The relationships between teacher and student understanding:

The conceptual field of combinatorics

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I. Introduction

The purpose of this review is to build a foundation from which to consider the relationships among the professional development activities of middle and secondary school mathematics teachers, the mathematical understanding and pedagogical content knowledge of the teachers, and the mathematical understanding of their students. The goal is to enable investigation of the following questions:

What are the relationships among:

(i) a teacher's understanding in combinatorics;

(ii) the teacher's pedagogical content knowledge in combinatorics;

(iii) their students' understanding in combinatorics?

Specifically, what is the relationship between the ways in which a teacher is able to solve and explain introductory problems in combinatorics and the ways in which their students respond to similar problems? Are there connections between a teacher's mathematical understanding and their students' mathematical understanding?

"Understanding" in mathematics will be considered as a term synonymous with mathematical reasoning, or deep understanding. This is in keeping with the idea put forth by Ma (1999), that even fundamental mathematical concepts have deep levels of comprehensibility. This is differentiated from procedural knowledge, which is exemplified by the recollection of formulae, rules, and algorithms. Note that the existence of procedural knowledge does not preclude the presence of deep understanding for the same individual on the same topic. For example, the ability and tendency to use the column-format algorithm for multi-digit subtraction does not imply that the individual is not capable of using other methods or of justifying the algorithm. Procedural knowledge and mathematical understanding may be linked together, depending on the individual and the topic (Carpenter & Moser, 1982; Vergnaud, 1982). This distinction is mentioned here to mark the goal of uncovering teacher and student mathematical cognition, rather than only recording procedural knowledge or correct answers.

The second term in use will be "pedagogical content knowledge" (PCK; Shulman, 1986, p. 9). As described by Shulman, this is knowledge of the subject matter that relates not just to the content itself, but that enables one to teach it. It includes knowing multiple representations and explanations for a particular topic, as well as understanding the student view of the subject area, including difficulties and common misconceptions.

The ultimate goal of this review will be to contribute to our knowledge of the impact of teacher mathematical understanding and pedagogical content knowledge on student learning and understanding. By evaluating this relationship, we can attempt to determine how the teacher's depth of understanding might affect their students. In the longer term, work on this topic could result in a clear goal of the mathematical understanding teachers should be helped to gain, allowing us to improve pre-service teacher education and in-service professional development with an eye toward ultimately improving mathematics learning and understanding for the students.

This paper will argue that little is known about the relationships among teachers' understanding, teachers' pedagogical content knowledge, and students' understanding in the area of introductory combinatorics. Existing theoretical and research work, both on combinatorics and on teacher understanding in mathematics, will lead to the claim that preliminary work to be done in this area is needed and should consist of:

(i) outlining the conceptual field of combinatorics;

(ii) collection and analysis of mathematical responses from teachers and their students on combinatorics problems;

(iii) collection and analysis of detailed explanations from teachers and their students about their responses;

(iv) consideration of similarities, differences, and connections between students' and teachers' understanding.

To provide a context in which to address the issues above, justification for examining them, and a foundation for the need for the work just stated, existing research and theory in relevant areas will be reviewed here. *First*, we will briefly evaluate some of the existing work on the value of mathematics courses for teachers. The main focus of this section will be to catalogue the ways in which past studies have measured the impact of either pre-service or in-service courses for mathematics teachers. Because of this focus, this section will also include studies that attempt to link teacher-level outcomes with student-level outcomes, regardless of whether these studies used a particular intervention. Studies will be restricted to mathematics, but not to combinatorics as few studies involving teachers specifically consider this area of mathematics.

Second, we will focus on just those studies that also consider student-level outcomes, and how the concepts involved in these relate to teachers' pedagogical content knowledge. This will include attempts to measure these teacher qualities, as well as attempts to determine the impact on students. In the existing research on these topics, different terminology is used across studies. An attempt will be made to form explicit connections to PCK.

Third, we will look at the mathematical landscape of introductory combinatorics. This includes combinations and permutations in the sense that they are used to count a number of possible outcomes. Probability will be included only in the sense of determining the likelihood of some subset of the possible outcomes. Statistics are not included. The purpose of looking at the mathematical landscape is to outline some of the typical problems and difficulties that are encountered when the topics are introduced to students. By doing this, we can attempt to understand the map of sub-topics and theoretical constructs, in order to design future methodology to cover this ground. The area of combinatorics has not been fully examined by existing research on teachers' understanding and pedagogical content knowledge. As a result, this section is included so that we can begin to think about this mathematical content area in conjunction with the research discussed in the prior two sections. This mapping approach follows the theory of conceptual fields, as outlined by Vergnaud (1996). Vergnaud emphasized the need to understand the area in which cognition occurs, defining a conceptual field as, "a set of situations, the mastering of which requires several interconnected concepts. It is at the same time a set of concepts, with different properties, the meaning of which is drawn from this variety of situations" (p. 225). Instructional approaches and studies describing student work will also be examined.

Fourth, we will use the existing studies and the mathematical landscape to consider the implications for studying links between the mathematical understanding of the teacher and the mathematical understanding of their students, as well as the pedagogical content knowledge of the teacher. Studying these links could include, for instance, examining whether teacher and student approaches to problems follow a set routine; whether more than one approach is used; whether explanations are procedural or conceptual; and whether individuals can identify and explain misconceptions related to the mathematics. These criteria within a subject area are related to the concept of pedagogical content knowledge as described by Shulman (1986) and defined above. This review will create a basis for conducting new work that addresses the questions above.

II. Measuring the impact of courses for mathematics teachers

As interest in the mathematical abilities of students in the United States has increased, attention has naturally turned to how these students are taught. Spurred by testing comparing student performance at the school, community, state, national, and international levels, communities and policy makers have focused on how to improve mathematics instruction. However, at the same time, there has been no consensus on how to accomplish this. While mathematics education in the past may have focused on facts and algorithmic competence, today many prefer to strive for student understanding that goes deeper than procedural ability (Ma, 1999; National Council of Teachers of Mathematics [NCTM], 2000). Standards for education have reflected these changes, and with new standards and expectations for mathematics learners come new expectations for mathematics teachers, which may include teaching in an exploratory framework, working with open-ended problems, analyzing and accepting students' use of alternative methods, and confronting students' existing mathematical ideas (NCTM, 2000). These are challenges for any teacher, and may present particular difficulty to those who have an existing method of teaching that does not align with these requirements. This also presents a unique set of difficulties for teachers whose own student experiences did not involve this approach to teaching. Ball (1988) cites a tendency for in-service elementary school teachers to revert to the ways they were taught as children despite intervention, perhaps because their own teacher education did not replace their existing conceptions.

In order to deal with this changing view of mathematical competence for students, educational researchers have considered how best to teach teachers. Some of the research studies discussed here have been conducted with pre-service teachers; these are generally university students working toward a career as a teacher. Others have been conducted with in-service teachers with varying levels of experience; these courses are intended for professional development.

Research on courses for pre-service or in-service mathematics teachers has focused on three broad categories: content courses, pedagogy courses, and pedagogical content knowledge courses. In the first case, content courses attempt to teach only the mathematics; there is no component that addresses the implementation of the mathematics in a future classroom. In contrast, pedagogy courses use only mathematics that is assumed to be known by the teacher participants. The courses, then, cover techniques and methods for teaching lessons in mathematics. In the third case, the course covers the landscape of instruction for a mathematical topic, as discussed above. A course of this type might include learning common misconceptions about the topic, as well as multiple methods of explaining a concept (Shulman, 1986).

Certainly, almost every course will traverse the convenient boundaries between the three categories created here. For example, a course that is focused on mathematics content is likely to elicit teachers' comments on how this topic would look in their classrooms and how they would implement a lesson on this material. As a result, pedagogy is brought into the course. Similarly, a course that focuses on pedagogical techniques in mathematics depends on the subject matter to make sense of the techniques themselves. Hence, while the primary objective is not to teach mathematical content, completion of the activities in the course depends on knowledge or reflection on the content itself. Recognizing, then, that classifying these courses for teachers may exclude the nuances, it is still possible to undertake this division by considering the main focus of the course as one of the three listed above.

Research perspectives

From a theoretical perspective, arguments have been made in favor of including the different categories of courses within teacher preparation. A strict focus on advanced mathematical content has been rejected by some educational researchers, such as Ball and Bass (2003), who cite Begle's (1979) study that found no positive effect on the achievement of students who had teachers with higher numbers of advanced mathematics courses. At the same time, the mathematics content is still viewed as relevant (Ball, 1990; Ball & Bass, 2003; Graeber, 1999). These may seem to be contradictory arguments, in which the mathematical content showed no benefit to students, yet we cannot imagine successful mathematics instruction without content knowledge. A possible solution to this is that advanced courses may not be the appropriate type of content to include in the course of instruction for future mathematics teachers. Instead, an alternative that has been discussed in past research is to offer courses in which math content that is part of the pre-college school curriculum is explored deeply by the teachers (Lubinski & Otto, 2004; Ma, 1999).

Courses on pedagogical tools, or teaching how to teach, have also been supported by researchers, as will be discussed below (Hadfield, Littleton, Steiner, & Woods, 1998; Huinker & Madison, 1997; Lowery, 2002). The third category, Shulman's (1986) pedagogical content knowledge (PCK), in some ways bridges the gap between an exclusive emphasis on content and an exclusive emphasis on pedagogy. As mentioned above, PCK refers to knowledge that is still content specific, but that relates not just to fluency in the content but to the ability to teach it as well. This might include knowing not just how to find the correct answer to a mathematics problem, but also being able to analyze the solution methods of students and determine if they are mathematically sound.

Others have followed up on Shulman's suggestion; Ball and Bass (2003) refer to breaking down one's mathematical content knowledge for use in instruction as having it "unpacked" (p. 11). This idea suggests that it is not only necessary to know the mechanics of the mathematics, but that it is also important to be able to break this knowledge down. Carpenter, Fennema, Peterson, Chiang, and Loef (1989) have also worked toward unpacking mathematical thought using the approach of analyzing children's thinking and sharing this information with teachers. These researchers argue that informing teachers about educational research on how children think about particular mathematical concepts will result in improved teaching. The supposition is that the information on children's thought processes and theories may help increase the teachers' pedagogical content knowledge and bring their informal understandings into their practice; the results of studies evaluating this approach have been positive, as is discussed in greater detail below. These different perspectives underpin some of the research studies described here.

Research on teacher-level outcomes

This sub-section looks at courses provided to both in-service and pre-service teachers; these courses vary across the three broad categories detailed above: content courses, pedagogy courses, and pedagogical content knowledge courses. The commonality of these studies is that they assess course impact by measuring teacher-level outcomes. In the category of pedagogy courses, multiple studies (Huinker & Madison, 1997; Lowery, 2002) have assessed courses that explicitly teach pedagogical techniques for mathematics instruction. The impact of the courses in Huinker and Madison's and Lowery's studies was measured through a pre-test and post-test of the prospective teachers, assessing topics such as their beliefs in their own abilities, their beliefs regarding mathematics instruction, and their own attitudes towards math. The tests were administered at the beginning and end of the semester in which the courses were taken. In both studies, positive changes were observed in these metrics. This is interpreted in these studies as indicating a positive outcome of providing courses in pedagogical techniques for pre-service teachers. Note that the measures were related to attitudes and beliefs, which are important for educators, but these measures do not substitute for assessing mathematical understanding.

Few studies have focused strictly on content-based courses for teachers, although Lubinski and Otto (2004) are an exception to this, examining a course which used exploratory problems to approach not advanced university level mathematical content, but rather to delve into concepts considered to be basic, in order to understand them more deeply. This mirrors the suggestions made by Ma (1999). The metric for evaluating the success of this course consisted of assessing the prospective student teachers' beliefs regarding mathematics education before and after the class, and finding positive changes. Here, the positive changes in the belief survey are again taken as support for this type of mathematics content course for pre-service teachers. As above, the importance of beliefs is not to be underestimated, but the teachers' mathematical understanding is not addressed here either.

A few studies have approached pedagogical content knowledge by recognizing the important role of understanding student work. Tirosh (2000) describes a course in which pre-service teachers completed math problems, but also focused on common errors and misconceptions for each concept. The teachers were again given a pre-test and a post-test at the beginning and end of the course. This time, rather than beliefs, tasks included solving a particular problem, then listing common student mistakes and possible sources for these mistakes. The results presented are qualitative in nature, but the researcher suggests that the pre-service teachers became familiar with sources of student errors. Tirosh takes this knowledge of student errors as a positive outcome for the class. Similar to the ideals of PCK, the premise of this study is that this kind of knowledge will be beneficial to the teachers and their teaching.

Philipp, Thanheiser, and Clement (2002) also attempt to stimulate consideration of children's mathematical thought through a course that combines examining mathematical ideas with tutoring and reflection on that tutoring. They believe that an early introduction to children's ideas will convince prospective teachers of the need to have a deep understanding of math. They use belief surveys similar to those described above (Huinker & Madison, 1997; Lowery, 2002; Lubinski & Otto, 2004), although here they compare the pre-service teachers to those in another type of preparation program. Their results show that the teachers in their program had beliefs more in line with the goals of current proposals for improving mathematics instruction (e.g., NCTM, 2000).

Hadfield et al. (1998) attempted a more comprehensive approach in which they did not consider a class that fit into one of our categories of pedagogy, mathematical content, or pedagogical content knowledge. Instead, they examined the effectiveness of pre-service elementary teachers and then looked for correlations to a number of elements, hoping to find possible *predictors* of success. In order to assess the effectiveness of the teachers, each pre-service teacher taught three brief lessons to a group of their peers, with each lesson focusing on a topic in elementary school mathematics. These lessons were videotaped and the tapes then assessed by three unidentified expert educators. Each lesson was rated on a researcher-designed scale that had a rubric for scores in accuracy of content, delivery, and methodology. This effectiveness score was then correlated to each of several possible predictors. The predictors were a researcher-designed mathematics content test, attitude towards mathematics as measured by an established mathematics anxiety metric, spatial ability as measured by an established aptitude test, and finally each pre-service teacher's quiz grade average in their current mathematical methods course. The result of this was that the only significant predictor was the quiz score in the mathematical methods course, accounting for 25% of the variance in the judged effectiveness of the teachers (Hadfield et al., 1998). Although the study conscientiously explores different reasons for the lack of more significant results, it does not fully consider that the rubric used to score effectiveness of the pre-service teachers is closely aligned with the teachings of the methodology course, particularly in heavily emphasizing the use of manipulatives.

As is clear from the studies above, determining the efficacy of instruction for teachers is not straightforward. With the exception of the last study mentioned, all other studies reviewed above relied on pre- and post-tests – several of them on attitudes and beliefs – administered to teachers taking part in specific courses. Several issues are raised with this approach. *First*, the evaluation of the teachers takes place immediately at the end of the course. This shows what they may have learned in the course or even beliefs that may have changed during that time; however, it does not show whether these changes are sustained later on, especially as the pre-service teachers begin their careers, or whether the practice and beliefs revert to their previous states.

Second, these studies found favorable changes as a result of the courses. However, they may not consider how closely related the elements of the course are to the assessment used. For example, while Tirosh (2000) assesses each teacher's ability to list common student misconceptions for a particular mathematical concept, the class had focused on explicitly identifying just these misconceptions. The possibility exists that the answers given by the teachers do not represent their own mathematically based understanding of why these problems might occur for students; rather, the assessment may only determine their recollection of facts. This is not necessarily or even likely to be the case for any of the studies discussed here, including the one put forth by Tirosh. However, by looking at Hadfield et al. (1998), we can see that the metric used to judge merit or progress of teachers can be unintentionally oriented to a specific perspective. That is, in this example, the rubric used to score the videotapes of the pre-service teachers was based on the ideals of the methodology course in which the teachers were enrolled. Thus, the rubric assessed how well they executed a particular type of teaching. We do not know, however, that that model of teaching is the best one for student learning; therefore the rubric does not assess the *effectiveness* of teaching in general.

Third, using pre- and post-tests tells us about the beliefs, knowledge, or understanding of teachers, but it does not tell us about their actual practice of teaching. The Hadfield et al. (1998) work goes beyond written assessment and recognizes the need to evaluate the teaching itself. However, since the teaching takes place with a peer group instead of students, and the use of the rubric is in question as discussed above, even in this attempt we are not able to look at the teaching itself. In the studies using pre- and post-tests, the implicit assumption is that a gain in the knowledge or beliefs they assess is directly related to a gain in teaching ability. However, the execution of teaching is complex and non-ideal, as any teacher will be able to confirm, so this implicit assumption may not be valid.

The *fourth* concern with the pre- and post-test model is that while it tells us about the teachers, it does not currently tell us anything about the students who hopefully benefit from the courses undertaken by the teachers. Ultimately, the goal of teachers, teacher educators, and educational researchers is to improve the learning of the students. This is not to suggest that studies based on teachers are not or could not be useful. In fact, in the following section, we will see that studies based on teachers may eventually enable us to make connections to their students. In addition to this potential benefit, these studies already provide insight into education from the perspective of an adult learner, and they also allow us to gauge the beliefs and mathematical conceptualizations of the teacher. Affect has shown to be a powerful factor in other areas related to mathematics, particularly for adults who may have had unpleasant experiences with math in the past (Burton, 1987). It is reasonable to consider that a teacher's attitude toward mathematics is important in the classroom, including both their personal feelings as well as their understanding of what mathematics is and how it should be taught. This is particularly relevant for elementary school teachers in the United States who typically teach mathematics as one of many subjects; thus, they have not necessarily chosen to teach mathematics, but rather must teach it as part of their choice to be a teacher.

The search for better evaluation of the impact of interventions aimed at improving teachers' preparation to teach mathematics is clearly a challenge. Compounding the problem is that studies that measure the impact of teacher-level factors on student-level outcomes are difficult to undertake, as will be discussed below. Because of this difficulty, the model pioneered by Hadfield et al. (1998), in which they look for the predictors of teaching success, is potentially invaluable to future research. Hadfield's study sought to correlate teaching effectiveness with a number of predictors. While nothing definitive can be concluded from the particular study, this idea could be taken further and used to evaluate the relationship between student performance and teacher participation in various courses. If this were to be done and done fully, then researchers could return again to measures that are easier to obtain, such as those used by Hadfield et al. (1998), including course grades, surveys, pre- and post-tests, or trial teaching on videotape, resting secure in the knowledge that established correlations to student outcomes do in fact exist.

Research on student-level outcomes

In the meantime, we can turn instead to a few research studies that do provide a look directly at student performance. The first of these is from Carpenter et al. (1989). This study comes from a group with significant research in the area of student cognition in elementary mathematics, particularly addition and subtraction (e.g. Carpenter et al., 1981; Carpenter & Moser, 1982, 1984). Drawing on their data about students' approaches to addition and subtraction problems, these researchers implemented a course in which in-service teachers were exposed to the research on children's cognitive processes. In doing this, they followed an approach they refer to as cognitively guided instruction (CGI) that is similar to some aspects of Shulman's pedagogical content knowledge, particularly in that it considers student explanations of concepts, and also the importance of existing student conceptions. No teaching methods were suggested to the teachers. They then undertook an extensive assessment of the impact of this course, using the teachers in the course, their students, a control group of 20 teachers who were not in the course, and these control group students. Data was collected through classroom observation, teacher and student surveys, and standardized math tests for the students. A number of results came of this, but two of them are significant for the current analysis. First, the teachers who received the course on cognition had significantly higher scores in knowledge of student strategies for particular students in their classes. Second, although the students of the teachers who had been in the experimental group spent significantly less time on number fact problems, they did significantly better than the students of the teachers in the control group on questions of this type on the standardized test. The depth of this assessment may afford us insight not just into this particular course but might also provide an additional benefit to those studies which cannot be so vast. In considering the numerous outcomes, it is possible that correlations may occur between the student performance measures that we hold in high regard, such as understanding and test scores, and other measures, which are more accessible to researchers. For example, Carpenter et al. (1989) show that the teachers in the experimental group have stronger beliefs regarding the importance of cognitively guided instruction as opposed to the teachers in the control group. The study also shows the experimental group teachers changing their classroom practice to reflect the ideals of cognitively guided instruction, and it shows the increase in student performance mentioned previously. While no correlations are made between these different outcome variables, it is possible that future large scale studies might find that they exist. If so, smaller studies of individual courses or programs could assess something easy to measure and have some degree of confidence that intermediary outcomes, assessed on elements such as teacher beliefs, teacher procedural knowledge, or teacher awareness of student thought, do correlate to gains in student understanding.

As with the above study, Saxe, Gearhart, and Nasir (2001) implemented a professional development workshop for teachers and measured student-level outcomes as a result. In their case, they focused on the mathematical topics of skill with fractions and understanding of fractions. Students of participating teachers completed pre- and post-assessments consisting of items intended to test for skill and others intended to test for understanding. While the test items were gathered from a mixture of sources, including existing curricula, the research team validated this assessment and the distinction between skill-oriented and understanding-oriented items through measures of internal consistency. The participating teachers were divided into three groups; the first group served as a control and used a traditional mathematics curriculum in the classroom. The second group used a reform curriculum and received a form of professional development in which they worked with a support group of other teachers to plan and discuss lesson topics; however, they were relatively self-directed. By contrast, the third group also worked with a reform curriculum, but received a program referred to as "Integrating Mathematics Assessment" (IMA). The IMA program was designed to address teachers' own mathematical understanding, their understanding of their students' work in mathematics, and their understanding of student motivation, in addition to providing a network of other teachers. Again, we can see that the second aim of the IMA program, understanding student work, is compatible with the ideas of PCK. Note also that in contrast to the program presented by Carpenter et al. (1989), this program worked with a specific curriculum. The findings from Saxe et al. (2001) show greater gains for the students of teachers in the IMA group, as opposed to those in the first or second group. However, this difference was only on the subset of items considered to test conceptual knowledge, and there was no significant difference between the IMA group and either the control group or the teacher support group on the items intended to test computational skill.

As Saxe et al. implemented their professional development course with an orientation toward reform curriculum, Simon and Schifter's (1993) professional development course focused on guiding teachers to a constructivist view of mathematics. In their case, they gauged the student outcomes by comparing those taught by the teachers after participating in the program to those taught by the same teachers prior to the program. This study also used teacher-level outcomes designed to look at classroom practices, but of particular interest here are the student-level outcomes. These included student attitudes and beliefs, the type of math activity in the classroom, as reported by the teachers for their own classrooms, as well as student performance on standardized tests appropriate for the different grade levels. Although Simon and Schifter did find changes in the student beliefs and attitudes towards mathematics, and these changes include increased perceived importance of creativity and trying new things in math, they did not find changes in the standardized test scores before and after the teacher had participated in the program. However, in keeping with the orientation of the study, the fact that beliefs changed with no accompanying decrease in test scores may be acceptable. It would require further investigation to determine whether the changes in student beliefs were matched by changes in conceptual understanding or actual approach to mathematics problems, as these student gains in mathematical ability would not necessarily have been captured by the standardized tests used by the researchers in this study.

Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti, and Perlwitz (1991) also included student-level outcomes in their study. They base their work on a particular theoretical orientation, considering both a constructivist perspective and the role of social interaction. Here, teachers participated in a professional development course and then received support during the school year. As with the work of Saxe et al. (2001), the students of the participant teachers received higher scores than those of their counterparts in the control group, but again, only on the portions of the test designed to assess conceptual knowledge as opposed to computational knowledge. The other factor to consider when looking at this work is that the participant teachers also implemented a curriculum designed by the research team, while the control teachers did not. Thus, while the effort as a whole can be examined, the specific effects of the professional development activity or of any particular resulting attribute of the teachers are obscured by the stark differences in classroom curriculum.

Another project looking directly at the impact on students is the work done by Hill, Rowan, and Ball (2005). This group has produced a significant body of research and reflection on teacher education and teaching practice. In this particular work, the researchers report on the findings of a study of first and third grade students and their teachers across 115 elementary schools. While there is work from this research initiative that includes the evaluation of professional development courses for teachers (e.g. Hill & Ball, 2004), no intervention occurred or was measured in the particular case described here. This is in contrast to those studies listed above. Instead, the mathematical performance of eight students from each participating classroom was assessed at the beginning and end of an academic year. During that year, the teachers kept a log of measures relating to their teaching practices, such as content covered and the duration of mathematics lessons. Teachers also completed a survey, once during the year, that included educational background, certification information, experience, and other potentially relevant items. In addition, each teacher survey had five to twelve multiple-choice questions that were designed to assess the mathematics needed for teaching. A full description of the development of these items is provided by the researchers in a separate publication (Hill, Schilling, & Ball, 2004).

In this particular study, focusing on the student outcomes, Hill, Rowan, and Ball (2005) look at "mathematical knowledge for teaching" (p. 373), or MKT. This included items that target two areas of MKT, referred to together as "content knowledge for teaching mathematics" (CKT-M; p. 387). The first of these is referred to as "'common' knowledge of content" (p. 387), and includes functional knowledge or what we might consider to be pure mathematical content; this is the knowledge of mathematics apart from the need to teach it. The example provided for this first area of content knowledge is the solution for x in the expression $10^x = 1$. The second area is the content knowledge that would be useful only to a teacher, referred to as "specialized content knowledge". The authors are careful to note that this second area is still mathematical knowledge, not pedagogy. For this area, the example provided requires the teachers to evaluate three methods for multiplying two digit numbers, and determine which of the methods are always mathematically valid. The knowledge used in completing an activity of this type has commonalities with pedagogical content knowledge (Shulman, 1986), in that it requires the teacher to recognize alternative solution strategies outside the traditional algorithm, and to reflect on the legitimacy of these mathematically. However, note that the study authors do not consider CKT-M or MKT to be contained in or equivalent to PCK, as will be discussed in the following section.

Hill et al. (2005) found that their measure for CKT-M was significantly correlated with student gains in both the first and third grades. They are careful to control for other variables, including socio-economic status, the time spent on mathematics in the classroom, and mathematics courses taken by the teacher. The diligence of the researchers lends credence to their analysis of the data, and they are justified in noting the correlation between the scores on their teacher assessment and the gains for the students, and in calling for courses that are focused on content of this type for teachers. Interestingly, they do offer a potential alternate explanation for the results. They suggest that the teachers who scored well on content knowledge for teaching mathematics might have some other, unknown, factor that truly impacts the student scores. They recommend an analysis of the practice of teachers that could potentially suggest factors which, while not necessarily independent of or dependent on mathematical knowledge for teaching, may be manifestations of some sort of teacher knowledge or practice that leads directly to student understanding.

Conclusions on background research

While some progress has been made in linking the outcomes, in terms of student performance, to factors connected to the teachers, no clear consensus exists on how this would translate into practice for teachers, or into preparation and professional development for teachers. One noticeable pattern in the studies above is that those that are able to directly measure student performance are quite large in scale, and are time- and fund-intensive projects (e.g., Carpenter et al., 1989; Hill et al., 2005). The smaller scale studies, including many that attempt to move directly to addressing the problem by working in courses with pre-service teachers, do not have measures that tell us about student performance (e.g., Hadfield et al., 1998; Huinker & Madison, 1997; Lowery, 2002; Lubinski & Otto, 2004; Philipp et al., 2002; Tirosh, 2000). These studies may have insights into key elements of teaching, or may describe courses for teachers which are highly beneficial to students in the long term, but we cannot assess at present what this benefit is.

III. Consideration of PCK in existing studies

Existing studies, including some of those described above, have attempted to clarify, specify, measure, or engender Shulman's PCK. However, as pointed out by Hill, Ball, and Schilling (2008), there is still little information showing how teachers' levels of PCK relate to student-level outcomes, or even about what constitutes PCK. The intent of this section is to look at those pieces of research and theory discussed above that measure student-level outcomes and also attempt to address elements related to PCK in mathematics. We will consider how these elements are linked to Shulman's original conception, and conclude by proposing ideas of how to consider PCK when constructing new measures, given the existing work in the field.

The original introduction to pedagogical content knowledge put it forth as a subset of content knowledge; that is, Shulman proposed "three categories of content knowledge: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge" (Shulman, 1986, p. 9). Pedagogical content knowledge is put forth as the knowledge, still particular to the content, that is specifically used for teaching. Inside PCK, Shulman includes representations, examples, and explanations, as well as common difficulties, common student preconceptions, and ways of changing incorrect student conceptions. Knowledge of the curriculum, though, including knowledge of the range of available materials, is not included in this initial outline of PCK.

We can see that many of the studies described in the previous section draw on ideas that are at least *closely related* to the theoretical form of PCK even if they are not designated by this term. Since PCK is by its very nature domain specific, for each area of mathematics we require a full description of all those items put forth by Shulman in order to say we have defined the PCK for this area. Since we do not have this clearly defined description of PCK, it is reasonable to choose to look at the studies that are both explicitly and implicitly connected to his theory and then to consider their commonalities. As shown in Table 1, five of the studies discussed above either engage teachers in activities that are part of PCK or else attempt to assess teacher qualities that are aligned with PCK.

Study	**Guiding principle**	**Mathematical topics**	**Similarities to PCK**	**Student performance outcomes**
Carpenter et al. (1989)	Cognitively guided instruction (CGI)	Number and operations	• Examining student explanations • Considering existing student conceptions	Gains in student performance, as compared to control group
Cobb et al. (1991)	Constructivist perspective	Number and operations	• Examining student explanations	Gains in student performance on conceptual items, as compared to control group
Hill et al. (2005)	Content knowledge for teaching mathematics (CKT-M)	Number concepts; operations; patterns, functions, and algebra	• Representations of mathematical ideas • Analyzing and evaluating student responses	Greater student gains in performance, as a function of teacher CKT-M
Saxe et al. (2001)	Integrating Mathematics Assessment (IMA)	Skills with fractions; understanding of fractions	• Understanding student work	Gains in student performance on conceptual items, as compared to control group
Simon and Schifter (1993)	Constructivist perspective	Varied	• Analyzing and evaluating student responses	No change on standardized tests, as compared to control group

Table 1. Summary of studies with activities similar to those that are part of PCK.

Note that while some of these studies cite the principles of PCK, none claim to wholly employ or measure it. This is at least in part because what constitutes PCK in mathematics has not been fully specified or agreed upon by the research community (Hill et al., 2008). Who, then, could claim to measure it? As the process of mapping PCK is domain-specific, while work in mathematics continues as discussed here, the same is true for teaching science (e.g. Gess-Newsome, 1999) as well for teaching teachers (e.g. Strauss, 1993).

In mathematics, Hill, Ball, and Schilling (2008) give the most comprehensive look at PCK. However, they also propose that PCK is part of a larger construct, mathematical knowledge for teaching (MKT). They separate the universe of MKT into subject matter knowledge on one side, and pedagogical content knowledge on the other. However, for them, the subject matter knowledge side includes both common content knowledge and specialized content knowledge – the two concepts that form the CKT-M described in their previous work (Hill et al., 2005). Note, then, that Hill et al. (2005) do not consider the intent or results of their work to be measurement of PCK. Instead, they propose that their assessment tasks measure the specialized content knowledge mentioned above. This specialized content knowledge sits next to PCK but does not contain it; neither is it contained by it (Hill et al., 2008). On the side of pedagogical content knowledge, they include a new term, knowledge of content and students (KCS), that more specifically includes "knowledge of how *students* think about, know, or learn this particular content" (Hill et al., 2008, p. 375 [italics added]). The intent is to define this area as a measurable domain of knowledge that is distinct from the specialized content knowledge in that it requires more knowledge of how students learn.

While the work on carving out the area of KCS continues, it is clear that the knowledge described by the researchers as being theoretically in this area is also part of Shulman's PCK, both as Shulman (1986) and Hill et al. (2008) have defined it. What remains to be seen is how separable it will be from common content knowledge and from specialized content knowledge. Note that the difficulty of making these distinctions is acknowledged by the researchers (Ball, Thames & Phelps, in press; Hill et al., 2008). The current work (Hill et al., 2008) supports the theoretical ideal of KCS, but has not demonstrated that this is quantitatively separable from specialized content knowledge through any form of assessment. In particular, specialized content knowledge as conceived of in the earlier work (see Hill et al., 2005) requires making judgments about alternative solution strategies. While this activity is undoubtedly mathematical, it sits tight against knowledge of how students think about the content, which is thought to be KCS. The distinction that led to the separation between common content knowledge and specialized content knowledge also makes more difficult the measurable distinction between specialized content knowledge and knowledge of content and students. However, Ball et al. (in press) do make the theoretical dividing line more clear by marking KCS as requiring some knowledge of students, while specialized content knowledge for teaching does not require knowledge of students.

So the work in conceiving of and measuring PCK continues as well. However, from the studies summarized above, it is fair to at least make the conjecture that teacher knowledge of mathematical thought is powerful. Carpenter et al. (1989) harness this through an intervention in which students' ideas are explicitly taught. As described above, Hill et al. (2005) do not claim to measure anything specifically related to student thought; instead they attempt to gauge each teacher's existing level of specialized content knowledge through their assessments. In both cases, positive connections to student performance are made. Though the distinctions between the specific areas contained in or bordering on PCK are not fully specified for mathematics, these two studies have shown successful results from requiring the teachers to reflect, not strictly on the responses to mathematics questions, but on the solution processes that ultimately connect learners to the mathematics.

One other issue that arises when considering the task of defining and refining PCK in mathematics is that mathematics itself is infinite and complex. Just as defining PCK in chemistry might not fully elucidate PCK in physics, PCK in arithmetic does not necessarily imply PCK in geometry. The knowledge is not just content-based at the level of subjects in school, but actually on concepts within that. Many of the existing studies have looked at number and operations (e.g. Carpenter et al., 1989; Cobb et al., 1991; Hill et al., 2005), which is not surprising given that these are foundations for later mathematical activity in and out of school and generally comprise a student's first exposure to mathematics. Other areas have not been addressed yet, with the exceptions of some work in algebra (Hill et al., 2005) and in fractions (Saxe et al., 2001).

If we wish to ultimately define PCK in multiple sub-areas, there is initial work to be done to define each of these areas of mathematics and generate a tentative framework of what the PCK for each one might look like. In addition, if we wish to consider student-level outcomes, another aspect of the link between student and teacher is the degree to which depth and mode of understanding is connected between both. We are currently dependent on test scores, which are partial measures of performance, to determine the impact on students. This is not unusual: it is consistent with the increased emphasis on standardized testing in the schools and it is the most realistic plan for looking at large numbers of teachers and students. Nevertheless, it does not generally allow us to see all relevant aspects of performance. Tests certainly seek to draw out and measure understanding on a topic, rather than necessarily focusing on procedural knowledge. However, we might consider that a concept is not fully understood until the individual can make it explicit (Karmiloff-Smith, 1992). As a result, eliciting student explanations on mathematical topics may elucidate the depth of their understanding. We could then, on a smaller scale, shift the question from the impact of teacher pedagogical content knowledge on student *performance*, to the impact of teacher PCK on student *understanding*. This indicates an opening in the field for a qualitative analysis of the connections between teacher and student understanding.

IV. <u>The Mathematical Space – Combinatorics</u>

As discussed above, there is room for exploration of teacher PCK in different mathematical topics. One of these untouched areas is combinatorics. For instance, an examination of the types of problems given at the middle school level yields simple combinations and permutations, together referred to as counting, and simple discrete probability (see Connected Mathematics 2, Grades 6, 7, 8; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006). Combinatorics, including the combinations and permutations mentioned above, deals with the ordering and grouping of fixed numbers of items. This topic is within the field of discrete mathematics, or the mathematics of unconnected elements (Rosen, 2003).

This mathematical area may prove interesting to study for three reasons. *First*, there are problems in this field that are confusing and counterintuitive to almost everyone. Our everyday notions of probability are often in conflict with the mathematical reality (Barnes, 1998). Even when we attempt to consider the mathematics, small shifts in problem situations create dramatically different answers, as we will see. All of this results in a complex mathematical landscape that offers challenges even for teachers who are already proficient in mathematics.

Second, even seemingly simple problems in counting and probability may have multiple solution strategies and multiple ways to consider what is happening in the situation. This makes it a ripe ground for considering pedagogical content knowledge. Assessing PCK could include looking at a teacher's use of multiple representations, ability to give different explanations for a single concept, and ability to evaluate the mathematical validity of solutions given by others. Since strategies are so varied in this area, a wide range of PCK might be observed.

Third, these concepts are addressed in the middle school curriculum of the Boston Public Schools (Lappan et al., 2006), and are also part of the grades 7 and 8 curriculum standards for the state of Massachusetts (Massachusetts Department of Education, 2000). This topic, then, is relevant to teachers and students.

Below, we will clarify the types of counting and probability problems that could be considered in a future study. The purpose of this discussion is to define a small segment of combinatorics for consideration; this in no way covers the breadth of these mathematical topics. As part of this purpose, we will outline for ourselves the relationships and connections among the questions. This is a first and rough attempt at examining the conceptual field of combinatorics (Vergnaud, 1996). Vergnaud had proposed a theory of conceptual fields based on the need to understand the mathematical area in which cognition occurs. He defines a conceptual field as, "a set of situations, the mastering of which requires several interconnected concepts. It is at the same time a set of concepts, with different properties, the meaning of which is drawn from this variety of situations" (p. 225).

For instance, questions about permutations and combinations in this small subset of combinatorics will refer to small arrays of objects. Within permutations, there could be two cases given initially. First, the case with n objects, where all n must be arranged. For example, if we have three different letters, how many ways can we arrange all three of them? Second, the case with n objects where some number less than n must be arranged. For example, given all 26 letters in the English language alphabet, how many three letter words can be formed? A case like this can be furthered by asking for the implications of allowing or disallowing repeat letters. This, then, leads to the more difficult cases of permutations, in which there are non-unique objects to be arranged. For example, if we have three letters, but two are identical and cannot be differentiated, and only the third is unique, how many ways can we arrange all three of them? This is a potentially more troubling case because it requires the individual to determine which, of the n! arrangements that would be present for unique items, would be duplicates in this new structure. This sort of problem can be solved by force (i.e. by listing all possible permutations and manually checking for those that appear identical) for small arrays of items, but even this technique can then be the source for conjecture on determining how many items would need to be removed.

Questions on combinations could follow a similar pattern. In this scenario, n objects, of which n are selected, results in one possible combination. This shift from the ordered permutations discussed above to the unordered groups here can actually be difficult to conceptualize; this first case is not trivial. From here, work can progress to choosing an unordered subset of fewer than n items, followed by consideration of what happens when some of the items are identical. A summary of these types of items is shown in Table 2. Brute force can solve the problem for small arrays and may also lead to fruitful discussions. Combinations seem like they should be easier than permutations, when considered from a non-mathematical standpoint. The complication of ordering has been removed, which makes it seem as if we should be able to breathe more easily. However, in the formulaic calculation, in the brute force solution methods, and in the conceptualization, it can be challenging to mark this distinction.

	Permutations	**Combinations**
Given:	**Determine:**	**Determine:**
n unique objects	number of arrangements of all n objects	number of groups of all n objects
n unique objects	number of arrangements of m objects for $m < n$	number of groups of m objects for $m < n$
n objects, of which some are not unique	number of distinguishable arrangements of all n objects	number of distinguishable groups of m objects for $m < n$

Table 2. Summary of types of simple permutations and combinations.

This will naturally lead to questions in which the individual needs to judge whether a permutation or combination is needed. However, by requiring this type of decision as part of the question, this suggests the use of contextualized problems. This is because problems stated in mathematical symbols and language, as seen in Table 2, specify directly whether they want the number of *arrangements* or the number of *groups*. This may be phrased differently, say by asking for the number of *permutations* or the number of *combinations*, or the number of *sets* or the number of *ordered lists*. However, if the reader has experience with this vocabulary, then the phrasing of the question betrays whether permutations or combinations are required. As a result, the need to judge which of the two to use is removed from the problem. Instead, a contextualized problem must be used in which the reader uses their knowledge of extra-mathematical topics to deem whether or not order matters. For example, a question might ask about the number of possible automobile license plates given a particular format of four numeric digits followed by two letters. Cultural knowledge of license plates tells us that the plate *1234 PK* is not the same as the plate *1234 KP*. As a result, someone responding to this question might deduce that a permutation is required to reach the correct answer, and not a combination.

Several established representations of combinatorics exist, and these are used for both instruction and understanding. One possibility is a list of all the outcomes. This brute force method is effective for small sets. Tree diagrams are also commonly used, particularly for permutations. The slot method is another option, and, of course, there are established mathematical formulae for problems of this type. One potential area for exploration in teacher and student understanding in combinatorics would be the relationships between these representations. In particular, it may occur that the use of one representation leads naturally to the adoption of another. Representations may also be invented, or they may have been explicitly taught to an individual.

Probability could also be included within this limited look at combinatorics, if only where it connects to permutations and combinations. That is, we could focus not on large-scale probability, but on simple cases of discrete probability, determining the probability of an outcome when it is necessary to use combinatorics to count all possible outcomes. For example, a permutation can be used to determine the number of possible sequences of raffle winners given a set pool of entrants. Probability could then be applied to find the likelihood that a particular person wins a prize.

In considering the developmental aspects of understanding chance, probability, and combinatorics, Piaget and Inhelder (1975) suggest that children and adolescents' understanding progresses through stages that correlate with other, more general developmental stages. Specifically, they suggest that young children do not appreciate the notion of chance, and instead seek causal explanations for events, both in the outside world and in the indoor world of dice games and coin flips. It is only as they reach the formal operations stage (12 to 13 years of age) that they are able to consider or enumerate a set of all possible outcomes and the likelihood of these various outcomes. For instance, in creating permutations of small sets of distinct objects, Piaget and Inhelder found that pre-operational children (before seven years of age) have no system for creating different arrangements or for considering how many arrangements are possible. As they grow older and reach the concrete operations stage (between ages seven and 11) they are able to create the different permutations more readily, but still do not use a consistent system to do so and often miss items or create the same item more than once. It was only in the third stage that students used a consistent system to create permutations or could make a conjecture on how many permutations were possible.

Schliemann and Acioly (1989) interviewed bookies with different levels of formal schooling, including those with no formal schooling at all, who were accustomed to taking bets that involved the determination of the number of permutations of a fixed set of digits. While the bookies used tables listing the number of permutations for different scenarios during their work, the researchers interviewed them about permutations of colored chips and alphabetic characters, finding that some of the subjects connected this activity to the way that numeric digits are permuted in their work, while others did not make this connection and even rejected it when it was suggested. Relating the responses to the stages suggested by Piaget and Inhelder (1975) described above, they found that the level of schooling was positively and significantly related to the stage suggested by the response: while none of the bookies had formal instruction on probability, those with some formal schooling were more able to make logical probabilistic arguments.

This work confirms the types of reasoning about permutations seen by Piaget and Inhelder (1975). However, the progression through stages is shown to depend on factors other than development, such as schooling, work, and cultural factors. Even without the added element of the bookies' work, an individual's level of understanding of combinatorics may be uneven. In another study, Fischbein and Schnarch (1997) hypothesized that the informal intuitions and ideas relating to probability would stabilize during the formal operations period in later adolescence. On the contrary, they found that some misconceptions did grow weaker, but some actually grew stronger.

Although their analysis is focused on children's justifications and proofs, rather than the mathematics of combinatorics, Maher and Martino (1996) show us young children engaged in simple problems of permutations. As part of a longitudinal study, students in fourth grade were asked to build all possible towers of blocks, given the height of the tower and two different colors of cubes to use in construction. Consistent with Piaget and Inhelder's (1975) theory regarding children in the concrete operations stage (between ages seven and 11), students often did not have a foolproof system for organizing the possible permutations. However, with Maher and Martino's emphasis on students proving their answers to an interlocutor, over time some students felt the need to create organizational schemes. In doing so, students created either *patterns* of the colored towers, or *categories* of the towers. Patterns were organized visually and often led the students to count the same permutation more than once. Categories, however, enabled students to prove that they had all possible permutations, as they were able to generate all the possibilities within a category. For example, one category could be thought of as "towers three cubes high with exactly two blue cubes", and students generated all three possibilities within this category. Aside from this increased organization in thinking about permutations, students also generated the beginnings of a recursive argument about the number of possible towers as a function of tower height, recognizing that the number doubled when the height was increased by one block. Their explanation of this suggests their reasoning is close to the classic permutation representation of a tree diagram, as they consider each existing tower with a height of $n-1$ blocks to branch into two possibilities for the nth block. This example shows the richness and variety in combinatorial techniques, even for very simple problems.

Literature on the practice of teaching combinatorics has emphasized the difficulties in understanding present in this area of mathematics. Borg (1998) suggests that cognitive conflict must occur in order to change the existing conceptions students already hold. However, Liu and Thompson (2007) looked at teacher understanding, and found uneven conceptual knowledge even within a single teacher. They also found that some teachers could not verbalize their solutions or think of a strategy to teach the topic; this ties again to Karmiloff-Smith's (1992) ideas of implicit knowledge, which the holder cannot make explicit to an external other.

The reviewed studies also speak to the challenge in learning and explaining combinatorics. Other literature has addressed common errors in the field. Some, such as Watson (1995), have looked at specific mathematical errors, such as double counting of possible cases. However, far more have considered the difficulties caused by strongly held misconceptions, and the connection between probability and everyday language and activities. Barnes (1998) refers to a subjective probability that reflects an individual's belief regarding the likelihood of an event. The mathematical truth, when in conflict with this subjective probability, seems unreasonable to the person. One of the strongest cases for this relates to the independence of events. The classic example of independence of events is a coin toss where the coin has come up with "tails" ten times in a row. Mathematically, this has no effect on the 50% chance of getting "tails" on the next coin toss, but psychologically it seems to a human observer that it is due time to get "heads" instead. This is directly related to the work of Konold, Pollatsek, Well, Lohmeier, and Lipson (1993), who found that student answers about successive independent events were inconsistent, and often varied based on how the problem was posed. Many of these studies suggest that our everyday language that relates to probability plays a part in our conceptualizations as well; we may hold ideas that do not connect well to mathematical reality. To counter this, Jones et al. (1999) suggest that invented language in the mathematical setting may help to break some of the connections to the everyday world and overcome misconceptions.

Combinatorics often receive short shrift in educational treatment, and may be peripheral to the other mathematics taught within the same school year. There is also little connection between the literature on teacher education, as discussed in the previous section, and the literature on learning and teaching combinatorics. Nevertheless, the principles of pedagogical content knowledge and careful examination of the impact on the student should still guide investigation in this mathematical arena.

V. Conclusions

With the discussion above in mind, we can return to the question of ultimate interest: what are the relationships among a teacher's understanding in combinatorics, the teacher's pedagogical content knowledge in combinatorics, and their students' understanding in combinatorics? The answer is not known; however, the existing research and theory provides the structure to plan the type of work that could illuminate this issue.

The research summarized above shows some of the approaches that are common in assessing the impact of courses for pre-service or in-service teachers, such as pre- and post-tests or pre- and post-surveys, as well as approaches to assessing teacher knowledge and understanding. These measures do not currently tell us about the learning of students, since we do not know how the teacher-level outcomes relate to the student-level outcomes. However, those few studies which can look at student-level outcomes have shown exciting positive results for student performance that are associated with teacher measures that have some connection to pedagogical content knowledge (Carpenter et al., 1989; Cobb et al., 1991; Hill et al., 2005; Saxe et al., 2001). Further, by looking for correlations between teacher-level outcomes and student-level outcomes, there is the possibility of ultimately finding teacher measures that allow us to make reasonable deductions about the impact on students (Hill et al., 2005). This goal is immensely important to studies that may wish to measure the impact of particular courses for teachers, but that do not have the resources or access to assess students directly.

However, just the task of distinguishing pedagogical content knowledge from purely mathematical knowledge is immense (Ball, Thames, & Phelps, in press). Yet, we also know that we must outline the conceptual field of combinatorics if we are to ever reach the point of outlining the pedagogical content knowledge in this subfield of mathematics. I propose, then, that work in this area, toward this larger goal, could begin through collection and analysis of mathematical responses, as well as detailed explanations from both teachers and their students on the types of combinatorics problems described above.

This approach would serve three purposes. *First*, it would begin to show common solution strategies and errors, an important part of the conceptual field. Some of this information already exists, of course, as described in the literature above about students working in combinatorics. *Second*, the responses from teachers would begin to reveal factors that might be part of PCK in combinatorics. Since there is no existing yardstick against which the teachers' responses could be measured, it would not be possible to make assertions or comparisons about levels of PCK. However, it would be possible to make qualitative comparisons between responses to questions and to catalogue the teacher activities as possible elements of PCK. To this end, questions for teachers would include solving mathematical problems in combinatorics, but then be followed by interview questions that are more closely related to the practices of teachers.

These first two purposes suggest some of the types of information that should be sought in examination of teacher and student responses. In order to build a basis for both the conceptual field of combinatorics and understanding PCK in combinatorics, elements that are part of either or both should be included. Table 3 shows how potential aspects of teacher and student responses connect to the goals stated here.

Item for analysis	Connection to goals
Solution to the mathematics question	Teacher and student mathematical understanding
Explanation of solution	Teacher and student mathematical understanding; information about conceptual field
Type of solution strategy	Teacher and student mathematical understanding; information about conceptual field
Presence of multiple explanations	Teacher PCK; information about conceptual field
Use of multiple representations	Teacher PCK; information about conceptual field
Finding errors in work of others	Teacher PCK; information about conceptual field
Explaining why errors might have occurred	Teacher PCK; information about conceptual field
Evaluation of alternative solution strategies	Teacher PCK; information about conceptual field

Table 3. Suggested elements for analysis of teacher and student responses.

The *third* reason to take this initial approach of looking closely at a small number of teachers and their students is that it would provide a qualitative look at the understanding of teachers and their own students. Although this will not lead to implications as clear as those from studies that include student-level outcomes for a large number of teachers and classrooms, it would still let us consider the student in concert with the teacher, and let us examine their understandings as necessarily intertwined.

Note that while these recommendations cover the next steps in analyzing teacher pedagogical content knowledge, teacher understanding, and student understanding in combinatorics, this would be only the beginning of turning this type of analysis into something of value. The work suggested here would create a foundation for the larger consideration of how to use this knowledge to improve teaching and learning. This topic is still under discussion for mathematics education as a whole, and has by no means been resolved to give a final judgment on how best to prepare mathematics teachers, so that they may help their students to learn mathematics. However, while this sort of long term research and questioning is underway, we may benefit from simultaneously beginning rigorous examination of mathematical topics such as combinatorics.

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