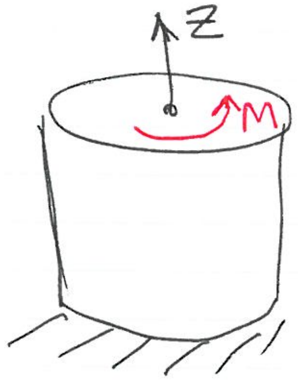
Torsion problem – 1

Torsion Problem



Guess : geometry of deformation:

- "Slices" normal to Z-axis remain planar.
- Angle of rotation of slic… proport. to distance from bas…

$$\theta = \alpha Z$$

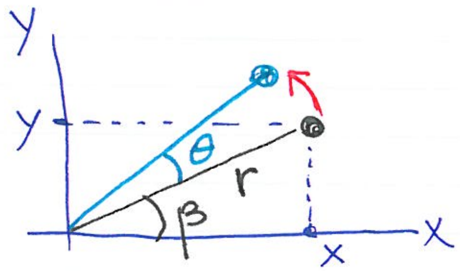
α — twist per unit length

Translate into : displacements → strains → stresses

Notations : $(X_1 X_2 X_3) \rightarrow (X, Y, Z)$

Obviously, $u_z = 0$

To find u_x, u_y: consider displacement of some point of cross-section



$$\begin{cases} u_x = r\cos(\theta+\beta) - r\cos\beta \\ u_y = r\sin(\theta+\beta) - r\sin\beta \end{cases}$$

(r, β) mark material point

θ - angle of rotation

or

$$\begin{cases} u_x = r(\cos\beta\cos\theta - \sin\beta\sin\theta) - r\cos\beta \\ u_y = r(\cos\beta\sin\theta + \cos\theta\sin\beta) - r\sin\beta \end{cases}$$

Use: $\left.\begin{array}{l} r\cos\beta = x \\ r\sin\beta = y \end{array}\right\} \Rightarrow$

$$\begin{cases} u_x = x\cos\theta - y\sin\theta - x \\ u_y = x\sin\theta + y\cos\theta - y \end{cases}$$

Rot. angle θ is <u>small</u> (small strains) $\Rightarrow$ $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$ (to within linear terms)

$$\Rightarrow \left.\begin{array}{l} u_x = -\theta y = -\alpha z y \\ u_y = \theta x = \alpha z x \end{array}\right\} \quad \text{displacements}$$

<u>Strains:</u>

$$\left.\begin{array}{l} \epsilon_{xz} = -\frac{1}{2}\alpha y \\ \epsilon_{yz} = \frac{1}{2}\alpha x \end{array}\right\} \Rightarrow \underline{\text{stresses}} \begin{cases} \sigma_{xz} = 2G\epsilon_{xz} = -G\alpha y \\ \sigma_{yz} = G\alpha x \end{cases}$$

(other $= 0$)

$$\underset{\sim}{\sigma} = G\alpha\left[-y(e_x e_z + e_z e_x) + x(e_y e_z + e_z e_y)\right]$$

check eq-ns

1. Equilibrium $\dfrac{\partial \sigma_{ij}}{\partial x_j} = 0 \quad (i = 1,2,3)$

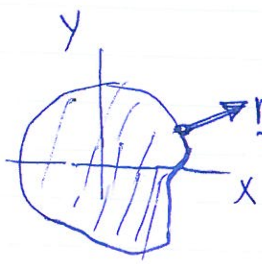
$$\left.\begin{array}{l} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + \dfrac{\partial \sigma_{xz}}{\partial z} = 0 \\ \text{— — — — —} \\ \text{— — — — —} \end{array}\right\}$$ each derivative $= 0$

2. Hooke's law: has been incorporated

3. Compatibility of strains: not to worry. ϵ's have been derived from displacements

Check b.c: (1) lateral surface & (2) bases

(a) Lateral surface: should be traction-free



$$\underset{\sim}{n} = n_x \underset{\sim}{e}_x + n_y \underset{\sim}{e}_y$$

$$t^{(n)} = \underset{\sim}{\sigma} \cdot \underset{\sim}{n} = \underbrace{(\sigma_{xz} n_x + \sigma_{yz} n_y)}\, \underset{\sim}{e}_z$$

$$= G\alpha \underbrace{(-y n_x + x n_y)}$$

$\longrightarrow$ generally, non-zero! B.C. violated!

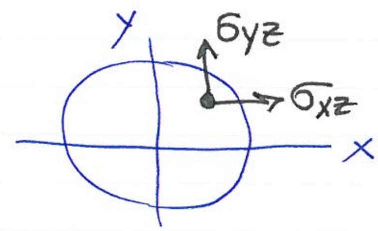
However: for the circular cylinder

$$\left.\begin{array}{ll} x = r\cos\beta & n_x = \cos\beta \\ y = r\sin\beta & n_y = \sin\beta \end{array}\right\} \Rightarrow \text{it } \underline{\text{is}} \text{ zero!}$$

Run into restriction: our guess works for the circular only

(b) <u>B.C. on bases</u> : distrib. of tractions should be equivalent to applied mom. $\underline{M}$; principal vector $= 0$

On base : $\underset{\sim}{n} = \underset{\sim}{e}_z$



$$\underset{\sim}{t}^{(n)} = \underset{\sim}{\sigma} \cdot \underset{\sim}{n} = \sigma_{xz}\, \underset{\sim}{e}_x + \sigma_{yz}\, \underset{\sim}{e}_y$$

Moment generated (about z-axis)

$$\int_F \Big(-\sigma_{zx} \underset{\uparrow}{y} + \sigma_{zy} \underset{\uparrow}{x}\Big)\, dF = G\alpha \underbrace{\int_F (x^2+y^2)\, dF}$$

area $\longrightarrow F$

mom. arms

I_o — polar mom. of inertia $= \int_0^R r^2\, 2\pi r\, dr = \frac{\pi R^4}{2}$

Equating to appl. mom. $\underline{M}$:

$$\alpha = M \cdot \frac{1}{(G\, I_o)}$$

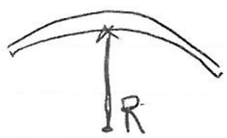
— torsional stiffness

intensity of twist

physical   geometrical

<u>Observation</u>: similarly to bending: b.c. provides <u>stiffness relation</u>

(in bending : $\frac{1}{R} = \frac{M}{\underbrace{EI_y}_{\text{bending stiffness}}}$).



Check that the princ. vector $= 0$ at base

$$\int_F \underbrace{\left(\sigma_{xz}\,\underline{e}_x + \sigma_{yz}\,\underline{e}_y\right)}_{\underline{\sigma}\cdot\underline{n}} dF \stackrel{?}{=} 0$$

$\underline{n} = \underline{e}_z$

e_x, e_y components. Each should be $= 0$

$$\left.\begin{aligned} \int_F \sigma_{xz}\, dF &= -G\alpha \int_F y\, dF \\ \int_F \sigma_{yz}\, dF &= G\alpha \int_F x\, dF \end{aligned}\right\}$$

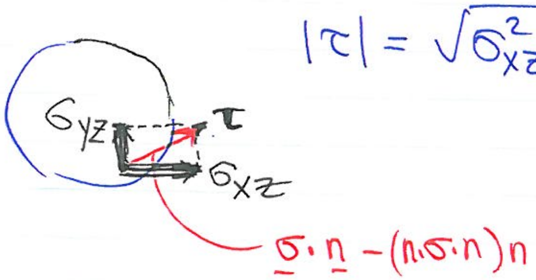
Both zero (coordinates of centroid) in coord system with origin at centroid

Summary: our guess works for the circular profile only, and

$$\begin{cases} \sigma_{xz} = -G\alpha y = -\dfrac{M}{I_0}\, y \\ \sigma_{yz} = \dfrac{M}{I_0}\, x \end{cases}$$

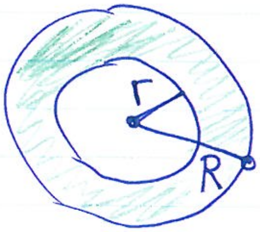
Magnitude of shear traction vector

$$|\tau| = \sqrt{\sigma_{xz}^2 + \sigma_{yz}^2} = \frac{M}{I_0} \underbrace{\sqrt{x^2+y^2}}_{r} \quad \text{— max on boundary}$$



Extend. to: circular pipe:  both boundaries circular → solution work

The difference with the solid profile:



$$I_0 = \int_{\text{ring}} (x^2+y^2)\,dF = \frac{\pi}{2}\left(R^4 - r^4\right) = \frac{\pi}{2} R^4 \left(1 - \frac{r^4}{R^4}\right)$$

⇒ economy of mat'l can be very substantial! (for the same stiffness) as compared to solid profile

Example: thin pipe, $r = 0.9\,R$

$$I_0 = \frac{\pi}{2} R^4 \left(1 - 0.9^4\right) = \frac{\pi}{2} \cdot 0.35\,R^4$$

Solid cylinder of radius a that has the same I_0:

$$\frac{\pi}{2} a^4 = \frac{\pi}{2} \cdot 0.35\,R^4 \Rightarrow a = \underbrace{\sqrt[4]{0.35}}_{0.78}\,R$$

Compare cross-sect. areas:

$$\left.\begin{aligned} \pi\left(R^2 - r^2\right) &= \pi\,(1 - 0.81)\,R^2 = 0.19 \cdot \pi R^2 \\ \pi a^2 &= \pi\sqrt{0.35}\,R^2 = 0.60\,\pi R^2 \end{aligned}\right\} \Rightarrow \underline{\underline{3 \text{ times}}}$$