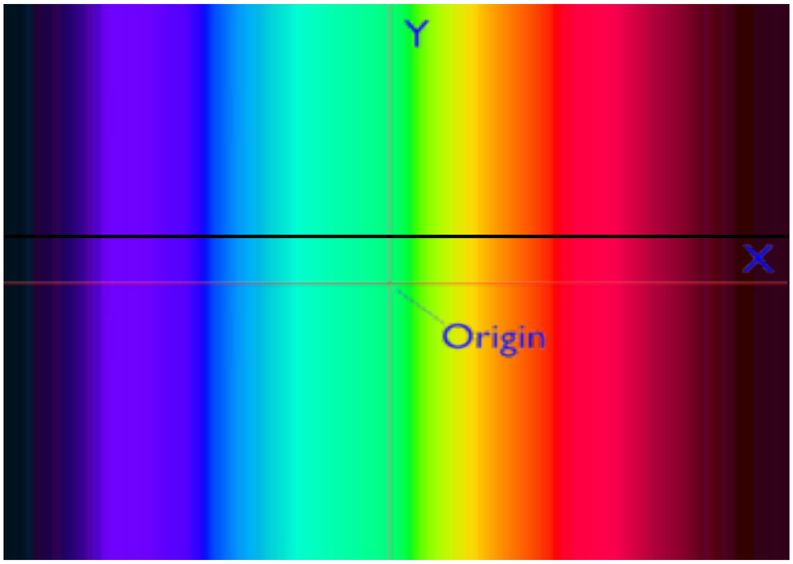
Figure 1.1: Examples of Voronoi Decompositions

(a) Real line with $d(x, y) = |x - y|$. Here, we have the set T consisting of three sites.

(b) Real plane with Euclidean metric. Here, T consists of five sites.



(c) Real plane with taxicab metric. In this case, the uncountable set T is the black horizontal line above the x−axis. With the taxicab metric, each Voronoi cell will be a vertical line $x = a$ for some $a \in \mathbb{R}$. Replacing the taxicab metric with a metric induced by any p-norm will yield the same Voronoi cells.



survival number is not invariant under quasi-isometries; polynomial growth implies the survival number is finite; and a graph with infinitely many ends has infinite survival number. She shows this first property by giving two generating sets of $\mathbb{Z}^2$ with survival numbers 3 and 4, respectively. After comparing graphs with finite and infinite survival numbers, she shows that exponential growth is not enough to ensure infinite survival number and that not all Liouville graphs have finite survival number. (A graph Γ is Liouville if all bounded harmonic functions $\gamma : \Gamma \to \mathbb{R}$ are constant.) She does this by showing the survival number of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ is finite and that of the lamplighter group $\mathbb{Z}_2 \wr (\mathbb{Z} \times \mathbb{Z})$ is infinite. (She examines $\mathbb{Z}_2 \wr \mathbb{Z}$ and $\mathbb{Z}_2 \wr (\mathbb{Z} \times \mathbb{Z})$ with specific generating sets.)

In this dissertation, we expand Finucane's examples of $\mathbb{Z}^2$ and $\mathbb{Z}_2 \wr \mathbb{Z}$. Our main results are the following two theorems: For all $d \geq 2$, let $S = -S$ be a finite generating set for $\mathbb{Z}^d$ and let Q be the convex hull of S in $\mathbb{R}^d$ and Q° its interior.

Theorem 1.0.1 *The survival number of $\mathbb{Z}^d$ equals the minimum amount of copies of Q° necessary to cover Q.*

Theorem 1.0.2 *For a Diestel-Leader graph formed from two infinite regular trees, its survival number is the sum of the trees' valences.*

The first theorem connects $\mathbb{Z}^d$ with an open problem in combinatorial geometry called Hadwiger's Conjecture. Proofs of both theorems provide different methods for computing the survival number of a group and provide such numbers for two classes of finitely generated groups, namely $\mathbb{Z}^d$ and $\mathbb{Z}_n \wr \mathbb{Z}$.

Chapter 2

Definitions of Survival Number

2.1 Definition using Voronoi cells

For a simple, infinite, and connected graph $\Gamma = (V, E)$ and finite subset $T \subset V$ with $v \in T$, let $C_T(v)$ denote the resulting Voronoi cell of v. Since the Γ is infinite and $\bigcup_{v \in T} C_T(v) = V$, at least one cell is infinite. The *survival number* $s(\Gamma)$ is maximum number of Voronoi cells that must be infinite as T ranges over all finite, sufficiently dispersed subsets of V. Formally, let $\Delta \in \mathbb{Z}^+$ and

$$\mathcal{T}(\Delta, n) = \{T \subset V : n \leq |T| < \infty \text{ and } d(v_1, v_2) > \Delta \text{ for all } v_1, v_2 \in T\}$$

and

$$I(T) = |\{v \in T : |C_T(v)| = \infty\}|. \tag{2.1}$$

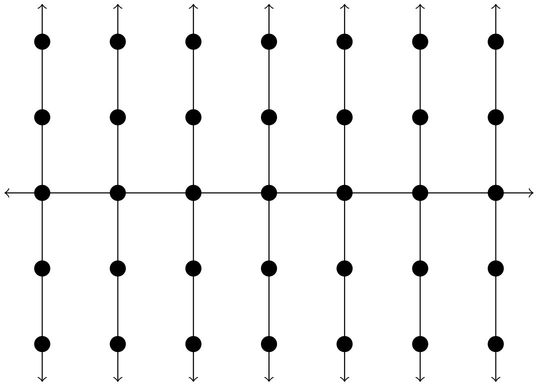
The survival number $s(\Gamma)$ is then

$$s(\Gamma) = \sup\{n : \exists \Delta \text{ with } I(T) \geq n \text{ for all } T \in \mathcal{T}(\Delta, n)\}.$$

Another way to view $s(\Gamma)$ is by letting m be the smallest value such that for all $\Delta > 0$, there is a $T \in \mathcal{T}(\Delta, m+1)$ with exactly $m+1$ vertices such that one vertex v is "blocked" by the remaining m vertices. (In other words, there is a maximum distance between vertices in $C_T(v)$.) Then $s(\Gamma) = m$, and if no such m exists, then $s(\Gamma) = \infty$.

Note that while the given definition is specific to graphs, survival number can be defined for any metric space. The only change necessary is rather than requiring that $|C_T(v)| = \infty$ in (2.1), we require that $C_T(v)$ is unbounded. The remainder of the above definition is unchanged.

Figure 2.1: Spiked linear graph



Example (Real line and cyclic groups): Consider the real line with $d(x, y) = |x - y|$. As Figure 1.1(A) suggests, its survival number is 2. For any $\Delta > 0$ and $x, y, z \in \mathbb{R}$ such that $x < y < z$ with pairwise distance at least Δ, the Voronoi cell of y is the closed interval $[\frac{x+y}{2}, \frac{y+z}{2}]$. Conversely, any pair $x, y \in \mathbb{R}$ with $x < y$ results in two unbounded Voronoi cells $(-\infty, \frac{x+y}{2}]$ and $[\frac{x+y}{2}, \infty)$.

By similar reasoning, any cyclic group $\langle a \rangle$ equipped with the word metric has survival number equal to 2. This is clear from $\langle a \rangle$ being isomorphic to $\mathbb{Z} = \langle 1 \rangle$, whose Cayley graph is a linear graph.

Example (Spiked linear graph): Let $\Gamma = (V, E)$ be the connected graph whose vertex set is $V = \mathbb{Z}^2$ and whose adjacency relation is $(x_1, y_1) \sim (x_2, y_2)$ if and only if either

1. $x_1 = x_2$ and $|y_1 - y_2| = 1$; or

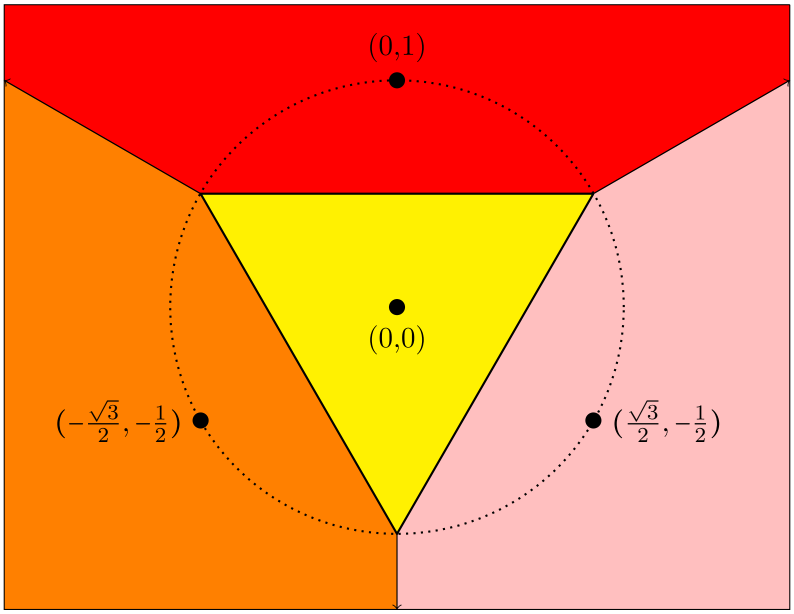
2. $|x_1 - x_2| = 1$ and $y_1 = y_2 = 0$.

See Figure 2.1.

For any Δ, there exist three vertices on the same vertical line which are pairwise at least Δ apart. This set is then in $\mathcal{T}(\Delta, 3)$, and one vertex is blocked by the other two. Hence, $s(\Gamma) \leq 2$. Also, any two vertices lie on a bi-infinite geodesic in Γ. The pair results in two infinite Voronoi cells, and so $s(\Gamma) \geq 2$.

Example (Real plane): The survival number of $\mathbb{R}^2$ with the Euclidean metric equals 3. To see that $s(\mathbb{R}^2) < 4$, consider the unit circle S^1. We select the origin $(0, 0)$

Figure 2.2: Voronoi cells of four points in $\mathbb{R}^2$



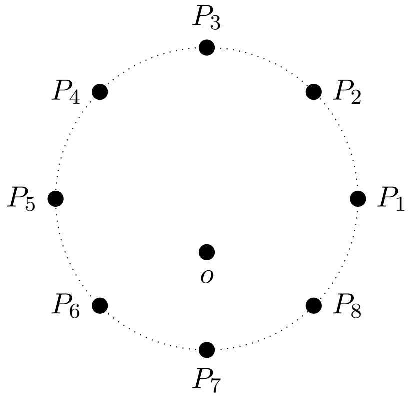
and three equally spaced points $(0,1)$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$, and $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$ in S^1:. With little effort, one can see the Voronoi cell centered at the origin is bounded by the triangle with vertices at $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, and $(0,-1)$. The remaining three Voronoi cells are each unbounded. Therefore, for any $\Delta > 0$, the value $I(T)$ in equation (2.1) for $T = \{(0,0), (0,\Delta), (\frac{\Delta\sqrt{3}}{2}, -\frac{\Delta}{2}), (-\frac{\Delta\sqrt{3}}{2}, -\frac{\Delta}{2})\}$ is strictly less than 4.

We next show $s(\mathbb{R}^2) \geq 3$. Let $T = \{a, b, c\}$, and consider whether or not these three points are collinear. If they are collinear, then the Voronoi cells $C_T(a)$, $C_T(b)$, and $C_T(c)$ are each unbounded. Hence $I(T) = 3$. If these points are not collinear, then we examine the (unique) circle containing a, b, and c. The three Voronoi cells each will have as its boundary two of the following three rays protruding from the center of this circle: (1) the ray bisecting arc $\widehat{\text{ab}}$, (2) the ray bisecting arc $\widehat{\text{bc}}$, and (3) the ray bisecting arc $\widehat{\text{ac}}$. Each of the three resulting Voronoi cells are therefore unbounded, and so $I(T) = 3$.

In general, the survival number $s(\mathbb{R}^d) = d + 1$ for any $d \geq 1$. This can be shown using Theorem 2.2.1 below and a topological argument about the covering of spheres in $\mathbb{R}^d$. (For details, see [10].)

The above examples are already known and simple to verify. Our next example

Figure 2.3: Circle in $\mathbb{H}^2$ centered at o



is new and inspired by our argument that $s(\mathbb{R}^2) = 3$.

Proposition 2.1.1 *The survival number of* $\mathbb{H}^2$ *is infinite.*

Proof: Our argument begins with examining a circle C of radius r in $\mathbb{H}^2$. Let o be the center of this circle and choose m points $P_1, \ldots, P_m$ evenly spaced around C. So for any two succeeding points, the angle between them is $\frac{2\pi}{m}$; next, let Q be the point on C midway between P_1 and P_2 and let $\theta = \frac{\pi}{m}$. We now show that $d(Q, o) < d(Q, P_1)$ for all large r.

To do this, consider the rays $\overrightarrow{oQ}$ and $\overrightarrow{oP_1}$. Define $D = D(\theta)$ be the maximum distance which these rays can fellow δ-travel. Our last labeling will be the inpoints, i.e. the three unique points on the triangle defined by o, Q, and P_1 which divide the three sides into three pairs of equal length segments. Let Q' be the inpoint for $\overline{oQ}$, P_1' the inpoint for $\overline{oP_1}$, and X the inpoint for $\overline{QP_1}$. Notice that the distances $d(o, Q')$ and $d(o, P_1')$ are both at most D and by definition, the maximum distance between any two inpoints is δ.

We now show that $d(Q, o) < d(Q, P_1)$ by finding a sufficiently large lower bound for $d(Q, P_1)$. Since we do not know the length of the geodesic path from Q to P_1, we examine the path from Q to Q' to X to P_1' to P_1. By the triangle inequality (used twice),

$$d(Q, P_1) \quad = \quad d(Q, X) + d(X, P_1)$$

$$\begin{aligned} &\geq [(r-D)-\delta]+[(r-D)-\delta] \\ &= 2r-2(D+\delta) \end{aligned}$$

Both D and δ are independent of r, and $r = d(Q,o)$. So for all large values of r we have $d(Q,o) < d(Q,P_1)$. By symmetry, we also have $d(Q,o) < d(Q,P_2)$

Next, we use this property to show that for $T = \{o, P_1, \ldots, P_m\}$, the Voronoi cell $C_T(o)$ contains every point on the ray $\overrightarrow{oQ}$ outside of C. Suppose there exists a point $S \in \overrightarrow{oQ}$ such that $d(o,S) = r+k$. Then $d(P_1,S) \geq d(P_1,Q)-d(Q,S) \geq [2r-2(D+\delta)]-k$. Since the value k is independent of r, we have $d(o,S) < d(P_1,S)$ for sufficiently large r. So, $S \in C_T(o)$.

So far, we have shown that for m evenly spaced points around a circle C, the center of that circle cannot be blocked by those m points. We now generalize this result to any set $T = \{o, P_1, \ldots, P_m\}$ which is sufficiently dispersed. First, notice that if $P_1, \ldots, P_m$ block o, then they also surround o. (i.e. The convex polygon with $P_1, \ldots, P_m$ as its vertices contains o in its interior.) This follows immediately from the Voronoi cell $C_T(o)$ having a boundary that encloses it. Every point on this boundary is equal distance from o and some other point P_i. Therefore, the polygon generated by $P_1, \ldots, P_m$ encloses $C_T(o)$.

Suppose now that $P_1, \ldots, P_m$ surround o in $\mathbb{H}^2$ and that these $m+1$ points are pairwise at least Δ apart from each other. Let C be the circle of radius $r = \Delta$ centered at o. There exist m points $\grave{P}_1, \ldots, \grave{P}_m$ on C such that $\grave{P}_i = \overrightarrow{oP_i} \cap C$ for each $i = 1, \ldots, m$. For each successive pair $\grave{P}_i$ and $\grave{P}_{i+1}$, the above argument can be applied to show $d(o,Q) < d(Q,\grave{P}_i)$ for $Q \in C$ midway between $\grave{P}_i$ and $\grave{P}_{I+1}$. It follows that $d(o,Q) < d(Q,P_i)$ and $C_T(o)$ contains an infinite ray. Therefore, $C_T(o)$ is unbounded and because m and o are arbitrary, $s(\mathbb{H}^2) = \infty$. $\square$

2.2 Alternate definition via covering properties

Finucane examines vertex transitive graphs in [10] and in doing so, she proves the following important theorem:

Theorem 2.2.1 (Finucane, [10]) *For a vertex transitive graph* Γ*,* $s(\Gamma) \geq n$ *if and only if there exist* $\Delta > 0$ *and an infinite set* $R \subset \mathbb{Z}^+$ *such that for any* $r \in R$ *and* $v \in \Gamma$*, the sphere* $S_r(v)$ *cannot be covered by* $n-1$ *balls of radius* $r-1$ *whose centers are pairwise at least* Δ *apart from each other and from* v*.*

Throughout this dissertation, we use Theorem 2.2.1 in both the above form and its contrapositive form: the survival number $s(\Gamma) \leq m$ if and only if for every $\Delta \in \mathbb{Z}$ and for all large r, the sphere $S_r(v)$ can be covered by m balls of smaller radius and whose centers are pairwise at least Δ apart from each other and from the origin.

We now look at a more complicated graph. In this example, we will use Theorem 2.2.1 to compute its survival number. Let $\Gamma = (V, E)$ be the graph in Figure 2.4. Then the vertex set $V = \mathbb{Z} \times \mathbb{Z}$, and two vertices (x_1, y_1) and (x_2, y_2) share an edge in E if and only if both $|x_1 - x_2| = 1$ and the vertices lie on one of the following lines (for some $k \in \mathbb{Z}$):

1. $y = k$
2. $x + y = 2k$
3. $x - y = 2k + 1$

Proposition 2.2.2 *For the graph* Γ *in Figure 2.4, the survival number* $s(\Gamma) = 4$*.*

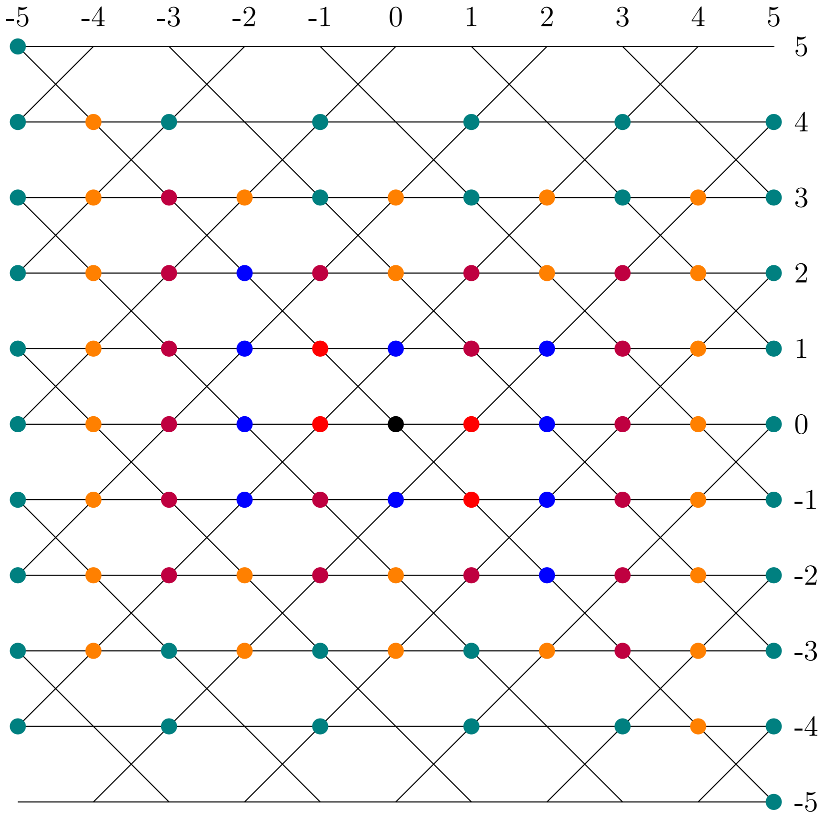
Proof: Before computing $s(\Gamma)$, we first examine distances in Γ with the path metric. For any two vertices $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, we will show

$$|x_1 - x_2| \leq d(P_1, P_2) \leq \max\{|x_1 - x_2|, |y_1 - y_2|\} + 2. \tag{2.2}$$

Clearly, if P_1 and P_2 satisfy $y = k$ for some k, then $d(P_1, P_2) = |x_1 - x_2|$. Because $x + y = 2k$ and $x - y = 2k + 1$ have slopes of -1 and 1 resp., for every change in y-coordinate, there is a change in x-coordinate of equal size. So, if P_1 and P_2 both satisfy either $x + y = 2k$ or $x - y = 2k + 1$, then $d(P_1, P_2) = |x_1 - x_2|$.

Choose vertices P_1 and P_2 that do not satisfy the above condition and that $y_1 < y_2$. We construct a path from P_1 to P_2 whose length satisfies (2.2):

Figure 2.4: Graph Γ with spheres of radius 1 through 5.



Case 1 Suppose $|x_1-x_2| = |y_1-y_2|$. Then P_1 and P_2 are corners (lower and upper) of a square. By our assumption, there is no diagonal path across the square's center connecting the two. Make a path from P_1 to P_2 by first moving horizontally along the square one vertex, then diagonally $|y_1-y_2|$ vertices to the point Q on line $y = y_2$. This point is either (x_2-1, y_2) or (x_2+1, y_2), depending on if the horizontal step is to the left or to the right of P_1. So, the path P_1QP_2 has length $1+|y_1-y_2|+1 = |y_1-y_2|+2$.

Case 2 Suppose $|x_1-x_2| > |y_1-y_2|$. Then P_1 is a lower corner of two squares; both squares have an upper corner on the line $y = y_2$. Let Q be the corner closer to P_2. If the diagonal from P_1 to Q consists of edges in Γ, the path P_1QP_2 has length $|y_1-y_2| + (|x_1-x_2| - |y_1-y_2|) = |x_1-x_2|$. Otherwise, the path has length $1+|y_1-y_2| + (|x_1-x_2| - 1 - |y_1-y_2|) = |x_1-x_2|$.

Case 3 Suppose $|x_1-x_2| < |y_1-y_2|$. Then there exists a particular square of side length $|y_1-y_2|$, containing both points P_1 and P_2. A diagonal across the center of this square either contains P_1 or P_2 or is a distance one from both points. Assume a lower corner is at P_1. (A similar argument holds if at P_2.) Then a path γ from P_1 across the center of the square to the opposite corner is of length $|y_1 - y_2|$. Since $|x_1 - x_2| < |y_1 - y_2|$, make a "u-turn" at some point Q in the square's interior. Choose Q so that an adjacent vertex lies on a line γ' which crosses $y = y_1$ at either P_2 or at a vertex adjacent to P_2. Such a path P_1QP_2 travels a vertical distance $|y_1 - y_2|$ plus at most two horizontal units.

Proof that $s(\Gamma) \leq 4$ For any $r > 2$, the points $P_1 = (-1, r-2), P_2 = (1, -r+2), P_3 = (r-1, r-2)$, and $P_4 = (-r+1, -r+2)$ are all at a distance $\geq r - 2$ from $(0,0)$. The points P_1 and P_3 lie on the horizontal line $y = r - 2$ and P_2 and P_4 lie on $y = -r + 2$, so

$$d(P_1, P_3) = d(P_2, P_4) = \quad r$$

Any path from P_1 to P_2 travels a vertical distance $|(r-2)-(-r+2)| = 2r-4$. Likewise for any path from P_3 to P_4. Hence

$$\begin{aligned} d(P_1, P_2) &\geq 2r - 4 \\ d(P_3, P_4) &\geq 2r - 4 \end{aligned}$$

So, let $\Delta = r$ for all large r. Each vertex in the sphere $S_r(0)$ centered at $(0,0)$ is within a distance $r-1$ from one of these four points. So, by Theorem 2.2.1, $s(\Gamma) \leq 4$.

Proof that $s(\Gamma) \geq 4$ The vertices $Q_1 = (-r, r)$, $Q_2 = (r, -r)$, $Q_3 = (-r, -r+1)$, and $Q_4 = (r, r-1)$ lie on $S_r(0)$. Also notice that $(-r, r)$ and $(r, -r)$ lie on the line $x + y = 0$. So,

$$d(Q_1, Q_2) = 2r$$

Next, for $i \neq j$, $d(Q_i, Q_j) \geq |x_i - x_j|$ in (2.2) implies

$$\begin{aligned} d(Q_1, Q_4) &\geq 2r \\ d(Q_2, Q_3) &\geq 2r \\ d(Q_2, Q_3) &\geq 2r \end{aligned}$$

To examine $d(Q_1, Q_3)$ and $d(Q_2, Q_4)$, notice the change in x-coordinates is less than the change in y-coordinates. Hence,

$$\begin{aligned} d(Q_1, Q_3) &\geq 2r - 1 \\ d(Q_2, Q_4) &\geq 2r - 1 \end{aligned}$$

So, no ball of radius $r-1$ covers more than one of these points. Therefore $s(\Gamma) \geq 4$.□

We end the chapter by extending the idea in Theorem 2.2.1 to graphs which are not vertex transitive. First, we define an equivalence relation on the vertex set V of Γ: let $v \sim w$ if and only if there exists an automorphism $\varphi \in \mathrm{Aut}(\Gamma)$ such that $\varphi(v) = w$. Under this relation, vertex transitive graphs have just one equivalence class. We now extend Theorem 2.2.1 to graphs with finitely many equivalence classes. Our argument is a slight modification of Finucane's proof of Theorem 2.2.1.

Proposition 2.2.3 *Let Γ be a graph with finitely many equivalence classes. Then $s(\Gamma) \geq n$ iff there exists $\Delta > 0$ and an infinite set $R \subset \mathbb{Z}$ such that for any $r \in R$, the sphere $S_r(v)$ cannot be covered by $n-1$ balls of radius $r-1$ whose centers are pairwise at least Δ apart from each other and from v.*

Proof: Suppose there exist $k < \infty$ equivalence classes and choose k vertices $v_1, \ldots, v_k$ such that pairwise $v_i \nsim v_j$. Further assume for each v_i, there exist a $\Delta_i \in \mathbb{Z}$ and an infinite set $R_i \subset \mathbb{Z}$ such that for each $r \in R_i$ the sphere $S_r(v_i)$ cannot be covered by $n-1$ balls of smaller radius whose centers are pairwise Δ_i apart from each other and v_i. This property then holds for every vertex in the equivalence class of v_i. Next, let $\Delta = \max_i \Delta_i$ and $R = \bigcap_i R_i$. Now we have for all $v \in V$ and $r \in R$, the sphere $S_r(v)$ cannot be covered by $n-1$ balls of smaller radius whose centers are pairwise Δ apart from each other and v. Then for any set $T \in \mathcal{T}(\Delta, n)$ with exactly n vertices and each $v \in T$, there exists a vertex $u \in S_r(v)$ with $d(u, v') \geq r$ for all $v' \in T$. Hence u is in the Voronoi cell $C(v,T)$ and because this is true for each $r \in R$, $|C(v,T)| = \infty$. So, $s(\Gamma) \geq n$.

Conversely, suppose that for all Δ, all v, and all but finitely many $r \in \mathbb{Z}$, the sphere $S_r(v)$ can be covered by $n-4$ balls whose centers $v_1, \ldots, v_{n-1}$ are pairwise Δ apart from each other and v and whose radii are less than r. Then for any particular r, for any $u \in S_r(v)$ we have $d(u, v_i) < r$ for some i. Also for every w outside $B_r(v)$, the shortest path from v to w must intersect $S_r(v)$. Hence, w is closer to some v_i than to v, so w is not in the Voronoi cell $C(v,T)$ where $T = \{v, v_1, \ldots, v_{n-1}\}$. Therefore, $|C(v,T)| < \infty$ and $s(\Gamma) < n$. $\square$

Remark It is an interesting question whether Proposition 2.2.3 requires that Γ have finitely many equivalence classes. Consider the spiked linear graph in Figure 2.1, for example. This graph has infinitely many equivalence classes. A sphere $S_r(0)$ centered at the origin contains the vertices $\pm(r,0)$ and $\pm(0,r)$. Any ball containing more than one of these four vertices must then also contain the origin. At least four $(r-1)$-balls are necessary to cover $S_r(0)$. However, the survival number of this graph equals two. Therefore, finitely many equivalence classes is indispensable.

Chapter 3

Pinching the Survival Numbers of $\mathbb{Z}^2$

In this chapter, we use tools from convex geometry to prove the survival number of $\mathbb{Z}^2 = \langle S \rangle$ is either three or four depending on the generating set S. We require that $S = -S$ and let $s(\mathbb{Z}^2, S)$ denote its survival number. Our key theorem is more general than $\mathbb{Z}^2$ and in Chapter 4, we further discuss the general case of $\mathbb{Z}^d$.

3.1 Using convex hulls to compute survival numbers

Definition 3.1.1 *A* convex polytope $Q \subset \mathbb{R}^d$ *is the convex hull of a finite set of vertices in* $\mathbb{R}^d$ *not lying in a common hyperplane. The convex polytope is called a* polygon *when* $d = 2$ *and a* polyhedron *when* $d = 3$.

We say that Q *is* covered by ℓ translates of Q° *if there are some* ℓ *unit vectors* $v_1, \ldots, v_\ell$ *such that for any sufficiently small* $\epsilon > 0$,

$$Q \subset \bigcup_{i=1}^{\ell} \left(Q^\circ + \epsilon v_i\right).$$

Definition 3.1.2 *For a symmetric generating set* S *of* $\mathbb{Z}^d$, *let* Q *denote its convex hull in* $\mathbb{R}^d$ *and let* L *denote the boundary of* Q. *The* Minkowski norm $\|\cdot\|_L$ *on* $\mathbb{R}^d$ *is the unique norm for which* L *is the unit sphere. (i.e.* $\|w\|_L = \lambda$ *if and only if* $\lambda^{-1} w \in L$*)*

In [8], Duchin, Lelievere, and Mooney show that there exists a minimum $K = K(S)$ such that for all $w \in \mathbb{Z}^d$

$$\|w\|_L \leq |w| < \|w\|_L + K \tag{3.1}$$

where $|w|$ is the geodesic word length with respect to S. This value K is the maximum word length in $Q \cap \mathbb{Z}^d$; and for all $r > K$, the sphere $S_r(0)$ in $\mathbb{Z}^d$ embeds into the annulus bounded by $(r - K)L$ and rL in $\mathbb{R}^d$. We use this fact now to prove our key theorem for $\mathbb{Z}^d$, and this theorem along with Theorem 2.2.1 provides a link between $s(\mathbb{Z}^d, S)$ and convex geometry. Unless otherwise specified, let $r \in \mathbb{R}$ vary continuously. In $\mathbb{Z}^d$, let $S_r(0) = S_{\lfloor r \rfloor}(0)$.

Theorem 3.1.3 *For all large r, there exist ℓ balls of radius $r-1$ covering $S_r(0)$ in the word metric if and only if there exist ℓ translates of Q^o in $\mathbb{R}^d$ covering Q.*

Proof: Suppose that ℓ balls in $\mathbb{Z}^d$ cover $S_r(0)$. Then the convex hull rQ has its extreme points $x_1, \dots, x_k \in \mathbb{Z}^d$ covered by ℓ translates of $(r-1)Q \subset rQ^\circ$. Choose an edge $[x_1, x_2]$ with midpoint m. Then for some unit vector v_1 and $\epsilon \in \mathbb{R}$, we have $x_1 \in (r-1)Q + \epsilon v_1 \subset rQ^\circ + \epsilon v_1$. Choose ϵ small enough that m is contained in $rQ^\circ + \epsilon v_1$. Then the segment from x_1 to m is contained in $rQ^\circ + \epsilon v_1$. Doing this with a small enough ϵ that each half-edge is contained in a translation $rQ^\circ + \epsilon v_j$ for some $j = 1, \dots, \ell$ covers the boundary L of Q. Next, choose a fixed point $x \in Q^\circ$. Then for sufficiently small ϵ, there is a half-edge $[x_i, m]$ such that the triangle formed by the convex hull of x, x_i, and m is in some $rQ^\circ + \epsilon v_j$. So, $rQ \subset \bigcup_{j=1}^{\ell} rQ^o + v_j$. Rescaling by $\frac{1}{r}$ gives the desired result.

Conversely, assume $Q \subset \bigcup_{i=1}^{\ell} Q^o + v_i$ for some $v_i \in \mathbb{R}^d$. Let K be as in (3.1), so the ball $B_K(0)$ contains $Q \cap \mathbb{Z}^d$. For $r > K$, we define $C_r(0) := (r-K)Q \cap \mathbb{Z}^d$. Then $C_{r-1}(0) \subset B_{r-1}(0)$ and if ℓ translates of $C_{r-1}(0)$ cover $S_r(0)$, then we are done. By our assumption and by compactness, for large r we have

$$Q \subset \bigcup_{i=1}^{\ell} \left(1 - \frac{1+K}{r}\right) Q + v_i. \tag{3.2}$$

If we rescale (3.2) by r and then take lattice points, we get

$$rQ \cap \mathbb{Z}^d \subset \left[\bigcup_{i=1}^{\ell} (r-1-K)\, Q + rv_i \right] \cap \mathbb{Z}^d. \tag{3.3}$$

Slightly enlarging Q if necessary, we may assume $rv_i \in \mathbb{Z}^d$. Last, we make two observations: $S_r(0) \subset rQ \cap \mathbb{Z}^d$ and the $C_{r-1}(rv_i)$'s cover the right side of (3.3). Therefore, $S_r(0) \subset \bigcup_{i=1}^{\ell} C_{r-1}(rv_i)$. □

In order to use Theorem 2.2.1, the ℓ balls in $\mathbb{Z}^d$ must have centers which are pairwise at least Δ apart. Recall that $\|rv_i - rv_j\|_L = r\|v_i - v_j\|_L$ for all $v_i, v_j \in \mathbb{Z}^d$ and $r \in \mathbb{R}^+$. So because of (3.1), for every $\Delta > 0$ there is a large enough r that pairwise $|rv_i - rv_j| > \Delta$ in $\mathbb{Z}^d$. To show that $s(\mathbb{Z}^2, S)$ is either three or four, we next prove four lemmas. In light of Theorem 2.2.1 and Theorem 3.1.3, Lemma 3.1.4 implies $s(\mathbb{Z}^2, S) \geq 3$; Lemma 3.1.5 implies $s(\mathbb{Z}^2, S) = 4$ if Q is a parallelogram; and Lemma 3.1.7 implies $s(\mathbb{Z}^2, S) = 3$ if Q is not a parallelogram.

Lemma 3.1.4 *No convex polygon is covered by two translates.*

Proof: This property is affine invariant. Without loss of generality assume that Q^o contains the origin. Choose two unit vectors v_1 and v_2 and consider two cases:

If $v_1 \neq -v_2$, then the projections of v_1 and v_2 onto $v_1 + v_2$ have the same direction. However, the line which passes through the origin and is parallel to $v_1 + v_2$ intersects L at two points in opposite directions from the origin. The translates $Q^o + \epsilon v_1$ and $Q^o + \epsilon v_2$ then cannot cover both intersection points.

Next, suppose $v_1 = -v_2$. Recall that if a line intersects the boundary L, it does so either once, twice, or infinitely many times. If some line parallel to v_1 intersects L at exactly one point, then that point must be a vertex of L. Neither ϵv_1 nor ϵv_2 is inward pointing at this vertex. So, neither translate contains it. If no parallel line intersects L exactly once, then some edge is parallel to v_1. This edge is therefore disjoint from both $Q^o + \epsilon v_1$ and $Q^o + \epsilon v_2$. $\square$

Lemma 3.1.5 *Every triangle T is covered by three translates. No parallelogram P is covered by three translates, but four translates suffice.*

Proof: Let w_1, w_2, w_3 be the vertices of T, and at each w_i, choose a unit vector v_i in an inward-pointing direction, i.e., such that $w_i + \epsilon v_i \in T^\circ$ for any sufficiently small $\epsilon > 0$. But then $w_i \in T^\circ - \epsilon v_i$. We claim that

$$T \subset \bigcup_{i=1,2,3} (T^\circ - \epsilon v_i) .$$

We have already seen that each vertex w_i is covered by the corresponding translate $T^\circ - \epsilon v_i$. Next, fix attention on the edge from w_1 to w_2 and let m be its midpoint. Choosing ϵ small enough that a ball of radius ϵ about m misses the other two sides of T, we are guaranteed that $m + \epsilon v_1 \in T^\circ$ and therefore that $x + \epsilon v_1 \in T^\circ$ for all $x \in [v_1, m]$. Then that half-edge is contained in the translate $T^\circ - \epsilon v_1$. If ϵ is chosen small enough that the corresponding property holds for all six half-edges of T, then the whole perimeter of T is covered. Choose a fixed point $t \in T^\circ$. Then for sufficiently small ϵ, each of our translates is a convex set covering a half-edge of T, and therefore covering the triangle formed by the convex hull of the half-edge with t. Therefore, between them, the three translates cover all six of these triangles, which is all of T.

The parallelogram statement follows from two observations: any translate of P° can cover no more than one of its vertices; and four translates corresponding to inward-pointing vectors suffice, following the same argument as above. $\square$

Lemma 3.1.6 *If Q is a convex polygon that is not a parallelogram, then some three of its edges can be extended to form a triangle that encloses Q.*

Proof: This property is affine invariant, so without loss of generality suppose that Q has one vertex at the origin, lies entirely in the first quadrant, and has edges along the x- and y-axes. Let us call those edges X and Y. We must find a triangle T whose three sides are extensions of edges of Q. If any edge E of Q has a negative slope, then extensions of X, Y, E form the needed triangle T. So suppose no such edge exists. There must be some edge with positive slope; otherwise, Q is a rectangle. Every edge of Q that is not vertical satisfies a linear equation of the form $y = mx + b$. Let us define an edge of Q to be an *upper bound* if all points of Q satisfy $y \leq mx + b$ and similarly a *lower bound* if all points satisfy $y \geq mx + b$, so that every non-vertical edge is one or the other. Moving clockwise from the y-axis, it follows from convexity of Q that the successive edges are upper bounds followed by a possible vertical edge followed by lower bounds until reaching the x-axis.

Suppose Q has a second horizontal edge (besides X); call it H. It is necessarily an upper bound. If there is a lower bound E of positive slope, then Y, H, E form

T. Otherwise H must be preceded by an edge of positive slope F and followed by a vertical edge V, in which case F, V, X form T. Clearly a symmetric argument handles the case of a vertical edge, so we may now assume that besides X and Y, the polygon only contains edges of positive slope.

Let E and F be the last upper bound and first lower bound, reading around the perimeter clockwise. Then they form a triangle with either X or Y. $\square$

Lemma 3.1.7 *If Q is a convex polygon that is not a parallelogram, then it is covered by three translates.*

Proof: By Lemma 3.1.6, there is a triangle T such that $Q \subset T$ and three edges of Q share edge segments with T. If the vertices of T are w_1, w_2, w_3 and unit vectors are chosen in inward-pointing directions for T, as above, then we will show that

$$Q \subset \bigcup_{i=1,2,3} (Q^\circ - \epsilon v_i)$$

for sufficiently small $\epsilon > 0$.

Fix attention on a single vertex w of T, with inward-pointing unit vector v, and let u and u' be the first vertices of Q that are encountered when traveling away from w along the two incident edges of T. Then it suffices to show that for all x along the perimeter of Q between u and u', we have $x + \epsilon v \in Q^\circ$, because (as above) this will show that that whole arc of the perimeter of Q is covered by the translate $Q^\circ - \epsilon v$.

By choosing ϵ sufficiently small, we can ensure that ϵv is inward-pointing along the whole segments $[wu]$ and $[wu']$. By convexity of Q, the whole line segment $[uu']$ is contained in Q and therefore the interior of that line segment is either in the interior of Q or along an edge, and ϵv is still inward-pointing along that segment. For any point along $[wu]$ or $[wu']$, if the direction v is followed, convexity again ensures that the trajectory must hit the boundary of Q at a unique point before the edge $[uu']$ is encountered, and then enters Q. Therefore ϵv is inward-pointing at all such boundary points. This completes the argument. $\square$

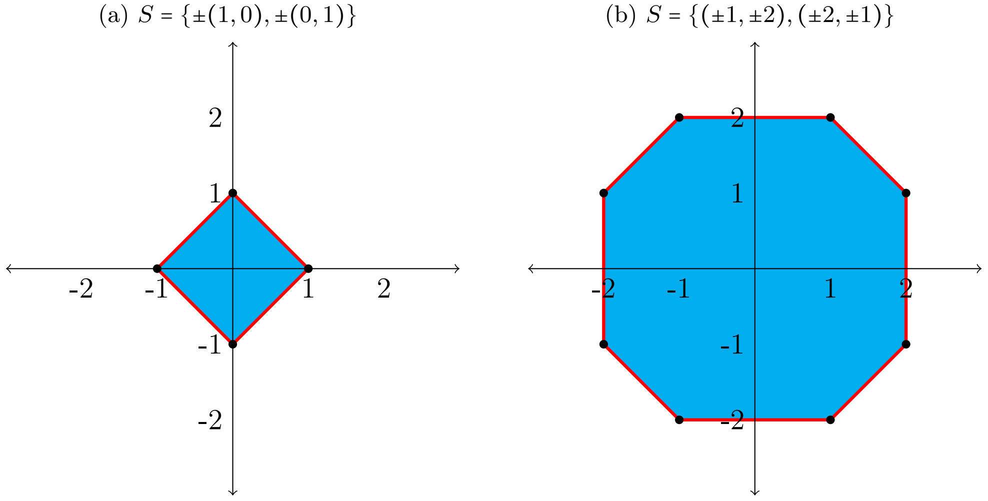
Corollary 3.1.8 *For any symmetric generating set S of $\mathbb{Z}^2$, the survival number $s(\mathbb{Z}^2, S) = 4$ if the polygon induced by S is a parallelogram, and $s(\mathbb{Z}^2, S) = 3$ otherwise.*

3.2 Examples with different generating sets for $\mathbb{Z}^2$

Example (Standard Generating Set): The convex hull of the generating set $S = \{(\pm 1, 0), (0, \pm 1)\}$ is a parallelogram. So, the survival number $s(\mathbb{Z}^2, S) = 4$. □

Example (Knight Generating Set): Consider the generating set $S = \{(\pm 1, \pm 2), (\pm 2, \pm 1)\}$. Its elements can viewed as the possible moves of a knight piece on a chessboard. The convex hull is an octagon, so $s(\mathbb{Z}^2, S) = 3$. □

Figure 3.1: Convex hulls of standard generating set and knight generating set for $\mathbb{Z}^2$


(a) $S = \{\pm(1,0), \pm(0,1)\}$

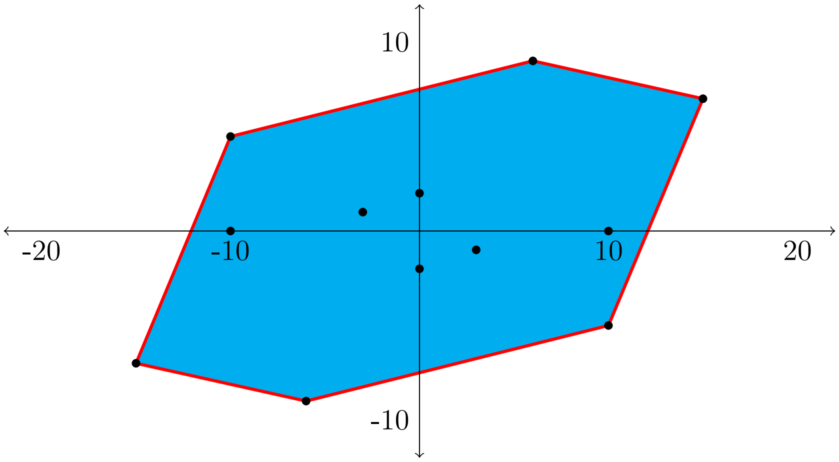
(b) $S = \{(\pm 1, \pm 2), (\pm 2, \pm 1)\}$

Example (A redundant generating set): For a generating set

$$S = \{\pm(0,2), \pm(3,-1), \pm(6,9), \pm(10,-5), \pm(10,0), \pm(15,7)\},$$

we can plot the points to see the convex hull is a hexagon. Hence, $s(\mathbb{Z}^2, S) = 3$. This

Figure 3.2: Convex hull of $S = \{\pm(0,2), \pm(3,-1), \pm(6,9), \pm(10,-5), \pm(10,0), \pm(15,7)\}$



generating set is "redundant" because

$$(1,0) = 1 \cdot (0,2) - 3 \cdot (3,-1) + 1 \cdot (10,-5)$$

$$(0,1) = -2 \cdot (0,2) + 10 \cdot (3,-1) - 3 \cdot (10,-5)$$

The vertices $\pm(10,0)$ in the interior can then be removed and S remains a generating set with $s(\mathbb{Z}^2, S) = 3$. If we remove vertices $\pm(6,9)$ or $\pm(15,7)$ as well, we still have a generating set but now $s(\mathbb{Z}^2, S) = 4$. □

Example (Class of Sequences of Generating Sets): Here we present a method to inductively generate a sequence $(S_i)_{i\geq 1}$ of generating sets such that the $s(\mathbb{Z}^2, S_i)$'s alternate between 4 and 3:

Step 1 Select a generating set S_1 whose convex hull is parallelogram.

Step i For $i > 1$

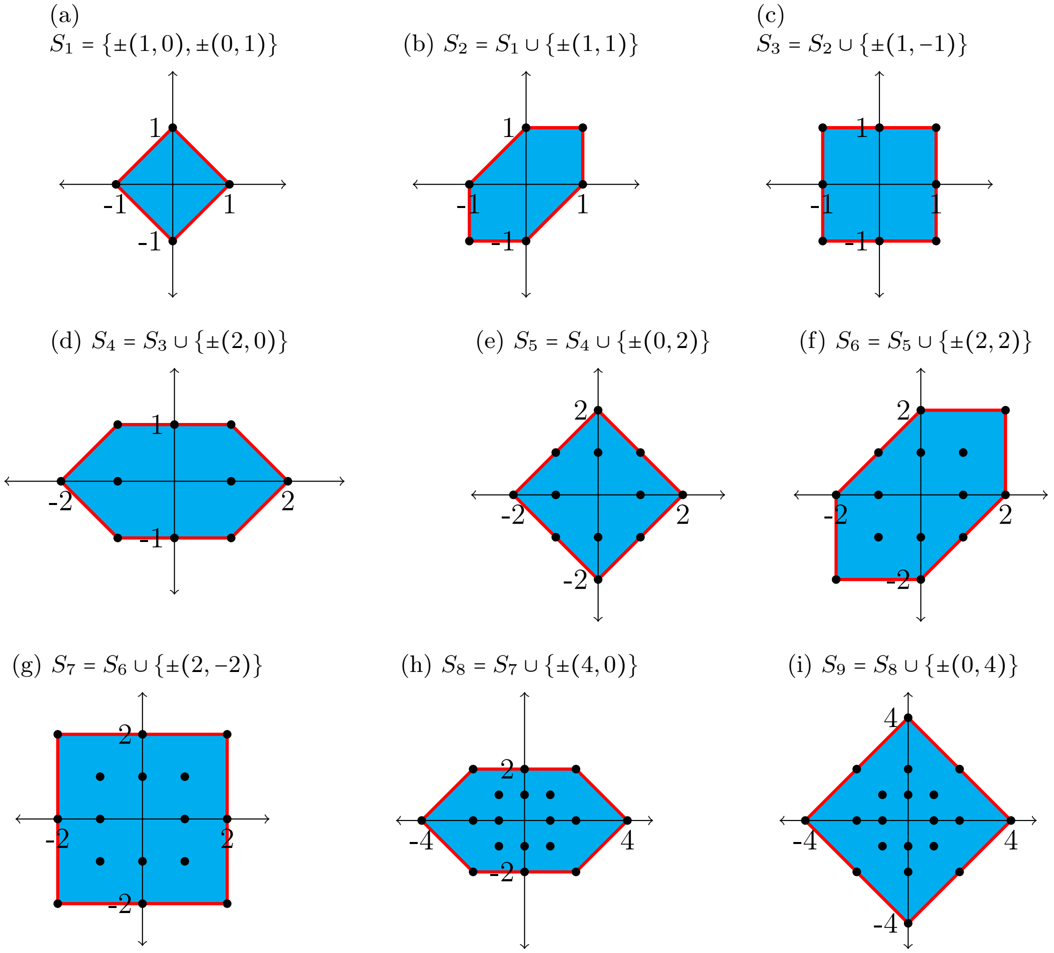
- if S_{i-1} is a parallelogram, select two vertices $\pm\mathbf{x} \in \mathbb{Z}^2$ so that $S_i = S_{i-1} \cup \{\pm\mathbf{x}\}$ has a hexagon as its convex hull; and
- if S_{i-1} is a hexagon, select two vertices $\pm\mathbf{x} \in \mathbb{Z}^2$ so that $S_i = S_{i-1} \cup \{\pm\mathbf{x}\}$ has a parallelogram as its convex hull.

From these instructions, we see the sequence (S_i) is well ordered by inclusion; hence each S_i is a generating set of $\mathbb{Z}^2$; and $s(S_i) = 4$ for odd i and $s(S_i) = 3$ for even i.

For example, let $S_1 = \{\pm(1,0), \pm(0,1)\}$ and for $i > 1$, apply division algorithm $i = 4q + r$ and then define S_i as follows:

- if $r = 1$, then $S_i = S_{i-1} \cup \{\pm(0, 2^q)\}$
- if $r = 2$, then $S_i = S_{i-1} \cup \{\pm(2^q, 2^q)\}$
- if $r = 3$, then $S_i = S_{i-1} \cup \{\pm(2^q, -2^q)\}$
- if $r = 0$, then $S_i = S_{i-1} \cup \{\pm(2^{q+1}, 0)\}$

Figure 3.3: Sequence $(S_i)_{i \geq 1}$


(a) $S_1 = \{\pm(1,0), \pm(0,1)\}$

(b) $S_2 = S_1 \cup \{\pm(1,1)\}$

(c) $S_3 = S_2 \cup \{\pm(1,-1)\}$

(d) $S_4 = S_3 \cup \{\pm(2,0)\}$

(e) $S_5 = S_4 \cup \{\pm(0,2)\}$

(f) $S_6 = S_5 \cup \{\pm(2,2)\}$

(g) $S_7 = S_6 \cup \{\pm(2,-2)\}$

(h) $S_8 = S_7 \cup \{\pm(4,0)\}$

(i) $S_9 = S_8 \cup \{\pm(0,4)\}$

Of the first six generating sets $S_1, \ldots, S_6$, only S_5 and S_6 are redundant. The vertices $\pm(1,0)$ and $\pm(0,1)$ cannot be removed from any of these sets and remain a generating set of $\mathbb{Z}^2$. However we can remove $\{\pm(1,1), \pm(1,-1)\}$ from S_5 and from S_6. In general, any S_i can be reduced to the generating set $kS_j \cup \{\pm(1,0), \pm(0,1)\}$ for some integers $k \geq 1$ and $1 \leq j \leq 4$ and retain its survival number. $\square$

Chapter 4

Survival Numbers of $\mathbb{Z}^d$ and Hadwiger's Conjecture

Theorem 3.1.3 equates $s(\mathbb{Z}^d, S)$ with the covering number of the unit sphere in the normed space $(\mathbb{R}^d, \|\cdot\|_L)$, i.e. the minimum number $c(Q) \in \mathbb{N} \cup \{\infty\}$ such that Q is the convex hull of S in $\mathbb{R}^d$ and there exist $c(Q)$ copies of Q° which cover the boundary $L = \partial Q$. We now explore $\mathbb{Z}^d$ for $d > 2$ by first establishing lower and upper bounds $s(\mathbb{Z}^d, S) \geq d + 1$ and $s(\mathbb{Z}^d, S) \leq \#\{v \in Q : v \text{ is an extreme point of } Q\} \leq |S|$. Then we discuss Hadwiger's Conjecture which, if true, gives the least upper bound $s(\mathbb{Z}^d, S) \leq 2^d$.

4.1 Lower and upper bounds for $s(\mathbb{Z}^d, S)$

Let $Q \subsetneq \mathbb{R}^d$ be a nonempty convex set with boundary L and let $x_0 \in L$. For $a \in \mathbb{R}^d \setminus Q$, the hyperplane $H_b = \{x : a \cdot x = b\}$ is a *supporting hyperplane of* Q *at* x_0 if both $x_0 \in H_b$ and $\sup_{x \in Q} a \cdot x \leq b$. So, Q intersects H_b in at least one point and lies inside one of the two half spaces defined by H_b.

Theorem 4.1.1 (Supporting Hyperplane Theorem, [5]) *Let $Q \subsetneq \mathbb{R}^d$ be a nonempty convex set with boundary L and let $x_0 \in L$. Then there exists at least one hyperplane passing through x_0 and containing Q in one of its half-spaces.*

The supporting hyperplane theorem is a basic property of convex sets.

Next, a point $x \in Q$ is called an *extreme point* if no line segment in Q contains x in its interior. We use this to prove our lower bound for $c(Q)$.

Proposition 4.1.2 *For any d and any convex polytope $Q \subset \mathbb{R}^d$, we have $c(Q) \geq d+1$.*

Proof: We show this by induction on d. If $d = 1$, then Q is a closed line segment in $\mathbb{R}$. No single translate of Q° can then cover both endpoints of Q and so $c(Q) \geq 2$.

(The first example in Section 2.1 shows that $c(Q) = 2$.) Lemma 3.1.4 shows the case where $d = 2$.

Next, suppose the proposition is true for all $d_0 < d$ for a given $d > 2$. Because $Q \subset \mathbb{R}$ is compact, there exists a hyperplane H such that the diameter of $H \cap Q$ is maximal. Our hypothesis requires d copies of $(H \cap Q)^{\circ}$ to cover $H \cap Q$ and since $H \cap Q$ has maximum diameter, it follows that d copies of Q° are necessary to cover $H \cap Q$. These d copies will not suffice to cover Q. Otherwise, for each pair of vertices $\{a, -a\} \in Q$ the convex hulls generated by $(H \cap Q) \cup \{a\}$ and by $(H \cap Q) \cup \{-a\}$ are covered by the same d translates. This is a contradiction, so at least $d + 1$ vertices are necessary. □

Proposition 4.1.3 *If Q has n vertices, then $c(Q) \leq n$.*

Proof: We can view a convex hull with n vertices as the image of a standard simplex $\Delta_{n-1} \subset \mathbb{R}^n$ under some linear map $\pi : \mathbb{R}^n \to \mathbb{R}^d$. Specifically, let $\mathbf{x}_0, \ldots \mathbf{x}_{n-1}$ be the vertices of Q, let $\mathbf{e}_0, \ldots, \mathbf{e}_{n-1}$ be the vertices of Δ_{n-1}, and let $\pi(\mathbf{e}_i) = \mathbf{x}_i$. The vectors $\{\mathbf{e}_1 - \mathbf{e}_0, \ldots, \mathbf{e}_{n-1} - \mathbf{e}_0\}$ are a basis for $\mathbb{R}^n$. So, extend π to a linear map. Translates of Δ_{n-1} will map to translates of Q.

It now suffices to show that n translates of Δ^{o}_{n-1} cover Δ_{n-1}. For each $\mathbf{e_i}$, choose an inward-pointing vector v_i. Then we have $\mathbf{e_i} + \epsilon v_i \in \Delta^{o}_{n-1}$ if and only if $\mathbf{e_i} \in \Delta^{o}_{n-1} - \epsilon v_i$ where $\epsilon > 0$ is sufficiently small. We claim that

$$\Delta_{n-1} \subset \bigcup_{i=0}^{n-1} \Delta^{o}_{n-1} - \epsilon v_i$$

The boundary of Δ_{n-1} has $\binom{n}{k+1}$ k-faces where $k = 0, \ldots, n-2$ and induction on k will show the desired translates cover the boundary. We already have seen that each vertex $\mathbf{e_i}$ is contained in $\Delta^{o}_{n-1} - \epsilon v_i$. Suppose all $(k-1)$-faces are covered. Focus attention on a k-face and let c be the corresponding centroid. Our assumption is that for each point x on the boundary of this k-face, $x \in \Delta^{o}_{n-1} - \epsilon v_i$ for some v_i. By choosing ϵ small enough $c \in \Delta^{o}_{n-1} - \epsilon v_i$. Hence, $[x, c] \subset \Delta^{o}_{n-1} - \epsilon v_i$ which implies the k-face is covered. Therefore, since we chose that face arbitrarily, the n translates cover all k-faces. Last, choose a fixed point $s \in \Delta^o_{n-1}$. Then s is in the intersection of n half-spaces and for small enough ϵ, it remains in the interior of some translate. Since these translates also cover the boundary, they cover Δ_{n-1}. □

Proposition 4.1.4 *If $Q \in \mathbb{R}^d$ is a parallelotope or a cross-polytope, then $c(Q) = 2^d$ or $c(Q) = 2d$, respectively.*

Proof: A parallelotope in $\mathbb{R}^d$ has 2^d vertices and a cross-polytope has $2d$ vertices. Because of Proposition 4.1.3, it then suffices to show the number of vertices of Q is a lower bound for $c(Q)$. If Q is a parallelotope or a cross-polytope, then for any pair of vertices, x_1 and x_2, there exist a pair of parallel hyperplanes H_1 and H_2, such that $x_1 \in H_1$ and $x_2 \in H_2$ and Q^o lies in the region bounded by H_1 and H_2. Any translate of Q^o containing x_1 must then be disjoint from H_2. So, no translate of Q^o contains more than one vertex; hence $c(Q) \geq 2^d$ or $c(Q) \geq 2d$. □

Brass, Moser, and Pach refer to the property "for any pair of vertices, x_1 and x_2, there exist a pair of parallel hyperplanes H_1 and H_2, such that $x_1 \in H_1$ and $x_2 \in H_2$ and Q^o lies in the region bounded by H_1 and H_2" as the "slab property". For any polytope with the slab property, $c(Q)$ must equal its number of vertices. In 1962, Danzer and Grünbaum proved that if Q has this property, then it has at most 2^d vertices. [6]

4.2 Hadwiger's conjecture

German mathematician Hugo Hadwiger (1908-1981) is associated with four famous unsolved problems in graph theory and geometry [11]. One of these problems concerns covering convex bodies (i.e., compact convex sets with nonempty interior) with smaller homothetic copies of itself. Given a d-dimensional convex body Q, a set of the form $\lambda Q + x$, where $0 < \lambda < 1$ and $x \in \mathbb{R}^d$, is called a *smaller homothetic copy of Q*.

Conjecture (Hadwiger, [6]) *Any convex body $Q \subset \mathbb{R}^d$ can be covered by at most 2^d of its smaller homothetic copies. Equality holds only for parallelotopes.*

Hadwiger made this conjecture in 1957. Let $H(Q)$ denote the minimum number of smaller homothetic copies of Q that can cover Q. In 1955, Levi proved the conjecture to be true for $d = 2$; Levi also showed in 1955 that if $Q \in \mathbb{R}^d$ has a smooth boundary, then $H(Q) = d + 1$. Hadwiger's conjecture remains unsolved for all but a few special cases. In 1984, Lassak confirmed the conjecture for centrally symmetric bodies in $\mathbb{R}^3$ [12]. Bezdek generalized Lassak's result to convex polytopes in $\mathbb{R}^3$ with any affine symmetry. [3]

The title of [3], "A note on the illumination of convex bodies", alludes to an alternate formulation of Hadwiger's Conjecture. We say a boundary point x of Q is *illuminated by another point* $t \notin Q$ if the ray from t to x passes through Q° but $[t, x)$ is disjoint from Q. Boltyanski in 1960 observed that for convex bodies in $\mathbb{R}^d$, the minimum number of points outside Q necessary to illuminate the boundary of Q equals $H(Q)$. Hence, Hadwiger's Conjecture is often rephrased as the *Illumination Conjecture*: Any d-dimensional convex body Q can be illuminated by at most 2^d light sources with equality if and only if Q is an affine d-cube [4]. This formulation was adopted by Lassak in proving the conjecture for centrally symmetric bodies in $\mathbb{R}^3$.

Our concern is when $Q \subset \mathbb{R}^d$ is a centrally symmetric convex polytope. This is because

$$Q^\circ = \bigcup_{0<\lambda<1} \lambda Q$$

and so $c(Q)$, the minimum number of copies of Q° that can cover Q, is less than or equal to $H(Q)$. It follows that Hadwiger's conjecture in the case of centrally symmetric convex polytopes suggests $s(\mathbb{Z}^d, S) \leq 2^d$ for all generating sets S of $\mathbb{Z}^d$.

In the case of centrally symmetric polytopes, Rogers proved in 1963 an upper bound for all d

$$H(Q) \leq 2^d (d \log d + d \log\log d + 5d).$$

This is the best known upper bound for such polytopes. The best known upper bound in the general case is $4^d(5d \ln d)$. [6] We will not explore these estimates further; instead, we simply record that for all generating sets S of $\mathbb{Z}^d$

$$s(\mathbb{Z}^d, S) \leq 2^d(d \log d + d \log \log d + 5d).$$

Chapter 5

Descriptions of $DL(m,n)$ and $\mathbb{Z}_n \wr \mathbb{Z}$

Let $(\mathbb{Z}_n, +)$ denote the group of integers modulo n, and let $\mathbb{T}_n$ denote the infinite $(n+1)$-valent tree.

Wolfgang Woess in his 1991 survey article "Topological Groups and Infinite Graphs" asks if all vertex transitive graphs "look like" Cayley graphs from a distance. Specifically, he gives the definition of a quasi-isometry between graphs and asks if there exists a vertex transitive graph which is not quasi-isometric to some Cayley graph. [13] Woess points out that if a graph has polynomial growth, then it is quasi-isometric to a Cayley graph. (He credits this result to the work of Vladimir Trofimov in the mid 1980's.) Twenty years later, Reinhard Diestel and Imre Leader construct a graph that they conjectured is not quasi-isometric to any Cayley graph. Now known as a Diestel-Leader graph, they initially construct it as the limit of a sequence of graphs. In the same article, they also give an explicit construction as a horocyclic product of two infinite trees. [7] If the two trees have the same valency, then the Diestel-Leader graph is a Cayley graph of a finitely generated group. [14] Eskin, Fisher, and Whyte proved in 2007 that if the valencies differ, then the Diestel-Leader graph is not quasi-isometric to any Cayley graph. [9]

The goal of this chapter is to define a coordinate system for the Diestel-Leader graph $DL(m,n)$. We do this by defining two coordinate systems for a planar embedding of the tree $\mathbb{T}_m$. One coordinate system was created by Woess and he used it to define a bijection between vertices in $DL(n,n)$ and words in the lamplighter group $\mathbb{Z}_n \wr \mathbb{Z}$. The other system on T_m will consist of ordered pairs $(x,k) \in \mathbb{Z}^2$. We will use this to define coordinates in $DL(m,n)$ as ordered triples $(x,y,k) \in \mathbb{Z}^3$. These triples will allow us to compute the survival numbers of Diestel-Leader graphs and lamplighter groups.

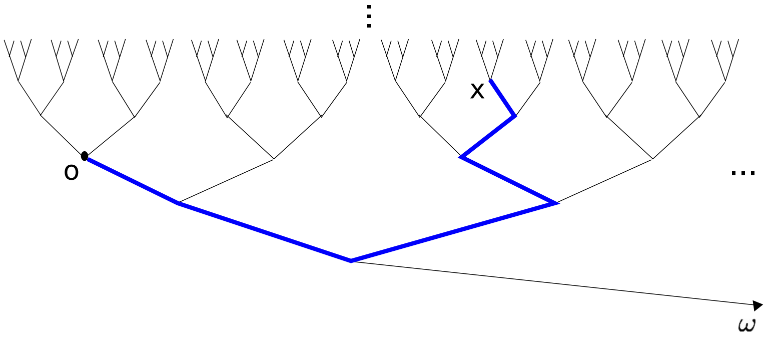


Figure 5.1: $\mathbb{T}_2$ with path $\overline{ox}$. The point x in counting coordinates is $(10,2)$. In Woess coordinates, it is $((\sigma_i),2)$ where $(\sigma_i)_{i\le 0} = (\dots,0,0,0,1,0,1,0)$.

5.1 Coordinates in $\mathbb{T}_m$ and $DL(m,n)$

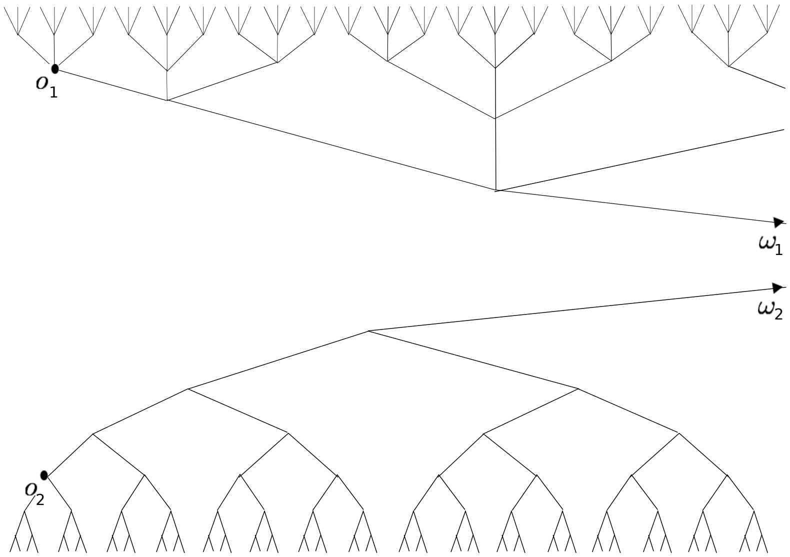
5.1.1 Two Coordinate Systems for $\mathbb{T}_m$

Fix an origin $o \in \mathbb{T}_m$ and choose a direction ω. Visually represent $\mathbb{T}_m$ in the plane so that the ray $\overline{o\omega}$ is the path furthest to the left and pointing downward. Also align the vertices horizontally according to height k with respect to ω. Technically, the height k of $x \in \mathbb{T}_m$ is defined by $\mathfrak{h}(x) = \lim_{y\to\omega} d(x,y) - d(o,y)$, called a *Busemann function*, and each horizontal row $H_k = \{x \in \mathbb{T}_m : \mathfrak{h}(x) = k\}$ is called a *horocycle.*

A vertex $x \in H_k$ has m successors in H_{k+1}. We can intuitively define a coordinate system on $\mathbb{T}_m$ as follows: Let $(x,k) \in \mathbb{Z}\times\mathbb{Z}$ be given so that x is nonnegative. We find the location of (x,k) by locating height k and counting from zero to the x-th vertex from the left. The vertex (x,k) then has m successors $\{(xm+i,k+1) : i = 0,\dots,m-1\} \subset H_{k+1}$ and succeeds $\left(\left\lfloor \frac{x}{m} \right\rfloor, k-1\right) \in H_{k-1}$.

In addition to the "counting" coordinate system defined above, we will use another system defined by Woess. For every vertex, index its m successors by $\mathbb{Z}_m$. Let Σ_m be the set of all sequences $(\sigma_i)_{i\le 0}$ in $\mathbb{Z}_m$ with finite support $\{i : \sigma_i \ne 0\}$. (It is important to notice these sequences are indexed by nonpositive integers.) We identify a pair $((\sigma_i),k) \in \Sigma_m \times \mathbb{Z}$ with a vertex $x \in H_k$ in the following way: Let $\ell = \min\{i : \sigma_i \ne 0\}$, and let $\ell = 1$ if $\sigma_i = 0$ for all i. Then a geodesic path $\overline{ox}$ first travels from o to the leftmost vertex $((\dots,0),k+\ell-1) \in H_{k+\ell-1}$. If $\ell = 1$, then x is this vertex. If $\ell \ne 1$, the path moves to the successor $\sigma_\ell \in \mathbb{Z}_m$ in $H_{k+\ell}$. The path continues moving upward to

Figure 5.2: DL(3,2)



$x \in H_k$ with each successor given by σ_i for $\ell < i \leq 0$. [14]

To convert from counting coordinates $(x, k) \in \mathbb{Z} \times \mathbb{Z}$ to Woess coordinates $((\sigma_i), k) \in \Sigma_m \times \mathbb{Z}$. We only need to find the base m expansion of $x \in \mathbb{Z}^+$. If

$$x = \alpha_0 + \alpha_1 m + \alpha_2 m^2 + \cdots + \alpha_{\bar{\ell}} m^{\bar{\ell}}$$

then let $\alpha_j = 0$ for all $j > \bar{\ell}$ and let the sequence $(\sigma_i)_{i \leq 0} = (\alpha_{-i})_{i \leq 0}$. Throughout this chapter and the next chapter, we will use both coordinate systems. Counting coordinates will be used to locate vertices in the Diestel-Leader graph, and Woess coordinates will be used to identify vertices with elements in the lamplighter group. We explain the identification in Section 5.2.

5.1.2 A Coordinate System for Diestel-Leader Graphs

Diestel and Leader give the definition of $DL(m, n)$ in [7] as a horocyclic product of T_m and T_n. The vertex set is defined by the

$$DL(m, n) = \{x_1 x_2 \in \mathbb{T}_m \times \mathbb{T}_n : \mathfrak{h}(x_1) + \mathfrak{h}(x_2) = 0\}$$

and the adjacency relation is

$$x_1x_2 \sim y_1y_2 \Longleftrightarrow x_1 \sim y_1 \text{ and } x_2 \sim y_2.$$

We visually represent $DL(m,n)$ by its projections as in Figure 5.2. A path in $DL(m,n)$ is represented by two paths, one in each tree. The two paths move up or down simultaneously. We give coordinates in $DL(m,n)$ as triples (a,b,k) for non-negative integers a and b and height k. These coordinates then project to (a,k) and $(b,-k)$ in $\mathbb{T}_m$ and $\mathbb{T}_n$, respectively. Observe that vertices (a,b,k) and (x,y,ℓ) are adjacent if and only if both

1. $|k-\ell| = 1$; and
2. either

$$\text{if } k-\ell = 1, \text{ then } \left\lfloor \frac{a}{m} \right\rfloor = x \text{ and } \left\lfloor \frac{y}{n} \right\rfloor = b$$

or

$$\text{if } \ell-k = 1, \text{ then } \left\lfloor \frac{x}{m} \right\rfloor = a \text{ and } \left\lfloor \frac{b}{n} \right\rfloor = y$$

So, a 1-neighborhood in $DL(m,n)$ projects onto a 1-neighborhood in each tree.

Bertacchi in [2] derives a formula relating distance in $DL(m,n)$ to distances in $\mathbb{T}_m$ and $\mathbb{T}_n$:

$$d(x_1x_2, y_1y_2) = d(x_1,y_1) + d(x_2,y_2) - |\mathfrak{h}(x_1) - \mathfrak{h}(y_1)| \tag{5.1}$$

We will use this formula heavily to compute the survival number of $DL(m,n)$.

5.2 Lamplighter groups

Definition 5.2.1 *The* lamplighter group $\mathbb{Z}_n \wr \mathbb{Z}$ *(also commonly denoted by* L_n*) is defined by the wreath product*

$$\mathbb{Z}_n \wr \mathbb{Z} := \left(\bigoplus_{\mathbb{Z}} \mathbb{Z}_n \right) \rtimes \mathbb{Z}.$$

An element in $\mathbb{Z}_n \wr \mathbb{Z}$ *is denoted* $[\eta, k]$ *for some* $k \in \mathbb{Z}$ *and some finitely supported configuration* $\eta : \mathbb{Z} \to \mathbb{Z}_n$*. The group* $\mathbb{Z}$ *acts on* $\bigoplus_{\mathbb{Z}} \mathbb{Z}_n$ *by the right shift* $k\eta(x) = \eta(x+k)$*. Multiplication in this group is defined as*

$$[\eta_1, k_1] \cdot [\eta_2, k_2] = [\eta_1 + k_1\eta_2, k_1 + k_2] \tag{5.2}$$

where for any $x \in \mathbb{Z}$ *we have* $(\eta_1 + k\eta_2)(x) = \eta_1(x) + \eta_2(x + k_1) \mod n$*. We let* $\mathbf{0}$ *denote the zero map from* $\mathbb{Z}$ *to* $\mathbb{Z}_n$*.*

We can represent η as a bi-infinite sequence indexed by $\mathbb{Z}$. For $x \in \mathbb{Z}$ and $a \in \mathbb{Z}_n$, we let a_x denote the equation $\eta(x) = a$. (e.g. 1_5 means $\eta(5) = 1$) Then

$$\eta = (\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots) \tag{5.3}$$

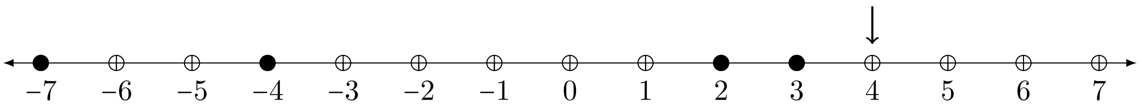
As an example of multiplication in $\mathbb{Z} \wr \mathbb{Z}_3$, we now compute $[\eta_1, 3] \cdot [\eta_2, 5]$ for

$$\begin{aligned} \eta_1 &= (\dots, 0_{-1}, 0_0, 1_1, 2_2, 1_3, 2_4, 0_5, 1_6, 0_7, 0_8, \dots) \\ \eta_2 &= (\dots, 0_{-3}, 0_{-2}, 2_{-1}, 1_0, 1_1, 0_2, 0_3, \dots) \end{aligned}$$

Our group action tells us that

$$3\eta_2 = (\dots, 0_0, 0_1, 2_2, 1_3, 1_4, 0_5, 0_6, \dots).$$

Figure 5.3: Element in the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$



So from (5.2) we get

$$\eta_1 + 3\eta_2 = (\dots, 0_{-1}, 0_0, 1_1, 1_2, 2_3, 0_4, 0_5, 1_6, 0_7, 0_8, \dots)$$

and $[\eta_1, 3] \cdot [\eta_2, 5] = [\eta_1 + 3\eta_2, 8]$.

In 2005, Woess observed that the Diestel-Leader graph $DL(n,n)$ is a Cayley graph for the lamplighter group. In doing so, he contsructs a bijection from $\mathbb{Z}_n \wr \mathbb{Z}$ to $DL(n,n)$ by "splitting" $[\eta, k]$ into two sequences ${}_k\eta^-$ and ${}_k\eta^+$ in Σ_n. Specifically, for η in (5.3)

$${}_k\eta^- = (\dots, a_{k-2}, a_{k-1}, a_k) \text{ and } {}_k\eta^+ = (\dots, a_{k+3}, a_{k+2}, a_{k+1})$$

and $[\eta, k]$ maps to $x_1 x_2$ for $x_1 = ({}_k\eta^-, k)$ and $x_2 = ({}_k\eta^+, -k)$ in $\mathbb{T}_n$ in Woess coordinates. (Recall, Woess coordinates are indexed by nonpositive integers. So, ${}_k\eta^- = (\sigma_i)_{i \le 0}$ with $\sigma_0 = a_k$ and ${}_k\eta^+ = (\sigma_i)_{i \le 0}$ with $\sigma_0 = a_{k+1}$.) The lamplighter group acts freely and transitively on $DL(n,n)$ by translating height and cyclically permuting the n successors of each vertex. [14]

This is called a lamplighter group because of a geometric interpretation of $[\eta, k]$. At each vertex in the Cayley graph for $\mathbb{Z} = \langle 1 \rangle$ is a "lamp" with settings $0, \dots, n-1$. Initially, each lamp is off, i.e. has the 0 setting. To change a lamp's setting, a lamplighter must first walk to that lamp and then change the setting. So, $[\eta, k]$ gives the settings of each lamp and the location of the lamplighter.

The generating set for $\mathbb{Z}_n \wr \mathbb{Z}$ that we are using is $S = \{ta^i, a^{-i}t^{-1} : i = 0, \dots, n-1\}$ for $t = [\mathbf{0}, 1]$ and $ta^i = [\eta^i, 1]$, where $\eta^i = (\dots, 0_{-1}, 0_0, i_1, 0_2, 0_3 \dots)$. We interpret right multiplication by t as the lamplighter taking one step in the positive direction and by ta^i as taking a step then increasing the lamp's intensity by i units modulo

n. In Figure 5.3, for example, there are two intensities, “on” and “off”. The lamps at -7, -4, 2, and 3 are on, every other lamp is off, and the lamplighter is located at 4. With our generating set S, this element is $t^{-8}(ta)t^2(ta)t^5(ta)^2t = t^{-7}at^3at^6atat$ and its word length is 20.

Chapter 6

Survival Numbers of $DL(m,n)$ and $\mathbb{Z}_n \wr \mathbb{Z}$

6.1 Survival Numbers of Diestel-Leader graphs

Throughout this chapter, we will denote a vertex in $DL(m,n)$ interchangeably as a pair $x_1x_2 \in \mathbb{T}_m \times \mathbb{T}_n$ and as a triple (a,b,k) as in Subsection 5.1.2. It will be clear when and why we switch the notation. We also let $S_r(\mathbf{0})$ denote the r-sphere in $DL(m,n)$ centered at the origin (denoted as both o_1o_2 and $(0,0,0)$) .

Lemma 6.1.1 *For all $r > 0$, we can choose $m+n$ vertices in $S_r(\mathbf{0})$ that are pairwise $2r$ apart from each other. Hence, $S_r(\mathbf{0})$ cannot be covered by less than $m+n$ balls of smaller radius.*

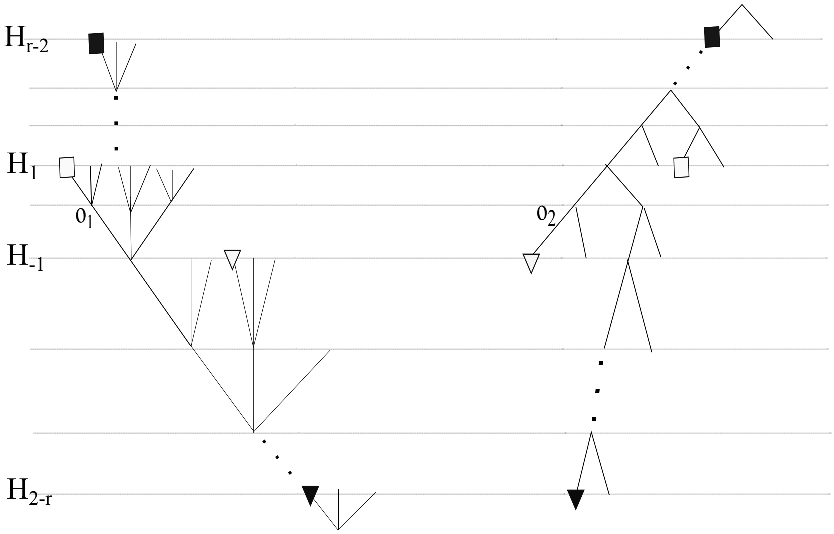
Proof: We choose m geodesic paths in $\mathbb{T}_m$ from the origin to a point in H_r in such a way that each path has a distinct initial edge. Take the inclusion of these m paths in $DL(m,n)$. Our "m-vertices" will be the set of m endpoints. We derive our "n-vertices" in the same manner.

The definition of $DL(m,n)$ requires that the pairwise distance between m-vertices and n-vertices is $2r$: A path between an m-vertex and an n-vertex projects onto two paths, one in each tree, and from height r to $-r$. Also, a path connecting two m-vertices or two n-vertices must cross the origin. Therefore, $S_r(\mathbf{0})$ cannot be covered by less than $m+n$ smaller balls. □

Theorem 6.1.2 *For all m and n, we have $s(DL(m,n)) \geq m+n+2$.*

Proof: We will first show that $m+n$ balls centered at the each of the vertices in the neighborhood of the origin do not cover all of $S_r(\mathbf{0})$. However, these $m+n$ balls are necessary to cover its maximum and minimum heights. Then we show that for two particular vertices, disjoint from the aforementioned balls, if they are contained in the

Figure 6.1: Subset of r-ball in $DL(3,2)$ centered at the origin.



same $(r-1)$-ball, then that ball's center is too close to origin. Hence, these two points must be in separate balls and, by Theorem 2.2.1, we conclude $s(DL(m,n)) \geq m+n+2$.

Choose $\square = (0,n,1)$ and $\triangledown = (m,0,-1)$. (See Figure 6.1.) Then $\square$ projects onto the leftmost successor of the origin o_1 in $\mathbb{T}_m$ and $\triangledown$ projects onto the leftmost successor of the origin o_2 in $\mathbb{T}_n$. Notice that the projection of $\square$ into $\mathbb{T}_n$ is $(n,-1)$ which two units away from the ray $o_2\omega_2$. Likewise, the projection of $\triangledown$ into $\mathbb{T}_m$ is two units away from $o_1\omega_1$. So using equation (5.1), we get distances

$$\begin{aligned} d(\mathbf{0},\square) &= \quad 1+5-1=5 \\ d(\mathbf{0},\triangledown) &= \quad 5+1-1=5 \\ d(\square,\triangledown) &= \quad (5+1)+(5+1)-2=10 \end{aligned}$$

So, $\mathbf{0}$, $\square$, and $\triangledown$ are pairwise at least $\Delta = 5$ apart from each other. Proof of Lemma 6.1.1 says tells us $\square$ and $\triangledown$ are necessary to cover $S_r(\mathbf{0})$.

We next find two vertices which are in neither $(r-1)$-ball centered at $\square$ nor at $\triangledown$. Let $\blacksquare \in S_r(\mathbf{0})$ be the vertex reached by first traveling from the origin $(0,0,0)$ to $(1,0,0)$ and then continuing up to level set $H_{r-2} \times H_{2-r}$ with each subsequent vertex projecting onto the leftmost successor in both trees. (See Figure 6.1.) We let $\blacktriangledown \in S_r(\mathbf{0})$ be the vertex in $H_{2-r} \times H_{r-2}$ reached in an analogous manner. The coordinates for these two vertices are $\blacksquare = (m^{r-2}, 0, r-2)$ and $\blacktriangledown = (0, n^{r-2}, 2-r)$.

$$\begin{aligned}
d(\square, \blacksquare) &= \quad (r+1) + (r-3) - (r-3) = r+1 \\
d(\triangledown, \blacksquare) &= \quad (5+r-2) + (r-1) - (r-1) = r+3 \\
d(\square, \blacktriangledown) &= \quad (r-1) + (5+r-2) - (r-1) = r+3 \\
d(\triangledown, \blacktriangledown) &= \quad (r-3) + (r+1) - (r-3) = r+1
\end{aligned}$$

Hence, $s(DL(m,n)) \geq m+n+1$.

Last, there exist three $(r-1)$-balls containing both $\blacksquare$ and $\blacktriangledown$. The centers are located at $(1,1,0)$, $(m,0,1)$ and $(0,n,-1)$; and they are located are at most $4 < \Delta$ away from $(0,0,0)$. So, $S_r(\mathbf{0})$ cannot be covered by $m+n+1$ sufficiently spaced smaller balls. $\square$

Theorem 6.1.3 *For all* m *and* n*, we also have* $s(DL(m,n)) \leq m+n+2$.

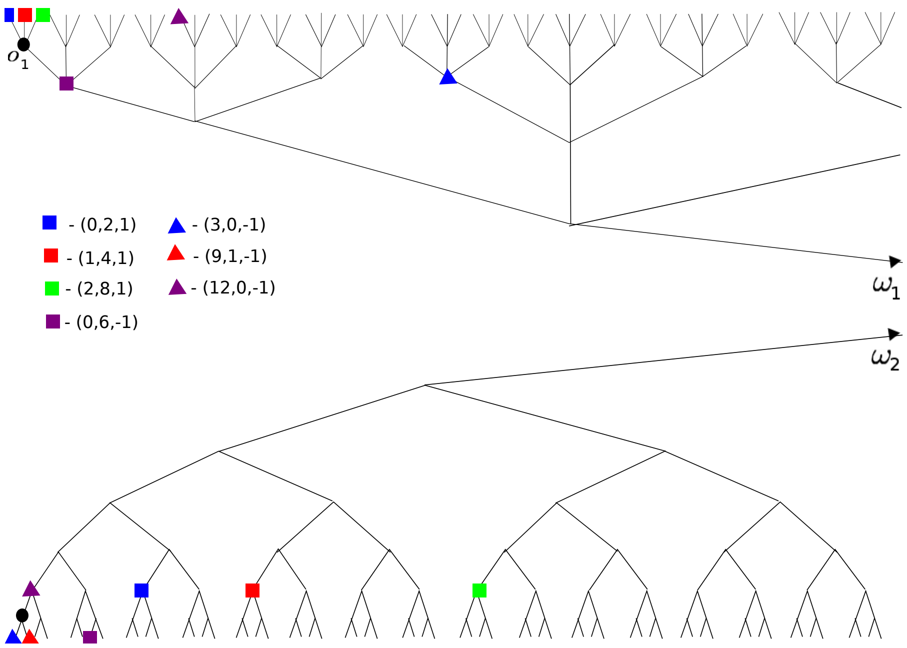
Proof: To prove that $S_r(\mathbf{0})$ can be covered by $m+n+2$ sufficiently spaced smaller balls, we surround the origin $o_1 o_2$ using vertices in $DL(m,n)$ gathered into two sets $\mathcal{U}$ and $\mathcal{L}$. The vertices in $\mathcal{U}$ project onto the neighborhood of o_1 in $\mathbb{T}_m$ and have sufficiently spaced projections in $\mathbb{T}_n$. Likewise, vertices in $\mathcal{L}$ project onto the neighborhood of o_2 in $\mathbb{T}_n$ and have sufficiently spaced projections in $\mathbb{T}_m$. For a given $\Delta \geq 5$, let

$$\begin{aligned}
\mathcal{U} &= \left\{(i, n^{\Delta-4+i}, 1) : i = 0, \ldots m-1\right\} \cup \left\{(0, n^{\Delta-4} + n^{\Delta-3}, -1)\right\} \\
\mathcal{L} &= \left\{(m^{\Delta-4+j}, j, -1) : j = 0, \ldots m-1\right\} \cup \left\{(m^{\Delta-4} + m^{\Delta-3}, 0, 1)\right\}
\end{aligned}$$

(See Figure 6.2.)

We must first show these $m+n+2$ vertices are pairwise at least $\Delta \geq 5$ apart from each other and $o_1 o_2 = (0,0,0)$. To begin, there exists a geodesic path of successors from $(n^{\Delta-4+i}, -1)$ to the ray $\overline{o_2 \omega_2}$ in $\mathbb{T}_n$. (Recall that $(n^{\Delta-4+i}, -1) \in \mathbb{T}_n$ succeeds

Figure 6.2: $DL(3,2)$ for $\Delta = 5$.



$(\frac{n^{\Delta-4+i}}{n}, -2)$ which in turn succeeds $(\frac{n^{\Delta-4+i}}{n^2}, -3)$. Continue inductively to get the path.) The length is $(\Delta - 4 + i) + 1 = \Delta - 3 + i$ and it intersects $\overline{o_2\omega_2}$ at the vertex $(0, -\Delta + 2 - i)$, and so by Equation (5.1)

$$\begin{aligned} d((0,0,0),(i,n^{\Delta-4+i},1)) &= d((0,0),(i,1)) + d((0,0),(n^{\Delta-4+i},-1)) - |0-(-1)| \\ &= 1 + [(\Delta-3+i) + (\Delta-2+i)] - 1 \\ &= 2\Delta - 5 + 2i \end{aligned}$$

Apply this reasoning to every possible pair of vertices in $\mathcal{U} \cup \mathcal{L} \cup \{o_1 o_2\}$. The distances are listed in Figure 6.3. (Two things to clarify if not obvious: This is a symmetric table, and distances between $(i_1, n^{\Delta-4+i_1}, 1)$ and $(i_2, n^{\Delta-4+i_2}, 1)$ assume $i_1 \neq i_2$.) Figure 6.3 shows that for all $\Delta \geq 5$, the minimum distance between two vertices is $2\Delta - 5$.

Now we show that $(r-1)$-balls centered at these $m + n + 2$ vertices cover $S_r(\mathbf{0})$. Three important facts to reiterate are (*) each vertex in $\mathcal{U}$ projects to a vertex in $\mathbb{T}_n$ whose distance from $\overline{o_2\omega_2}$ is at most $\Delta - 3 + (m-1) = \Delta - 4 + m$; (**) similarly,

Figure 6.3: Pairwise distances between vertices in $\mathcal{U} \cup \mathcal{L} \cup \{o_1 o_2\}$

d	$(0,0,0)$	$(i, n^{\Delta-4+i}, 1)$	$(0, n^{\Delta-4} + n^{\Delta-3}, -1)$	$(m^{\Delta-4+j}, j, -1)$	$(m^{\Delta-4} + m^{\Delta-3}, 0, 1)$
$(0,0,0)$	0	$2\Delta - 5 + 2i$	$2\Delta - 5$	$2\Delta - 5 + 2j$	$2\Delta - 5$
$(i, n^{\Delta-4+i}, 1)$	-	$2\Delta - 4 + 2\max\{i_1, i_2\}$	$2\Delta - 4 + 2i$	$4\Delta - 10 + 2(i+j)$	$4\Delta - 10 + 2i$
$(0, n^{\Delta-4} + n^{\Delta-3}, -1)$	-	-	0	$4\Delta - 10 + 2j$	$4\Delta - 14$
$(m^{\Delta-4+i}, 1)$	-	-	-	$2\Delta - 4 + 2\max\{j_1, j_2\}$	$2\Delta - 4 + 2j$
$(m^{\Delta-4} + m^{\Delta-3}, 0, 1)$	-	-	-	-	0

the projections of vertices in $\mathcal{L}$ into $\mathbb{T}_m$ are distance at most $\Delta - 4 + n$ from $\overline{o_1\omega_1}$; and (***) the vertices whose projections are furthest from $\overline{o_1\omega_1}$ and $\overline{o_2\omega_2}$ are located furthest to the right in each tree. For a given $x_1x_2 \in S_r(\mathbf{0})$, consider three cases based on the height $\mathfrak{h}(x_1)$:

Case 1 Suppose $\mathfrak{h}(x_1) > \Delta - 4 + m$. Choose a vertex $y_1y_2 \in \mathcal{U}$ whose coordinate y_1 coincides with the initial step of the projection of a geodesic from o_1o_2 to x_1x_2. Using equation (5.1), we have

$$|\mathfrak{h}(x_1) - \mathfrak{h}(y_1)| = d(x_1, y_1) + d(x_2, y_2) - d(x_1x_2, y_1y_2) \tag{6.1}$$

Our choice of y_1y_2 admits $d(x_1, y_1) = d(o_1, x_1) - 1 < r$. Because $\mathfrak{h}(x_1) > \Delta - 4 + m$ and because of (*), we have $d(x_2, y_2) \leq \Delta - 4 + m < \mathfrak{h}(x_1) \leq r$. Let $\ell = |\mathfrak{h}(x_1) - \mathfrak{h}(y_1)|$, which is at most $r - 1$. Combining all this with (6.1) yields the inequality

$$\ell < r < 2r - d(x_1x_2, y_1y_2), \tag{6.2}$$

and so $d(x_1x_2, y_1y_2) \leq r - 1$.

Case 2 If $\mathfrak{h}(x_1) < -(\Delta - 4 + n)$, then $\mathfrak{h}(x_2) > \Delta - 4 + n$. Choose a vertex $y_1y_2 \in \mathcal{L}$ whose coordinate y_2 coincides with the initial step of the projection of a geodesic from o_1o_2 to x_1x_2. The same argument as above with some switching of indices gives $d(x_1x_2, y_1y_2) \leq r - 1$.

Case 3 If $-(\Delta - 4 + n) \leq \mathfrak{h}(x_1) \leq \Delta - 4 + m$, then for large r, a geodesic path from o_1o_2 to x_1x_2 contains a vertex $\overline{x_1x_2}$ with $\mathfrak{h}(\overline{x_1}) > \Delta - 4 + m$ or $\mathfrak{h}(\overline{x_1}) < -(\Delta - 4 + n)$. This is because a geodesic path in $DL(m, n)$ changes direction at most twice. So, if this was not true, then the distance r from o_1o_2 to x_1x_2 would be bounded above by $4\Delta - 12 + 2(m+n)$, the sum of maximum lengths in each direction. Let $\overline{r} = d(o_1o_2, \overline{x_1x_2})$, and apply Case 1 or Case 2 for $\overline{x_1x_2} \in S_{\overline{r}}(0)$ and $\mathfrak{h}(\overline{x_1})$. The conclusion of $d(\overline{x_1x_2}, y_1y_2) \leq \overline{r} - 1$ extends to $d(x_1x_2, y_1y_2) \leq r - 1$ after using the triangle inequality with $d(\overline{x_1x_2}, x_1x_2) = r - \overline{r}$. □

6.2 Survival number of the lamplighter group $\mathbb{Z}_n \wr \mathbb{Z}$

Recall that for $m = n$, there exists a Cayley graph for $\mathbb{Z}_n \wr \mathbb{Z}$ isomorphic to $DL(m,n)$. In this section, we express the set $\mathcal{U} \cup \mathcal{L}$ as elements of the lamplighter group. To determine the corresponding elements in $\mathbb{Z}_n \wr \mathbb{Z}$, we must find the configurations $\eta : \mathbb{Z} \to \mathbb{Z}_n$ that "split" appropriately. (Splitting is defined in Section 5.2.)

1. First, consider the element $(i, n^{\Delta-4+i}, 1) \in \mathcal{U}$. Convert to Woess coordinates to get x_1x_2 for $x_1 = ((\ldots,0,i),1)$ (since $i = in^0$) and $x_2 = ((\ldots,0,1,0,\ldots,0),-1)$ with $1 = \sigma_{-(\Delta-4+i)}$. So, $k=1$, ${}_1\eta^- = (\sigma_i)_{i\leq 0}$ with $\sigma_0 = a_1$, and ${}_1\eta^+ = (\sigma_i)_{i\leq 0}$ with $\sigma_0 = a_2$. Therefore

   $$\begin{aligned} {}_1\eta^- &= (\ldots, 0_0, i_1) \\ {}_1\eta^+ &= (\ldots, 0_{\Delta-1+i}, 1_{\Delta-2+i}, 0_{\Delta-3+i}, \ldots, 0_2) \\ \eta &= (\ldots, 0_0, i_1, 0_2, \ldots, 0_{\Delta-3+i}, 1_{\Delta-2+i}, 0_{\Delta-1+i} \ldots) \end{aligned}$$

   and $[\eta, 1]$ is the corresponding lamplighter element.

2. Next, consider $(0, n^{\Delta-4} + n^{\Delta-3}, -1) \in \mathcal{U}$. Its Woess coordinates are x_1x_2 for $x_1 = ((\ldots,0),-1)$ and $x_2 = ((\ldots,0,1,1,0,\ldots,0),1)$ with $1 = \sigma_{4-\Delta} = \sigma_{3-\Delta}$. Then $k = -1$, ${}_{-1}\eta^- = (\sigma_i)_{i\leq 0}$ with $\sigma_0 = a_{-1}$, and ${}_{-1}\eta^+ = (\sigma_i)_{i\leq 0}$ with $\sigma_0 = a_0$. So

   $$\begin{aligned} {}_{-1}\eta^- &= (\ldots, 0_{-1}) \\ {}_{-1}\eta^+ &= (\ldots, 0_{\Delta-2}, 1_{\Delta-3}, 1_{\Delta-4}, 0_{\Delta-5}, \ldots, 0_0) \\ \eta &= (\ldots, 0_{\Delta-5}, 1_{\Delta-4}, 1_{\Delta-3}, 0_{\Delta-2}, \ldots) \end{aligned}$$

   and $[\eta, -1]$ is the corresponding lamplighter element.

3. Third, consider $(m^{\Delta-4+j}, j, -1) \in \mathcal{L}$. Its Woess coordinates are x_1x_2 for $x_1 = ((\dots, 0, 1, 0, \dots, 0), -1)$ with $1 = \sigma_{-(\Delta-4+j)}$ and $x_2 = ((\dots, 0, j), 1)$. Then $k = -1$, ${}_{-1}\eta^- = (\sigma_i)_{i\le 0}$ with $\sigma_0 = a_{-1}$, and ${}_{-1}\eta^+ = (\sigma_i)_{i\le 0}$ with $\sigma_0 = a_0$. Therefore,

$$\begin{aligned} {}_{-1}\eta^- &= (\dots, 0_{2-\Delta-j}, 1_{3-\Delta-j}, 0_{4-\Delta-j}, \dots, 0_{-1}) \\ {}_{-1}\eta^+ &= (\dots, 0_1, j_0) \\ \eta &= (\dots, 0_{2-\Delta-j}, 1_{3-\Delta-j}, 0_{4-\Delta-j}, \dots, j_0, 0_1, \dots) \end{aligned}$$

and $[\eta, -1]$ is the corresponding lamplighter element.

4. Last, consider $(m^{\Delta-4}+m^{\Delta-3}, 0, 1)$. Its Woess coordinates are x_1x_2 for $x_1 = ((\dots, 0, 1, 1, 0 \dots, 0), 1)$ with $1 = \sigma_{4-\Delta} = \sigma_{3-\Delta}$ and $x_2 = ((\dots, 0), -1)$. So, $k = 1$, ${}_1\eta^- = (\sigma_i)_{i\le 0}$ with $\sigma_0 = a_1$, and ${}_1\eta^+ = (\sigma_i)_{i\le 0}$ with $\sigma_0 = a_2$. Therefore

$$\begin{aligned} {}_1\eta^- &= (\dots, 0_{3-\Delta}, 1_{4-\Delta}, 1_{5-\Delta}, 0_{6-\Delta}, \dots, 0_1) \\ {}_1\eta^+ &= (\dots, 0_2) \\ \eta &= (\dots, 0_{3-\Delta}, 1_{4-\Delta}, 1_{5-\Delta}, 0_{6-\Delta}, \dots) \end{aligned}$$

and $[\eta, 1]$ is the corresponding lamplighter element.

Recall the generating set that we are using is $S = \{ta^i, a^{-i}t^{-1} : i = 0, \dots, n-1\}$ for $t = [\mathbf{0}, 1]$ and $ta^i = [\eta^i, 1]$, where $\eta^i = (\dots, 0_{-1}, 0_0, i_1, 0_2, 0_3 \dots)$. The above elements therefore have the following lamplighter interpretations:

1. Move to position 1 and turn lamp to state i. Then move to position $\Delta - 2 + i$ and turn lamp to state 1. Return to position 1.

2. Move to position $\Delta - 4$ and turn lamp to state 1. Move forward one step and turn lamp to state 1. Return to position -1.

3. At position 0, turn lamp to state j, and then move to position $3 - \Delta - j$. Turn lamp to state 1 and move back one step. Return to position -1.

4. Move to position $5 - \Delta$. Turn lamp to state 1 and move back one step. Turn lamp to state 1 and move back another step. Return to position 1.

Since the Cayley graph of $\mathbb{Z}_n \wr \mathbb{Z}$ with generating set S is isomorphic to $DL(n, n)$, $(r-1)$-balls in the word metric centered at the above elements cover the r-sphere centered at the identity element. Therefore, for a lamp with n states, the survival number of lamplighter groups with the "walk-switch" generating set is $2n + 2$.

Chapter 7

Conclusion

Our first main result generalizes Finucane's example of generating sets S for $\mathbb{Z}^2$. This result links $s(\mathbb{Z}^d, S)$ and convex geometry. We now know that for any $d \geq 2$ and generating set $S = -S$, the survival number $s(\mathbb{Z}^d, S)$ can be determined by examining the convex hull of S.

For $d = 2$, this result leads to a complete classification, and determining the survival number becomes simple. It equals 4 if the convex hull of S is a parallelogram and 3 otherwise. For $d \geq 3$, the problem of classifying and determining all possible survival numbers is more difficult. The lower bound $s(\mathbb{Z}^d, S) \geq d + 1$ results from basic convex geometry. While $s(\mathbb{Z}^d, S)$ is bounded above by the number of vertices in the convex hull, the least upper bound is unclear.

Hadwiger's Conjecture presents 2^d as the least upper bound. This value is realizable for any generating set whose convex hull is a parallelotope. Even if true, though, creating a complete classification of $s(\mathbb{Z}^d, S)$ for all $d \geq 3$ appears infeasible. The range of possible values would be from $d+1$ to 2^d with as of now no clear method to determine all possible values.

Our second result was also inspired by a Finucane example. Finucane showed the survival number of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ with a "switch-walk-switch" generating set is at most eight. By examining Diestel-Leader graphs, we showed this value with a "walk-switch" set is exactly six. More generally, we discovered that for any $m, n \in \mathbb{Z}^+$ the survival number $s(DL(m, n))$ equals $m + n + 2$. We proved this by exploiting the fact that a set of vertices in $DL(m, n)$ can be sufficiently spaced while a projection of that set is not.

These results add two new examples to a new statistic. It remains to see the significance of a group's survival number to other areas of geometric group theory. But without more examples, this significance will remain hidden.

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