c. Notice that Galileo's theory is what licenses a height to be a measure of speed -- a theory-mediated measure, using a proxy for speed (squared)!

(1) Same height of fall, same speed -- a prerequisite

(2) Regardless of path, regardless of weight, regardless of shape of the movable: all those Galilean principles, and not just the principle of free fall, needed to license the use of height as a proxy for speed squared; so more thoroughly theory-mediated than one might notice

(3) Further claim: invariant over the surface of earth

3. Given some specified height as a unit of measure of speed and impetus, then measure of speed and impetus generally will be specified in terms of the square root (i.e. the mean proportional) of distance an object would have to fall to gain impetus in question

a. I.e. Given AB as basic measure of impetus, and want to know impetus acquired in fall through AC; measure is the length that is the mean proportional between AB and AC

b. Because, of course, speed = $\sqrt{(2*g*h)}$, so that ratio of speeds is as sqrt of ratios of lengths: AD = $\sqrt{(AB*AC)}$, so that AD/AC = $\sqrt{(AB/AC)}$

c. (Mean proportional determinable by ruler and compass; see Euclid, VI, 13)

4. Using vertical distance in this way provides a conveniently commensurate measure for comparing horizontal and vertical impetus and speed even when the vertical acceleration g -- the fall in one second -- is unknown

a. Horizontal distance traversed in time required for free fall from specified height = twice that height

b. Thus earlier theorem provides way of expressing uniform horizontal speed and impetus in terms of speed and impetus gained in free fall without as such requiring a specific value of g!

5. Note that Galileo might have tried to use impact measurements for impetus, as suggested by the discussion on [291-293]

a. Fortunately he did not, but even if he had, the rules of his mathematics would have forced him into some form of extensive representation of the measure, some form of proxy

b. This way not only gives him the latter, but also avoids problems of measuring percussive force, a parameter of great interest in military ballistics (see "Added Day")

E. Galileo's Determination of the Parabola

1. Suppose now we are given a projectile with uniform horizontal motion of a specified speed and impetus

a. Let the vertical distance p be that required for projectile to gain this speed and impetus in free fall -- i.e. the sublimity

b. But then the parabola is the one obtained by allowing the projectile to fall a further distance p while proceeding horizontally a distance 2p

c. This uniquely determines the parabolic trajectory in question

2. In effect the sublimity locates the directrix and focus of the parabola (see Galileo's figure for Proposition IV, or the Appendix)
   a. Directrix is horizontal through A, and focus is at D, where AD = 2p
   b. Point of tangency the point where horizontal impetus exactly equals the vertical impetus -- i.e. slope = 45 deg
3. Thus the parabola in question has three distinct parameters -- the sublimity, the altitude, and the amplitude, any two of which are sufficient to define the third and the parabola
   a. Corollary to Proposition V: $1/2 * amp = \sqrt{(p * alt)}$
   b. Can therefore infer p from amplitude and altitude, etc.
   c. Though recognize that Galileo does not do this algebraically, but under the tighter constraint that the length representing each be constructible via compass and ruler from the other two
4. Proposition IV then gives a geometric construction to determine compounded speed and impetus everywhere along the trajectory
   a. Algebraic equivalent: (*speed*/*initial speed*) = $\sqrt{(1+y/p)}$
   b. Therefore, (*impetus at impact*/*initial impetus*) proportional to $\sqrt{(1+h/p)}$ and thus greater than purely horizontal impetus
   c. Impact angle = $\arctan(\sqrt{(h/p)}) = \arctan(amp/(2p))$
5. Thus a totally unique determination of the projectile trajectory, given any two of the three quantities
   a. But this is just as it should be, for since g fixed, the parabolic trajectory depends only on the initial horizontal speed
   b. And p has the virtue of combining this initial horizontal speed and the acceleration g into the single parameter p needed to define the parabola (see Appendix)
   c. Only other thing special here is that corresponding parabola geometrically constructible, using compass and ruler, given any two of the quantities, represented of course as lengths

F. The Significance of Galileo's Approach

1. Galileo is thus able to define not only the path, but the speed and the direction everywhere along the path -- i.e. the trajectory, in the same sense as Kepler -- without having to resort to calculus
   a. Indeed, without having to resort to algebra (or analytic geometry, which was just beginning to take shape at the time)
   b. Constructive Euclidean geometry throughout: every quantity determinable via compass and ruler
2. More important, he manages to bypass both a theory of forces and a specific value of the acceleration of gravity in his choice of parameters determining the parabola
   a. Galileo's "sublimity" the perfect device for doing this since it absorbs the acceleration of gravity within it, in effect normalizing everything with respect to a unit of free fall

b. Galileo thus provides a purely "kinematic" account of projectile motion, in contrast to a "dynamic" account (to use phrasing that emerged in the nineteenth century)

c. Anyone inclined to sell Galileo short should give some thought to the remarkable ingenuity (and depth) of this solution!

3. Still more important from a methodological standpoint, he shows a thorough understanding of the constraints that have to be satisfied in coming up with measures of speed and impetus

   a. We of course define speed in terms of distance divided by time, and hence we can bypass the problem he faced in this case

   b. But we cannot bypass it generally, and his treatment of it is a model that others could follow

4. An extraordinarily impressive achievement at the time, though well within the reach of others, like Kepler and Descartes

   a. Tartaglia had examined projectile motion empirically roughly 100 years before, but had not begun to reduce it to a mathematical account

   b. And even Kepler's account is not strictly within the confines of Euclidean geometry – area rule, in particular, is non-Euclidean

5. By contrast Galileo was never able to construct such a theory of circular pendular motion

   a. That is, he was unable to construct a kinematic account of pendular motion using uniform acceleration, for not uniformly accelerated at all, nor finitely compounded from uniformly accelerated motions: different conceptual obstacles

   b. In one respect, then, not really the same category of phenomenon, and hence not so inappropriate that Galileo had trouble

IV. "The Fourth Day": The Theory of Projectile Motion

A. The Basic Claims Made by Galileo's Theory

1. So far have been considering only the manner in which Galileo conceptualized the problem of projectile motion; time now to turn to the theory itself and the empirical evidence for it

2. Basic claim: the semi-parabola is uniquely determined, given the sublimity p and the height h where the compound motion begins

   a. The sublimity in effect compounds the initial horizontal speed and the uniform (universal) acceleration of gravity, g, into a single parameter that suffices to determine the shape of the semi-parabola

   b. And the height then determines where the semi-parabola ends in impact with the ground

   c. As Appendix indicates, h and p are then sufficient to determine all other quantities (geometrically as well as algebraically)

3. Further claim: by a symmetry argument, this semi-parabola can be extended into the full parabolic trajectory that occurs when a projectile is launched from the ground with an initial speed at some angle of projection θ

a. I.e. determine a full vertical parabola given angle of projection θ and the initial speed, stated in terms of a sublimity
b. If not stated in terms of a sublimity, then require specific value of vertical acceleration g to determine the parabola
c. $\mathrm{Tan}(\theta) = 2*alt/amp = 4*alt/range = range/(4p)$
d. {The mathematics required for oblique projection may have given Galileo trouble, necessitating his appeal to symmetry here without proof; Toricelli supplied the missing proof in the early 1640s, published in his *De Motu Gravium Naturaliter Descendentium et Projectorem*}

4. Galileo has two ways of specifying the full parabola without having to know the acceleration g:
   a. Given θ and *p*, with the latter representing the initial horizontal speed and impetus
   b. Or given range and θ, can infer *p* from above
   c. (All in the absence of resistance effects)
5. The analogy with Kepler is now complete: for a repeatable initial impetus, can measure angle of launch and range, from which the complete trajectory can be determined
   a. Can even recover time if measure time until projectile lands -- the correlate of the Period -- so that the complete trajectory in time is determined for a given initial angle and unknown, but uniformly repeatable impetus -- e.g. the impetus from a given amount of powder in a cannon
   b. Indeed, if measure time as well as range and initial angle, can infer acceleration of gravity g -- i.e. fall in first second -- (from *h*) and the unknown initial speed of projection (from *p* or from *a*)
   c. Thus a way of measuring fall in first second, if resistance effects are negligible

B. Mathematical Consequences of the Theory
1. The mathematical development of the theory of projectile motion, although somewhat more limited in terms of the number of results, is even more impressive in some of the entailed claims
2. Proposition VII and Corollary: for a given initial impetus, the maximum range is achieved when θ = 45 deg
   a. Something that had been observed in practice, and had been noted by Tartaglia
   b. As Galileo remarks, now being explained (and shown to be nomological)
   c. And explanation shows that it has nothing as such to do with air resistance effects, but instead comes from uniform horizontal and uniformly accelerated vertical compounded
   d. Notice how much evidential weight Galileo attaches to the result in these remarks
3. Proposition VIII: for a given initial impetus, the range varies symmetrically as θ varies on either side of 45 deg
   a. E.g. get the same range for θ=30 and θ=60 deg
   b. Empirical claim that Galileo says had not yet been observed in practice
   c. A potential confirming experiment, if can achieve repeatable initial impetuses