d. More precisely the force = $(k/r)^2$ times this, where k is Kepler's constant ($r^2 * d\theta/dt$), the areal velocity

5. The question this theorem answers is how the force -- i.e. the change of motion -- varies along non-circular trajectories dictated by centripetal forces -- i.e. when all changes of motion are directed toward a single point
   a. In the case of a circle so dictated, the force does not vary since equal areas entail uniform motion
   b. The key to obtaining it is the realization that time can be represented by the area swept out
   c. This is why I suspect that the question answered in Theorem 3 was posed first, leading to the question answered in Theorem 1; which then unlocked the door to everything else
6. Theorem 3 should be seen as a generalization of Theorem 2, employing essentially the same geometric construction to infer a measure of force from departures from inertial motion
   a. Old approach, using curvatures and Theorem 2 directly, had not yielded a well-behaved measure of force: two unknowns, the normal "force "and the tangential "force"
   b. Added constraint of directed toward a center reduces the number of separate unknowns
7. The corollary to Theorem 3 simply states that the result gives a means for determining the rule of force for any point along a given trajectory governed by centripetal forces
8. A subtle, but radical step has been taken with Theorem 3, for force is now being treated in the abstract, as a mere magnitude, divorced from any question of mechanism!
   a. I.e. in contrast to Huygens's treatment of the static centrifugal force in a string or on a wall
   b. Now just a mathematically characterizable force taken to be acting on the moving object
   c. In other words, Newton's talk of forces in the abstract here parallels his talk of rays in the abstract in his earlier work on optics (which people had objected to)

E. Problems 1 & 2: Two Applications of Theorem 3

1. The precise reason unclear why Newton included Problems 1 and 2, both of which involve direct applications of Theorem 3, but to questions that there is no immediately obvious reason to be asking
   a. At first glance mere mathematical curiosities, serving to illustrate the thrust of Theorem 3 before turning to its key application (but only at first glance)
   b. Notice in manuscript here and elsewhere, "*gravitas*" was replaced by "*vis centripeta*"
2. Problem 1: the rule of force for a body going through a circular arc under centripetal forces directed to a point on that arc
   a. Akin to Galileo's rejected claim in *Two New Sciences* that a circular arc all the way to the center of the earth
   b. Symbolically, this case has $1/r = 1/D * \sec(\theta)$
   c. Of course, a singularity at the center of force, so solution only up to that point
3. Solution: force varies as $1/r^5$, where r is distance from the force center

a. Because triangles ZQR, ZTP, and SPA are similar, $RP^2/QT^2 = SA^2/SP^2$, where $RP^2 = QR*LR$ (by Euclid's Prop. 36 again)

b. Applying Theorem 3, $\lim(QT^2/QR) = SP^3/SA^2$

c. Hence force varies as $1/SP^5$

4. (My guess is that Newton had first solved for the Ptolemaic eccentric circle, discovering that force varies as the product of two varying geometric magnitudes and then decided to present this result which follows from that one when the eccentricity becomes 1; see Appendix)

5. Problem 2: the rule of force for a body in an ellipse under centripetal forces directed to the center of the ellipse, not a focus

6. Solution: force varies with distance from center -- a linear relationship

a. Since an ellipse, $PV*VG/QV^2 = PC^2/CD^2$ (the counterpart to Euclid's Book 3 Prop. 36 for ellipses) and $QV^2/Qt^2 = PC^2/PF^2$ -- facts about conjugate diameters from Apollonius (derived in the Appendix)

b. Therefore, $PV*VG/Qt^2 = (PC^2/CD^2)*(PC^2/PF^2)$

c. But QR=PV and BC∗CA = CD∗PF, and in the limit 2PC = VG

d. Thus, $Qt^2*PC^2/QR = 2*BC^2 * CA^2/PC$ -- q.e.d.

III. The Results on Keplerian Elliptical Orbits

A. Problem 3: The Inverse-Square Rule of Force

1. Suppose now that the trajectory is an ellipse governed by centripetal forces aimed at a focus -- i.e. the converse of the problem Hooke posed in his letter

a. Solution turns on proving that $\lim(QT^2/QR)$ for any such ellipse varies simply as the latus rectum of the ellipse, and hence a constant

b. But then the force varies as $1/SP^2$ -- i.e. as $1/r^2$, or more fully as $1/(L*r^2)$

2. Newton's proof is comparatively intricate, turning on a number of properties of ellipses

a. EP = AC, the semi-major axis (something Newton "discovered" in solving this problem)

b. By a complex chain of ratios,

$$L*QR/QT^2 = (2*PC/GV)*(M/N), \text{ where } M/N = QV^2/QX^2$$

(1) Using $(GV*VP/QV^2) = CP^2/CD^2$ -- as above

(2) And using lemma ii to obtain 4∗CD∗PF = 4∗CB∗CA

c. But as Q approaches P, (2∗PC/GV) and (M/N) approach 1, so that $(SP^2 * QT^2)/QR = L * SP^2$

3. Newton's solution is more general and perhaps easier in analytical form, as developed by Johann Bernoulli around 1710

a. Equation for any conic: $1/r = A + B*\cos(\theta)$, where $e = B/A$, $B = CS/CB^2$

b. An ellipse so long as $0 < B/A < 1$; a parabola when A=B; and a hyperbola when $B/A > 1$

c. $\lim(QR/QT^2) = 1/2[(A+B*\cos(\theta)) + d^2/d\theta^2\,(A+B\cos(\theta))] = A/2$