Torsion problem - 2

Torsion of Non-Circular bars

Solution for circle does not work! (B.C. violated)

Try:

To the displacements for circle

$$\left.\begin{aligned} u_x &= -\alpha y z \\ u_y &= \alpha x z \end{aligned}\right\}$$

Add: warping displacement $u_z = u_z(x, y)$

Strains:

$$\left\{\begin{aligned} \epsilon_{xz} &= \frac{1}{2}\left(-\alpha y + \frac{\partial u_z}{\partial x}\right) \\ \epsilon_{yz} &= \frac{1}{2}\left(\alpha x + \frac{\partial u_z}{\partial y}\right) \end{aligned}\right.$$

other $\epsilon_{ij} = 0$

Stresses:

$$\left\{\begin{aligned} \sigma_{xz} &= G\left(-\alpha y + \frac{\partial u_z}{\partial x}\right) \\ \sigma_{yz} &= G\left(\alpha x + \frac{\partial u_z}{\partial y}\right) \end{aligned}\right.$$

other $\sigma_{ij} = 0$

Check :

Equations : (1) equilibrium : the third eq-n of eq.

$$\sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} = \nabla^2 u_z = 0}$$ - warping u_z is a harmonic function

(2) compatibility : don't have to worry :
strains derived from displacements

(3) Hooke's law : has been incorporated

B. C. (1) lateral surface should be traction-free

$$\underset{\sim}{t}^{(n)} = \underset{\sim}{n} \cdot \underset{\sim}{\sigma} = (n_x \sigma_{xz} + n_y \sigma_{yz}) \underset{\sim}{e}_z =$$ substituting stresses

$$= G \underbrace{\left(n_x \frac{\partial u_z}{\partial x} + n_y \frac{\partial u_z}{\partial y} \right)}_{\underset{\sim}{n} \cdot \nabla u_z} + G\alpha \underbrace{(x n_y - y n_x)}_{\text{would be 0 for circle}} \overset{?}{=} 0$$

rate of change in n-direction (of u_z)

$$\boxed{\underset{\sim}{n} \cdot \nabla u_z = -\alpha (x n_y - y n_x)}$$ on boundary

Thus, the following problem emerges for the warping displacement :

$$\begin{cases} \nabla^2 u_z = 0 & \text{in } F \\ \underset{\sim}{n} \cdot \nabla u_z = -\alpha (x n_y - y n_x) & \text{on the boundary} \end{cases}$$

Difficult problem.

Try: Formulation in Stresses

From

$$\begin{cases} \sigma_{xz} = -G\alpha y + G \dfrac{\partial u_z}{\partial x} \\ \sigma_{yz} = G\alpha x + G \dfrac{\partial u_z}{\partial y} \end{cases}$$

eliminate u_z by differentiating (producing $\partial^2 u_z / \partial x \partial y$):

$$\left.\begin{aligned} -\frac{\partial \sigma_{zx}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial x} &= 2G\alpha \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} &= 0 \end{aligned}\right\}$$

plus equilibrium

two eq-ns for two stresses

Approach: satisfy equilibrium first

Then focus on the remaining eq-n

Eq-m satisfied by introducing stress f-n $\Psi(x,y)$ such that

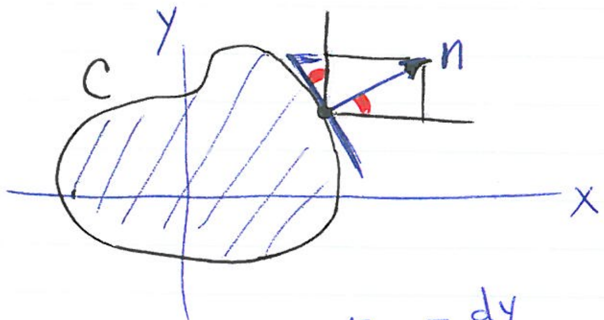
$$\sigma_{xz} = \frac{\partial \psi}{\partial y}, \quad \sigma_{yz} = -\frac{\partial \psi}{\partial x}$$

The remaining eq-n for Ψ

$$\underbrace{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}}_{\nabla^2 \Psi} = -2G\alpha$$

Boundary conditions - in terms of ψ .

On contour (lateral side) $\sigma \cdot n = 0$



$$\sigma \cdot n = \left(n_x \underbrace{\sigma_{xz}}_{\frac{\partial \psi}{\partial y}} + n_y \underbrace{\sigma_{yz}}_{-\frac{\partial \psi}{\partial x}} \right) \underline{e}_z \overset{?}{=} 0$$

$$n_x = \frac{dy}{ds}$$

$$n_y = -\frac{dx}{ds}$$

$$\Rightarrow \frac{\partial \psi}{\partial y} \frac{dy}{ds} + \frac{\partial \psi}{\partial x} \frac{dx}{ds} = \frac{d\psi}{ds} \overset{?}{=} 0$$

$$\Downarrow$$

$$\boxed{\psi = \text{const} \quad \text{on } C}$$

Since ψ is introduced through its derivatives (stresses)

it is defined to within const. $\Rightarrow$ set const $= 0$

$$\Rightarrow \boxed{\psi = 0 \quad \text{on } C}$$

- ensures that lateral surface traction-free

B.C. on bases

1. $$\int_F (\sigma_{yz}\, x - \sigma_{xz}\, y)\, dF = M_{applied}$$

expect this b.c. to yield
α vs M
(stiffness relation)

2. $$\int_F \sigma_{xz}\, dF = \int_F \sigma_{yz}\, dF = 0$$ (princ. vector $= 0$)

In terms of ψ:
$$\sigma_{xz} = \partial\psi/\partial y$$
$$\sigma_{yz} = -\partial\psi/\partial x$$

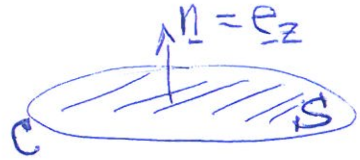
Want to utilize $\psi\big|_{boundary} = 0$

Difficulty:

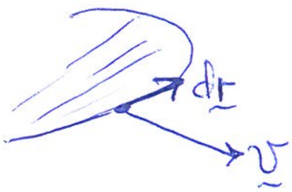
integrals are over F, not boundary C

$\rightarrow$ transform $\int_F \rightarrow \int_C$ via Stokes' theorem

Stokes theorem (in case of flat surface)



For any vector field $\underset{\sim}{v} = \underset{\sim}{v}(x,y)$



$$\underbrace{\int_C \underset{\sim}{v} \cdot d\underset{\sim}{r}}_{\text{circulation of } \underset{\sim}{v} \text{ around } C} = \iint_S \underbrace{\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)}_{\substack{\text{z-comp. of curl } \underset{\sim}{v} \\ \text{flux of curl across } S}} dS$$

$$\int_C v_x\, dx + v_y\, dy$$

(- interpretion in fluid mech
$\underset{\sim}{v}$ - velocities of fluid particles)

In particular, choosing $\begin{cases} v_y = 0 \\ v_x = \text{some } f(x,y) \end{cases}$

obtain

$$\boxed{\int_S \frac{\partial f}{\partial y}\, dS = -\int_C f\, dx}$$

– for any function $f(x,y)$

Similarly, choosing

$$\begin{cases} v_x = 0 \\ v_y = f(x,y) \end{cases}$$

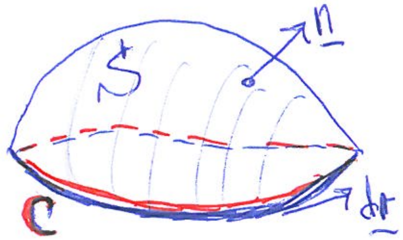
$$\boxed{\int_S \frac{\partial f}{\partial x}\, dS = \int_C f\, dy}$$

These relations transform integrals over area into integrals over its edge line

Note: more general Stokes theorem

(fluid mech., heat transfer)

– for non-flat surface S, with edge C:



$$\int_C \underset{\sim}{v} \cdot d\underset{\sim}{r} = \int_S \underset{\sim}{n} \cdot \operatorname{curl} \underset{\sim}{v} \, dS$$

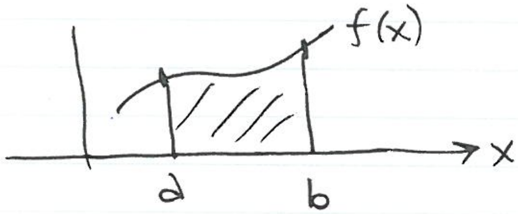
circulation of vector field $\underset{\sim}{v}$ along C

flux of curl $\underset{\sim}{v}$ across S

Comment on Stokes theorem:

belongs to the class of statements that relate:

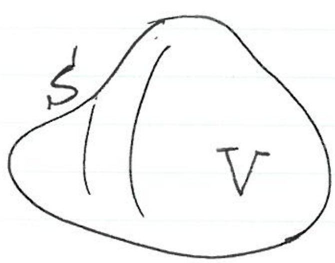
- Integral over interval $\longleftrightarrow$ boundary values:



$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- Integral over volume $\longleftrightarrow$ integral over boundary (surface)

(Div. theorem)



$$\int_S \underset{\sim}{v} \cdot \underset{\sim}{n}\, dS = \int_V \operatorname{div} \underset{\sim}{v}\, dV$$

Back to B.C. (at base of the cylinder)

(A) Princ. vector of tractions $\overset{?}{=} 0$:

$$\int_F \sigma_{xz}\, dF = \int_F \frac{\partial \psi}{\partial y}\, dF \underset{\text{Stokes th.}}{=} -\int_C \underbrace{\psi}_{=0 \text{ on } C}\, dx = 0$$

$$\int_F \sigma_{yz}\, dF = -\int_F \frac{\partial \psi}{\partial x}\, dF = -\int_C \underbrace{\psi}_{=0 \text{ on } C}\, dy = 0$$

(B) Distribution of tractions amounts to appl. mom. M :

$$\int_F \underbrace{(x\,\sigma_{yz} - y\,\sigma_{xz})}_{= -x\frac{\partial \psi}{\partial x} - y\frac{\partial \psi}{\partial y} \overset{\text{identity}}{=} -\frac{\partial (x\psi)}{\partial x} - \frac{\partial (y\psi)}{\partial y} + 2\psi}\, dF \overset{\text{must}}{=} M$$

Stokes' theorem $\Rightarrow$

$$\int_F = -\int_C (x\underbrace{\psi}_{0}\, dy + y\underbrace{\psi}_{0}\, dx) + 2\int_F \psi\, dF$$

Thus :

$$\boxed{2\int_F \psi\, dF = M}$$

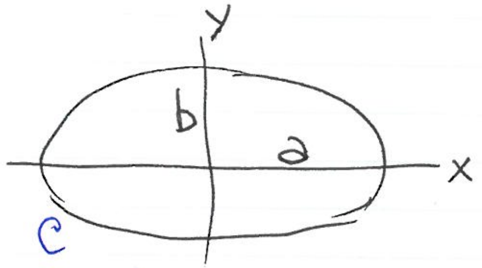
$\Rightarrow$ · in terms of stress f-n Ψ, the torsion problem is:

$$\begin{cases} \left.\begin{aligned} \nabla^2 \psi &= -2G\alpha \quad \text{in } F \\ \psi &= 0 \quad \text{on } C \end{aligned}\right] \Rightarrow \text{find } \Psi \text{ (dependent on } \alpha) \\ \left. 2\int_F \psi \, dF = M \right] \Rightarrow \alpha \text{ vs } M \text{ relation (torsional stiffness)} \end{cases}$$

Procedure :

- guess ψ that $=0$ on C retaining some flexibility (constants)
- adjust constants to satisfy $\nabla^2 \psi = -2G\alpha$
- determine α vs M from 3rd eq-n

Elliptical bar



covers $\left\{\begin{matrix}\text{thin plates} \\ \text{circles}\end{matrix}\right\}$ as special cases

try: $\Psi = A\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$

A ← some constant

reasons:
- $\psi = 0$ on C
- $\nabla^2\psi = \text{const} = -A\left(\frac{2}{a^2} + \frac{2}{b^2}\right)$

hence $\nabla^2\psi = -2G\alpha$ can be satisfied by choice of A

$$\Downarrow$$

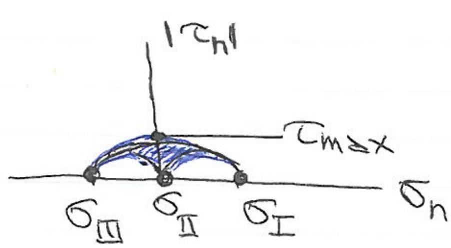
$$A = \frac{a^2 b^2}{a^2+b^2} G\alpha$$

Stresses derived from Ψ:

$$\begin{cases} \sigma_{zx} = \dfrac{\partial\psi}{\partial y} = -2G\alpha \dfrac{a^2}{a^2+b^2} y \\[2ex] \sigma_{yz} = -\dfrac{\partial\psi}{\partial x} = 2G\alpha \dfrac{b^2}{a^2+b^2} x \end{cases}$$

Princ. stresses, max shear stress? Eigenvalue problem

$$\text{Det}\begin{vmatrix} 0-\lambda & 0 & \sigma_{xz} \\ 0 & 0-\lambda & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & 0-\lambda \end{vmatrix} = 0 \Rightarrow \begin{cases} \sigma_{I} = \sqrt{\sigma_{xz}^2 + \sigma_{yz}^2} \\ \sigma_{II} = 0 \\ \sigma_{III} = -\sqrt{\quad} \end{cases}$$



Torsional stiffness :

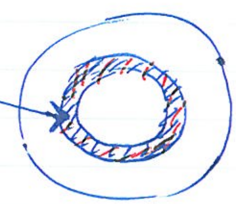
$$M = 2\int_S \Psi\, dS = 2A\int_S \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \underbrace{dS}_{dx\,dy}$$

$$= 2A \cdot ab \cdot \int_S \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \frac{dx}{a}\,\frac{dy}{b}$$

integration over unit circle since $\frac{x}{a}$, $\frac{y}{b}$ vary in $(-1, 1)$

in polar coord:

$$= 2\underbrace{A}_{\frac{a^2b^2}{a^2+b^2}G\alpha}\, ab \int_0^1 (1-\rho^2)\, \underbrace{dS}_{2\pi\rho\, d\rho}$$



$$= \underbrace{\pi \frac{a^3 b^3}{a^2+b^2} G}_{\text{torsional stiffness}} \cdot \alpha$$

area $= \pi ab$

$$= \frac{1}{2\pi} G\, (\text{area})^2\, \frac{2ab}{a^2+b^2}$$

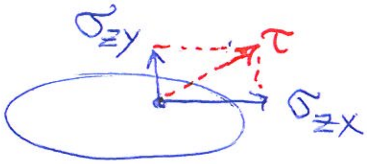
shape factor $(0 \to 1)$

Shape factor is max if $a=b$ (circle)

(elongated shapes are softer)
at the same area

Local stresses: what's the max. shear stress? And where will it occur?

↑ relevant for metals



$$\tau^2 = \sigma_{xz}^2 + \sigma_{yz}^2 =$$

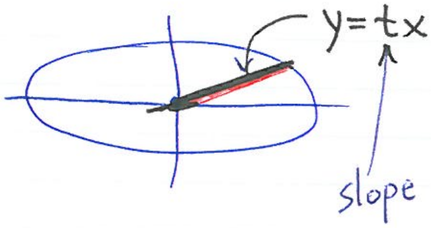
$$= \frac{4M^2}{(\pi ab)^2}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right) \qquad (*)$$

Its max. value?

(relevant for onset of plasticity in metals)

formal approach: f-n of two variables, find max

Less formally: look along the straight line



$$\tau^2 = \frac{4M^2}{(\pi ab)^2}\left(\frac{x^2 t^2}{b^4} + \frac{x^2}{a^4}\right) \text{ – increasing f-n of } x$$

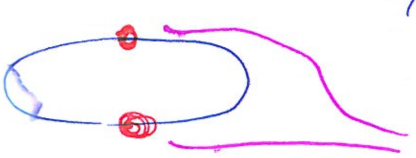
$\Rightarrow$ max. is on the boundary

Along the boundary: substitute $y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$ into τ^2 given by (*)

$$\tau^2 = \frac{4M^2}{(\pi ab)^2}\underbrace{\left[\frac{x^2}{a^4} + \frac{b^2}{b^4} - \frac{x^2}{a^2 b^2}\right]}_{\frac{1}{b^2} + x^2 \frac{1}{a^2}\underbrace{\left(\frac{1}{a^2} - \frac{1}{b^2}\right)}_{<0}}$$

$\Rightarrow$ max is at $x = 0$

$\Rightarrow y = \pm b$



At these points: $\tau_{max} = \frac{2M}{\pi a b^2}$

(plastic deform. will start there)

Try to extend results to other shapes,

(in an approximate way)

Torsional stiffness of ellipse

$$C = \frac{M}{\alpha} = G\pi \cdot \frac{a^3 b^3}{a^2+b^2}$$

- in terms of area F and mom. of inertia I:

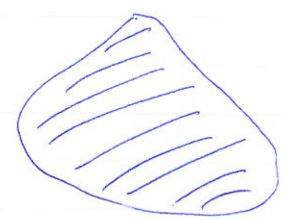
$F = \pi a b$, $\quad I = \frac{\pi}{4} ab(a^2+b^2)$

$$\boxed{C = \frac{1}{4\pi^2} G \frac{F^4}{I}}$$ — does not refer to ellipse geometr

Hypothesis: can be applied, as approximation,

to other shapes, in terms of their F, I

Computations show: reasonably accurate for convex profiles



worse for concave: