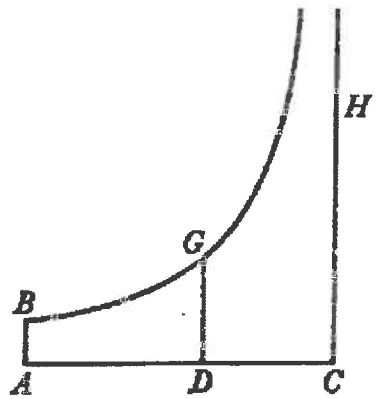
Problem 6. Horizontal Motion with Resistance $\propto v$



In the figure, represent

time **by the increasing area under the rectangular hyperbola BG**

distance **by the increasing length AD**

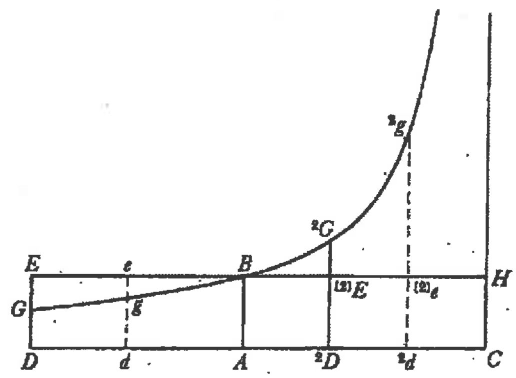
velocity **(and *resistance*) by the decreasing length DC**

i.e.

$$x = \left(\frac{u_0}{k}\right)(1 - e^{-kt})$$

insofar as the *velocity* and hence the *resistance* decrease in a *geometrical progression* as the *time* increases in an *arithmetical progression*

Problem 7. Vertical Motion with Resistance $\propto v$



<u>In the figure</u>, in ascent represent

***centripetal force* by the area of the rectangle ABHC**

***resistance at the start of ascent* by the area ABED taken in the opposite way**

***time* by the increasing area DGgd**

***distance* by the increasing area EGge**

***velocity* (and *resistance*) by the decreasing area ABed**

<u>In the figure</u>, in descent represent

***time* by the increasing area AB^2G^2D**

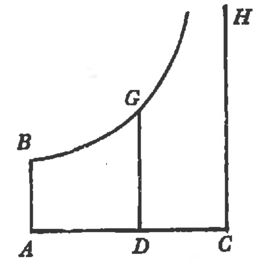
***distance* by the increasing area B^2E^2G**

***velocity* (and *resistance*) by the increasing area AB^2E^2D**

***terminal velocity* by the area BACH**

In the figure, $AC = u/\lambda$, which 'expresses' (is proportional to) both the initial resistance $u\lambda$ and the velocity u: and $DC = a - x = \dot{x}/\lambda$ expresses the velocity at time t. From (i) the time is given by

$$t = \frac{1}{\lambda}\log\frac{u}{u-\lambda x} = \frac{1}{\lambda}\log\frac{a}{a-x}$$



which is the area $ADGB$ of the hyperbola, if $\lambda = 1/ab$.

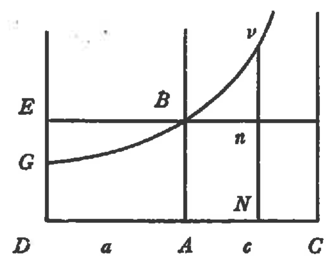
The figure is therefore a graph of the *reciprocal* of the velocity co-ordinated with the space described ($y = ab/\dot{x}$, $x =$ space described), so that the area $\int y\,dx$ of the graph expresses the time t.

(4) This statement is the heart of the argument: the increment dt of the time is constant, so that $-d\dot{x}$, the decrement of the velocity, satisfies

$$-d\dot{x} = \lambda\dot{x}dt,$$

by the Second Law of motion, the operating force being the resistance $\lambda\dot{x}$.

(5) This is known in either of two ways: (i) from Napier's original definition of a logarithm. If D moves from A to C, with a velocity always proportional to DC, and simultaneously if d moves from a in another straight line, but with a constant velocity, then ad is proportional to log AD. Also (ii) from Grégoire de St Vincent's geometrical discovery (1647), which, in effect, replaces the second motion, that of d, by the hyperbolic graph. In fact D moves 'geometrically', and d 'arithmetically' (in Napier's phraseology) and the area $ADGB$ increases 'arithmetically'. Cf. Napier, *Mirifici Logarithmorum Canonis descriptio* (Edinburgh, 1614), or English Translation (1616), p. 2; and Grégoire de Saint Vincent, *Geometricum Quadraturæ Circuli et Sectionum Coni* (Antwerp, 1647).

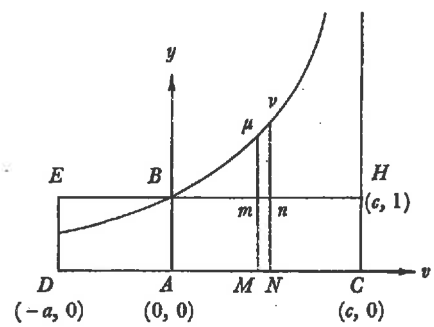


(6) The graph $B\mu\nu$ is the hyperbola $y(c-v) = c$, with A as origin, $AM = v$, $M\mu = y$, $MN = dv$, $AC = c$, and $AB = 1$. Newton invokes the unusual combination of plotting y, proportional to the reciprocal of the acceleration, against v, the velocity: for then at a time t, proportional to the area $AB\mu M$, the space s traversed by the projectile is proportional to the area $B\mu m$. This follows from the equation of motion (reckoned downward)

$$dv/dt = g - \lambda v, \qquad (1)$$

where g is the 'ever-equal centripetal force'—gravity, and λv is the resistance ($\lambda =$ constant). This integrates as

$$v + \lambda s = gt, \qquad (2)$$



where s, t, v are the space traversed, the time, and the velocity, all reckoned from the highest point of flight at which v, s, t vanish. If $c = g/\lambda$ and $dt/dv = y/g$, then (1) becomes the equation $y(c-v) = c$ of the hyperbola. But $g\,dt = y\,dv$; so that $\int y\,dv = gt = v + \lambda s$ by (2): that is the area $AB\mu M = v + \lambda s$. Hence this whole area is gt and the parts, BM and $B\mu m$, are v and λs.

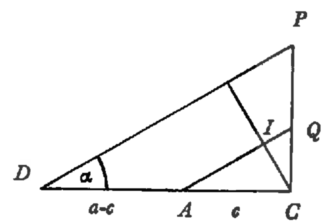
Since $MC = c - v = (g - \lambda v)/\lambda$, MC is proportional to the 'absolute force', the resultant of gravity and resistance at the time t. If $DA = a$, then a is the initial velocity of upward ascent (at a time when t is negative). In Fig. 2, p. 457, Newton marked F as E (ULC. Add. 3965 (7), fo. 62).

(7) The curve $DarFK$ is the trajectory of a body moving, under constant acceleration g vertically downwards, against a resisting force λv directly opposed to the motion. Take D as origin, $DR = x$, $Rr = y$, u = initial velocity of projection at an elevation α, the angle ADP. Then

$$\ddot{x} = -\lambda \dot{x}, \quad \ddot{y} = -g - \lambda \dot{y}, \quad \text{so that} \quad \dot{x} = -\lambda x + u \cos\alpha \quad \text{and} \quad \dot{y} = -gt - \lambda y + u \sin\alpha. \qquad \text{(i)}$$

Let $DC = a$, $AC = c$, $CH = 1$, so that $DP = a \sec\alpha$, which 'represents' the initial velocity (in direction and magnitude): say

$$u = \lambda a \sec\alpha. \qquad \text{(ii)}$$



Newton defines CI by $DA:CI::\lambda u:g$, giving $CI = (a-c)g/u\lambda$. (There is an error in placing I on DP in the figure: it should be on a line through A parallel to DP, so that $CI = c \sin\alpha$.) Hence

$$(a-c)g/u\lambda = c \sin\alpha. \qquad \text{(iii)}$$

He takes the hyperbola

$$y(a-x) = c \qquad \text{(iv)}$$

which passes through the point $B(a-c, 1)$. Then $EG = 1 - c/a$. He then takes

$$N = EG.DC/CP = (1 - c/a) \cot\alpha,$$

on using (ii) and (iii).

His equation for the trajectory is

$$y = \frac{1}{N}\left(x - c \log \frac{a}{a-x}\right) \qquad \text{(v)}$$

and for the time is

$$t = c \log\{a/(a-x)\}; \qquad \text{(vi)}$$

so that $Ny = x - t$. This requires that $c = 1/\lambda$ and $N = \lambda/g$: for then the equations (v) and (vi) agree with the equations (i). Also (iii) is verified so that $CI = c \sin\alpha$ (see second figure).

The maximum of y (when $x = DA = a - c$) follows from $\dot{y} = 0$, so that $\dot{x} = 1$: but $\dot{x} = \lambda(a-x)$ by (i) and (ii). Hence, at A, $x = a - 1/\lambda = a - c$. At F, $y = 0$ and $x = t$, so that the areas $DFsE$, $DFSBG$ are equal. (vii)

The velocity at the point r is represented in magnitude and position by the vector rL, that is $v = \lambda.rL$, which is true since $\dot{x}$, its horizontal component, is $\lambda(a-x) = \lambda.RC$. (viii)

Finally, if $DF = x'$, the gradient of the curve (v) at F is, by differentiation,

$$\frac{1}{N}\left(1 - \frac{c}{a-x'}\right) = -\frac{sS}{N}, \quad \text{since } Fs = 1 \text{ and } FS = \frac{c}{a-x'}$$

by (iv), and at D(where $x = 0$) the gradient is

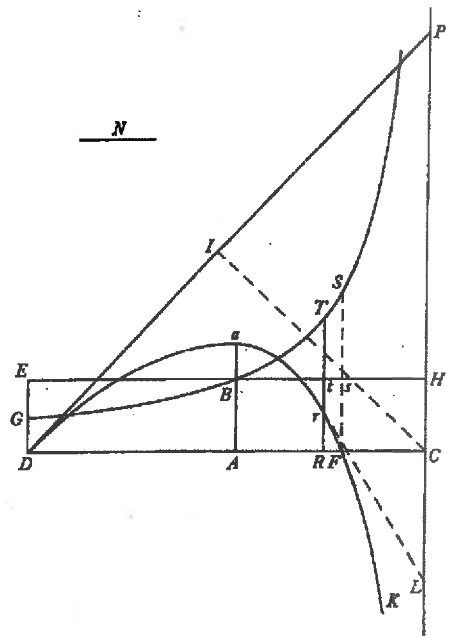
$$\frac{1}{N}\left(1 - \frac{c}{a}\right) = \frac{EG}{N};$$

which verifies Newton's statement concerning the ratio of sS to EG. (ix)

(8) See note (7) equation (ii).

(9) See (iii).

Solution for Projectile Motion with Resistance ∝ *v*



<u>In the figure</u>, represent

The initial velocity by DP, and the ratio of the initial resistance to gravity by DA/CI.

Erect the perpendicular AB of any length, complete the rectangle DABE, and construct a rectangular hyperbola through B with asymptotes DC and CP.

Define N such that N/EG = DC/CP.

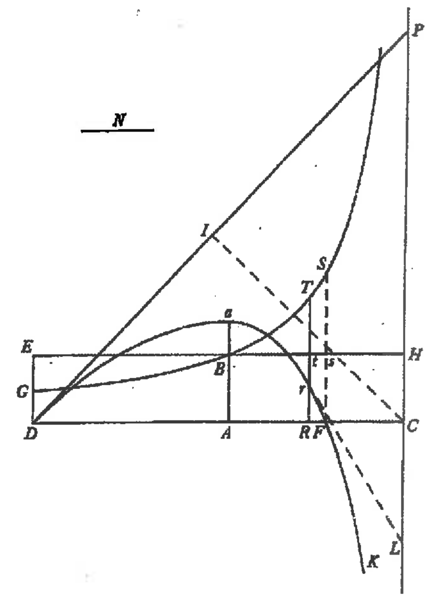
Then at any point R on the horizontal DC, r is given by

Rr = (area-DRtE – area-DRTBG)/N

The projectile will reach point r along the trajectory DarFK at *time* DRTBG, reaching the horizontal at F where the two areas become equal.

The *velocity* at every point is given by the tangent to the curve extended to CP, e.g. rL

To Determine Resistance for a Given Sphere and *v*



Measure the angles ADP and the angle AF*r* at impact

On DF form the rectangle DFsE of any height.

Construct a rectangular hyperbola with asymptotes DC and CP such that DfsE = DFSBG, sS/EG = tan(AF*r*) / tan(ADP)

From point B where the hyperbola crosses the line Es, draw a line perpendicular to DC intersecting it at A.

Then DA/CI gives the ratio of the initial resistance to gravity.

<u>To extend this result to other spheres</u>, let

The *deceleration from resistance* vary directly as the *density* of the fluid, the *surface area* of the sphere, and its *velocity*, and inversely as the *weight* of the sphere